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ABSTRACT

In this paper, three special pursuit games on cyclic graphs are solved. These games are relevant to unsolved problems initiated by a game without a value presented at the International Workshop on Game Theory held at Cornell University in 1978.

We propose thereto the following problems :

(i) For a finite or infinite game without a saddle point, how should the players make their decisions? Here, we suppose that the game is played once. Thus mixed strategies are not considered.

(ii) For an infinite game without a value v, i.e., $v_1 < v_2$, how should the players make their decisions?

To answer these questions, new concepts of "quasi-equilibrium" and "pseudo-equilibrium" are defined and it is shown that a game MDCPG has a quasi-equilibrium pair.

KEY WORDS

Pursuit games

Geometric games

Cyclic graphs

Quasi-equilibrium

Pseudo-equilibrium

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INTRODUCTION

The value of game theory applied to problems of pursuit, search and ambush has been recognized since the Second World War. Many of the games which arise in such situations are of the type called "geometric games" by W.H.Ruckle [R(W)1]. A geometric game is an example of a two-person zero-sum game (see section 2.0). Namely, it is a game having the following form: Two antagonists, known hereafter as Red and Blue, choose subsets r and b respectively of a set S. Blue then receives from Red a payoff which is a function of the triple r,b and $r \cap b$. In general, Red and Blue may not choose any subset of S, but rather Red must select from a collection R of admissible subsets (pure strategies) for Red and Blue from a collection B. Usually, the collections R, B and the payoff are determined within a geometric structure on S. The term "geometric" is used in a wide sense and includes finite structures on S. A geometric game is described by specifying S, R,B and the payoff.

Geometric games may be divided into three categories: (a) finite games, for which the set S is finite, (b) continuous games, for which the set S is a line segment, circle or region in Euclidean space of two or more dimensions and (c) the case in which S is a countable set, such as the countable product of finite graphs. A pursuit game over a discrete graph belongs to category (c).

In section 1 of this paper, games and actions will be described either in subset terms or in terms of functions.

Suppose G is a graph, N a set of positive integers. Let B as well as R consist of functions ρ from N into nodes of G such that $\rho(i+1)$ is connected to $\rho(i)$ by an edge of G (i.e., is "adjacent") in the nondirected graph case and at the end of a directed edge leading from $\rho(i)$ in the directed case. If Red chooses the pure strategy ρ in R and Blue chooses the pure strategy β in B, the payoff to Blue is the smallest integer m such that $\rho(m)=\beta(m)$, or ∞ if $\rho(k)\neq\beta(k)$, for all k in N. The study of these games leads to mathematical difficulties which have not yet been mastered. [R(W)2] gave solutions to some such problems.

(i) PURSUIT ON A COMPLETE GRAPH (PCG)

A graph of n points is called complete if every pair of vertices in the graph G are adjacent (connected by an arc). The value of the PCG to Blue for complete graphs is n. An optimal strategy for both Red and Blue is to choose the kth point with independent probability 1/n from among all n points of G.

(ii) PURSUIT WITH COMPARTMENTS (PC)

Consider a pursuit game on a graph G in which Red, the pursuer, and Blue, the pursued, have different capabilities. As far as Red is concerned, the graph is complete. On the other hand, Blue's freedom of motion is more restricted. From Blue's point of view, the Graph G is the union of m disjoint complete graphs.

$$G = G_1 \cup G_2 \cup \ldots \cup G_m$$

such that G_i contains k_i points. This means that at time 1 Blue may choose any point of G, but if he chooses his initial position in G_i he must thereafter remain in G_i. This game is called Pursuit with Compartments (PC).

When $k_i = k$ for each i in the PC, the value of the game is $(m+1)/2+(k-1)m$. An optimal strategy for Blue is to choose at time 1 any vertex in G with probability 1/n and thereafter choose with probability 1/k one of the k vertices in the subgraph G_i in which he finds himself. An optimal strategy for Red is the following: (a) at times gm+1, g=0,1,2,... visit any vertex with probability 1/n; (b) at times qm+r, q=0,1,2,...,where $1<r\leq m$ visit a vertex in a subgraph not visited from time qm+1 through time qm+(r-1) with probability 1/(n-(r-1)k).

The approximate value of the general PC is

$$\sum_{0}^{\infty} r_q,$$

where

$$r_q = \exp \frac{q}{\sum_{j=1}^{m} \frac{1}{\log(1-1/k_j)}}.$$

A rule of thumb strategy for Red is any pure strategy f for which

$c_q(i)$= number of integers $j \leq i$ for which $f(j)=q$.

It has the value

$$\frac{1/\log(1-1/k_i)}{\sum_{j=1}^{m} 1/\log(1-1/k_j)}$$

or

$$\frac{1/\log(1-1/k_i)}{\sum_{j=1}^{m} 1/\log(1-1/k_j)} + 1 .$$

A rule of thumb strategy for Blue is to initially hide at any vertex with probability $1/n$ and thereafter to move to any vertex in the subgraph G_i in which he finds himself with probability $1/k_i$.

(iii) PURSUIT ON A DIRECTED LINEAR GRAPH (PDL)

The value of the PDL (game) to Blue is n. An ε-optimal strategy for Blue (see section 1.0) is to choose the initial point x_1 with probability $\varepsilon/(n-1)$ and remain there or choose the final point x_n with probability $1-\varepsilon/(n-1)$ and (of necessity) remain there. An optimal strategy for Red is to proceed directly from x_1 to x_n.

(iv) PURSUIT ON A CYCLIC GRAPH WHEN THE PLAYERS ARE RESTRICTED TO SYMMETRIC AND STOCHASTIC STRATEGIES (MCPG)

Solution of the pursuit game on a cyclic graph with n vertices, CPG, was described as an unsolved problem at the International Workshop on Game Theory held at Cornell University in 1978. Dr. Ruckle found the solution of the CPG in the special case when Blue and Red both are limited to symmetric Markovian strategies (MCPG) in 1981 [R(W)1]. That is, at the beginning of the game Blue chooses a number p and Red chooses a number q between 0 and 1/2. At each turn, Blue moves left with probability p, right with probability p, and remains in place with probability 1-2p while Red exhibits the same behavior with q replacing p. Assume $n \geq 4$, since when n=1,2,3 the CPG coincides with the PCG which are solved.

Let q_0 be the number between 0 and 1/2 for which $f(0,q_0)=f(1/2, q_0)$, where the value of the MCPG is $f(0,q_0)$. An optimal strategy for Red, the pursuer, is the pure strategy q_0. An optimal strategy for Blue, the evader, is to choose p=0 with probability t_0 and 1/2 with probability $1-t_0$ where

$$t_0=\frac{-(df(1/2,q)/dq)_{q=q_0}}{(df(0,q)/dq)_{q=q_0}-(df(1/2,q)/dq)_{q=q_0}}.$$

If the Markovian strategies are not symmetric, the arguments in [R(W)1] no longer hold. The situation becomes much more complicated[CZ1]. Here,in section 1 we discuss the pursuit game on a discrete cyclic graph of n vertices [R(W)2] when Blue and Red are restricted to stochastic strategies.

Also we present here in section 1 the solution of CPG when its payoff function is modified, i.e., on replacing the payoff to Blue

$$(r,b)=\begin{cases} \min i: r(i)=b(i), & i=1,2,3,\ldots \\ \infty, & r(i)\neq b(i),\ i=1,2,3,\ldots \end{cases}$$

by the payoff to Blue

$$(r,b)=\begin{cases} -1, & r(i)=b(i), \text{for some } i=1,2,3,\ldots \\ 1 \text{ (or } \infty), & r(i)\neq b(i),\ i=1,2,3,\ldots \end{cases}$$

We also obtain similar results for solution of the pursuit game on a directed cyclic graph with n vertices, DCPG [R(W)2].

As will be seen,these modifications are reasonable and interesting. The result obtained here could help us in understanding the complexity of CPG itself.

In section 2 of the paper, the equilibrium concept is discussed and a new concept "quasi-equilibrium" is proposed. The concept of optimal solutions to a game, hence, is generalized.

1. PURSUIT ON A CYCLIC GRAPH -- SOME MODIFIED CASES

1.0 TWO-PERSON ZERO-SUM GAMES [O(G)1]

A pursuit game on a cyclic graph is an example of a two-person zero-sum games. This section describes the essentials of their theory employed in this paper.

A two-person zero-sum game is defined here as a triple (B,R,f) where B and R are sets and f is a real valued function defined on the cartesian product B×R. The set B is called the set of admissible pure strategies for Blue and the set R is called the set of admissible pure strategies for Red. The function f is called the payoff function. The underlying ideas in describing a two person game in this way are: (a) if Blue chooses b from B and Red chooses r from R then Blue receives f(b,r) from Red or equivalently Blue pays -f(b,r) to Red; (b) Blue knows exactly what is in R and Red knows exactly what is in B, but neither have any further knowledge of what element their opponent will choose; (c) both Blue and Red know f.

If both B and R are finite then (B,R, f) is called a finite game. Otherwise, it is called infinite.

If the following two conditions hold:

$$f(b_0,r_0)\geq f(b,r_0) \qquad \text{for all } b\in B,$$
$$f(b_0,r_0)\leq f(b_0,r) \qquad \text{for all } r\in R,$$

Then the pair $(b_0,r_0)\in B\times R$ is called a saddle point of the game (B,R,f). These two inequalities are equivalent to the relation

$$\min_{r\in R}\ \max_{b\in B}\ (f(b,r))=f(b_0,r_0)=\max_{b\in B}\ \min_{r\in R}\ (f(b,r)).$$

If a game (B,R,f) has a saddle point (b_0,r_0) the number $f(b_0,r_0)$ is called the value of the game and the pure strategies b_0 and r_0 are called optimal strategies for Blue and Red, respectively. Not every finite game has a saddle point.

A mixed strategy β for Blue is a real valued function defined on B with the following two properties

$$\beta(b)\geq 0 \qquad \text{for each } b\in B,$$
$$\sum_{b\in B}\beta(b)=1.$$

In other words, β is a probability distribution or probability measure on B. A mixed strategy for Red is similarly defined as a probability measure on R. The expected payoff for the pair of strategies (β,ρ) is given by the formula

$$f(\beta,\rho)=\sum_{b\in B}\sum_{r\in R}\beta(b)\rho(r)f(b,r).$$

A triple (β_0,ρ_0,v) is called a solution for the game (B,R,f) if

$$f(b,\rho_0)\le v\le f(\beta_0,r) \qquad \text{for each } (b,r)\in B\times R,$$

where v is a real number and β_0,ρ_0 are mixed strategies for Blue and Red, respectively. The value v is called the value of the game and β_0 and ρ_0 the optimal strategies for Blue and Red respectively.

Minimax Theorem Every finite two-person zero-sum game has a solution in the sense of mixed strategies.

However, the Minimax Theorem is not valid for infinite games. In that case, we can define

$$v_1=\sup_{\beta}\ \inf_{\rho} f(\beta,\rho),$$

$$v_2=\inf_{\rho}\ \sup_{\beta} f(\beta,\rho).$$

If $v_1=v_2$ and supinf and infsup can be replaced by maxmin and minmax, respectively, optimal mixed strategies will exist; If these can be found, the game is as well determined as the finite games are. If only $v_1=v_2$, the game has a value $v(v_1=v_2)$ but may have no optimal strategies. It will have an "ε-optimal strategies", however; if given any $\varepsilon>0$,there exist mixed strategies, β and ρ for Blue and Red, respectively, such that

$$f(\beta,r)>v-\varepsilon \quad \text{and} \quad f(b,\rho)<v+\varepsilon$$

for any $(b,r)\in B\times R$ Thus, although such games are not as well determined as the finite games, they do seem to show a type of stability in their ε-optimal solution strategies.

The solution of a game in terms of mixed strategies was defined from an entirely formal point of view. Besides the pure mathematical interest, the fundamental treatise of Von Neumann and Morgenstern [VM1] presents an eloquent case in favor of the optimal strategy concept. From a practical viewpoint, such optimal strategies have been observed to develop (a) in animal behavior by naturalists; (b) in contrived children's games by child psychologists and sociologists and (c) in repeated game playing on a computer by computer scientists.

1.1. DCPG WHEN THE PLAYERS ARE RESTRICTED TO STOCHASTIC STRATEGIES

A directed cyclic graph with n vertices is shown in Figure 1. A hider, Blue, chooses a point on the directed cyclic graph, so does a pursuer, Red. Blue and Red both move along the graph, i.e. they remain at their vertices or each moves to one adjacent vertex pointed to by an arrow. The payoff to Blue equals the number of moves required by Red to find (or catch) Blue, or is ∞ otherwise. The collection B of pure strategies for Blue consists of all functions β from the set $\{1, 2, \ldots\}$ into the nodes of the directed cyclic graph such that $\beta(i+1) = \beta(i)$ or $\beta(i) + 1$ (1 if $\beta(i) = n$). The collection R of pure strategies for Red is the same, i.e., R = B. This is the pursuit game on a directed cyclic graph of n vertices, DCPG. It is unsolved [R(W)2], i.e.,it is an open question whether or not it has a solution.

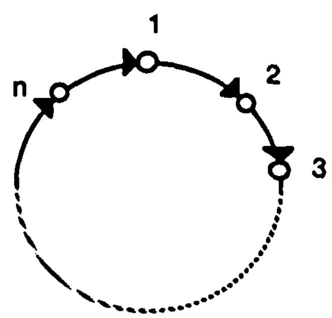


Figure 1

We discuss here <u>the pursuit game on a directed cyclic graph of n vertices when players Blue and Red are restricted to stochastic strategies</u>. That is, at the beginning of the game Blue chooses a number p and Red chooses a number q between 0 and 1. At each turn, Blue moves ahead with probability p, and remains in place with probability 1 - p while Red exhibits the same behavior with q replacing p. We mainly consider $n \geq 3$. When $n = 1$ or 2, the DCPG coincides with the PCG which is solved [R(W)1, 2].

In the stochastic DCPG on an n vertex graph a mixed strategy for Blue is a probability distribution defined on the set of all pairs (k, p) where $k = 0, 1, \ldots,$ or $n - 1$ and $0 \leq p \leq 1$. The integer k represents the initial position taken by Blue. Let $Tk = k + 1$ for $k = 0, 1, \ldots, n - 2$ and $T(n - 1) = 0$. While β is an "optimal" strategy for Blue (for which Blue gets at least the gain floor) then $T\beta$, defined by $T\beta(A) = \beta(\{(Tk, p): (k, p) \in A\})$ is also "optimal" since T is a renumbering of the vertices which retains their order. Hence, the strategy $\beta_o = \frac{1}{n}\sum_{k=0}^{n-1} T^k \beta$ is also "optimal" for Blue. The distribution β_o, therefore, has the property

$$\beta_o(\{(k, p): 0 \leq p \leq 1\}) = 1/n$$

for each fixed k. So, we may assume Blue and by the same reasoning Red chooses his initial vertex with probability $1/n$ from the set of all vertices.

To solve the problem, let us consider a random walk MDCPG equivalent to the above stochastic DCPG. In the MDCPG, a single counter is placed with probability $1/n$ on one of the vertices of an n-point directed cyclic graph. The counter indicates the distance (modulo n) from Blue to Red along the clockwise direction. If the initial vertex is zero, Blue receives one and the game ends. Otherwise, at each stage Blue and Red, without knowing the position of the counter, each vote on whether it is to move ahead or remain fixed. Their votes are combined so that if both Red and Blue decide upon one clockwise or both Red and Blue remain in place the counter remains fixed. If Red (Blue) votes one clockwise and Blue (Red) remains the counter is increased (decreased) one. The payoff to Blue is the number of stages until the counter reaches zero.

Suppose at the outset of MDCPG, Blue, the hider, chooses the strategy of moving the counter (-1) with probability p and staying it with probability $1 - p$. Red, the pursuer, chooses a strategy of moving the counter (+1) with probability q and staying it counter with probability $1 - q$. Thus the counter moves in a random walk with absorbing state at zero. Assume

$$a_{ij} = \text{prob. } \{\text{the counter moves to } j \text{ (mod n)} \mid \text{the counter is at } i\}$$

where

$i, j = 0, 1, 2, \ldots, n-1,$

so that $A = (a_{ij})$ is the transition matrix.

We have

$$a_{ij} = \begin{cases} 1, & i = j = 0, \\ 0, & i = 0; j = 1, 2, \ldots, n-1 \\ p(1-q), & j = i-1 \pmod n) \\ (1-p)(1-q)+pq, & j = i \neq 0 \\ q(1-p), & j = i+1 \pmod n), i \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

For $n = 3$,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ p(1-q) & (1-p)(1-q)+pq & q(1-p) \\ q(1-p) & p(1-q) & (1-p)(1-q)+pq \end{bmatrix}.$$

and for $n = 4$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ p(1-q) & (1-p)(1-q)+pq & q(1-p) & 0 \\ 0 & p(1-q) & (1-p)(1-q)+pq & q(1-p) \\ q(1-p) & 0 & p1-q) & (1-p)(1-q)+pq \end{bmatrix}.$$

Let e_j $(j = 0, 1, \ldots, n-1)$ denote the $n \times 1$ matrix having a one in the jth row and zeros elsewise, and let $e = e_0 + e_1 + \ldots + e_{n-1}$. The initial probability distribution of the counter's location is $(1/n)e^T$. At time k the distribution of the counter's location is $(1/n)e^TA^{k-1}$ because of the Chapman-Kolmogorov equations [R(S)1]. The probability that the counter is at vertex j at time k is $(1/n)e^TA^{k-1}e_j$. The expected payoff to Blue, $f(p, q)$ when Blue adopts the parameter p and Red adopts the parameter q is

$$f(p,q) = \frac{1}{n} + \sum_{k=2}^{\infty} k\left(\frac{1}{n}e^TA^{k-1}e_o - \frac{1}{n}e^TA^{k-2}e_o\right)$$

$$= 1 + \sum_{j=1}^{n-1}\sum_{k=1}^{\infty} \frac{1}{n}e^TA^{k-1}e_j = 1 + \sum_{k=1}^{\infty}\frac{1}{n}e^TA^{k-1}(e - e_o),$$

where the sum in the equation can diverge to infinity.

Let $B = (b_{ij})$ to be a $(n - 1) \times (n - 1)$ matrix, where $b_{ij} = a_{ij}$ $(i, j = 1, 2, \ldots, n - 1)$, i.e.,

B=

$$B = \begin{bmatrix} (1-p)(1-q)+pq & q(1-p) & 0 & & & \\ p(1-q) & (1-p)(1-q)+pq & q(1-p) & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & p(1-q) & (1-p)(1-q)+pq & q(1-p) \\ & & & 0 & p(1-q) & (1-p)(1-q)+pq \end{bmatrix} \tag{1}$$

and d is the $n-1$ column matrix consisting entirely of one's. By considering partitions of the matrices we conclude $e^T A^{k-1}(e - e_0) = d^T B^{k-1} d$ for each $k = 1, 2, \ldots$. From this we get the following conclusion,

$$f(p, q) = 1 + \frac{1}{n} \sum_{k=0}^{\infty} d^T B^k d. \tag{2}$$

Lemma 1: In either (i) $p = 0$ and $q \neq 0$, (ii) $p \neq 1$ and $q = 1$ or (iii) $p \neq 0$ and $q \neq 1$,

$$|\lambda| < 1$$

for each eigenvalue λ of B.

Proof. (i) $p = 0$ and $q \neq 0$. Then B has an eigenvalue $1 - q$ of multiplicity $(n-1)$ (which is less than 1 since $0 < q \leq 1$) since

$$B = \begin{bmatrix} 1-q & & * \\ & 1-q & \\ 0 & & 1-q \end{bmatrix}$$

(ii) $p \neq 1$ and $q = 1$. Then B has an $(n-1)$ - multiple eigenvalue $p < 1$ since

$$B = \begin{bmatrix} p & & * \\ & \ddots \; p \; \ddots & \\ 0 & & p \end{bmatrix}$$

(iii) $p \neq 0$ and $q \neq 1$.

Suppose $|\lambda| \geq 1$, where λ is an eigenvalue of B, and u is an associated eigenvector such that $\max_{1 \leq i \leq n-1} |u_i| = 1$.

We have

$$\sum_{j=1}^{n-1} b_{ij} u_j = \lambda u_i, \qquad i = 1, 2, \ldots, n-1.$$

Because A is a stochastic matrix and $a_{10} = p(1-q)$, it follows that

$$p(1-q) + \sum_{j=1}^{n-1} b_{1j} = 1.$$

Since $0 < p$ and $q < 1$, $p(1-q) > 0$ so that $\sum_{j=1}^{n-1} b_{1j} < 1$. Hence

$$|u_1| \leq |\lambda u_1| \leq \sum_{j=1}^{n-1} b_{1j} |u_j| < 1.$$

If $|u_{k-1}| < 1$, then we have

$$|u_k| \leq |\lambda u_k| \leq \sum_{j=1}^{n-1} b_{kj} |u_j| \leq \sum_{j=1}^{k-2} b_{kj} + |u_{k-1}| b_{k\,k-1} + \sum_{j=k}^{n-1} b_{kj} < \sum_{j=1}^{n-1} b_{kj} \leq 1.$$

So, $|u_k| < 1$. By induction we get $|u_i| < 1$ for each $i = 1, 2, \ldots, n-1$ contradicting our hypothesis that $\max_{1 \leq i \leq n-1} |u_i| = 1$. Hence $|\lambda| < 1$.

Lemma 2: When Blue chooses the parameter p and Red chooses the parameter q in the MDCPG, the payoff to Blue is

$$f(p,q) = \begin{cases} \infty, & p = q = 0 \text{ or } p = q = 1, \\ 1 + \frac{1}{n} d^T (I - B)^{-1} d, & \text{otherwise.} \end{cases} \tag{3}$$

Proof. When $p = q = 0$ or $p = q = 1$, $B = I$, the unit matrix. Using (2), we get

$$f(p, q) = 1 + \sum_{k=0}^{\infty} \frac{n-1}{n} = \infty.$$

In other cases, each eigenvalue λ of B has absolute value less than 1 by Lemma 1. Using the Jordan canonical form of B, we get that $\sum_{k=0}^{\infty} B^k$ converges absolutely to $(I - B)^{-1}$ [G(F)1]. Then because of (2), the lemma holds.

Q.E.D.

Proposition 1: For the MDCPG, or the stochastic pursuit game on a directed cyclic graph with n vertices, $v_1 < v_2$ and

$$v_1 = \sup_P \inf_q f(p, q) = f(1/2, 0) = n,$$

$$v_2 = \inf_Q \sup_p f(p, q) = f(1/2, 1/2) = \frac{n^2+2}{3},$$

where P, Q are mixed strategies taken by Blue and Red, respectively. That is, Blue can get a payoff which is not less than n if he takes $p = 1/2$ as his strategy. And Red can pay what is not greater than $\frac{n^2+2}{3}$ if he takes $q = 1/2$ as his strategy. This game has no equilibrium value.

Proof. First we claim that the payoff function $f(p, q)$, (3), has the following properties.

(i) $f(p, q) = f(q, p)$, $f(p, q) = f(1 - q, 1 - p)$.

(ii) $f(0, q)$, $f(1, q)$, $f(p, 0)$, $f(p, 1)$, all are monotone strictly as being showed by arrows in Figure 2.

(iii) $f(0, 0) = f(1, 1) = \infty$,

$$f(1, 0) = f(0, 1) = \frac{n+1}{2},$$

$$f(1/2, 0) = f(1/2, 1) = f(0, 1/2) = f(1, 1/2) = n,$$

$$f(1/2, 1/2) = \frac{n^2+2}{3}.$$

(iv) $f(p, p)$ and $f(p, 1/2)$ are both strictly monotone on the intervals $[0, 1/2]$ and $[1/2, 1]$.

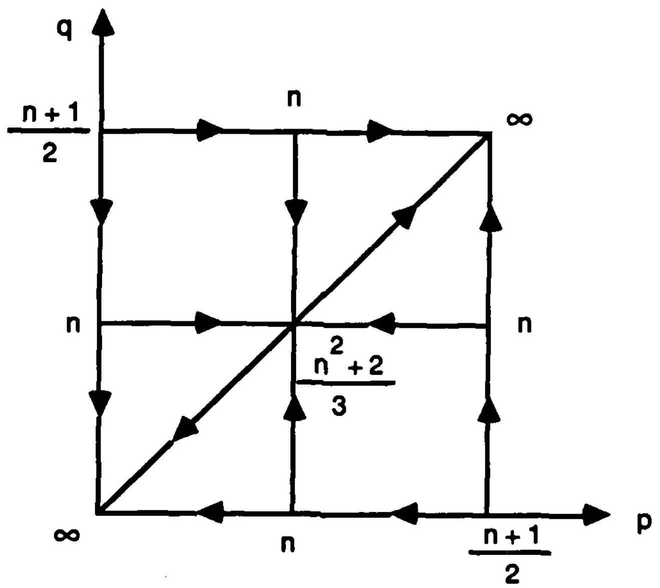


Figure 2

As examples, we calculate f(1/2, 1/2) and prove f(p, p) is monotone on both intervals [0, 1/2] and [1/2, 1], respectively.

From (1), we get (n - 1) x (n - 1) matrix (I -B) at p = 1/2, q= 1/2 as follows.

$$(I-B)_{n-1}\left(\frac{1}{2},\frac{1}{2}\right) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & & & & \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & & & \\ & -\frac{1}{4} & \frac{1}{2} & \ddots & & \\ & & \ddots & \frac{1}{2} & -\frac{1}{4} & \\ & & & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ & & & & -\frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

So, the inverse of (I - B)(1/2, 1/2) is

$$(I-B)^{-1}_{n-1}\left(\frac{1}{2},\frac{1}{2}\right)=$$

$$(I-B)^{-1}_{n-1}\left(\frac{1}{2},\frac{1}{2}\right)=4\left[\begin{array}{cccccccc}
\frac{n-1}{n} & \frac{n-2}{n} & \frac{n-3}{n} & \frac{n-4}{n} & \cdots & \frac{3}{n} & \frac{2}{n} & \frac{1}{n} \\
\frac{n-2}{n} & \frac{2(n-2)}{n} & \frac{2(n-3)}{n} & \frac{2(n-4)}{n} & \cdots & \frac{2\cdot 3}{n} & \frac{2\cdot 2}{n} & \frac{2\cdot 1}{n} \\
\frac{n-3}{n} & \frac{2(n-3)}{n} & \frac{3(n-3)}{n} & \frac{3(n-4)}{n} & \cdots & \frac{3\cdot 3}{n} & \frac{3\cdot 2}{n} & \frac{3\cdot 1}{n} \\
\frac{n-4}{n} & \frac{2(n-4)}{n} & \frac{3(n-4)}{n} & \frac{4(n-4)}{n} & \cdots & \frac{4\cdot 3}{n} & \frac{4\cdot 2}{n} & \frac{4\cdot 1}{n} \\
\hdashline
\frac{1\cdot 3}{n} & \frac{2\cdot 3}{n} & \frac{3\cdot 3}{n} & \frac{4\cdot 3}{n} & & \frac{(n-3)\cdot 3}{n} & \frac{(n-3)\cdot 2}{n} & \frac{(n-3)\cdot 1}{n} \\
\frac{1\cdot 2}{n} & \frac{2\cdot 2}{n} & \frac{3\cdot 2}{n} & \frac{4\cdot 2}{n} & & \frac{(n-3)\cdot 2}{n} & \frac{(n-2)\cdot 2}{n} & \frac{(n-2)\cdot 1}{n} \\
\frac{1\cdot 1}{n} & \frac{2\cdot 1}{n} & \frac{3\cdot 1}{n} & \frac{4\cdot 1}{n} & & \frac{(n-3)\cdot 1}{n} & \frac{n-2}{n} & \frac{n-1}{n}
\end{array}\right]$$

Hence

$$\begin{aligned}
f\left(\frac{1}{2},\frac{1}{2}\right) &= 1+\frac{1}{n}d^T(I-B)^{-1}\left(\frac{1}{2},\frac{1}{2}\right)d \\
&= 1+\frac{4}{n^2}\left[\sum_{i=1}^{n-1} i\,(n-i)+2\sum_{j=1}^{n-2}\ \sum_{i=1}^{n-(j+1)} i\,j\right] \\
&= \frac{n^2+2}{3}\,.
\end{aligned}$$

When $p = q$,

$$I-B=\begin{bmatrix}
2p(1-p) & -p(1-p) & & & & \\
-p(1-p) & 2p(1-p) & -p(1-p) & & & \\
 & -p(1-p) & 2p(1-p) & -p(1-p) & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & -p(1-p) & 2p(1-p) & -p(1-p) \\
 & & & & -p(1-p) & 2p(1-p)
\end{bmatrix},$$

$$|I-B|_{n-1}=n\,p^{n-1}(1-p)^{n-1},$$

$$(I-B)^*_{n-1} = \left(a_{ij}p^{n-2}(1-p)^{n-2}\right)_{n-1},$$

$$(I-B)^{-1}_{n-1} = \frac{1}{p(1-p)}\left(a_{ij}/n\right)_{n-1},$$

where a_{ij} is a constant for given n, $(i, j = 1, 2, \dots, n-1)$. So,

$$f(p,p) = 1 + \frac{1}{n} d^T (I-B)^{-1} d = 1 + \frac{c}{p(1-p)},$$

where c is a constant for given n. From $f(1/2, 1/2)$, $c \neq 0$. Hence

$$\frac{df(p,p)}{dp} = \frac{c(2p-1)}{p^2(1-p)^2}$$

is strictly increasing (decreasing) on interval [1/2, 1] ([0, 1/2]). Since $f(0, 0) = f(1, 1) = \infty$ and $f(1/2, 1/2) < \infty$.

Now, we calculate v_1 and v_2. Suppose Red takes $q = 1/2$ as his strategy, then

$$\max_p f(p, 1/2) = f(1/2, 1/2) .$$

For any q,

$$\max_p f(p,q) \geq f(q,q) \geq f(1/2, 1/2) .$$

So,

$$v_2 = \min_Q \max_p f(p,Q) = f(1/2, 1/2) = \frac{n^2+2}{3}.$$

Then suppose Blue takes $p = 1/2$ as his strategy. Then

$$\min_q f(1/2, q) = f(1/2, 0) .$$

For any p,

$$\min_q f(p,q) \leq \begin{cases} f(p,0) < f(1/2,0), & p > 1/2 \\ f(p,1) < f(1/2,0), & p < 1/2 \end{cases}$$

So,

$$v_1 = \max_P \min_q f(P,q) = f(1/2, 0) = n.$$

Finally, $v_1 < v_2$ since $v_2 - v_1 = (n-2)(n-1)$ and $n \geq 3$.

Q.E.D.

1.2. PURSUIT ON A CYCLIC GRAPH

Given a cyclic graph of n vertices as in Figure 3. Both a hider, Blue, and a pursuer, Red, can at each stage remain at a vertex i, move to i + 1 (1 if i = n) or i - 1 (n if i = 1). The payoff to Blue is -1 if Red catches (or finds) Blue, or 1 (or ∞) otherwise. The collection B of pure strategies for Blue consists of all functions β from the set {1, 2, 3, . . . } into the cyclic graph such that $\beta(i + 1) = \beta(i) - 1$, $\beta(i)$ or $\beta(i) + 1$ (mod n). The collection R of pure strategies for Red is the same, i.e., R = B. This two-person zero-sum game is called <u>a pursuit game with modified payoff on a cyclic graph of n vertices, CMPG.</u>

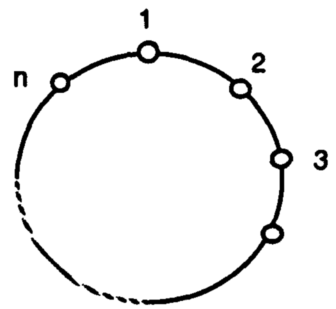


Figure 3

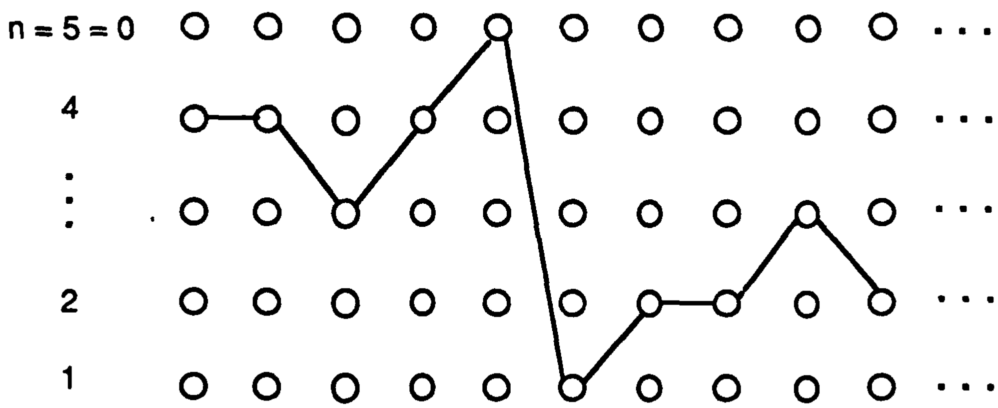


Figure 4 (n = 5 = 0 (mod 5))

To solve the problem, first let us consider a problem equivalent to SMPG. Take an infinite set L having an underlying structure which limits the B and R as follows. L is a "semi-infinite rectangular" array of lattice points having n rows and countable columns; Thus,

$$L = \{(i, j) : i = 1, 2, \ldots ; j = 1, 2, \ldots, n\}, n \geq 3.$$

And, let

$$B = R = \{\beta : \beta \text{ satisfies some condition as follows}\},$$

e.g., β is a function from $\{1, 2, \ldots\}$ into $\{1, 2, \ldots, n = 0 \pmod n\}$ (or a path on L) such that

$$\beta(i+1) = \begin{cases} \beta(i) - 1 & \text{or} \\ \beta(i) & \text{or} \qquad \pmod n \\ \beta(i) + 1 \end{cases}$$

The payoff to Blue is

$$(\beta, \rho) = \begin{cases} -1, & \rho(i) = \beta(i), \text{ for some } i = 1, 2, 3, \ldots \\ 1 \text{ (or } \infty), & \text{otherwise,} \end{cases}$$

where β, ρ are pure strategies of Blue and Red, respectively.

Let $G_C = R = B$. If $\rho \in G_C$, then

$$\rho(i+1) = \rho(i) + \bar{\rho}(i+1) \qquad \pmod n,$$

where $\bar{\rho}(i+1) = 0, -1$ or $+1$. Let ρ, $\sigma \in G_c$. Define

$$(\rho + \sigma)(1) = \rho(1) + \sigma(1) \qquad \pmod n$$

$$(\rho + \sigma)(i+1) = (\rho + \sigma)(i) + (\bar{\rho}(i+1) + \bar{\sigma}(i+1)) \qquad \pmod n,$$

where $(\bar{\rho}(i+1) + \bar{\sigma}(i+1)) = -1, 0$ or $+1 \pmod 3$. Then we know G_c is a commutative group isomorphic to the group

$$M = Z_n + Z_3 + Z_3 + Z_3 + \ldots,$$

where Z_n, Z_3 are cyclic groups of order n and 3, respectively.

On the other hand, impose the discrete topology on both Z_n and Z_3. Then regard M as the product of Z_n, Z_3, Z_3, Z_3, ... Using the above mapping (isomorphism) between G_c and M, Gc is a discrete topological space, too. Since Z_n, Z_3 are compact, G_c is a compact topological space.

Suppose M_{Z_n}, M_Z, are Haar measure on groups Z_n and Z_3, respectively. That is

$$M_{Z_n}(j) = \frac{1}{n}, \quad j = 0, 1, 2, \ldots, n-1,$$

$$M_{Z_3}(j) = \frac{1}{3}, \quad j = 0, 1.$$

Then the product of M_{G_c} of M_{Z_n}, M_{Z_3}, M_{Z_3}, . . . is the Haar measure on $G_c = M = Z_n + Z_3 + Z_3 + \ldots$. That is

$$M_{G_c}([r]_k) = \frac{1}{n \cdot 3^{k-1}}, \quad k = 1, 2, 3, \ldots,$$

where

$$[r]_k = \{\rho : \rho(i) = r(i), \ 1 \le i \le k\}.$$

Let

$$G_{cr} = \{\rho : \rho(i) = r(i) \ \text{ for some } \ i = 1, 2, \ldots\}.$$

$$\overline{G_{cr}} = \{\rho : \rho(i) \neq r(i), \ i = 1, 2, \ldots\} = G_c \backslash G_{cr}.$$

And let

$$(r, k) = \{\rho : \rho(i) \neq r(i), \ \ i = 1, 2, \ldots, k\}.$$

Then

$$M_{G_c}(\overline{G_{cr}}) = \lim_{k \to \infty} M_{G_c}((r, k)),$$

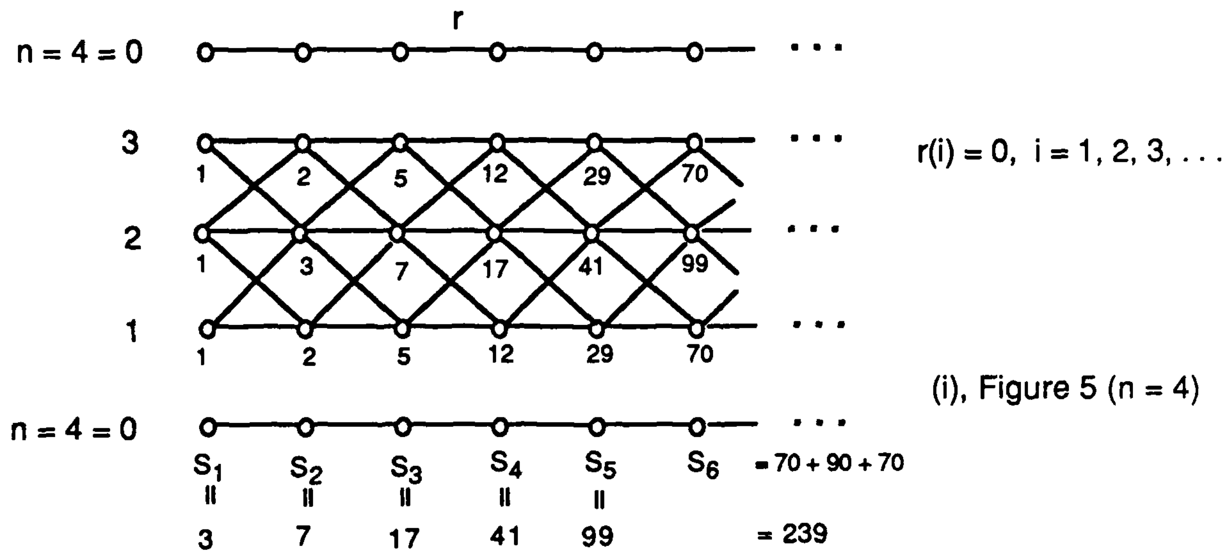
and

$$M_{G_c}((r, k)) = \frac{s_k}{n \cdot 3^{k-1}},$$

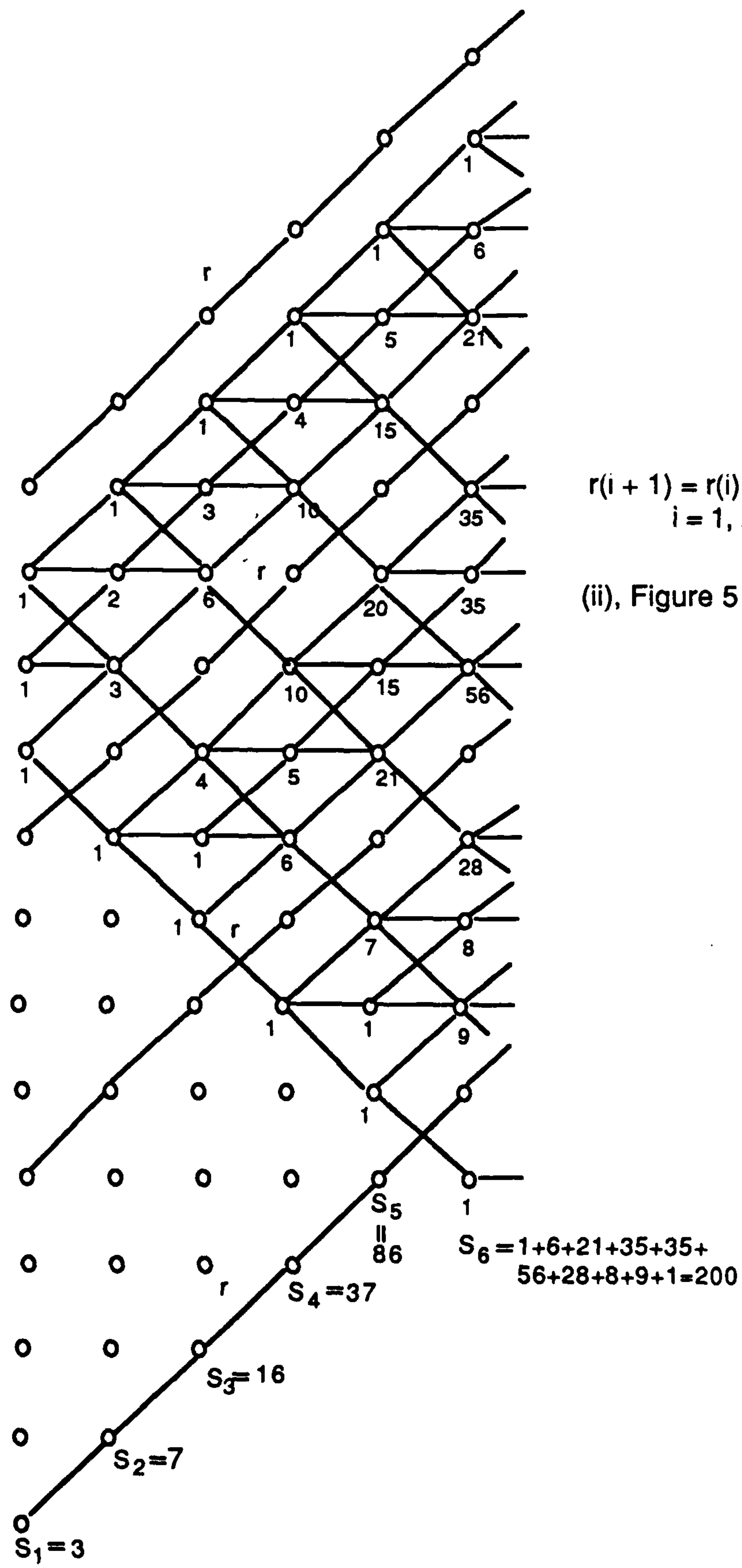
where s_k is the number of different heads cut down of paths in (r, k) between column k and k + 1.

<u>Lemma 3</u>: $M_{G_c}(\overline{G_{cr}}) = 0$ for any $r \in G_c$. Hence $M_{G_c}(G_{cr}) = 1$.

Proof. Let us look at Figure 5. In (i) of Figure 5, n = 4, r(i) = 0, i = 1, 2, 3, ... All heads cut down of paths in (r, 6) are shown in the figure. Attach a number to a node as follows. 1 to each node in the first column, but not on r. At each node from column 2 which is not on r, the attached number is the sum of numbers located on the left side and connected up by a line. The iteration process looks like that of the

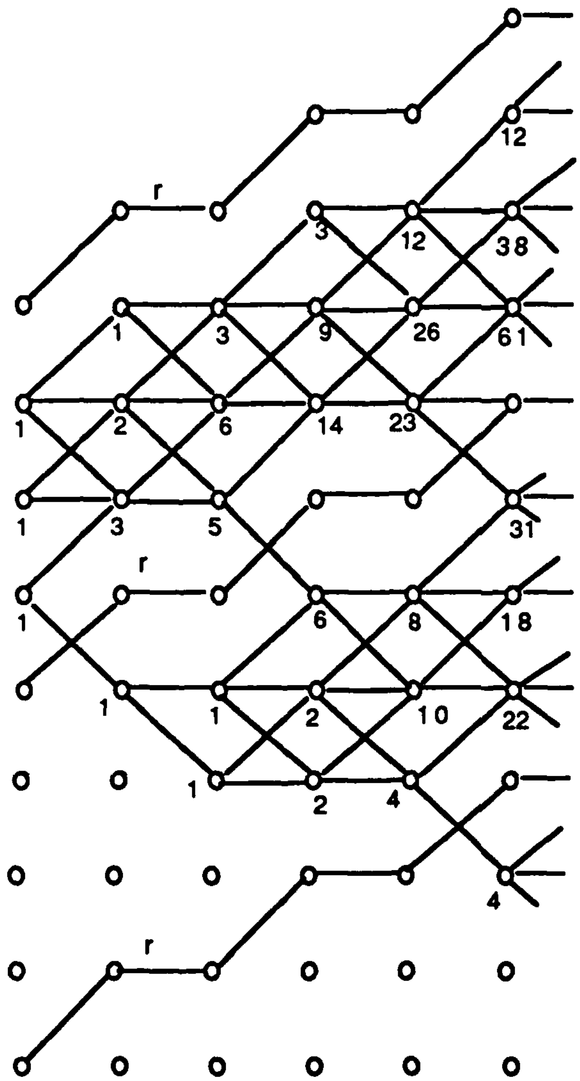

r
n = 4 = 0
3
2
1
n = 4 = 0
1 2 5 12 29 70
1 3 7 17 41 99
1 2 5 12 29 70
. . .
S1 S2 S3 S4 S5 S6 = 70 + 90 + 70
3 7 17 41 99 = 239
r(i) = 0, i = 1, 2, 3, . . .

(i), Figure 5 (n = 4)



$r(i + 1) = r(i) + 1,$
$i = 1, 2, 3, \ldots$

(ii), Figure 5 (n = 4)



$S_1 = 3$, $S_2 = 7$, $S_3 = 16$, $S_4 = 36$, $S_5 = 83$

$S_6 = 12 + 38 + 61 + 31 + 18 + 22 + 4 = 186$

$r(2i) = r(2i - 1) + 1$, $r(2i + 1) = r(2i)$ $\quad i = 1, 2, 3, \ldots$

(iii), Figure 5 (n = 4)

Pascal or H. Yang Triangle. Then s_k is the sum of all numbers on the kth column. For another example, in (ii) of Figure 5, S2 = 1 + 2 + 3 + 1 = 7.

For given k, different r corresponds to different s_k. However, s_k of (i) for n = 4 is the greatest. For (i), figure 5, suppose $s_i = a + b + a$. Then $s_{i+1} = (a + b) + (a + b + a) + (b + a) = 2(a + b + a) + b = 2s_i + b \leq 2s_i + (1/2)s_i = (5/2)s_i$. And

$$\frac{s_{i+1}/4 \cdot 3^{i}}{s_i/4 \cdot 3^{i-1}} \leq \frac{1}{3} \cdot \frac{5}{2} = \frac{5}{6} < 1.$$

Therefore,

$$M_{G_c}(\overline{G}_{cr}) = \lim_{k \to \infty} \frac{s_i}{4 \cdot 3^{i-1}} = 0.$$

For general n, the proof is similar.

Since $M_{G_C}(G_C) = 1$, we get

$$M_{G_C}(G_{CT}) = 1 - M_{G_C}(\overline{G}_{cr}) = 1 - 0 = 1.$$

Q.E.D.

Proposition 2: For a CMPG, the Haar measure M_{Gc} is an optimal strategy for both Red and Blue. And the value to Blue is -1.

Proof. If Blue uses the Haar measure x as his strategy and Red chooses r, then Blue can expect

$$\int_{G_c} f(x, r)\, d\, M_{G_c}(x) = (-1)\, M_{G_c}(G_{cr}) + 1 \cdot M_{G_c}(\overline{G}_{cr}) = -1$$

by lemma 3. Hence, $v_1 \geq -1$.

By symmetry, if Red uses the Haar measure he can hold Blue to the expected payoff -1, i.e., $v_2 \leq -1$. From $-1 \leq v_1 \leq v_2 \leq -1$, we get $v = v_1 = v_2 = -1$.

Q.E.D.

Corollary 1: For CMPG, the probability that Blue escapes is zero if Red takes Haar measure as his mixed strategy.

Similarly, we can discuss the pursuit game on a directed cyclic graph of n vertices whose payoff is modified, DCMPG. The payoff to Blue is -1 if Red catches (or finds) Blue, or 1 (or ∞) otherwise. $G_d = R = B$ consists of all functions β from the set $\{1, 2, 3, \ldots\}$ into the directed cyclic graph such that $\beta(i + 1) = \beta(i)$ or $\beta(i) + 1 \pmod n$.

Replacing figure 3, 4, 5, G_c above by figure 1, 6, 7, G_d, separately, we get proposition 3 below. Note

$$G_d \cong M = Z_n + Z_2 + Z_2 + Z_2 + \cdots.$$

And

$$M_{G_d}([r]_k) = \frac{1}{n \cdot 2^{k-1}}, \quad k = 1, 2, 3, \ldots .$$

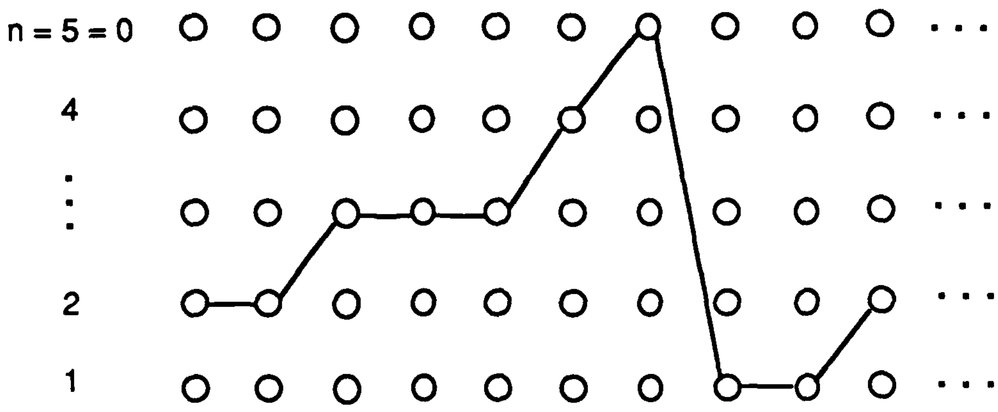


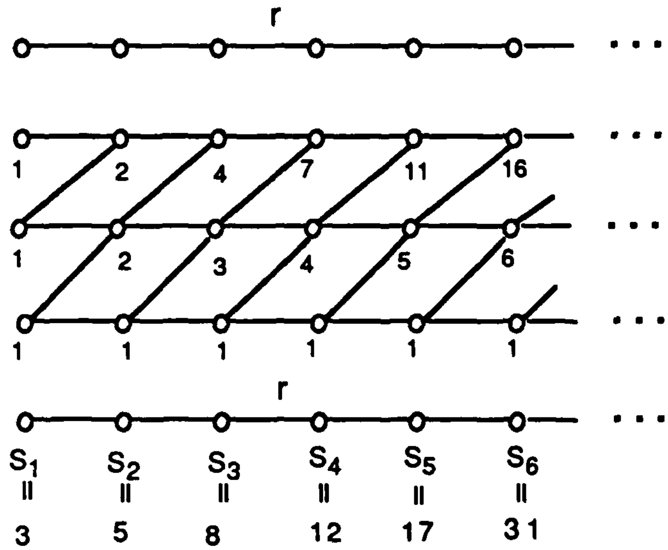
Figure 6 (n = 5 = 0 (mod 5))

Lemma 4: $M_{G_d}(\overline{G}_{dr}) = 0$ for any $r \in G_d$. And $M_{G_d}(G_{dr}) = 1$.

Proposition 3: For a DCMPG, the Haar Measure M_{G_d} is an optimal strategy for both Red and Blue. And the value to Blue is -1.

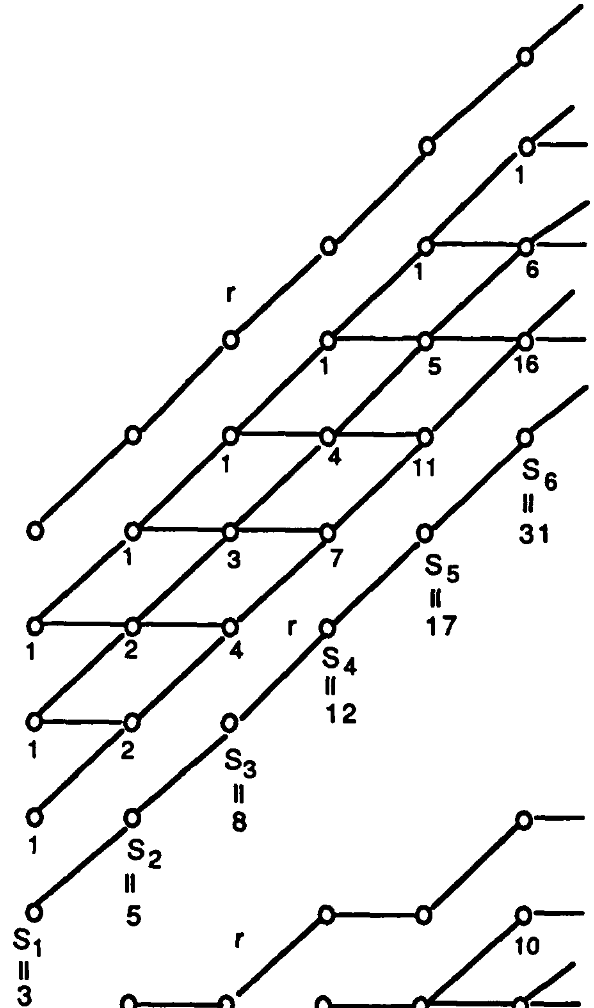
Corollary 2: For DCMPG, the probability that Blue escapes is zero if Red takes Haar Measure as his mixed strategy.

When we prove lemma 4, we note that r in (iii), Figure 7 corresponds to the largest s_k for given k. (See proof of lemma 3.)



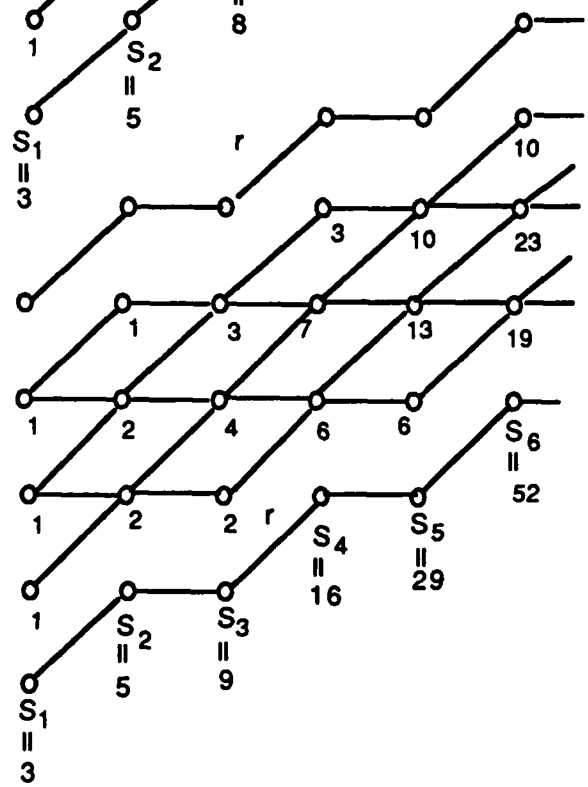
r(i) = constant, i = 1, 2, 3, . . .

(i), Figure 7



$$r(i+1) = r(i) + 1, \quad i = 1, 2, 3, \ldots$$

(ii), Figure 7



$$r(2i) = r(2i-1) + 1 \quad i = 1, 2, 3, \ldots$$
$$r(2i+1) = r(2i),$$

(iii), Figure 7

2. QUASI-EQUILIBRIUM . PSEUDO-EQUILIBRIUM

As we know, if there exists a saddle point (b^*,r^*) for a game, then (b^*,r^*) is an equilibrium pair of the game. In the case

$$f(b,r^*)\leq f(b^*,r^*), \qquad f(b^*,r)\geq f(b^*,r^*)$$

for any pure strategy b of Blue and any pure strategy r of Red. In other words, a pair of strategies is said to be in equilibrium if no player has any positive reason for changing his strategy, assuming that the opponent is not going to change strategies. If in such a case, each player knows what his opponent will play, then he has reason to play the strategy which will give such an equilibrium pair. Thus the game becomes very stable.

If there does not exist a saddle point (b^*,r^*), we consider mixed strategies. When the game is finite there exists a value $v(=v_1=v_2)$ and mixed strategy β^* for Blue and mixed strategy ρ^* for Red, respectively,which yield this value. The pair (β^*,ρ^*) is said to be in equilibrium, or an equilibrium pair of the game. In this case,

$$f(\beta,\rho^*)\leq f(\beta^*,\rho^*), \qquad f(\beta^*,\rho)\geq f(\beta^*,\rho^*)$$

for any mixed strategy β for Blue and any mixed strategy ρ for Red. When the game is infinite and

$$v_1=\sup_{\beta}\inf_{\rho} f(\beta,\rho)=\inf_{\rho}\sup_{\beta} f(\beta,\rho)=v_2,$$

then either there exists (β^*,ρ^*) such that

$$f(\beta,\rho^*)\leq \sup_{\beta} f(\beta,\rho) \qquad \text{for any } \rho$$

and

$$f(\beta^*,\rho)\geq \inf_{\rho} f(\beta,\rho) \qquad \text{for any } \beta,$$

and the pair (β^*,ρ^*) is said to be in equilibrium, or there exists (β^*,ρ^*) for any given $\varepsilon>0$ such that

$$f(\beta^*,r)>v-\varepsilon$$

and

$$f(b,\rho^*)<v+\varepsilon$$

for any $(b,r)\in B\times R$ and the strategy pair $(\beta*,\rho*)$ is said to be in ε-equilibrium, or an ε-equilibrium pair of the game. The strategies $\beta*$ and $\rho*$ are called an ε-optimal strategy for Blue and an ε-optimal strategy for Red, respectively.

In theory and practice, the following problems arise

1. For a finite or infinite game without a saddle point, how should the players make their decisions? Here, we suppose that the game is played once. Thus mixed strategies are not considered.

2. For an infinite game without a value v, i.e., $v_1<v_2$, how should the players make their decisions?

To answer the first question, it is necessary to state the assumption clearly. In fact, the model satisfies the following assumptions.

(a) There are two players and the playoff function is zero sum.

(b) The two players both are proficient in playing the game. Namely, they know what decisions to make if they know what the opponent does.

(c) The players are both conservatives. Namely, they always suppose that their opponents are so smart that they know beforehand what decisions their opponents will make.

(d) The players have played the game one or more times.

Consider finite games,at first. In such a case, there exists a strategy pair (b^*,r^*) and two strategies r_0 and b_0 such that

$$f(b^*,r_0)=\max_b \min_r f(b,r) \tag{1a}$$

and

$$f(b_0,r^*)=\min_r \max_b f(b,r) \tag{1b}$$

for all $(b, r)\in B\times R$. If $(b^*,r_0)=(b_0,r^*)$, i.e., $b^*=b_0$, $r^*=r_0$, then the pair $(b^*,r^*)(=(b^*,r_0)=(b_0,r^*))$ is a saddle point of the game. It is an equilibrium. If $(b^*,r_0)\neq(b_0,r^*)$, we treat the game as follows.

First, suppose the pair (b^*,r^*) satisfying conditions (1_a and 1_b) is unique. Then the game has an equilibrium "to some degree" at the pure strategy pair (b^*,r^*) because

$$f(b^*,r)\geq \min_r f(b,r), \qquad \text{and} \qquad f(b,r^*)\leq \max_b f(b,r)$$

for all b,r. In other words, no player has a positive reason for changing his strategy because any change could cause an appropriate change by the opponent so that the player could get less or lose more. We call such a pure strategy pair a "quasi-saddle point" of the game. If the game is played once or a few times, the saddle point or quasi-equilibrium pair should be the optimal or quasi-optimal strategy pair for players Blue and Red. If the game is played a large number times, the players Blue and Red should choose the saddle point or, generally, the optimal mixed strategies.

Second, suppose the strategy b* satisfying (1_a) is unique, and there are several pure strategies $r_1, r_2, ..., r_k$ satisfying (1_b) for player Red. Select a strategy r* from $r_1,r_2,...,r_k$ such that

$$f(b^*,r^*)= \min_{1\leq i\leq k} f(b^*,r_i).$$

Then the strategy pair (b*,r*) is said to be a "quasi-saddle point" of the game.

Similarly, if there are several pure strategies $b_1,b_2,...,b_k$ satisfying(1_a) for player Blue, and the strategy r* satisfying (1_b) is unique, then the strategy pair (b*,r*) is said to be a "quasi-saddle point", where b* is one of $b_1,b_2,...,b_k$ such that

$$f(b^*,r^*)= \max_{1\leq i\leq k} f(b_i,r^*).$$

Third, suppose there are k strategies $b_1,b_2,...,b_k$ satisfying (1_a) and l strategies $r_1,r_2,...,r_l$ satisfying (1_b). Then Blue should choose pure strategies from only $b_1,b_2,...,b_k$, Red from only $r_1,r_2,...,r_l$. Thus, the original game is reduced to a new one which is simpler than the original generally. Repeating the reducing process, the original game might be reduced to one which either is the case discussed above, i.e., there is single b (or r) satisfying (1_a) (or(1_b)), or cannot be reduced further.

For example, a game with payoff matrix

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

cannot be reduced as above. Call it an irreducible game.

In the later case, Blue should choose any one of $b_1,b_2,...,b_k$ as an "optimal" strategy, and Red should choose any one of $r_1,r_2,...,r_l$ as an "optimal" strategy, where $\{b_1,b_2,...,b_k;r_1,r_2,...,r_l\}$ cannot be reduced further. Such a pure strategy pair

(b^*,r^*) is said to be a "pseudo-saddle point" of the original game if $b\in\{b_1,b_2,\ldots,b_k\}$, $r\in\{r_1,r_2,\ldots,r_l\}$.

For infinite games, assume that there exists a strategy pair (b^*,r^*) such that

$$f(b^*,r)\geq \inf_r f(b,r) \quad \text{and} \quad f(b,r^*)\leq \sup_b f(b,r)$$

for all b,r. Then we can define concepts of "quasi-saddle point" and "pseudo-saddle point" similarly.

Now, let us consider problem 2 proposed at the beginning of this section. Besides the hypotheses ((a),(b),(c),(d)), we add one more as follows.

(e) v_1 can be reached by a mixed strategy β^* for player Blue; v_2 can be reached by a mixed strategy ρ^* for player Red in any game concerned in the discussed process. Namely, there exist β^* and ρ^* such that

$$f(\beta^*,\rho)\geq \inf_\rho f(\beta,\rho) \quad \text{for any } \beta \text{ for Blue} \tag{2a}$$

and

$$f(\beta,\rho^*)\leq \sup_\beta f(\beta,\rho) \quad \text{for any } \rho \text{ for Red} \tag{2b}$$

Then,when the pair (β^*,ρ^*) satisfying (2_a) and (2_b) is unique, the pair (β^*,ρ^*) is said to be in "quasi-mixed equilibrium", or a "quasi-mixed equilibrium" pair of the game.

When the strategy β^* satisfying (2_a) is unique and there exists a set R_1 of ρ satisfying (2_b), there are two cases. First, there exists ρ^* such that

$$f(\beta^*,\rho^*)=\inf_\rho f(\beta^*,\rho).$$

Then pair (β^*,ρ^*) is then in quasi-mixed equilibrium. Second, there is no such ρ^*. Then there exists ρ^* such that

$$f(\beta^*,\rho^*)\leq \inf_\rho f(\beta^*,\rho)+\varepsilon$$

for any given $\varepsilon>0$. (β^*,ρ^*) is said to be in "quasi-mixed ε-equilibrium".

Similarly, when there exists a set B_1 of b satisfying (2_a) and the strategy ρ^* satisfying (2_b) is unique, there are two cases. First, there exists β^* such that

$$f(\beta^*,\rho^*)= \sup_\beta f(\beta,\rho^*).$$

The pair $(\beta*,\rho*)$ is then in "quasi-mixed equilibrium". Second, there is no such $\beta*$. Then there exists $\beta*$ such that

$$f(\beta*,\rho*)\geq \sup_{\beta} f(\beta,\rho*)-\varepsilon$$

for any given $\varepsilon>0$. $(\beta*,\rho*)$ is said to be in "quasi-mixed ε-equilibrium".

If there exists a set B_1 of strategies β satisfying (2_a) and a set R1 of strategies r satisfying (2_b). Then Blue should choose a mixed strategy only from B_1, Red only from R_1. The original game is thereby reduced to a new one which is much simpler than the original one generally. Repeating the reducing process on the reduced games, the original game either can be reduced to one which is the case discussed as above or can be reduced to an irreducible game in a finite number of steps, or can be reduced without stopping i.e., the reduced game which is gotten in any finite number of steps can be reduced further. In the latter two cases, a pair $(\beta*,\rho*)$ is said to be <u>in "pseudo-mixed equilibrium"</u> or <u>an pseudo-mixed equilibrium pair</u> of the game, where $(\beta*,\rho*)\in B_1\times R_1$, and $B_1\times R_1$ either is irreducible or can be reduced without stopping.

<u>Proposition 1'.</u> for the MDCPG in section 2, $v_1<v_2$. In this case, there is a single $\beta*$ and a single $\rho*$ satisfying (2_a) and (2_b),respectively, where $\beta*,\rho*$ correspond p=1/2 and q=1/2. Thus, (p,q)=(1/2,1/2) is a quasi-mixed equilibrium pair of the game DPCG.

Finally, it should be pointed out that when there exists a saddle point in problem 1 (the value $v(v_1=v_2)$ in problem 2), hypothesis (c) can be replaced by the following hypothesis.

(c') One of the players is conservative, and the other is astute, i.e., he knows that his opponent is conservative.

However, in the case, the theory deduced above about quasi- equilibrium and pseudo-equilibrium does not apply.

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