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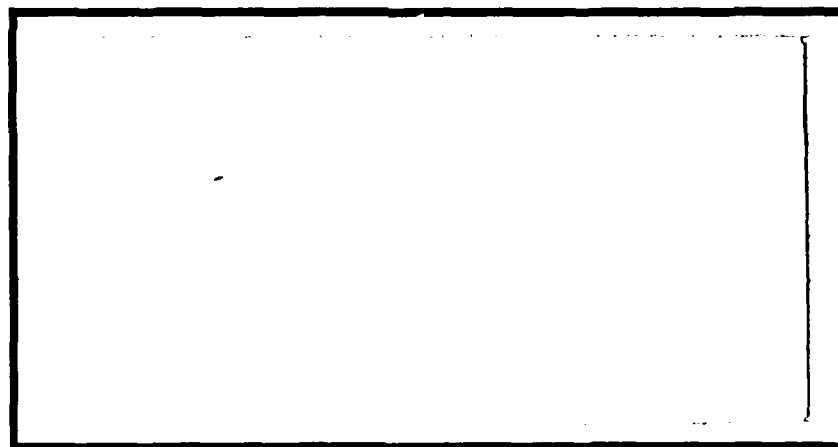
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A COMPARISON OF VARIABLE SELECTION
CRITERIA FOR MULTIPLE LINEAR
REGRESSION: A SIMULATION STUDY

THESIS

Ross J. Hansen
First Lieutenant, USAF

AFIT/GOR/MA/88D-3

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A COMPARISON OF VARIABLE SELECTION CRITERIA
FOR MULTIPLE LINEAR REGRESSION: A SIMULATION STUDY

THESIS

Presented to the Faculty of the School of Engineering
of the Air Force Institute of Technology

Air University

In Partial Fulfillment of the
Requirements for the Degree of
Master of Science in Operations Research

Ross J. Hansen, B.S.

First Lieutenant USAF

December 1988

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Ross J. Hansen



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Abstract

The purpose of this thesis was to identify three promising least squares selection procedures discussed in the literature during the previous decade and then test them using simulation. The three criteria chosen for this study were minimum mean square error (Min MSE), minimum S_p , and minimum C_p .

Most of the previous simulations in this area are limited to investigating the usefulness of variable selection criteria when all relevant regressors and some noise variables are available. It is questionable whether all relevant variables will be included. This research has examined the effects of not including a significant variable in the variable pool.

In examining each criterion, emphasis was placed on the technique's performance under varying amounts of multicollinearity, variable variation, number of variables, and sample size. Response Surface Methodology was used to determine the effects of varying these factors. A comparison was then made using the results from the Response Surface.

To supplement the simulation research a comprehensive literature review of the most current journal articles

dealing with several least squares criteria has been provided. This review includes a discussion of each technique's strengths and weaknesses. Since many of the least squares variable selection criteria are addressed, this thesis serves as a useful starting place for various regression questions.

**A COMPARISON OF VARIABLE SELECTION CRITERIA
FOR MULTIPLE LINEAR REGRESSION: A SIMULATION STUDY**

I. Introduction

Background

Linear regression is a statistical tool used to fit data to a surface. From the surface, predicted values for the dependent variable are determined with confidence. Unfortunately, a difficult aspect of regression analysis is determining the best set of independent variables to include in the linear regression model.

Before the wide use of computers, the analyst was forced to rely on his intuition for determining which variables should be included in the model. Variable selection techniques were available but difficult to implement. It was important for the analyst to "screen" the variables first by a reasonability test. For example, say the dependent variable is the height of a man. Possible independent variables are: size of hand, size of feet, and blood-type. It seems reasonable to assume that the height of a man is dependent on the size of his hands and feet. But, it does not make sense to assume his blood-type significantly contributes to his height. Therefore, the

analyst would include only feet and hand size into his variable pool. The analyst could perform his analysis on all the variables in the variable pool, which is ordinary least squares (OLS), or he could use a variable selection method to find the most significant variables. If he simply calculates the regression equation containing both feet size and hand size, the method is called OLS. A better model might include only one of these variables, so the analyst might want to use one of the selection criteria on the variable pool.

Due to computer limitations, only a few criteria for selecting variables were available, and the literature outpaced implementation. With computer advancements, implementation of new techniques should be increasing; however, this is not necessarily true. Often, these published techniques are forgotten. Rarely are the "new techniques" tested and implemented. In fact, most of the techniques currently in use are 1970 vintage, or earlier. To be sure, some of these "new techniques" are superior to those in use. Since these criteria are lost in the literature, research may be suffering.

Several problems can occur if the variable selection is left solely to the computer. First, there are literally hundreds of variable selection criteria available. Under certain circumstances, one method may be superior to another. By mindlessly using a technique, the resulting

model might be worthless. Certainly there are methods which "screen" better than others. However, there is not a method which absolutely picks only true variables. During a simulation study, Flack and Chang demonstrated that when the variable pool contains true variables, those which significantly contribute to the dependent variable, and random noise variables, random noise variables were often chosen (8:85). Only under the most ideal circumstance were a fair amount of true variables chosen.

The problem of selecting noise terms was further illustrated by Freedman. (9) By constructing simulated data containing only noise variables, Freedman demonstrated that high R^2 values could result when a model contained variables which are theoretically independent of the dependent variable. (9:153) The point being, selection criteria are not fail proof. The model chosen could possibly contain several noise variables. The more noise variables in the variable pool (those not screened by other means), the more likely one or more will be included in the model.

A subsequent simulation study performed by Hoerl, Hoerl, and Schuenemeyer identified and tested the effects of sample size, random noise variables and correlated data on biased regression techniques and least squares regression. (13) Their results indicated that both biased and least squares regression performed equally well except for stepwise

regression and principle component regression which performed poorly. (13:369)

Objective

The objective of this research is to identify three promising least squares selection procedures discussed in the literature during the previous decade and then test them using simulation. The simulation was similar to Flack and Chang's in the sense that the effects of noise variables and sample size were examined (8:84-86), but did not make the same assumptions and used different criteria. Flack and Chang tested the R^2 and stepwise procedures. Like several other simulations, they included all the true variables and some noise variables and then determined how many noise variables were selected. This research examined three criteria, minimum mean square error (Min MSE), minimum S_p , and minimum C_p and included random noise variables. An extension of Flack and Chang's work has been made. Flack and Chang's research did not examine the problem of not including all the significant variables. In the previous example, data on the height, size of feet, size of hands, and blood-type of a man was collected. It is possible that the weight of a man is also a significant factor in determining the man's height, but that no information has been collected on the man's weight. Most criteria are based on the assumption that all relevant variables are included in the variable pool; however, the fact is, all relevant

variables may not be included. This research has examined the effects of not including a significant variable in the variable pool.

In examining each criterion, emphasis was placed on the technique's performance under varying amounts of multicollinearity, variable variation, number of variables, and sample size. Response Surface Methodology was used to determine the effects of varying these factors. A comparison was then made using the results from the Response Surface.

Previous simulations have based their comparisons solely on the number of noise variables chosen. (2;8;12) The weakness of this performance measure lies in the fact that as the number of variables in the variable pool changes, the likelihood of choosing solely the correct variables also changes. In this study an alternative performance measure was be used for comparison.

To supplement the simulation research a comprehensive literature review of the most current journal articles dealing with several least squares criteria is provided. This review includes a discussion of each technique's strengths and weaknesses. Since many of the least squares variable selection criteria are addressed, this thesis serves as a useful starting point for various regression questions.

II. Concept Overview

Least Squares Regression

Assumptions. Assumptions are made prior to constructing a least squares linear regression. First, assume the collected data represents the population it came from. That is, the data reflects the normal case of the variable. Second, assume the error terms are independent and identically distributed, from a normal distribution, with a mean of zero and variance σ^2 .

Notation. The goal in linear regression is to find the "best-subset" of independent variables to include in the model which adequately predicts the value of the dependent variable. In general, the linear least squares regression equation is written in the following manner:

$$Y = B_0 + B_1X_1 + B_2X_2 + \dots + B_kX_k + \epsilon \quad (1)$$

where Y is the observed value of the independent variable.

B_0 is the constant term.

$B_1, B_2, \dots, B_k$ are the constant terms for the dependent variables $X_1, X_2, \dots, X_k$.

k is the number of independent variables included in the model.

ϵ is the error term.

If there are n observations, or data points, the above equation may be written as:

$$\sum Y_i = \sum (B_{0i} + B_{1i}X_{1i} + B_{2i}X_{2i} + \dots + B_{ki}X_{ki} + \epsilon_i) \quad (2)$$

For convenience, the above equation can be written in matrix notation.

$$Y = X B + \epsilon \quad (3)$$

where Y is a $(n \times 1)$ column vector:

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}$$

and X is a $(n \times (k+1))$ matrix.

$$X = \begin{bmatrix} 1 & X_{11} & X_{12} & \cdot & \cdot & \cdot & X_{1k} \\ 1 & X_{21} & X_{22} & \cdot & \cdot & \cdot & X_{2k} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & X_{n1} & X_{n2} & \cdot & \cdot & \cdot & X_{nk} \end{bmatrix}$$

The first column contains all ones for the constant terms.
The remaining columns contain the X_{ij} independent variables.
The X matrix is commonly referred to as the design matrix.
 B is a $(k \times 1)$ column vector:

$$B = \begin{bmatrix} B_0 \\ B_1 \\ \cdot \\ \cdot \\ B_k \end{bmatrix}$$

and ϵ is a $(n \times 1)$ column vector:

$$\epsilon = \begin{bmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ e_n \end{bmatrix}$$

In least squares regression, each subset of regression variables generates a surface which minimizes the squared distance (error) between the observed values for the dependent variables, Y , and the predicted values for the dependent variable $\hat{Y}$.

$$\min \sum \epsilon^2 = \min \sum (\hat{Y}_i - Y_i)^2 \quad (4)$$

The goal is to find the subset of variables which minimizes the squared distances between the actual values observed and the fitted surface. The sum of the squared-error values is commonly referred to as the sum-of-squares error (SSE).

Graphically, in two dimensions a regression resembles the following:

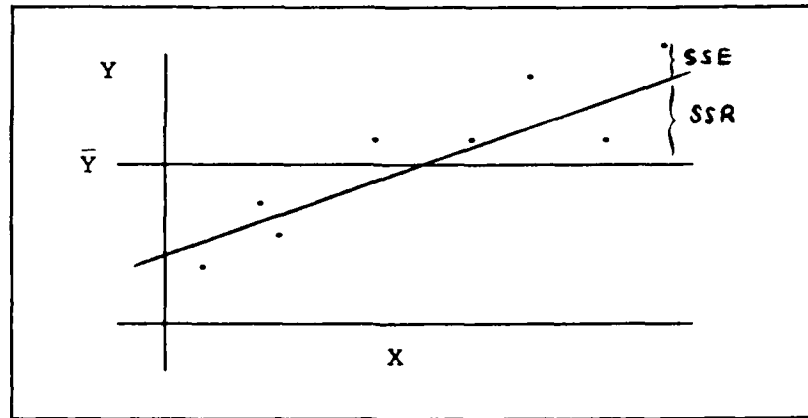


Figure 1. Two-dimensional representation of linear least squares regression

where SSR is the sum of the squared distances from the mean to the regression line and is called Regression Sum of Squares.

SSE is the sum of the squared distances from the point to the regression line and is called Sum of Squares Error.

SST is the sum of SSR and SSE and called Sum of Squares Total.

$$SST = SSR + SSE \quad (5)$$

III. Review of Past Research

Scope

The main focus of this literature search is limited to variable selection techniques published since 1978. This limitation was chosen for three reasons. First, two GOR students, Captain Larry J. Pulcher (24:1-137), and Second Lieutenant Joseph R. Cafarella, Jr. (4:1-127), performed literature reviews on variable selection techniques for their theses, written in 1978 and 1979 respectively. Second, most of the implemented techniques were published prior to 1978. Third, an excellent review of the pre-1980 techniques can be found in Hocking's 1976 paper (12).

Method of Treatment and Organization

This literature review reports methods for selecting the number of variables to include in a linear regression equation, as well as specific techniques for selecting the "best-subset" of variables to include in the regression equation.

Review

If the true variance, σ^2 , of the independent variables is known, selecting the proper regression model is relatively simple (7:294). However, the true variance is rarely known a priori. It is more likely that the underlying variance will have to be estimated. Therefore, techniques must be

available for the analyst so that he may properly select the variables to include in a linear regression equation.

Number of Variables. Draper and Smith (7:294) discuss two opposing viewpoints concerning the proper number of independent variables (or regressors) to include in a linear regression equation. One viewpoint is all variables should be included for both a reliable fit and predictive purposes. The opposite point-of-view is to include as few variables as possible, which adequately predict, in the model. The latter point-of-view is commonly referred to as parsimony.

For predictive purposes, Bayesian statisticians recommend including all possible variables (28:1553). They argue the cost associated with collecting and maintaining the data must be considered. If these costs are zero, than every variable should be included. Only by including as many variables as possible can one be certain to obtain the best prediction.

Cafarella (4:14) points out the major flaw of including every possible variable. The resulting model over-fits the data. That is, the independent variables not only predict the dependent variable, but also predict the variation, or noise. The resulting equation is useful for predicting the past, but not very useful for predicting the future. Thus, it is arguable whether including all possible regressors is the best course of action.

Even the Bayesian statisticians agree that if the costs of collecting and maintaining data are high, then one should include as few variables as possible in the model. However, some believe in the concept of parsimony regardless of the costs. Parsimony is predicting as much as possible, with as little as possible. Trader states, "Variable selection techniques, which facilitate parsimony, have been viewed primarily as providing approximate representations of a true underlying process." (28:1553) The advantages of parsimony is that model does not over-fit the data, and it does not cost as much as a larger model to collect and maintain. However, the question remains: does the model contain enough variables to predict adequately?

Recent literature (23:509-16; 3:131-6; 24:45-54) suggests a method for determining the optimal number of regressors for a linear regression equation. The two driving factors for determining the optimal number of regressors are sample size and the mean square error of prediction (MSEP).

(23:514) MSEP is the squared difference between the predicted value, $\hat{Y}_{n+1}$, and the actual value, Y_{n+1} , divided by the degrees of freedom. The MSEP has been decomposed by Breiman and Freedman (3:131) into the conditional MSEP, and the unconditional MSEP. The conditional MSEP is

$$M = M_{np} = \sum \{ (\hat{Y}_{n+1} - Y_{n+1})^2 / Y_i \text{ and } X_{ij} \text{ for all } j \text{ and } i = 1, \dots, n \} \quad (6)$$

while the unconditional MSEP is defined as

$$U = U_{rp} = E[M_{rp}] \quad (7)$$

With the mean square error of prediction and a given sample size, one can find the optimal number of parameters, p^* , to include in the model.

$$p^* = k + 1 \quad (8)$$

where

p^* is the number of parameters in the model
with Min MSEP.

k is the number of independent variables.

When either M or U from Eq (5) and Eq (6) are minimized, the corresponding p^* is the optimal number of parameters for the regression model. It is apparent that when the sample size is small, the optimal number of parameters tends to be small (28:1562).

Choosing the Best Subset of Variables. Regardless of whether the optimal number of parameters has been determined, a criterion must be used to select which variables should be included in the regression equation. Miller separates possible criteria into "(i) those which guarantee to find the best fitting subset of some of all sizes, and (ii) the 'cheap' methods which sometimes find the best-fitting subsets." (20:391)

"Guaranteed Methods". Miller specifically mentions two methods which guarantee choosing the best-fitting

variables. These two methods are all-subset and branch-and-bound.

All-Subsets. In the 1984 article, "Selection of Subsets of Regression Variables" (20:391), Miller claims that only by an exhaustive search of all $2^k - 1$ possible subsets can one be assured of finding the best-fitting model. He further suggests the all-subset procedure is practical only when the number of possible variables is less than twenty. With the improvement of computers over the last five years, it might be feasible to test every subset for more than twenty variables. However, the necessary number of regressions increases rapidly. For instance, with twenty variables, over one million subsets must be fit. By increasing the number of variables by just five to twenty-five, the number of necessary regressions increases to roughly 33.5 million.

The all-subset procedure is not without criticism. One of the underlying assumptions in regression is the variable pool contains all the relevant variables and some extraneous variables. (21:160) However, it is possible some of the relevant variables are not included in the variable pool. Because of this, the all-subset procedure may not find the best model in population terms, but always finds the best model within the variable pool (2:3). In fact, backward elimination, (one of the "cheap methods," which will be discussed later), does slightly better than all-subsets

under certain circumstances. Although Berk criticizes the all-subset procedure under certain circumstances, he notes that it usually does well (2:3). With large sample sizes (approaching the population size) all-subset is the best.

Branch-and-Bound. Miller includes the branch-and-bound technique as one of the guaranteed methods when the optimal number of variables has been determined (20:391). The branch-and-bound technique computes only a fraction of the possible models. It eliminates subsets from consideration if a calculated subset fits better than the eliminated subset could possibly obtain. The eliminated subsets are said to be "dominated." With some of the models "dominated," the all-possible subset procedure can be implemented on the subsets that are not eliminated by branch-and-bound. The branch-and-bound technique is not only useful in eliminating possible subsets, but can be modified to be an efficient all-subset procedure (10:510). The necessary modification requires that the "testing" mechanism be turned off.

"Cheap-Methods." Miller conveniently labels all techniques other than the all-subset and the branch-and-bound procedures as "cheap." These techniques are cheap in the sense that they do not require as many calculations for finding a solution. Consequently, none of the cheap-methods are guaranteed to find the best-fitting model.

Backward Elimination. Backward elimination is a technique which considers all of the possible variables in the original model, then eliminates, one at a time, those variables which do not significantly contribute to the regression. According to Draper and Smith, "this [backward elimination] is a satisfactory procedure, especially for statisticians who like to see all the variables in the equation once in order 'not to miss anything'." (7:307) This procedure is satisfactory if the data is not highly correlated.

Nonetheless, backward elimination has its drawbacks. One major drawback is once a variable is eliminated, it is not considered again. It is possible that an eliminated variable, which did not look significant when it was eliminated, will look significant after other variables are eliminated. Another criticism of backward elimination is that it is not useable when the number of possible variables exceeds the number of data points. Due to the matrix manipulations necessary in regression, this procedure cannot be used when variables exceed data points.

Forward Selection. Forward Selection is similar to backward elimination. However, instead of all of the variables being included in the original model, then eliminating one variable at a time, forward selection starts with one variable, and includes a variable-at-a-time. The major advantage of this technique is that it can be

implemented when the number of possible variables exceeds the sample size. This technique can be used since the original model has one variable, to which variables are subsequently added. (20:392).

The disadvantages of forward selection are similar to backward elimination. Like backward elimination, forward selection performs poorly when variables are highly correlated. Also, once a variable enters the regression equation, it is no longer considered for elimination.

Stepwise Regression. Stepwise regression is a procedure which combines both the forward and backward selection procedures. Like forward selection, variables are selected to enter one at a time. However, after a variable enters the regression equation, all variables in the subset are reconsidered for elimination. This improvement allows for the deletion of those variables which do not significantly contribute to the model.

Draper and Smith recommend this technique. However, they add the caveat that it is not be used blindly (7:310). If stepwise regression is used with little thought, many random noise variables can be selected. As is the case with all of the techniques, the likelihood of selecting a noise variable is fairly high if the number of noise variables is high.

To combat this problem Miller suggests an alternative stopping criterion for stepwise regression. His suggestion is to add "known" extraneous variables to the variable pool

(randomly generated noise) then execute the stepwise regression procedure. Once one of the "known" extraneous variables is selected, terminate the selection process. (20:395) The concept behind his suggestion is that once a "known" extraneous variable (one that was augmented to the variable pool) is selected, there is no more useful information to be gained by adding another variable. The model selected will include all variables picked not including the known extraneous variable. For the technique to be useful, Miller recommends that the number of added extraneous variables should be approximately equal to number of variables already in the variable pool. (20:395)

Excluding its weakness when several extraneous variables are in the variable pool, stepwise regression performs well when there is little correlation. Many computer packages use a significance level of 0.05 for forward, backward, and stepwise procedure. The choice of 0.05 seems to be arbitrary and unjustified. In fact, the results of a simulation study conducted by Hoerl, Hoerl, and Schuenemeyer indicate that a significance level of 0.05 often leads to poor results (13:378). They demonstrated that many of the true variables are not selected when the significance level is 0.05. However, most variables are picked using a significance level between 0.15 and 0.25

Near-Optimal-Model. Narula and Wellington (21:169) propose the near-best-model for Mean Square

Absolute Errors (MSAE) in 1983. MSAE is the sum of the absolute difference between the predicted value of Y and the actual value of Y, divided by the degrees of freedom.

$$\frac{\sum | \hat{Y}_i - Y_i |}{(n - p)} \quad (9)$$

where

n is the sample size.

p is the number of parameters in the model.

$\hat{Y}$ is the predicted value for Y.

Y is the actual value of Y.

Narula and Wellington put a "lower and an upper limit on the number of variables to be included in an effort to accelerate the implicit enumeration algorithms and aid the investigator in selecting a model." (21:169) Although they apply the near-optimal technique to MSAE, they suggest it may be effective in the least squares case.

Similar to Narula and Wellington's proposed near-best-model for Mean Square Absolute Errors, Huang and Panchapakesan have implemented a procedure to eliminate inferior models with some guaranteed probability (14:753). Their procedure is based on the residual sums of squares and is similar to Furnival and Wilson's branch-and-bound technique. The near-best-model for Mean Square Absolute Errors is based on SSR making it easy to implement. What makes this technique especially appealing is that once a

model is rejected, then all combinations of its variables are also eliminated. (14:758).

An example of how the near-best-method works is the following:

Suppose the subset of variables under consideration is $\{X_1, X_2, X_3\}$. If this subset of variables is rejected, then all subsets of X_1, X_2 , and X_3 would also be rejected (e.g. $\{X_1\}$, $\{X_2\}$, $\{X_3\}$, $\{X_1, X_2\}$, etc.).

Mallows C_p . Mallows C_p is a statistic used to determine the best model when the independent variables are fixed. C_p is an approximation of MSEP.

$$C_p = \frac{SSR}{s^2} + 2p - n \quad (10)$$

where SSR is the Regression Sum of Squares

s^2 is the estimate for the variance

p is the number of parameters

n is the number of data points

Theoretically the value of C_p is p . Therefore, when C_p is approximately equal to p , the model is good. Draper and Smith suggest using this criterion in conjunction with stepwise regression to obtain the best subset (7:341). It should be noted, however, as the variance approaches zero, the C_p statistic can not be calculated. Therefore this method has limitations especially when the fit is perfect.

Barr pointed out a weakness of Mallows C_p . Since s^2 , used in the C_p statistic, is estimated from the original variable pool, it could be biased and larger than the true variance. (1:5) If this is case, the C_p statistic will be deflated causing the wrong model to be selected.

A limitation of C_p , as well as many other statistics, is that it "depend[s] on the observed data only through sufficient statistics, so they model average behavior of the fit of a model to the data." (30:27) Weisberg developed a procedure which allocates the C_p statistic to individual cases. The advantage of Weisberg's procedure is if the model under consideration is biased, it provides a means to determine the bias of using a subset model instead of the entire model (30:28)

Another application of the C_p statistic is to choose the model which has the smallest C_p value. (15:863) By choosing the model with the minimum C_p , it is believed that one is choosing the model with the minimum prediction error. This is appealing, especially when it is difficult to determine the optimal subset using the C_p close to p criterion. Since the Min C_p criterion is based on minimum prediction error it is based on a sound principle. However, like the C_p close to p criterion, Min C_p is derived under the assumption that the independent variables are fixed. Since this rarely happens in practice, there is some question to the usefulness of the Min C_p criterion. Judge,

Griffiths, Carter, Lutkepohl, and Lee recommend that the Min C_p procedure should not be used in any applied work.

(15:864)

Coefficient of Determination. The coefficient of determination, R^2 , is a statistic which gives an estimation of the amount of variation about the mean which is explained by the model.

$$R^2 = \frac{\sum (\hat{Y}_i - \bar{Y})^2}{\sum (Y_i - \bar{Y})^2} \quad (11)$$

where

$\hat{Y}_i$ is the predicted value of Y_i .

Y_i is the actual value of Y_i .

$\bar{Y}$ is the mean of Y .

At first one might believe that it is desirable to find the model which has the maximum R^2 , since it explains the most variation about the mean. However, this is not necessarily the best true. Certainly when we look at the R^2 value we would like to see a large value, but it should not be used as the only measure for subset selection. Maximum R^2 receives little praise as far as its usefulness in determining a good fit. The major pitfall of using R^2 is that whenever a variable is added, it will increase R^2 . R^2 will increase regardless of whether the variable has anything to do with the dependent variable. According to Healy 1986, "In particular, the multiple correlation coefficient is not really a regression-related concept at

all. It is basically defined to be the largest possible correlation between the y-variate and any linear function of the x's and this only makes sense when y and x's have a joint probability distribution." (11:1984) If maximum R^2 is used as the selection criterion, the model containing all variables will always be selected.

Maximum Adjusted R^2 or Minimum MSE. For simplicity only maximum Adjusted R^2 will be discussed. However, maximum Adjusted R^2 and Minimum MSE test exactly the same thing.

Adjusted R^2 is related to R^2 , but an adjustment has been made for the degrees of freedom. The following equation shows the relationship between R^2 and Adjusted R^2 .

$$\text{Adjusted } R^2 = 1 - \frac{(1-R^2)(n-1)}{(n-p)} \quad (12)$$

According to Draper and Smith, the adjusted R^2 statistic can be used not only to compare models for the same data set (the same variable selection discussed in all other sections of this literature review), but also to compare models taken from two entirely different data sets (7:92). However, they do not recommend using the Adjusted R^2 statistic in the latter role.

The Adjusted R^2 statistic (or the minimum MSE criterion) is still widely used in practice.

S_p . The S_p criterion, originally proposed by Hocking in 1976 (12:20), has considerable appeal and

consequently receives praise in recent years. The S_p statistic is an approximation of the MSEP based solely on the data and number of variables. As is the case with MSEP, the goal of this criterion is to find the minimum value.

$$S_p = \frac{SSE}{(n-p)(n-p-2)} \quad (13)$$

Breiman and Freedman point out that the S_p statistic does not necessarily provide an accurate approximation of MSEP, but works none the less. (3:132)

The advantages of this method are numerous. Looking at Eq (12) gives the reader an idea of the relative ease with which S_p is calculated. What makes S_p even more appealing is it is based on MSEP. Thompson points out, "This method [S_p] is based on a sound criterion - that of minimizing the expected squared distance between the true and predicted values of the dependent variable, y ." (27:6) Since S_p is an approximation of MSEP, it can be used like MSEP to determine the optimal number of regressors to include in the model. (3:132)

S_p is not without its disadvantages. It must be calculated for all $2^k - 1$ possible subsets. (27:6) Even though it requires relatively little computation effort, it does require that many regressions be run. Through counter examples Breiman and Freedman show that when true variance due to prediction equals zero, the S_p criterion fails to

pick the optimal number of variables to include in the model. (3:132)

Conclusion

A considerable amount of literature has been devoted to linear regression in the last decade. This review has covered a few of the techniques mentioned in the literature. By no means should this review be considered exhaustive. However, it is intended to provide an excellent review of many least squares regression techniques from the last decade.

Since the calculations required for the "guaranteed" all-subsets method virtually make it impossible to use, much of the effort in the field is to find techniques which require far less calculation. Unfortunately, the savings in computer time is off-set by the results. None of the techniques mentioned in the "cheap" section of this report guarantee the best-fitting subset of regressors will be chosen.

Biased Regressors

Up to this point the discussion of literature has been restricted to selection techniques for least squares. The advantage of least squares over the techniques discussed in this section is that the estimators are best linear unbiased estimators, or BLUE. However, under certain circumstances

the estimators from least squares can be inferior to biased regressors.

An estimator is said to be unbiased if its expected value of the estimator is the parameter value itself.

$$E(b) = B \quad (14)$$

Conversely, if the expected value of the estimator is not equal to the parameter value, it is said to be biased.

The following techniques are intended to provide the reader with some insight into when the use of biased estimators may be a better course of action than least squares. Rather than discuss several biased techniques, this discussion will be limited to two methods which have received a great deal of attention in recent literature. Both ridge regression and principle components, the two methods to be discussed, are methods for dealing with multicollinearity.

Ridge Regression. There has been a significant amount of attention given to Ridge Regression in the past decade. Ridge Regression is a procedure which attempts to overcome the effects of highly correlated data. As mentioned before, many of the least squares selection criteria perform poorly under such conditions.

There are two situations for which Draper and Smith "wholeheartedly recommend ridge regression." The first situation is when there is "a Bayesian formulation of a regression problem with specific prior knowledge of a

certain type on the parameters." (7:319) The advantage over least squares in this situation is that the B's are restricted a priori. The second situation in which Draper and Smith recommend Ridge Regression is "a formulation of a regression problem as one of least squares subject to a specific type of restriction on the parameters." (7:320) Draper and Smith comment that under the two previous situations ridge regression is absolutely the correct model to use. However, they also point out that under any other situation, there needs to be further investigation to determine the usefulness of ridge regression.

Dempster, Schatzoff, and Wermuth performed a simulation study on alternatives to least squares regression in 1977. They believe ridge regression offers a drastic improvement over least squares regression when there is a great deal of correlation between the independent variables (5:77). An interesting comment by the authors is, "From a frequentist standpoint, it has long been recognized that good mean squared error properties do not necessarily follow from the celebrated minimum variance unbiasedness properties of least squares, since in certain regions of the parameter space the loss from increasing the squared bias can be overcompensated by reducing variance." (5:77)

In a subsequent study (19), Makin tested the ridge regression procedure and found it performed well. Even

though his models were fairly trivial, he concluded that ridge regression is, in fact, a good technique.

It should not, however, be misconstrued that ridge regression is a cure-all. If many of the variables under consideration are not related to the dependent variable ridge regression does not perform very well (13:375). A suggestion to counter this drawback is to delete a portion of the variables if there is reason to believe many have little to do with the independent variable. Another caveat of using ridge regression is to recognize there is an entire family of ridge regression techniques. One in particular, REGF, is not recommended by Dempster, Schatzoff, Wermuth as a practical tool.

Principle Components. Principle components is another technique used to combat collinearity and has received a considerable amount of attention recently. Collinearity tends to yield highly unstable estimates because the design matrix is nearly singular. The goal of principle components is similar to ridge regression in the sense that the goal is to produce estimates which are not greatly effected by collinear data. However, unlike ridge regression which assumes nonsample information (15:909), principle components does not assume any nonsample information is available.

In principle components, only subsets of the variables are considered. The choice of subsets can be made "by economic theory, previous statistical results, or ad hoc

dimensionality reduction procedures." (15:909) By reducing the number of variables under consideration, principle components attempts to eliminate variables which do not necessarily contribute to the model and cause unnecessary collinearity.

There is some criticism of the method of principle components. One criticism by Judge, Griffiths, Hill, Lutkepohl, and Lee, is that this method relies on the data in hand, which is the same weaknesses of every search method. It is not guaranteed to find the best subset of regressors to include in the model (15:910). It is quite possible that the subset of variables chosen does not contain all of the true regressors. In their simulation study, Hoerl, Hoerl, and Schuenemeyer demonstrated that principle components performs poorly. (13:375) "It was rarely noticeably better than LS [least squares], often worse than LS and uniformly inferior to RRB [a ridge regression technique] for all criteria." (13:375)

Conclusion

If a situation arises when significant multicollinearity exists, and it is not appropriate to use least squares, it may be appropriate to use a biased regression technique. Two of the methods which have received recent attention are ridge regression and principle components. However, like least squares techniques, these techniques give guarantees for finding the best model. Also like least squares, both

of these techniques require certain conditions to perform optimally. Ridge regression requires some type of nonsample information while principle components does not. However, principle components does not perform nearly as well as ridge regression and even least squares. These trade-offs must be considered when deciding on a technique to combat collinearity.

IV. Methodology and Model Development

The three techniques selected for this research were minimum MSE, minimum C_p , and minimum S_p . These techniques were chosen for the following reasons:

(1) Each of these techniques are absolute criterion. That is, they is no "art" necessary. The S_p , C_p and MSE statistic for each subset of variables is obtainable under some all-subset procedures, such as PROC R^2 on SAS. One only needs to find the minimum value for each technique and the corresponding set of variables to implement these techniques. Contrasted with the C_p close to p technique which leaves confusion as to whether a model is superior to another simply on the basis of the difference between C_p and p .

(2) All three techniques appear in the last decade's literature. The Minimum MSE procedure used to be one of the most widely used methods. Its appeal over techniques such as Max R^2 stems from its adjustment for degrees of freedom. More recently, S_p seems has become the most popular technique. Its appeal is based on the principle of minimizing mean square errors of prediction.

(3) The C_p criterion is also based on MSE, and some of authors praise this criterion.

Overview

With any procedure, certain assumptions must hold to insure proper execution and least squares is no exception. As mentioned, by using least squares one assumes the data is typical of the population, and the error terms are independent and identically distributed from a normal population with an expected value of zero and a constant variance σ^2 . Another assumption often made, yet not entirely justified, is that all the variables collected relate to the dependent variable.

Objective

The goal of this thesis is to determine the behavior of the Min MSE, Min C_p , and Min S_p variable selection criteria. To do this, response surface techniques was implemented.

Response Surface Methodology

When selecting variables to include in the original variable pool for the least squares regression equation, it is desirable to select only those variables which significantly contribute to the model. However, there is no guaranteed method to screen the extraneous variables (random noise terms which do not contribute at all to the model) from the variable pool. Even more discouraging, once in the variable pool there is no criterion which guarantees that no extraneous variables will be chosen for the model. The problem of selecting extraneous variables is not restricted to the "cheap" methods. In fact, one of Miller's guaranteed

methods, the all-subset procedure, which is suppose to perform quite well, also occasionally chooses extraneous variables. The limitations of variable selection criteria, including the "guaranteed" techniques will be emphasized since all three of the criteria under consideration are all-subset procedures.

Since there really are no "guaranteed methods" for capturing all and only the true variables, it is advantageous to strive for the highest percentage of those variables chosen correctly. Therefore the performance measure for the Response Surface is the following:

$$PM = \frac{(\text{number of correct variables chosen})}{(\text{number chosen})} \quad (15)$$

PM is the logical choice for two reasons. First, the best model may not include all variables it is generated from, but only the most significant. Even though an independent variable may have been generated from ten variables, the best model of reality using, say MSE, may only contain two of those variables. Therefore, PM compensates by determining the percentage of correct variables chosen. Second, PM is influenced by the number of extraneous variables chosen. If only two variables are chosen, and one of them is an extraneous variable, it is worse than when ten

variables are chosen and only one of them is extraneous. PM allows for such an adjustment.

By using Response Surface Methodology, equations made up of significant factors (multicollinearity, sample size, etc.) for each of the three criteria can be fitted. Using these equations, a comparison of all three criteria under best-case and worst-case scenarios can be completed.

Data Generation For Response Surface. The data for this study was generated from the following equation:

$$Y_i = X_{1i} + X_{2i} + X_{3i} + X_{4i} + \epsilon_i \quad (16)$$

where

Y_i is the dependent variable.

$X_{1i}, \dots, X_{4i}$ are correlated, randomly generated dependent variables.

ϵ_i is a noise term to create variance in the model.

Most simulation studies investigate variable selection techniques with all relevant dependent variables plus some extraneous variables included in the variable pool. This study attempts to show what happens when one of the relevant variables is not included in the variable pool.

After the data is created from equation (16), the independent variable, X_4 , is dropped from consideration. This simulates the situation which arises when a significant variable is not included in the variable pool. In addition

either one or three noise variables are included in the variable pool to simulate data collection on extraneous variables.

Factors. An equation made up of significant factors and factor interactions which adequately predicts the usefulness of each of the three techniques must be found. Ideally, the significant factors are observable a priori (e.g. sample size, correlation, etc.), thus, allowing for compensation prior to use. The following are the factors for this study:

(1) The number of extraneous variables (EX_1, EX_2, EX_3) in the original variable pool. These variables are noise, and therefore are theoretically independent from the dependent variable. In this study, at the low setting the number of extraneous variables is 1, and 3 for the high setting.

(2) The amount of the correlation between the dependent variables, X_1, X_2, X_3 , and X_4 . The low setting is orthogonal, or zero correlation, while the high setting is 0.9, which is highly correlated.

(3) The variance of the extraneous variables. The low setting for the variance is 1, and the high setting is 100.

(4) The variance of the independent variables. The low setting for the variance is 1, and the high setting is 100.

(5) The sample size. The low setting for the sample size is deliberately near the threshold of the usefulness of the S_p criterion. A requirement of the S_p statistic is $(n-p-2)$ should be greater than zero. At the low setting the sample

size is 10. The maximum value for p is 7, in the three extraneous variable case). When n equals 10 and p equals 7, the denominator of the S_0 statistic is one. Therefore, the threshold for sample size is 10. If the sample size were any smaller, S_p could not be found for all possible subsets. Thus, the low setting for sample size is 10, while the high setting is 20.

(6) The variance of the ϵ term. The low setting for the variance of ϵ is 0.0625, and the high setting is 0.25. The computer program used to create the correlated data can be found in Appendix B.

When using Response Surface Methodology it is convenient to work with coded factors (-1, 1 variables) for the following reasons:

(1) By coding the factors, the resulting variables are of the same magnitude.

(2) The calculations necessary for the estimates are simplified.

(3) The resulting design matrix, Z , is orthogonal. As a result, the stepwise procedure can be used to find the significant factors with confidence.

The following equations are necessary to code the variables:

$$Z_1 = \text{number of extraneous vars} - 2$$

$$Z_2 = (\text{correlation} - 0.45) / 0.45$$

$$Z_3 = (\text{variance of extraneous vars} - 50.5) / 49.5$$

$$Z_4 = (\text{variance of independent vars} - 50.5) / 49.5$$

$$Z_5 = (\text{sample size} - 15) / 5$$

$$Z_6 = (\text{variance of } \epsilon \text{ term} - 0.15625) / 0.09375$$

where $Z_1, \dots, Z_6$ are the coded variables.

Table I.
Variable Coding for Response Surface Methodology

Non-Coded			Coded		
Variable	Low	High	Variable	Low	High
Number of extraneous vars	1.0	3.0	Z_1	-1	1
Correlation	0.0	0.9	Z_2	-1	1
Variance of extraneous vars	1.0	100.0	Z_3	-1	1
Variance of independent vars	1.0	100.0	Z_4	-1	1
Sample Size	10.0	20.0	Z_5	-1	1
variance of ϵ term	0.0625	0.25	Z_6	-1	1

Experiment. It seems reasonable to assume significance in interactions between factors. For example, there might be a significant interaction between two of the main factors mentioned above. To insure the accurate calculation of estimates for both the main factors and

factor interactions, a full 2^6 factorial design is necessary. To construct the design matrix for a full factorial design, the main factors are varied from their low settings to their high settings in binary fashion. The interaction terms are simply the product of the corresponding main factors. An example of this process using a 2^2 full factorial design is summarized in the table below.

Table II.
Example of Coding Interaction Variables

Z_1	Z_2	$Z_1 Z_2$ (interaction)
-1	-1	1
-1	1	-1
1	-1	-1
1	1	1

If a design with less than 2^6 runs is used, information on some of the interactions would be unobtainable.

Each of the 2^6 runs contains 60 replications. The raw data was obtained by running regressions on unprocessed data sets from Appendix B. The code necessary to perform the regression analysis is found in Appendix C. A FORTRAN program was written to accumulate the statistics. The program and a sample output can be found in Appendix D and E respectively. The following statistics were collected for each of the 64 runs.

(1) The average number of variables chosen using each technique.

(2) The average number of correct and extraneous variables of those chosen by the technique.

Results

Since the design matrix for this experiment is orthogonal, stepwise regression is used to select the significant factors. Resulting equations strictly screen factors which are most significant to their respective response. The equations should not be used for predicting the percentage of correct variables chosen. Several unsuccessful transformations were attempted to obtain a reasonable model, however none made common sense. Many models appeared to fit well using several common criteria (R^2 , Adj R^2 , residual plots, etc), however, the models were not logical. Using stepwise regression with a significance level of 0.01, yielded the equations given below. Again, the role of these equations is restricted to determining the most significant factors for a comparison of the three criteria.

Min MSE. For the Minimum MSE case, only two main factors were significant. They are the number of extraneous variables in the variable pool and the sample size. Since the variables are coded and the values of the standard errors for all the estimates are equal, only one value is printed for the standard errors.

$$\begin{aligned} R^2 &= .9411 \\ \text{Adj } R^2 &= .9392 \end{aligned}$$

$$\text{ymse}(\% \text{ correct}) = .64448797 - .120997 (\# \text{ ex}) + .010415(\text{sample})$$

$$\text{standard errors} = .0039$$

where

ex is the number of extraneous variables allowed in the variable pool.

sample is sample size of each replication for a given run.

This equation provides insight into the usefulness of the Min MSE criterion. Within the data region, the effect of increasing the sample size can be observed. For every "extra" data point, an increase of 1.04 percent of correct variables chosen is expected. This is useful information since sample size can be moderately controlled. The equation also indicates a decrease of 12 percent of correct variables chosen for every extraneous variable allowed into the variable pool. At first this information seems irrelevant since there is no way of telling a priori if a variable is relevant. However, it does emphasize the need to screen the variable pool before using the Min MSE criterion. It also shows the danger of the kitchen sink approach. The kitchen sink approach is where every possible

variable is tossed into the variable pool and the computer is allowed to make all the decisions. Therefore, the two significant main factors using the Min MSE criterion are sample size and the number of extraneous variables, and the number of extraneous variables is the most significant.

Min S_p . For the S_p case, two main factors and one three-way interaction are significant.

$$\begin{array}{rcl} \text{Adj } R^2 & = & .8811 \\ R^2 & = & .8751 \end{array}$$

$$\text{y sp}(\% \text{ correct}) = .65598 - .1148(\# \text{ ex}) - .0184 (\epsilon \text{ var}) - .0168(\text{int})$$

$$\text{standard errors} = .0056$$

where # ex is number of extraneous variables.

ϵ var is variance of the ϵ term.

int is the three-way interaction term.

The three terms are: number of extraneous variables, the ϵ variance and the variance of the X variables.

The information obtained here is acceptable but not quite as useful as in the MSE case. Apparently the most significant factors which effect S_p within the observed data space is the number of extraneous variables and the variation of Y. Again, the point to emphasize is as few extraneous variables as possible should be allowed in the variable pool. It is interesting to note that sample size

is insignificant at the $\alpha=.01$ level. The low level set for sample size was near the threshold of S_p 's usefulness on purpose. Consequently, S_p might be a good criterion to use when the sample size is small.

Min C_p . Unfortunately the resulting equation for the C_p case is not as descriptive as the two previous cases. The C_p equation included two main factors, one three way interaction, two four-way interactions, and a five-way interaction.

$$\begin{array}{rcl} R^2 & = & .9280 \\ \text{Adj } R^2 & = & .9202 \end{array}$$

$$\begin{aligned} \text{ycp}(\% \text{correct}) &= .65 - .11(\# \text{ex}) - .01 (\epsilon \text{var}) \\ &\quad - .01(\# \text{ex}, \text{xvar}, \epsilon \text{var}) \\ &\quad + .02(\text{cor}, \text{xvar}, \text{exvar}, \epsilon \text{var}) \\ &\quad + .01(\# \text{ex}, \text{cor}, \text{xvar}, \text{exvar}, \epsilon \text{var}) \end{aligned}$$

$$\text{standard error} = .0045$$

where

#ex is the number of extraneous variables in the variable pool.

cor is the amount of correlation between the independent variables.

exvar is the variance of the extraneous variables.

xvar is the variance of the independent variables.

ϵ var is the variance of the ϵ term.

Only in this equation is the amount of correlation among the X 's significant. Even though it appears in the equation, it

is not one of the two main factors, but does appear in the four and five way interaction terms.

As is the case with the MSE equation and the S_p equation, the number of extraneous variables included in the variable pool is the most significant factor. However, probably the most notable result of this experiment is the repetition of variance factors appearing in the interactions. Actually this is expected, since the C_p statistic is heavily dependent on the total variation of the variable pool.

Comparison of Techniques

Up to this point, analysis has been limited to intra-technique. That is to say, the models constructed in the Response Surface phase show the effect of varying factors using a specific technique. Determining which technique actually outperforms the other two techniques is more useful.

Choice of a Comparison Criterion. Deciding which performance measure to use for comparing the Min MSE, Min C_p , and Min S_p variable selection criteria is not trivial. Due to the mechanics of the Min MSE procedure, the resulting subset of variables is always a super-set of both the Min C_p and Min S_p procedures. Also, in each of the sixty-four replications, the Min MSE procedure picked the highest percentage of correct variables of those chosen; additionally it picked the most correct variables in absolute terms. It seems the Min MSE criterion is superior

to the other two criterion. However, these statistics are a bit misleading.

In each of the sixty-four replications, the Min MSE criterion chose the most extraneous variables in absolute terms. Remembering the concept of best model versus true model, it is probably worse to have more extraneous variables in the model than the highest percentage of correct variables of those chosen. Therefore, comparing percentages to determine the "best technique" is not used.

Since there is no clear-cut criterion for comparing variable selection techniques, it was decided to compare how well each criterion performs using theoretical minimum mean square error of prediction (TMSEP).

$$\text{TMSEP} = \frac{\sum (Y_t - Y_p)^2}{(n - p)} \quad (17)$$

where

Y_t is the theoretical value of the independent variable Y .

Y_p is the predicted value of Y , using one of the three criteria.

n is the sample size.

p is the number of parameters in the predicting equation.

There are two reasons for picking TMSEP as the performance measure. First, MSEP has received considerable attention during the past decade as the most promising criterion for variable selection. Second, the S_p criterion, which is based on MSEP, is praised and considered by many as the method which insures the model selected has the Min MSEP.

The TMSEP criterion is a variation of MSEP. Like MSEP, TMSEP calculates the squared difference between the predicted value and the actual value of the independent variable and adjusts the value for degrees of freedom. However, TMSEP differs from MSEP in its calculation. TMSEP is the squared difference between the actual value, from the underlying equation that is generating the independent variable (not including the error term, ϵ), and the predicted value using the model picked by the variable selection procedure. The resulting value is the theoretical mean square error of prediction (TMSEP).

At first, it appears this performance measure unfairly favors the Minimum C_p and Minimum S_p criteria. Both of these criteria are based on minimum MSEP. The difference between the two lies in the assumption that the regressors are fixed in the C_p case and are random in the S_p case. Since the regressors in this study are randomly generated, it is then assumed that the S_p criterion would outperform the other two. However, this is not necessarily true.

It is assumed when calculating the S_p and C_p statistics that all relevant variables are included in the variable pool. It is also assumed that variable pool does not contain extraneous variables. In this study both of these assumptions are violated. Therefore, it is possible that either the C_p or MSE criterion could outperform the S_p criterion. Since the MSE criterion is not designed to minimize MSEP, it is doubtful whether it would outperform the other two criteria using TMSEP as the performance measure.

Performance of the Selection Technique using TMSEP.

Measurement of each criterion's performance is recorded under best-case and worst-case factor settings. If a technique outperformed the other two in both the best-case scenario and worst-case scenario, it is superior to the other two.

Since determining the necessary sample size is difficult, runs were made until the distributions for each technique stabilized. The number of necessary number of runs was 180 for both the best-case and worst-case scenarios.

Method of Comparison. For each of the 360 runs, the TMSEP value was calculated for every subset of variables. For example, when the factor setting for extraneous variables was one (best-case factor setting), there were four possible regressors. Since there were $2^k - 1$ subsets possible, in our example, fifteen TMSEP values were

calculated. In the three extraneous variable case the number of values increases to sixty-three. The 2^k-1 TMSEP values were then ordered from lowest to highest; lowest being more desirable. The TMSEP values are then calculated for the models chosen by each of the three techniques. Then, the values calculated from each technique's choice was compared to the rank ordered values and its ranking was recorded. For instance, if the value chosen by a technique was identical to the lowest value of TMSEPs, then a "one" was recorded. Additionally, the ranking of the TMSEP value for each technique was compared to the other two and recorded. A FORTRAN program was written to execute the procedure outlined in this section. The code for this program can be found in Appendix F, and a sample output can be found in Appendix G.

Best-Case Comparison. Using the equations found in the Response Surface phase, a common set of factors was obtained which constitutes a best-case scenario. The following settings maximize each techniques performance:

Table III.
Factor Settings For Best-Case Scenario

variable	Non-Coded value	coded value
Number of extraneous variables	1.0	-1.0
Correlation	0.9	+1.0
Variance of extraneous variables	100.0	+1.0
Variance of independent variables	1.0	-1.0
Sample size	20.0	+1.0
Variance of ϵ term	0.0625	-1.0

The Min S_p criterion was derived under the assumption that the regressors are random, while the Min C_p criterion was derived under the assumption that the regressors are fixed. Since the regressors are random in this experiment, it was expected that there would be a significant difference between the two. The following two figures summarize the results from the best-case comparison.

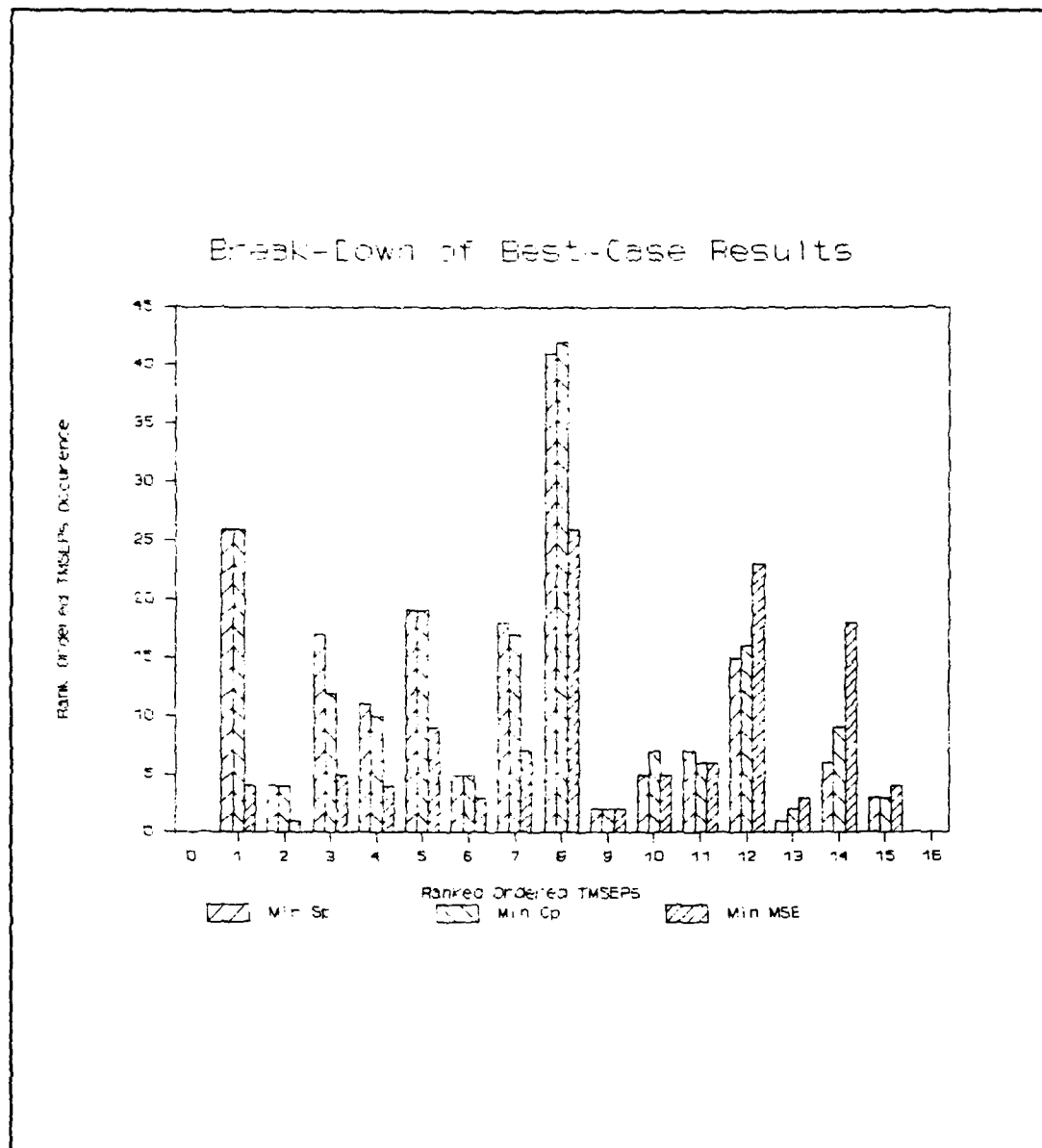


Figure 2. Bar Graph for Best-Case Comparison Results

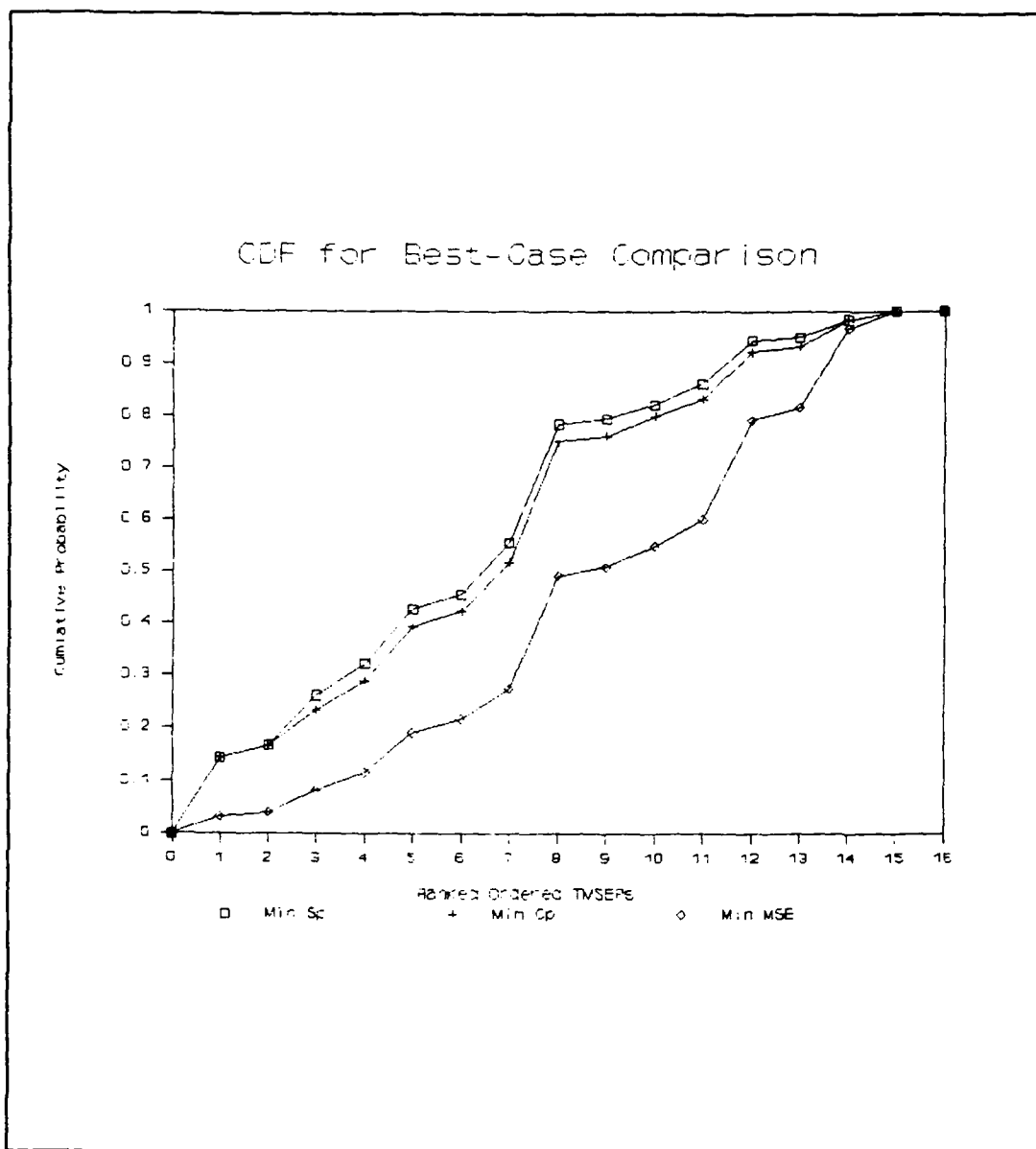


Figure 3. Cumulative Distribution Function for Best-Case Comparison

The cumulative distribution function of each technique indicates there probably is a significant difference between the two MSEP procedures, the difference between MSE and the other two is obvious. The chart below summarizes the number of times each technique chose the lowest TMSEP value amongst the three.

Table IV.
Best-Case Scenario Results

Criteria	Number of times Percentage Lowest Value	
C_p only	1	0.5%
S_p only	9	5.0%
MSE only	1	0.5%
C_p and S_p tied	58	32.0%
C_p , S_p , and MSE tied	111	62.0%

Allowing for ties, the following are the percentages that each technique found the lowest TMSEP value amongst the three.

Table V.
Best-Case Scenario Results

Criterion	Percentage of time lowest, or tied for lowest amongst the three
C_p	94.0%
S_p	99.0%
MSE	62.0%

The table above reaffirms our expectation that the Min S_p criterion is superior to the Min C_p criterion under a best-case scenario. To determine if the difference actually exists a hypothesis test is performed. The null hypothesis, H_0 , is: the percentage of lowest TMSEP values chosen solely by the Min S_p criterion equals the percentage chosen solely by the Min C_p criterion. The alternative hypothesis, H_a , is: the percentage chosen solely by the Min S_p criterion is not equal to the percentage chosen solely by the Min C_p criterion.

$$\begin{aligned} H_0: P_{cp} &= P_{sp} \\ H_a: P_{cp} &\neq P_{sp} \end{aligned}$$

$$\begin{aligned} \alpha &= 0.05 \\ \text{critical probability} &= 0.039 \end{aligned}$$

The following is the equation used to calculate the critical probability:

$$2 \sum_{i=0}^1 \binom{9}{i} \left(\frac{1}{2} \right)^i \left(\frac{1}{2} \right)^{9-i}$$

where i is number of occurrences out of 9 trials, that C_p chose a model with a lower TMSEP value.

9 is the total number of times the two MSE criteria differed.

This equation is based on the binomial distribution with a probability of 0.5. That is, the chance of one criterion picking the lowest TMSEP value is equal to the other.

Since the significance level is larger than the critical probability, the null hypothesis is rejected. The conclusion is there is a significant difference between the two criteria under the a best-case scenario, and the Min S_o is superior.

Worst-Case Comparison. By reversing the factor settings of the best-case scenario, a worst-case scenario is obtained.

Table VI.
Factor Settings For Worst-Case Scenario

variable	Non-Coded value	coded value
Number of extraneous variables	3.0	+1.0
Correlation	0.0	-1.0
Variance of extraneous variables	1.0	-1.0
Variance of independent variables	100.0	+1.0
Sample size	10.0	-1.0
Variance of ϵ term	0.25	+1.0

Similar to the best-case scenario results, the Min MSE criterion did significantly worse than the other two techniques. The graph below shows the cumulative distribution function for each of the criteria (the worst-case bar graphs are located in Appendix A).

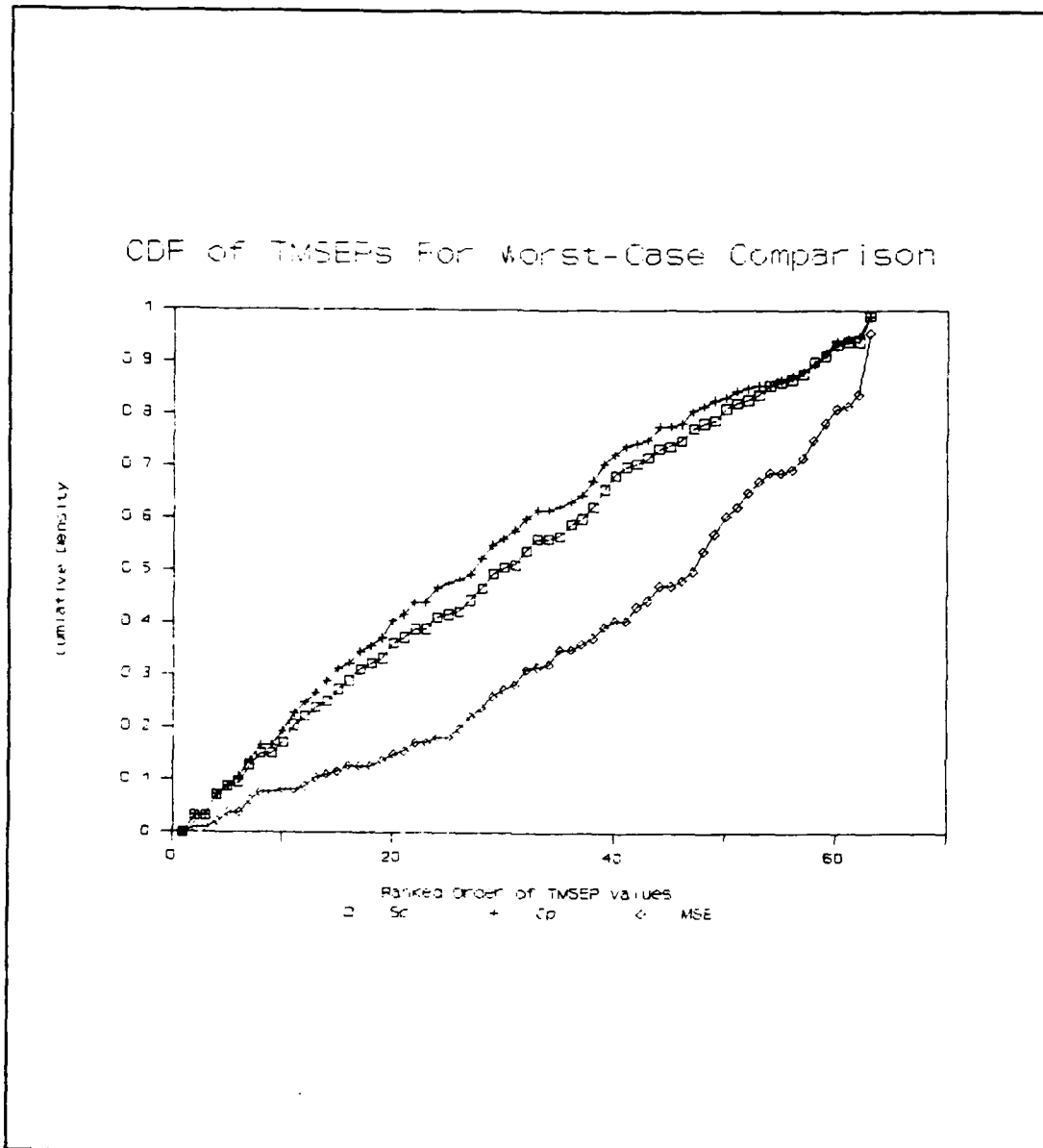


Figure 4. Cumulative Distribution Function for Worst-Case Comparison

Notice the cumulative distribution for C_p is larger than the other two criteria for most rank ordered TMSEP values. This is surprising since the factors are random. The question is whether the difference between the two MSEP criteria is significant. If so, the usefulness of the minimum S_p criterion would be questionable.

Table VII.
Worst-Case Scenario Results

Criteria	Number of times Lowest Value	Percentage
C_p only	28	16.0%
S_p only	4	2.0%
MSE only	0	0.0%
C_p and S_p tied	81	45.0%
C_p , S_p , and MSE tied	67	37.0%

Allowing for ties, the percentage each technique found the lowest TMSEP value amongst the three are summarized in the table below.

Table VIII.
Worst-Case Scenario Results

Criterion	Percentage of time lowest, or tied for lowest amongst the three
C_p	98.0%
S_p	91.0%
MSE	67.0%

Since the Min C_p criterion appears to be significantly better than the Min S_p criterion, a two tailed hypothesis test at the $\alpha=0.05$ level was performed. This test was identical to the test in the best case scenario. The null hypothesis was: the percentage of lowest TMSEP values chosen solely by the Min S_p criterion is equal to the percentage chosen by the Min C_p criterion. While the alternative hypothesis was: the percentage chosen by the Min S_p criterion is not equal to the percentage chosen by the Min C_p criterion.

$$\begin{aligned} H_0: P_{cp} &= P_{sp} \\ H_a: P_{cp} &\neq P_{sp} \end{aligned}$$

$$\begin{aligned} \alpha &= 0.05 \\ \text{critical probability} &= 0.00002 \end{aligned}$$

The critical probability for this hypothesis test was calculated in exactly the same manner as in the best-case scenario. The equation necessary to find the critical probability is:

$$2 \sum_{i=0}^4 \binom{32}{i} \binom{1}{-}^i \left(1 - \binom{1}{-}^{32-i} \right)$$

where

i is number of occurrences out of 32 trials, that

C_p chose a model with a lower TMSEP value.

32 is the total number of times the two MSEP
criteria differed.

Since the critical probability is less than the
significance level, the null hypothesis is rejected. Thus,
the conclusion is the Min C_p criterion significantly
outperforms the Min S_p criterion in the worst-case scenario.

V. Conclusions and Further Research

The objectives of this research were: (1) identify some promising least squares selection procedures discussed in the literature during the previous decade, (2) use Response Surface Methodology to find the factors which most significantly drive the selection process for each of the these techniques, and (3) make a best-case and worst-case scenario comparison of the techniques.

Conclusion

The three techniques chosen, Min MSE, Min C_p , and Min S_p , have received praise in the last decade's literature. About ten years ago Min MSE was considered a favorable technique because of its similarity to the Max R^2 criterion, with an adjustment for degrees of freedom. In recent literature, the Min S_p and Min C_p criteria have received the majority of praise. Both techniques are based on the principle of minimizing the mean square errors of prediction. Of the two, the Min S_p criterion has received more attention. Its uses vary from finding the optimal number of variables to include in the model, to finding the best subset of variables. Unlike Min C_p , Min S_p is designed for random regressors, which is more practical.

During the Response Surface phase, emphasis was placed on the technique's performance under varying degrees of multicollinearity, variable variance, number of variables,

and sample size. A full 2^6 factorial design was constructed to find the most significant factors and their interactions. The three equations developed from the Response Surface phase show the relative weights of the most significant variables. These equations serve as a screening method, and should not be used to predict the percentage of correct variables of those chosen for any of the techniques.

The most noticeable result from the Response Surface phase dealt with the number of extraneous variables which were included in the variable pool. This factor was overwhelmingly the most significant in determining the percentage of correct variables chosen by a technique. In fact, if the number of extraneous variables was the only variable in any of the Response Surface equations, it would account for over 85% of the variation about the mean. Therefore, it is very important to screen the variables used in a regression analysis by some other means than just letting a computer program do the work.

The conclusion dealing with extraneous variables is consistent with Freedman (9), Flack and Chang (8), and Hoerl, Hoerl, Schuenemeyer (13), as well as Miller's suggested technique for a stepwise regression stopping-rule. It can not be over-emphasized that the "kitchen sink" approach is not the best way to go. One must decide which variables seem relevant to the dependent variable. If a

good reason for including a variable can not be determined, then it should not be included in the model.

Using common factors of the three equations found in the Response Surface phase, a best-case and worst-case scenario comparison was made using the theoretical mean square errors of prediction (TMSEP). Since the information necessary to find TMSEP is not available in practice, this statistic's usefulness is restricted to the simulation arena. However, if a technique such as ($\text{Min } S_c$) is suppose to minimize MSEP, then it is expected that it will also have one of the lowest TMSEPs of all possible models. Likewise, since $\text{Min } C_p$ is also based on minimizing MSEP, it was expected to do fairly well.

The results of the best-case comparison indicate the $\text{Min } S_p$ criterion performs better than $\text{Min } C_p$. Since the $\text{Min } S_p$ criterion is by definition made for random regressors, one would expect it to do significantly better than $\text{Min } C_p$. Both criteria did very well in absolute terms. Over 75% of the TMSEP values calculated from the models chosen by S_p and C_p were in the upper half of rank ordered TMSEP values. 14% of the values found from the MSEP criteria's model were the absolute lowest TMSEP value possible. This is extremely good considering a fundamental regressor was dropped from the variable pool prior to using the variable selection criterion.

The results from the worst-case comparison were shocking. Surprisingly, Min C_p outperformed S_p . What makes this so surprising is the assumption that the regressors must be fixed to implement C_p . The difference between the C_p and S_p could be attributed to many things. First, when these two criteria were derived there was an assumption that all relevant regressors were included, and no extraneous variables were included in the variable pool. Both of these assumptions have been violated in this research. Under these circumstances, it is possible C_p outperforms S_p regardless of the nature of the regressors. In the best-case scenario, the MSE criterion performed significantly worse than the other two criteria.

The results of this thesis provide insight into the usefulness of the Min MSE, Min S_p , and Min C_p criteria under realistic conditions. It is possible that variables which significantly contribute to a model may be left out of the variable pool. It is also possible some extraneous variables will be included. The most important lesson of this research is: the number of extraneous variables which are in the variable pool significantly detracts the selection process. There must be some thought, commonly referred to by Operation Research analysts as the "art", used when picking variables to include in the variable pool. Not even the state-of-the-art variable selection criteria are able to perform well when the variables have not been

properly screened before implementation. This information may be useful to those who make a practice of collecting data on everything and letting the computer pick the "optimal model."

Another useful result of this research is the comparison of two MSEP criteria. A great deal of praise has been given to the minimum S_p criterion in the past decade. It has been identified as "one of the most promising" when the regressors are random, and one desires to minimize the mean square error of

prediction. The minimum C_p criterion has also received praise for minimizing mean square error of prediction, but its usefulness is limited to cases where the regressors are fixed. Some have recommended that the Min C_p criterion should not be used in practice.

The results of this thesis indicate that the Min S_p criterion does, in fact, perform well when the circumstances are "nice." However, when circumstance are poor, the Min C_p criterion does better than S_p . Ironically, there is no way to be sure whether the conditions the variable selection techniques must perform under are best-case or worst-case, yet it seems like such information is necessary if one intends to minimize MSEP.

Recommendations for Further Research

A study of this intent lends itself to many follow-on studies. The methodology and groundwork are established but

embellishments will be necessary. Using more complex models would be one example. The model used in this study was simple: four regressors created the data with one eliminated from further consideration.

A full-blown simulation could be implemented to construct guidelines for usage of these criteria. For instance in this study, results indicate that if the circumstances are favorable, use the Min S_p criterion. However, if the circumstances are unfavorable, use the Min C_p criterion. By expanding the simulation to randomly generate many true variables and many extraneous variables and then eliminate a random number of variables, guidelines could be established for which technique should be used under various conditions that seem most likely. That is, if it is likely to be a situation where there are many extraneous variables and many variables omitted, use one technique, otherwise use another. Even though it is impossible to determine whether there are going to be many extraneous variables, one can assume if the dependent variable is something entirely new, like a flight characteristic of the Soviet's new Blackjack bomber, information on significant variables will be unobtainable. Likewise, some data collected probably will have little to do with the dependent variable. In such circumstances it would be useful to have a guideline for which variable selection criterion should be implemented. To completely examine the usefulness of each criterion and establish

guidelines, further research should include interaction terms and indicator variables.

One area which leads to further research deals with Response Surface. The data range used for this thesis was limited. It may be useful to expanding the Response Surface region to include negative correlation and larger sample sizes. If an indicator variable was used to contrast the effect of dropping a variable, the loop would be complete. That is, by including a variable to keep track of the difference between the full model and a model where a variable is dropped, one could quantify the effects of failing to collect data on all the significant variables. In this thesis, only information from dropping a variable was collected. It was assumed that if all variables were present, the techniques would perform better; plus similar simulation studies recorded results without dropping variables. However, it would be worthwhile to quantify the effects not including all significant variables into the variable pool.

This research has skimmed the surface of many myths associated with variable selection techniques, and in particular the usefulness of the C_p criterion. Some authors regard the Min C_p criterion as secondary to the Min S_p criterion. However, the results of this simulation do not support such ranking. The two criteria perform differently under certain circumstances. Under the best-case scenario,

the Min S_p criterion significantly outperform the Min C_p criterion. However, under the worst-case scenario, the Min C_p criterion significantly outperforms the S_p criterion.

Other simulations deal with the number of correct variables chosen of those available in absolute terms. No provisions are made for circumstances in which significant regressors are not included in the variable pool.

Therefore, techniques praised as good variable selection techniques may not be as appealing as originally thought.

This research indicates this is the case with the Min S_p criterion.

Appendix A: Bar Graphs For Worst-Case Comparison

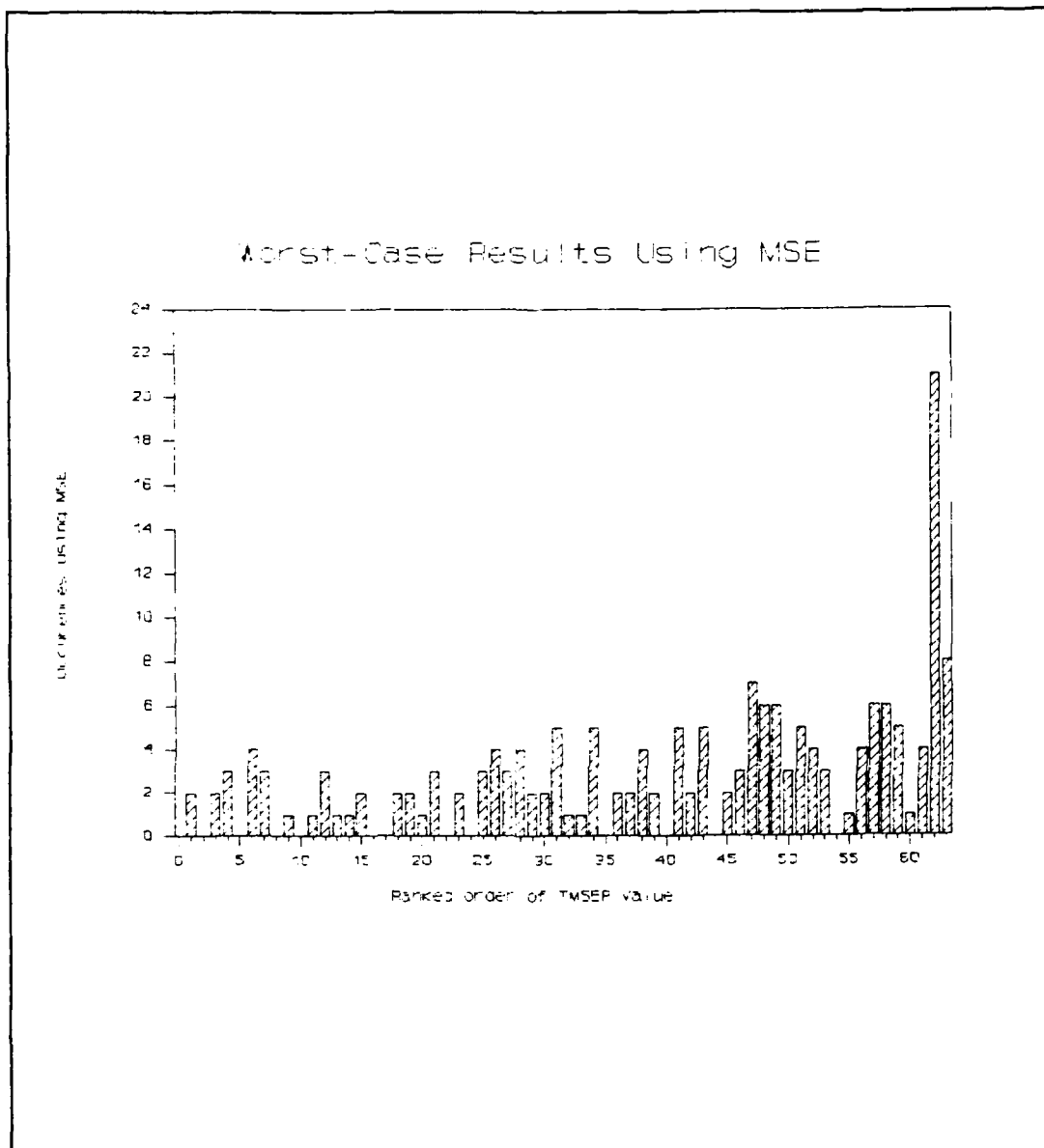


Figure 5. Bar Graph for results from MSE under a Worst-Case Scenario

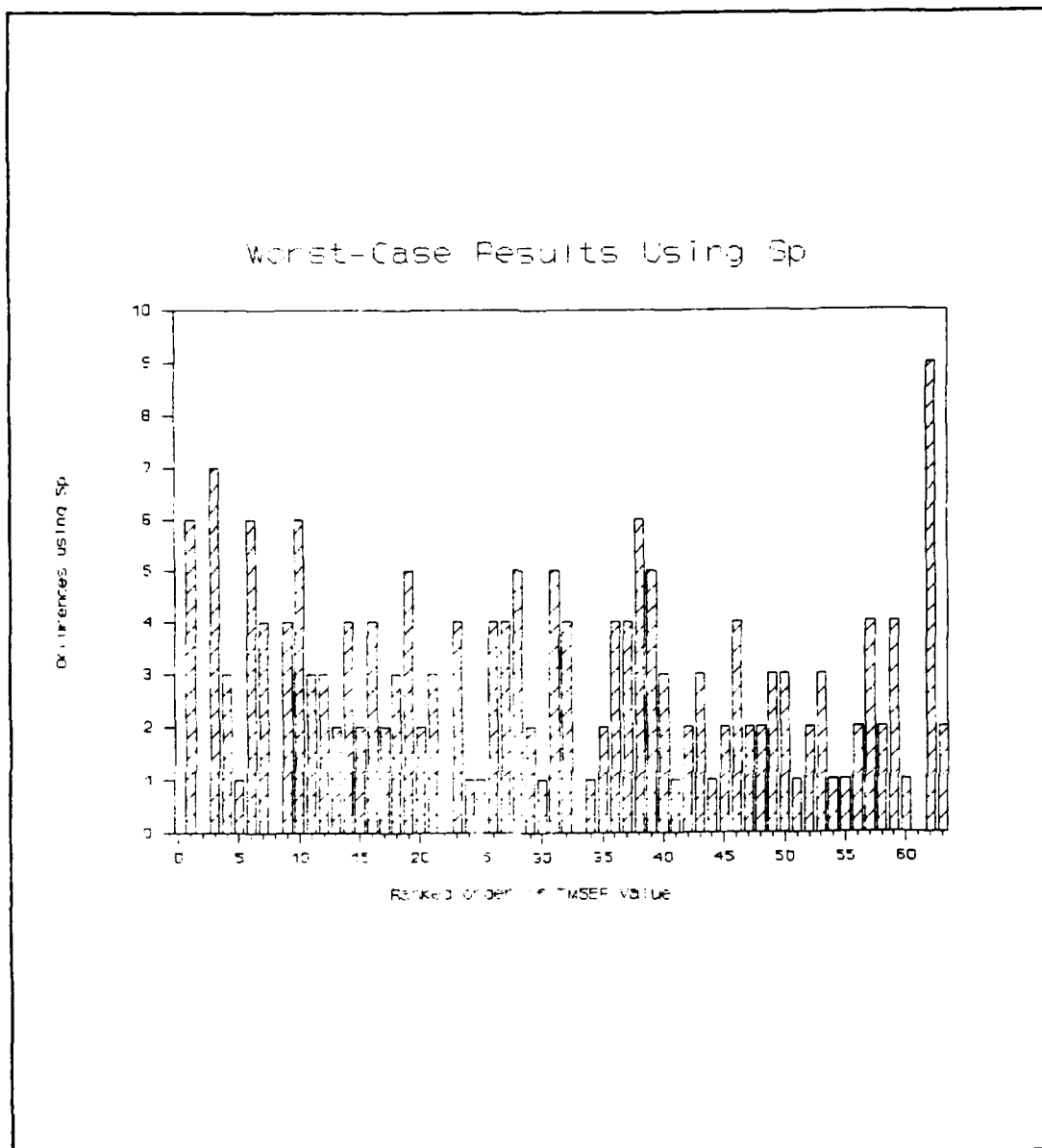


Figure 6. Bar Graph for Results from S_p under a Worst-Case Scenario

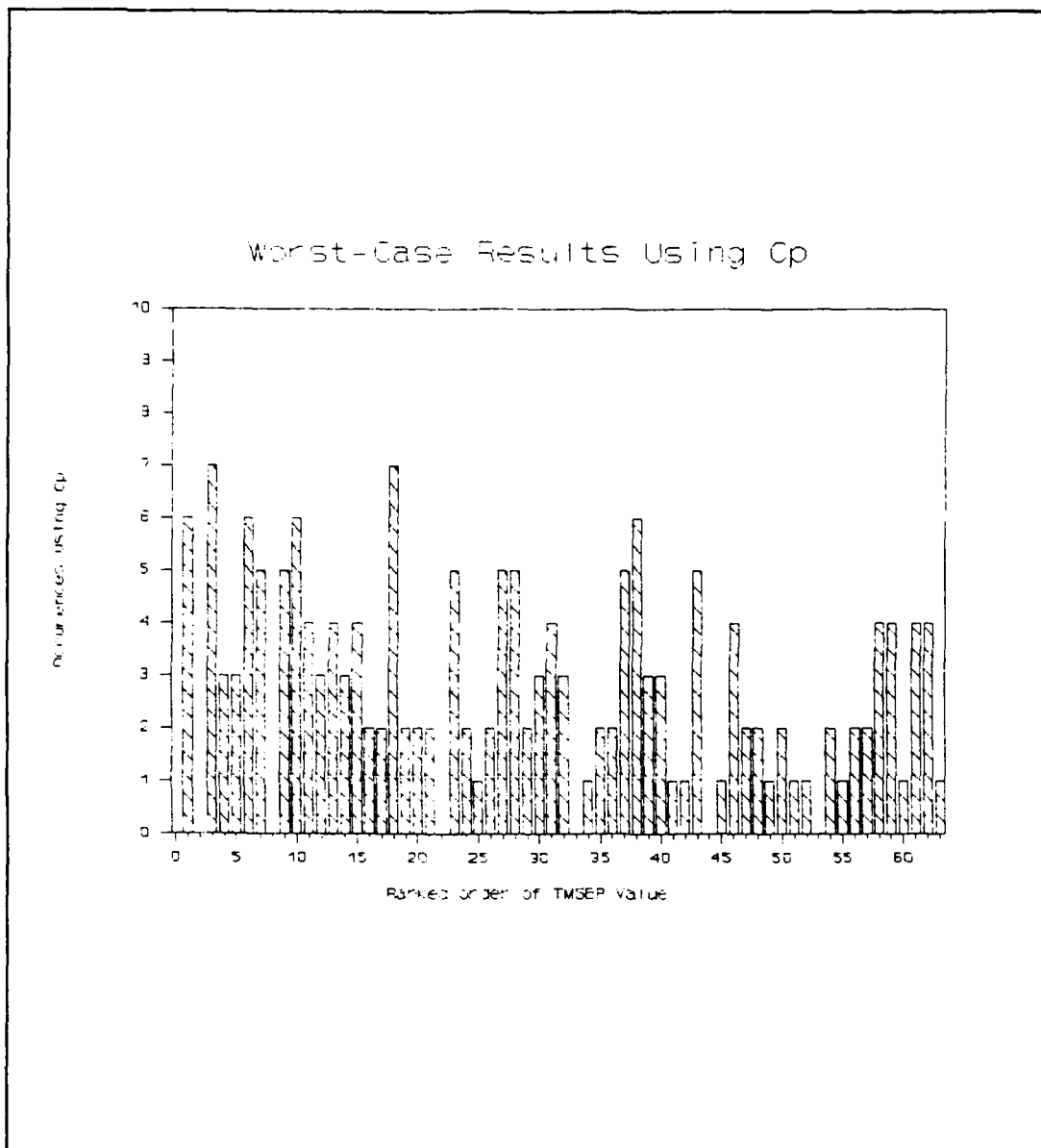


Figure 7. Bar Graph for Results from C_p under a Worst-Case Scenario

Appendix B: Macros to Generate Correlated Data
Using S on the ASC or Success on the CSC

```
#####
#
# This macro is designed to calculate correlated data from
# a random sample. This macro is for the 1 extraneous
# variable case. There is also a macro found for the 3
# extraneous variable case.
#
# To implement this program, the user must enter the S
# computer package by typing S on the ASC or Success on the
# CSC.
#
# Once in S, the following inputs must be made:
#
# 1. The 4x4 correlation matrix, K
#    > K_matrix(read(),4,4,byrow=T)
#    > "now input the rows of the matrix"
#
# 2. The standard deviation of the extraneous variables
#    > exerrstd_"value"
#
# 3. The standard deviation of the independent variables
#    > xerrstd_"value"
#
# 4. The standard deviation of the  $\epsilon$  term
#    > yerrstd_"value"
#
# 5. The sample size (this is a two step procedure)
#    > ptsperset_"sample size"
#    > samplesize_"total number of points desired"
#        samplesize*number of runs
#
# The macros can be implemented by the following commands
#    >source "make.1error"      for 1 ex vars case
#    >source "make.3error"      for 3 ex vars case
#
# The files which are implemented by these macros are the
# following:
#
# 1. make.1error
# 2. make.3error
# 3. batch
# 4. four
# 5. matgen
# 6. qmake
#
# The data that is generated has an appended column of
# integers which is used in the SAS runs.
#####
```

```
#####
#
# The following must be typed to write the data to a file.
#
#   > write(t(Z),"filename",#)
#       filename must be in quotes and # is the number of
#       columns of the data set. In the 1ex case it is 7
#       in the 3ex case it is 9.
#
#####

> # u an v are random noise, mean 0, and std exerrstd
> v_rnorm(samplesize,0,exerrstd)
> u_rnorm(samplesize,0,exerrstd)

> # x1..x4 are the independent variables

> x1_rnorm(samplesize,0,xerrstd)
> x2_rnorm(samplesize,0,xerrstd)
> x3_rnorm(samplesize,0,xerrstd)
> x4_rnorm(samplesize,0,xerrstd)
>
> X_cbind(x1,x2,x3,x4)
> e_eigen(K)
> lamb_diag(e$values)
> P_e$vector
> XTX_t(X)%*X
> qq_e$values
> sig_diag(XTX)
> q1_qq/sig
> Q1_diag(sqrt(q1))
> Q_Q1%*t(P)
> Z_X%*Q
> op_rnorm(samplesize,0,yerrstd)
> y_Z[,1]+Z[,2]+Z[,3]+Z[,4]+op
> error_rnorm(samplesize,0,exerrstd)
> Z_cbind(y,Z,error)
>
># The following statements create the appended integers
>
> g_diag(diag(ptsperset))
> gg_c(g,g*2,g*3,g*4,g*5,g*6,g*7,g*8,g*9,g*10)
> gg_c(gg,gg+10,gg+20,gg+30,gg+40,gg+50)
> Z_cbind(gg,Z,v,u)

```

Appendix C: An All-Subsets SAS Program
To Calculate all S_p , C_p , and MSE values

```
option linesize=80;
filename new 'msep.dat';
data new;
infile new;
input set y x1 x2 x3 x4 e1;
proc rsquare data=new mse sp cp b;
by set;
model y= x1 x2 x3 e1 ;
```

Appendix D :Fortran Program to Find
the Min MSE, Min C_p and Min S_p Values and Models

```

*****
* This program is designed to take a modified SAS listing
* from Appendix C (the modification is simply deleting
* all lines except for the raw data i.e. the number of
* variables, the  $R^2$ , the MSE, the  $S_p$ , and  $C_p$  values, as
* well as the variables printed. The program finds the
* correct model using the Min MSE, Min  $S_p$ , and Min  $C_p$ 
* criteria. In each case, the model chosen by each
* technique is printed. The values for each criteria are
* also printed.
*
* Statistics are collected on the average number of
* variables chosen and the number of extraneous variables
* chosen
*
* variables num is the number of variables in the model
* this is the first column of the SAS
* listing
*
* i,j,k are counters for do loops.
*
* ptrmse,ptrsp,ptrcp are variables used to keep
* track of the min values.
*
* numcp,numsp,nummse are the average number
* of variables chosen by their
* respective technique.
*
* cperr,sperr,mseeer are the average number
* of extraneous variables chosen
* by their respective technique.
*
*
*****

integer num(63),i,j,k,ptrmse,ptrsp
integer ptrcp,varsmse,varssp,varscp
integer check(6)
integer n,emse,esp,ecp
integer ccp,cmse,csp
integer chartmse(0:3,0:3),chartcp(0:3,0:3)
integer chartsp(0:3,0:3)
real MSE(63),Sp(63),cp(63),r2(63)
real minmse,minsp,mincp,nummse,numcp,numsp

```

```

real mseeer,cpeer,speer
character*2 m(6,63)
check(1)=6
check(2)=21
check(3)=41
check(4)=56
check(5)=62
check(6)=63

      do 7 i = 0,3
        do 3 k = 0,3
          chartmse(i,k)=0
          chartcp(i,k)=0
          chartsp(i,k)=0
3          continue
7          continue

varsmse=0
varssp =0
varscp =0
cumemse=0
cumesp=0
cumeclp=0
open (unit=10,file="temp",status='old')
open (unit=11,file='OUT',status='new')

do 20 k=1,60
do 10 i=1,63
  emse=0
  esp=0
  eclp=0
  read(10,*)
num(i),r2(i),MSE(i),Sp(i),cp(i)
+      ,(m(j,i),j=1,num(i))
  minmse=10000
  minsp =10000
  minclp =10000
  ptrmse=0
  ptrcp =0
  ptrsp =0
  do 30 j= 1,6
    if(mse(check(j)).lt.minmse) then
      minmse=mse(check(j))
      ptrmse=check(j)
    endif
    if(sp(check(j)).lt.minsp) then
      minsp=sp(check(j))
      ptrsp=check(j)
    endif
    if(cp(check(j)).lt.minclp) then
      minclp=cp(check(j))
      ptrcp=check(j)

```

```

endif

30  continue
10  continue
    varsmse=varsmse+num(ptrmse)
    varssp =varssp +num(ptrsp)
    varscp =varscp +num(ptrcp)

do 70 n=1,num(ptrmse)
    if(m(n,ptrmse).EQ.'E1') then
        emse=emse+1
    elseif(m(n,ptrmse).eq.'E2') then
        emse=emse+1
    elseif(m(n,ptrmse).eq.'E3') then
        emse=emse+1
    else
        continue
    endif

70  continue
do 80 n=1,num(ptrsp)
    if(m(n,ptrsp).eq.'E1') then
        esp=esp+1
    elseif(m(n,ptrsp).eq.'E2') then
        esp=esp+1
    elseif(m(n,ptrsp).eq.'E3') then
        esp=esp+1
    else
        continue
    endif

80  continue
do 90 n=1,num(ptrcp)
    if(m(n,ptrcp).eq.'E1') then
        ecp=ecp+1
    elseif(m(n,ptrcp).eq.'E2') then
        ecp=ecp+1
    elseif(m(n,ptrcp).eq.'E3') then
        ecp=ecp+1
    else
        continue
    endif

90  continue

    cumemse=cumemse+emse
    cumesp=cumesp+esp
    cumeclp=cumeclp+ecp
    cmse=num(ptrmse)-emse
    ccp=num(ptrcp)-ecp
    csp=num(ptrsp)-esp
    chartmse(cmse,emse)=chartmse(cmse,emse)+1
    chartcp(ccp,ecp)=chartcp(ccp,ecp)+1

```

```

chartsp(csp,esp)=chartsp(csp,esp)+1

write(11,*) 'MSE',num(ptrmse),mse(ptrmse)
+   ,Sp(ptrmse),cp(ptrmse)
+   ,(m(j,ptrmse),j=1,num(ptrmse))
write(11,*) 'Sp',num(ptrsp),mse(ptrsp)
+   ,Sp(ptrsp),cp(ptrsp),
+   ,(m(j,ptrsp),j=1,num(ptrsp))
write(11,*) 'Cp',num(ptrcp),mse(ptrcp)
+   ,Sp(ptrcp),cp(ptrcp),
+   ,(m(j,ptrcp),j=1,num(ptrcp))
write(11,*) '*****'
write(11,*) ' '
write(11,*) ' '
20 continue
   nummse = real(varsmse)/60.0
   numsp  = real(varssp)/60.0
   numcp  = real(varscp)/60.0
   mseeer= real(cumemse)/60.0
   cpeer  = real(cumecp) /60.0
   speer  = real(cumesp) /60.0
write(11,*) 'The avg number of vars using MSE
              was ', nummse
write(11,*) 'The avg number of errors from MSE
              was', mseeer
write(11,*) ' '
write(11,*) 'The avg number of vars using Sp was',
              numsp
write(11,*) 'The avg number of errors from Sp was
              ', speer
write(11,*) ' '
write(11,*) 'The avg number of vars using Cp was
              ', numcp
write(11,*) 'The avg number of errors from Cp was
              ', cpeer
write(11,*) ' '
write(11,*) ' '
write(11,*) ' MSE TABLE'
write(11,*) ' '
   do 100 i=0,3
       write(11,*) (chartmse(i,j),j=0,3)
100 continue
   write(11,*) ' '
   write(11,*) ' '
   write(11,*) 'Sp TABLE'
   write(11,*)
   do 110 i=0,3
       write(11,*) (chartsp(i,j),j=0,3)

```

```

110          continue

          write(11,*) ' '
          write(11,*) ' '
          write(11,*) 'Cp TABLE'
          do 120 i=0,3
            write(11,*) (chartcp(i,j),j=0,3)
120          continue

          END

```

Appendix E: Sample Output for Phase One of Analysis

The following is a sample of the output for one run (with 60 replications) for the Response Surface Methodology Phase (program found in Appendix D). The first column contains the technique used. The second column is the number of variables chosen. The third, fourth, and fifth columns are the values for Min MSE, Min C_p , and Min S_p respectively. The last column is the actual model chosen by the technique.

MSE	1	0.100654	1.43791e-02	-0.266093	X1
Sp	1	0.100654	1.43791e-02	-0.266093	X1
Cp	1	0.100654	1.43791e-02	-0.266093	X1

MSE	2	5.32424e-02	8.87374e-03	1.35403	X2 E1
Sp	1	6.00226e-02	8.57466e-03	0.898098	E1
Cp	1	6.00226e-02	8.57466e-03	0.898098	E1

MSE	3	7.63447e-02	1.52689e-02	3.17057	X1 X2 E1
Sp	1	7.64603e-02	1.09229e-02	0.904536	X2
Cp	1	7.64603e-02	1.09229e-02	0.904536	X2

MSE	1	2.95358e-02	4.21940e-03	-0.285088	X3
Sp	1	2.95358e-02	4.21940e-03	-0.285088	X3
Cp	1	2.95358e-02	4.21940e-03	-0.285088	X3

MSE	2	4.95487e-02	8.25812e-03	1.35192	X2 X3
Sp	2	4.95487e-02	8.25812e-03	1.35192	X2 X3
Cp	1	5.93601e-02	8.48001e-03	1.32763	X2

MSE	2	6.12631e-02	1.02105e-02	1.09398	X2 E1
Sp	1	6.46866e-02	9.24094e-03	0.147028	X2
Cp	1	6.46866e-02	9.24094e-03	0.147028	X2

MSE	2	3.19134e-02	5.31889e-03	1.45195	X2	X3
Sp	1	3.24903e-02	4.64148e-03	0.343450	X2	
Cp	1	3.24903e-02	4.64148e-03	0.343450	X2	

MSE	1	6.24057e-02	8.91510e-03	-0.299912	X3	
Sp	1	6.24057e-02	8.91510e-03	-0.299912	X3	
Cp	1	6.24057e-02	8.91510e-03	-0.299912	X3	

MSE	1	3.94331e-02	5.63330e-03	-0.741483	X2	
Sp	1	3.94331e-02	5.63330e-03	-0.741483	X2	
Cp	1	3.94331e-02	5.63330e-03	-0.741483	X2	

MSE	1	2.25172e-02	3.21674e-03	1.02995	E1	
Sp	1	2.25172e-02	3.21674e-03	1.02995	E1	
Cp	1	2.25172e-02	3.21674e-03	1.02995	E1	

MSE	2	2.43904e-02	4.06507e-03	1.66245	X2	X3
Sp	1	2.54640e-02	3.63772e-03	0.756217	X2	
Cp	1	2.54640e-02	3.63772e-03	0.756217	X2	

MSE	2	3.30050e-02	5.50083e-03	1.29611	X1	X2
Sp	2	3.30050e-02	5.50083e-03	1.29611	X1	X2
Cp	2	3.30050e-02	5.50083e-03	1.29611	X1	X2

MSE	2	3.95879e-02	6.59799e-03	1.55024	X1	X2
Sp	2	3.95879e-02	6.59799e-03	1.55024	X1	X2
Cp	2	3.95879e-02	6.59799e-03	1.55024	X1	X2

MSE	1	8.97254e-02	1.28179e-02	-0.889888	X1	
Sp	1	8.97254e-02	1.28179e-02	-0.889888	X1	
Cp	1	8.97254e-02	1.28179e-02	-0.889888	X1	

MSE	1	3.95394e-02	5.64849e-03	-0.658592	X2	
Sp	1	3.95394e-02	5.64849e-03	-0.658592	X2	
Cp	1	3.95394e-02	5.64849e-03	-0.658592	X2	

```

*****
MSE  2    3.60757e-02    6.01262e-03    1.61985    X2    E1
Sp   1    3.66450e-02    5.23500e-03    0.524038    E1
Cp   1    3.66450e-02    5.23500e-03    0.524038    E1
*****

```

```

*****
MSE  2    5.85844e-02    9.76407e-03    2.28246    X1    X2
Sp   2    5.85844e-02    9.76407e-03    2.28246    X1    X2
Cp   2    5.85844e-02    9.76407e-03    2.28246    X1    X2
*****

```

```

*****
MSE  2    4.41387e-02    7.35645e-03    1.80789    X2    X3
Sp   1    4.42947e-02    6.32782e-03    0.661062    X2
Cp   1    4.42947e-02    6.32782e-03    0.661062    X2
*****

```

```

*****
MSE  1    0.100894    1.44134e-02    -0.614306    X3
Sp   1    0.100894    1.44134e-02    -0.614306    X3
Cp   1    0.100894    1.44134e-02    -0.614306    X3
*****

```

```

*****
MSE  1    3.93769e-02    5.62527e-03    -0.790119    X3
Sp   1    3.93769e-02    5.62527e-03    -0.790119    X3
Cp   1    3.93769e-02    5.62527e-03    -0.790119    X3
*****

```

```

*****
MSE  1    4.66999e-02    6.67142e-03    -0.101598    E1
Sp   1    4.66999e-02    6.67142e-03    -0.101598    E1
Cp   1    4.66999e-02    6.67142e-03    -0.101598    E1
*****

```

```

*****
MSE  1    4.52970e-02    6.47101e-03    -0.946171    X2
Sp   1    4.52970e-02    6.47101e-03    -0.946171    X2
Cp   1    4.52970e-02    6.47101e-03    -0.946171    X2
*****

```

```

*****
MSE  3    2.84832e-02    5.69664e-03    3.30669    X1    X2    X3
Sp   1    3.08908e-02    4.41297e-03    1.67367    X2
Cp   1    3.08908e-02    4.41297e-03    1.67367    X2
*****

```

```

*****
MSE  1    4.47498e-02    6.39283e-03    -0.492574    E1
Sp   1    4.47498e-02    6.39283e-03    -0.492574    E1
Cp   1    4.47498e-02    6.39283e-03    -0.492574    E1
*****

```

MSE	1	6.10468e-02	8.72097e-03	-0.754845	X2
Sp	1	6.10468e-02	8.72097e-03	-0.754845	X2
Cp	1	6.10468e-02	8.72097e-03	-0.754845	X2

MSE	1	2.40736e-02	3.43908e-03	1.06484	X3
Sp	1	2.40736e-02	3.43908e-03	1.06484	X3
Cp	1	2.40736e-02	3.43908e-03	1.06484	X3

MSE	1	4.34820e-02	6.21172e-03	0.251277	X1
Sp	1	4.34820e-02	6.21172e-03	0.251277	X1
Cp	1	4.34820e-02	6.21172e-03	0.251277	X1

MSE	2	2.60387e-02	4.33979e-03	1.65346	X2	X3
Sp	2	2.60387e-02	4.33979e-03	1.65346	X2	X3
Cp	2	2.60387e-02	4.33979e-03	1.65346	X2	X3

MSE	3	1.14088e-02	2.28177e-03	3.04763	X1	X3	E1
Sp	3	1.14088e-02	2.28177e-03	3.04763	X1	X3	E1
Cp	3	1.14088e-02	2.28177e-03	3.04763	X1	X3	E1

MSE	4	1.11825e-02	2.79562e-03	5.00000	X1	X2	X3	E1
Sp	2	1.66321e-02	2.77202e-03	6.41136	X1	E1		
Cp	4	1.11825e-02	2.79562e-03	5.00000	X1	X2	X3	E1

MSE	3	1.24700e-02	2.49400e-03	3.16464	X2	X3	E1
Sp	1	1.44043e-02	2.05776e-03	1.95435	X2		
Cp	1	1.44043e-02	2.05776e-03	1.95435	X2		

MSE	1	3.17437e-02	4.53481e-03	-0.727781	X3
Sp	1	3.17437e-02	4.53481e-03	-0.727781	X3
Cp	1	3.17437e-02	4.53481e-03	-0.727781	X3

MSE	3	3.45629e-02	6.91257e-03	3.52295	X1	X2	X3
Sp	1	4.07384e-02	5.81977e-03	2.67969	X3		
Cp	1	4.07384e-02	5.81977e-03	2.67969	X3		

MSE	1	6.70823e-02	9.58318e-03	-0.670246	X3	
Sp	1	6.70823e-02	9.58318e-03	-0.670246	X3	
Cp	1	6.70823e-02	9.58318e-03	-0.670246	X3	

MSE	2	3.87667e-02	6.46111e-03	1.12307	X3	E1
Sp	1	4.21533e-02	6.02190e-03	0.366424	E1	
Cp	1	4.21533e-02	6.02190e-03	0.366424	E1	

MSE	1	4.98360e-02	7.11942e-03	-0.172478	X2	
Sp	1	4.98360e-02	7.11942e-03	-0.172478	X2	
Cp	1	4.98360e-02	7.11942e-03	-0.172478	X2	

MSE	1	1.98619e-02	2.83742e-03	-0.108219	E1	
Sp	1	1.98619e-02	2.83742e-03	-0.108219	E1	
Cp	1	1.98619e-02	2.83742e-03	-0.108219	E1	

MSE	2	2.27566e-02	3.79276e-03	1.19987	X1	X2
Sp	2	2.27566e-02	3.79276e-03	1.19987	X1	X2
Cp	1	2.67326e-02	3.81894e-03	0.981005	X2	

MSE	1	2.31557e-02	3.30796e-03	-0.325469	X2	
Sp	1	2.31557e-02	3.30796e-03	-0.325469	X2	
Cp	1	2.31557e-02	3.30796e-03	-0.325469	X2	

MSE	2	2.96808e-02	4.94681e-03	1.36022	X1	X3
Sp	2	2.96808e-02	4.94681e-03	1.36022	X1	X3
Cp	2	2.96808e-02	4.94681e-03	1.36022	X1	X3

MSE	3	2.72695e-02	5.45390e-03	3.00697	X1	X2	E1
Sp	3	2.72695e-02	5.45390e-03	3.00697	X1	X2	E1
Cp	3	2.72695e-02	5.45390e-03	3.00697	X1	X2	E1

MSE	2	4.81207e-02	8.02012e-03	1.77244	X3	E1
Sp	1	5.14718e-02	7.35311e-03	1.05649	X3	
Cp	1	5.14718e-02	7.35311e-03	1.05649	X3	

MSE 1 0.124977 1.78539e-02 -0.908955 X2
 Sp 1 0.124977 1.78539e-02 -0.908955 X2
 Cp 1 0.124977 1.78539e-02 -0.908955 X2

MSE 1 3.86961e-02 5.52802e-03 -0.554765 X3
 Sp 1 3.86961e-02 5.52802e-03 -0.554765 X3
 Cp 1 3.86961e-02 5.52802e-03 -0.554765 X3

MSE 2 1.07762e-02 1.79603e-03 1.05727 X3 E1
 Sp 1 1.25208e-02 1.78869e-03 0.715458 E1
 Cp 1 1.25208e-02 1.78869e-03 0.715458 E1

MSE 4 2.31882e-02 5.79705e-03 5.00000 X1 X2 X3
 E1
 Sp 2 2.46525e-02 4.10876e-03 3.44206 X1 E1
 Cp 2 2.46525e-02 4.10876e-03 3.44206 X1 E1

MSE 1 6.04122e-02 8.63031e-03 -0.792417 X1
 Sp 1 6.04122e-02 8.63031e-03 -0.792417 X1
 Cp 1 6.04122e-02 8.63031e-03 -0.792417 X1

MSE 1 4.62591e-02 6.60844e-03 0.381250 E1
 Sp 1 4.62591e-02 6.60844e-03 0.381250 E1
 Cp 1 4.62591e-02 6.60844e-03 0.381250 E1

MSE 1 0.110044 1.57205e-02 -0.301020 X1
 Sp 1 0.110044 1.57205e-02 -0.301020 X1
 Cp 1 0.110044 1.57205e-02 -0.301020 X1

MSE 1 0.116894 1.66991e-02 -0.512063 E1
 Sp 1 0.116894 1.66991e-02 -0.512063 E1
 Cp 1 0.116894 1.66991e-02 -0.512063 E1

MSE 3 5.91980e-02 1.18396e-02 3.09286 X1 X3 E1
 Sp 1 6.13972e-02 8.77102e-03 1.04274 X2
 Cp 1 6.13972e-02 8.77102e-03 1.04274 X2

MSE	3	1.67376e-02	3.34753e-03	3.37179	X2	X3	E1
Sp	3	1.67376e-02	3.34753e-03	3.37179	X2	X3	E1
Cp	3	1.67376e-02	3.34753e-03	3.37179	X2	X3	E1

MSE	1	8.60897e-02	1.22985e-02	0.261788	E1
Sp	1	8.60897e-02	1.22985e-02	0.261788	E1
Cp	1	8.60897e-02	1.22985e-02	0.261788	E1

MSE	3	1.19855e-02	2.39709e-03	3.05798	X1	X2	X3
Sp	2	1.26033e-02	2.10055e-03	2.20518	X1	X3	
Cp	2	1.26033e-02	2.10055e-03	2.20518	X1	X3	

MSE	2	3.42009e-02	5.70015e-03	1.36503	X1	E1
Sp	1	3.86254e-02	5.51791e-03	0.924681	E1	
Cp	1	3.86254e-02	5.51791e-03	0.924681	E1	

MSE	1	3.65205e-02	5.21721e-03	-9.18206e-02	X3
Sp	1	3.65205e-02	5.21721e-03	-9.18206e-02	X3
Cp	1	3.65205e-02	5.21721e-03	-9.18206e-02	X3

MSE	2	2.50195e-02	4.16992e-03	1.49327	X2	E1
Sp	1	2.60450e-02	3.72071e-03	0.535332	E1	
Cp	1	2.60450e-02	3.72071e-03	0.535332	E1	

MSE	1	4.73777e-02	6.76825e-03	-0.485272	E1
Sp	1	4.73777e-02	6.76825e-03	-0.485272	E1
Cp	1	4.73777e-02	6.76825e-03	-0.485272	E1

MSE	2	4.89740e-02	8.16234e-03	1.74329	X1	X3
Sp	1	5.48795e-02	7.83993e-03	1.35524	X1	
Cp	1	5.48795e-02	7.83993e-03	1.35524	X1	

MSE	1	2.25924e-02	3.22748e-03	0.729942	E1
Sp	1	2.25924e-02	3.22748e-03	0.729942	E1
Cp	1	2.25924e-02	3.22748e-03	0.729942	E1

The avg number of vars using MSE was 1.71667
The avg number of errors from MSE was 0.416667

The avg number of vars using Sp was 1.26667
The avg number of errors from Sp was 0.333333

The avg number of vars using Cp was 1.26667
The avg number of errors from Cp was 0.333333

where

errors stands for extraneous terms

vars stands for the number of variables chosen.

To find the response in the Response Surface Phase, the following equation was used.

$$(1 - \text{avg errors}) / (\text{avg vars})$$

For example, the following is the calculation for the C_c case.

$$(1 - .33333) / 1.2667 = .5263$$

This value was used as one of the 64 C_p response values in the response surface phase.

Appendix F: Fortran Program for finding
TMSEP values, rank ordering them and
Comparing the three criteria

```

*****
*
*
* This program is designed to take a modified SAS program
* and an existing data set and find the TMSEP the models.
* The modifications necessary are the same as discussed in
* Appendix D.
*
* The one exception is that the SAS listing prints out the
* coefficient values. When the variable is not included in
* the model a "." is left. This symbol must be replaced
* with a "0".
* To accomplish this the following UNIX editing command can
* be used.
*
*   g/ \. /s// 0 /g
*
* This command will make the appropriate substitution.
*
* In this program all integer values are counters (for do
* loops except for the arrays.
*
* variable ptrmse,ptrsp,ptrcp are used to keep track
*           of the lowest value found using
*           MSE,  $S_p$ , and  $C_p$  respectively.
*
*           spcount,equalcount,spcpct,cpcount are all
*           counters for keeping track of the
*           number of times the lowest TMSEP value
*           was found by Min  $S_p$ , all techniques,
*           Min  $S_p$  and Min  $C_p$ , and Min  $C_p$  only.
*
*           r2,mse,sp,cp are the arrays for  $r^2$ , mse,  $S_p$ 
*           and  $C_p$ .
*
*           b0 and betas are the values for the
*           coefficients
*
*           yssepmse,yssepsp,yssepcp are the Sum-of
*           -Squares for the respective technique.
*
*           ymsepmse,ymsepsp,ymsepcp are the MSEP
*           values using the respective technique.
*
*****

```

```

integer i,k,j,p,h,num(15),ptrmse,ptrsp,ptrcp

```

```

integer spcount,equalcnt,set,check(4),spcpct,cpcount
real b0(15),r2(15),mse(15),sp(15)
real betas(4,15),minmse,mincp,minsp
real x(4),ex,x3ex3(4),b0mse
real b0sp,b0cp,ypredmse,ypredsp
real ypredcp,ymsepmse,ymsepsp,ymsepcp
real yssepmse,yssepsp,yssepcp,y
real cp(15)

spcpct=0
equalcnt=0
spcount=0
cpcount=0
msecount=0
check(1)=4
check(2)=10
check(3)=14
check(4)=15

open(unit=12,file='msepl.lis',status='old')
open(unit=10,file='msepl.dat',status='old')
open(unit=11,file='msepl.out',status='new')

do 20 k=1,60
do 10 i=1,15
read (12,*) num(i), r2(i), mse(i), sp(i), cp(i),
b0(i)
+ ,betas(1,i),betas(2,i),betas(3,i),betas(4,i)
minmse = 10000
minsp = 10000
mincp = 10000
yssepmse=0
yssepsp =0
yssepcp =0
ptrmse = 0
ptrsp = 0
ptrcp = 0
do 30 j=1,4
if(mse(check(j)).lt.minmse) then
minmse = mse(check(j))
ptrmse = check(j)
endif
if (sp(check(j)).lt.minsp) then
minsp = sp(check(j))
ptrsp = check(j)
endif
if (cp(check(j)).lt.mincp) then
mincp = cp(check(j))
ptrcp = check(j)
endif
30 continue
10 continue

```

```

do 50 h= 1,20
  read (10,*) set,y,x(1),x(2),x(3),x(4),ex
  ypredmse= 0
  ypredsp = 0
  ypredcp = 0
  yactual = 0
  b0mse   = b0(ptrmse)
  b0sp    = b0(ptrsp)
  b0cp    = b0(ptrcp)
  ypredmse= b0mse
  ypredsp = b0sp
  ypredcp = b0cp
  x3ex3(1)= x(1)
  x3ex3(2)= x(2)
  x3ex3(3)= x(3)
  x3ex3(4)= ex

  do 60 p=1,4
    yactual = yactual+x(p)
60  continue
    do 70 p=1,4
      ypredmse = ypredmse + betas(p,ptrmse)*x3ex3(p)
      ypredsp  = ypredsp  + betas(p,ptrsp) *x3ex3(p)
      ypredcp  = ypredcp  + betas(p,ptrcp) *x3ex3(p)
70  continue
      yssepmse = ((ypredmse-yactual)**real(2)) + yssepmse

      yssepsp  = ((ypredsp -yactual)**real(2)) + yssepsp
      yssepcp  = ((ypredcp -yactual)**real(2)) + yssepcp
50  continue
      ymsepmse = yssepmse / (20-num(ptrmse))
      ymsepsp  = yssepsp  / (20-num(ptrsp))
      ymsepcp  = yssepcp  / (20-num(ptrcp))

      if(ymsepsp.lt.ymsepcp) spcount=spcount+1
      if(ymsepsp.eq.ymsepmse) equalcnt=equalcnt+1
      if(ymsepcp.lt.ymsepsp) cpcount=cpcount+1
      if(ymsepmse.lt.ymsepsp) msecout=msecout+1
      if(ymsepsp.eq.ymsepcp.and.ymsepmse.gt.ymsepsp)
        spcpct=spcpct+1
      write(11,*) 'run          ',k
      write(11,*) ' '
      write(11,*) ' '
      write(11,*) ' Sp  MSEP  = ', ymsepsp
      write(11,*) ' Cp  MSEP  = ', ymsepcp
      write(11,*) ' MSE MSEP = ', ymsepmse
      write(11,*) ' '
      write(11,*) ' '
20  continue
      write(11,*) ' '
      write(11,*) ' '
      write(11,*) ' '

```

```
write(11,*) ' Number of times Sp  had smallest MSEP  
' ,spcount  
write(11,*) ' Number of times Cp  had smallest MSEP  
' ,cpcount  
write(11,*) ' Number of times MSE had smallest MSEP  
' ,msecount  
write(11,*) ' Number of times Cp and Sp were equal  
' ,spcpct  
write(11,*) ' All MSEPs were equal',equalcnt  
END
```

Appendix G: Sample Output for Phase Two of Analysis

The following is a sample output for phase two of the analysis. This output is from the program found in Appendix F. The TMSEP values from each method is given, then the rank ordered TMSEP values for each subset of values is given.

run 1

Sp TMSEP = 8.1625051E-04
Cp TMSEP = 8.1625051E-04
MSE TMSEP = 8.1625051E-04

TMSEPs FOR RUN 1

1	5.3875311E-04
2	6.2458171E-04
3	6.2816928E-04
4	7.8544696E-04
5	8.1625051E-04
6	8.6704991E-04
7	8.7211549E-04
8	8.9747010E-04
9	9.1317965E-04
10	9.5657917E-04
11	1.3182798E-03
12	1.4625470E-03
13	1.7037858E-03
14	1.8694992E-03
15	9.8459609E-03

run 2

Sp TMSEP = 3.9654067E-03
Cp TMSEP = 3.9654067E-03
MSE TMSEP = 3.9654067E-03

TMSEPs FOR RUN 2

1	2.0111492E-03
2	2.2425181E-03
3	2.2925893E-03
4	3.9654067E-03
5	4.0591503E-03
6	4.1853175E-03
7	4.3666265E-03
8	4.4225999E-03
9	4.4807326E-03
10	4.4842619E-03
11	5.8024856E-03
12	5.8971639E-03
13	6.2541468E-03
14	6.5273000E-03
15	1.0629051E-02

run 3

Sp	TMSEP	=	1.7187677E-02
Cp	TMSEP	=	1.7187677E-02
MSE	TMSEP	=	1.7187677E-02

TMSEPs FOR RUN 3

1	1.0465025E-02
2	1.2726073E-02
3	1.3100259E-02
4	1.6458001E-02
5	1.7187677E-02
6	1.7703351E-02
7	1.9149231E-02
8	2.0130737E-02
9	2.1833103E-02
10	2.2608828E-02
11	2.3035588E-02
12	2.3823561E-02
13	2.5661029E-02
14	2.7239855E-02
15	3.0690275E-02

run 4

Sp	TMSEP	=	2.8920151E-02
Cp	TMSEP	=	2.8920151E-02
MSE	TMSEP	=	2.8920151E-02

TMSEPs FOR RUN

4

1	1.8989012E-02
2	1.9804025E-02
3	2.0740228E-02
4	2.1199832E-02
5	2.1934517E-02
6	2.2168593E-02
7	2.2491228E-02
8	2.3455916E-02
9	2.3542039E-02
10	2.4646813E-02
11	2.5090951E-02
12	2.6086060E-02
13	2.6278302E-02
14	2.8023606E-02
15	2.8920151E-02

run

5

Sp	TMSEP	=	2.5275916E-02
Cp	TMSEP	=	2.5275916E-02
MSE	TMSEP	=	2.5275916E-02

TMSEPs FOR RUN

5

1	1.1425762E-02
2	1.2566670E-02
3	1.2761764E-02
4	1.3455779E-02
5	1.4553362E-02
6	1.5473412E-02
7	1.5937347E-02
8	1.6735807E-02
9	1.7853793E-02
10	1.9062478E-02
11	2.1006519E-02
12	2.5275916E-02
13	2.7103966E-02
14	2.7148893E-02
15	2.9038427E-02

run

6

Sp	TMSEP	=	1.5230293E-02
Cp	TMSEP	=	1.5230293E-02
MSE	TMSEP	=	1.5230293E-02

TMSEPs FOR RUN

6

1	1.1189967E-02
2	1.1914390E-02
3	1.2381560E-02
4	1.2513387E-02
5	1.3219284E-02
6	1.3444155E-02
7	1.5230293E-02
8	1.5570977E-02
9	1.5827768E-02
10	1.6145628E-02
11	1.6554100E-02
12	1.6587460E-02
13	1.6841978E-02
14	1.7723817E-02
15	2.0091623E-02

run

7

Sp	TMSEP	=	8.4279571E-03
Cp	TMSEP	=	8.4279571E-03
MSE	TMSEP	=	8.4279571E-03

TMSEPs FOR RUN

7

1	8.4279571E-03
2	8.6327204E-03
3	8.6681442E-03
4	8.7661548E-03
5	8.7986309E-03
6	9.2602726E-03
7	9.5974831E-03
8	1.0461051E-02
9	1.0792202E-02
10	1.0847315E-02
11	1.0878014E-02
12	1.0926300E-02
13	1.1085336E-02
14	1.1660442E-02
15	3.0683922E-02

run

8

Sp	TMSEP	=	1.9174700E-03
Cp	TMSEP	=	1.9174700E-03
MSE	TMSEP	=	1.0391608E-02

TMSEPs FOR RUN

8

1	1.9174700E-03
2	2.1675059E-03
3	2.4701473E-03
4	5.9353570E-03
5	8.1103193E-03
6	1.0391608E-02
7	1.0793708E-02
8	1.2050519E-02
9	1.2089508E-02
10	1.2366496E-02
11	1.2528760E-02
12	1.2863966E-02
13	1.6300550E-02
14	1.7629199E-02
15	1.9998444E-02

run

9

Sp	TMSEP	=	2.9792758E-03
Cp	TMSEP	=	2.9792758E-03
MSE	TMSEP	=	2.9792758E-03

TMSEPs FOR RUN

9

1	2.5938307E-03
2	2.8756007E-03
3	2.9792758E-03
4	3.0860456E-03
5	3.8913097E-03
6	4.2111641E-03
7	4.4655493E-03
8	4.7994438E-03
9	5.4206038E-03
10	7.9978053E-03
11	8.4198406E-03
12	9.3529476E-03
13	9.8962309E-03
14	1.1325269E-02
15	1.5649561E-02

run

10

Sp	TMSEP	=	3.4058213E-03
Cp	TMSEP	=	3.4058213E-03
MSE	TMSEP	=	9.9750860E-03

TMSEPs FOR RUN

10

1	2.1047422E-03
2	2.5313916E-03
3	3.4058213E-03
4	4.1502793E-03
5	4.9941130E-03
6	9.2351809E-03
7	9.4529847E-03
8	9.9750860E-03
9	1.0221747E-02
10	1.0269716E-02
11	1.1170504E-02
12	1.4728523E-02
13	1.6914524E-02
14	1.9482933E-02
15	2.2141431E-02

run 11

Sp	TMSEP	=	1.8264394E-02
Cp	TMSEP	=	1.8264394E-02
MSE	TMSEP	=	2.6563916E-02

TMSEPs FOR RUN 11

1	5.1681511E-03
2	6.1819288E-03
3	6.7232223E-03
4	9.9291867E-03
5	1.0711449E-02
6	1.0777702E-02
7	1.1229556E-02
8	1.3848314E-02
9	1.7541861E-02
10	1.8264394E-02
11	1.8279061E-02
12	1.9208413E-02
13	2.3745688E-02
14	2.6563916E-02
15	2.8645089E-02

run 12

Sp	TMSEP	=	6.1405003E-03
Cp	TMSEP	=	6.1405003E-03
MSE	TMSEP	=	1.3503841E-02

TMSEPs FOR RUN 12

1	1.8639894E-03
2	1.9315871E-03
3	2.0497970E-03
4	2.2980263E-03
5	2.4225425E-03
6	2.7120986E-03
7	6.1405003E-03
8	7.1428162E-03
9	7.2628092E-03
10	8.3331009E-03
11	8.9695957E-03
12	1.3503841E-02
13	1.4081514E-02
14	1.4754524E-02
15	1.5624065E-02

run 13

Sp	TMSEP	=	3.2588169E-02
Cp	TMSEP	=	3.2588169E-02
MSE	TMSEP	=	4.0740110E-02

TMSEPs FOR RUN 13

1	7.9008838E-04
2	2.3885965E-03
3	4.2190668E-03
4	4.7011017E-03
5	6.7453287E-03
6	8.6223893E-03
7	9.0424791E-03
8	1.4171562E-02
9	1.5555089E-02
10	1.6289905E-02
11	2.1787029E-02
12	3.2588169E-02
13	3.4396078E-02
14	4.0740110E-02
15	4.3201827E-02

run 14

Sp	TMSEP	=	1.1849147E-02
Cp	TMSEP	=	1.1849147E-02
MSE	TMSEP	=	1.1849147E-02

TMSEPs FOR RUN 14

1	4.8623332E-03
2	4.9072048E-03
3	5.1156250E-03
4	5.8454480E-03
5	7.1764030E-03
6	7.8983689E-03
7	8.7281801E-03
8	1.1528674E-02
9	1.1675153E-02
10	1.1849147E-02
11	1.2552307E-02
12	1.3260029E-02
13	1.5406845E-02
14	1.6281059E-02
15	1.9996813E-02

run

15

Sp	TMSEP	=	7.4793403E-03
Cp	TMSEP	=	1.7116755E-02
MSE	TMSEP	=	1.7116755E-02

TMSEPs FOR RUN

15

1	6.6161933E-03
2	6.8892450E-03
3	7.4793408E-03
4	7.7769053E-03
5	8.7918332E-03
6	9.0924781E-03
7	9.1773802E-03
8	9.7412989E-03
9	9.8114768E-03
10	1.1472788E-02
11	1.7001675E-02
12	1.7116755E-02
13	1.7942483E-02
14	1.8224856E-02
15	1.9186204E-02

run

16

Sp	TMSEP	=	2.4616949E-02
Cp	TMSEP	=	2.4616949E-02
MSE	TMSEP	=	2.4616949E-02

TMSEPs FOR RUN

16

1	2.5192862E-03
2	6.2425360E-03
3	6.7220321E-03
4	8.6688502E-03
5	1.0253071E-02
6	1.1382802E-02
7	1.3577277E-02
8	1.5074901E-02
9	1.5540805E-02
10	1.5885668E-02
11	1.7365742E-02
12	2.4616949E-02
13	2.6280256E-02
14	2.8023722E-02
15	2.9615028E-02

run 17

Sp	TMSEP	=	3.4056220E-02
Cp	TMSEP	=	3.4056220E-02
MSE	TMSEP	=	3.4056220E-02

TMSEPs FOR RUN 17

1	2.3214938E-03
2	2.5332391E-03
3	2.6155079E-03
4	3.1588618E-03
5	7.8641446E-03
6	1.2681485E-02
7	1.5965864E-02
8	2.8908700E-02
9	3.0276716E-02
10	3.2774188E-02
11	3.3868384E-02
12	3.4056220E-02
13	3.9103031E-02
14	4.3489311E-02
15	4.7426648E-02

run 18

Sp	TMSEP	=	1.1656721E-02
Cp	TMSEP	=	1.1656721E-02
MSE	TMSEP	=	1.1656721E-02

TMSEPs FOR RUN 18

1	8.8144913E-03
2	9.2402538E-03
3	9.3459627E-03
4	1.0430153E-02
5	1.0963820E-02
6	1.1204778E-02
7	1.1656721E-02
8	1.3512224E-02
9	1.3699356E-02
10	1.5192608E-02
11	1.5793409E-02
12	1.6079456E-02
13	1.7682591E-02
14	1.8760057E-02
15	2.5110584E-02

run 19

Sp	TMSEP	=	2.8525121E-03
Cp	TMSEP	=	2.8525121E-03
MSE	TMSEP	=	8.1014968E-03

TMSEPs FOR RUN 19

1	2.8525121E-03
2	4.4923639E-03
3	4.9251076E-03
4	4.9598929E-03
5	8.1014968E-03
6	8.1187468E-03
7	8.4805330E-03
8	8.7108472E-03
9	8.9461999E-03
10	1.0312763E-02
11	1.2119596E-02
12	1.3127306E-02
13	1.3978862E-02
14	1.6505282E-02
15	2.0519499E-02

run 20

Sp	TMSEP	=	3.8725424E-03
Cp	TMSEP	=	3.8725424E-03
MSE	TMSEP	=	2.3732318E-02

TMSEPs FOR RUN 20

1	2.5775689E-03
2	2.8687967E-03
3	3.0975479E-03
4	3.8725424E-03
5	5.0329203E-03
6	7.2322008E-03
7	8.9591537E-03
8	9.2313904E-03
9	1.3217278E-02
10	1.6817946E-02
11	1.7074332E-02
12	1.9801777E-02
13	2.1266723E-02
14	2.3732318E-02
15	2.6426652E-02

run 21

Sp	TMSEP	=	1.7052114E-02
Cp	TMSEP	=	1.7052114E-02
MSE	TMSEP	=	1.7052114E-02

TMSEPs FOR RUN 21

1	1.7052114E-02
2	2.2108998E-02
3	2.2773303E-02
4	2.3853436E-02
5	2.3881707E-02
6	2.5029171E-02
7	2.5230104E-02
8	2.6611436E-02
9	2.6725085E-02
10	2.7688080E-02
11	2.7753545E-02
12	2.9282084E-02
13	2.9369567E-02
14	2.9715974E-02
15	3.1632055E-02

run 22

Sp	TMSEP	=	5.5181938E-03
Cp	TMSEP	=	5.5181938E-03
MSE	TMSEP	=	5.5181938E-03

TMSEPs FOR RUN 22

1	3.6380207E-03
2	3.9318283E-03
3	3.9480580E-03
4	4.1614547E-03
5	4.2129597E-03
6	4.3571047E-03
7	5.5181938E-03
8	5.7162573E-03
9	5.8647827E-03
10	6.0956995E-03
11	6.8830424E-03
12	7.0184362E-03
13	7.4183247E-03
14	7.5743883E-03
15	1.9468121E-02

run 23

Sp	TMSEP	=	1.2474651E-02
Cp	TMSEP	=	1.2474651E-02
MSE	TMSEP	=	1.6586190E-02

TMSEPs FOR RUN 23

1	1.7154425E-03
2	1.8433610E-03
3	2.1247640E-03
4	3.7305583E-03
5	7.3689125E-03
6	7.6402198E-03
7	1.0252496E-02
8	1.0763265E-02
9	1.1819728E-02
10	1.2474651E-02
11	1.3044675E-02
12	1.5219216E-02
13	1.6586190E-02
14	1.7523689E-02
15	1.8898807E-02

run 24

Sp	TMSEP	=	2.0092988E-02
Cp	TMSEP	=	2.0092988E-02
MSE	TMSEP	=	2.0092988E-02

TMSEPs FOR RUN 24

1	3.4344932E-03
2	6.0286978E-03
3	6.0683978E-03
4	6.0691400E-03
5	6.2191375E-03
6	7.2317598E-03
7	8.2883285E-03
8	1.1216477E-02
9	1.4045537E-02
10	1.4915518E-02
11	1.5253766E-02
12	1.5417388E-02
13	1.9777540E-02
14	2.0092988E-02
15	2.1272380E-02

run 25

Sp	TMSEP	=	2.0874506E-03
Cp	TMSEP	=	2.0874506E-03
MSE	TMSEP	=	2.0009512E-02

TMSEPs FOR RUN 25

1	2.0874506E-03
2	3.2037264E-03
3	3.9723651E-03
4	4.2093988E-03
5	5.0320607E-03
6	5.2119563E-03
7	8.8053672E-03
8	1.0451779E-02
9	1.2837918E-02
10	1.3056729E-02
11	1.4630186E-02
12	2.0009512E-02
13	2.1082407E-02
14	2.1697044E-02
15	2.2916408E-02

run 26

Sp	TMSEP	=	9.8276651E-04
Cp	TMSEP	=	9.8276651E-04
MSE	TMSEP	=	9.8276651E-04

TMSEPs FOR RUN 26

1	3.6615500E-04
2	4.0323741E-04
3	9.8276651E-04
4	1.0234589E-03
5	1.0259064E-03
6	1.0351705E-03
7	1.1282420E-03
8	1.2075821E-03
9	1.2196909E-03
10	1.3010738E-03
11	1.5202522E-03
12	1.5768777E-03
13	1.7284348E-03
14	1.9685118E-03
15	1.2083132E-02

run 27

Sp	TMSEP	=	6.8846606E-03
Cp	TMSEP	=	6.8846606E-03
MSE	TMSEP	=	1.9304380E-02

TMSEPs FOR RUN 27

1	6.8846606E-03
2	7.2969943E-03
3	9.8057995E-03
4	1.0382390E-02
5	1.2498997E-02
6	1.3136022E-02
7	1.3230429E-02
8	1.3853890E-02
9	1.3952672E-02
10	1.4731946E-02
11	1.7043175E-02
12	1.9304380E-02
13	2.0270826E-02
14	2.0551469E-02
15	2.1667840E-02

run 28

Sp	TMSEP	=	3.7091917E-03
Cp	TMSEP	=	3.7091917E-03
MSE	TMSEP	=	3.7091917E-03

TMSEPs FOR RUN 28

1	3.5498959E-03
2	3.7091917E-03
3	3.8102050E-03
4	3.9531672E-03
5	4.0174657E-03
6	4.0629935E-03
7	4.0745367E-03
8	4.0909247E-03
9	4.2000455E-03
10	4.2432775E-03
11	4.3620034E-03
12	4.4159661E-03
13	4.4462909E-03
14	4.5218416E-03
15	1.1852265E-02

run 29

Sp	TMSEP	=	8.8697784E-03
Cp	TMSEP	=	8.8697784E-03
MSE	TMSEP	=	8.8697784E-03

TMSEPs FOR RUN 29

1	8.1481282E-03
2	8.2919262E-03
3	8.8697784E-03
4	8.9680441E-03
5	9.1282884E-03
6	9.2851929E-03
7	9.3137734E-03
8	9.5947059E-03
9	9.8677045E-03
10	9.9192383E-03
11	1.0009223E-02
12	1.0221915E-02
13	1.0479798E-02
14	1.0976000E-02
15	1.7647961E-02

run 30

Sp	TMSEP	=	3.5457979E-03
Cp	TMSEP	=	3.5457979E-03
MSE	TMSEP	=	1.3879534E-02

TMSEPs FOR RUN 30

1	3.5457979E-03
2	4.6322457E-03
3	4.8427694E-03
4	5.4951324E-03
5	6.3336785E-03
6	6.3531036E-03
7	7.4292999E-03
8	8.7361382E-03
9	9.9864267E-03
10	1.1301067E-02
11	1.3879534E-02
12	1.4671805E-02
13	1.4762038E-02
14	1.5681023E-02
15	2.0051859E-02

run 31

Sp	TMSEP	=	1.9329343E-02
Cp	TMSEP	=	1.9329343E-02
MSE	TMSEP	=	2.2040404E-02

TMSEPs FOR RUN 31

1	1.3012470E-03
2	1.3328010E-03
3	1.6352949E-03
4	1.7551928E-03
5	1.8518085E-03
6	2.0177322E-03
7	4.5819799E-03
8	4.9842852E-03
9	7.8484677E-03
10	1.1772109E-02
11	1.2443145E-02
12	1.9329343E-02
13	2.1689299E-02
14	2.2040404E-02
15	2.4905885E-02

run 32

Sp	TMSEP	=	5.0999749E-02
Cp	TMSEP	=	5.6429233E-02
MSE	TMSEP	=	6.2712573E-02

TMSEPs FOR RUN 32

1	7.7922344E-03
2	9.5617203E-03
3	9.6836761E-03
4	1.1141410E-02
5	1.4167450E-02
6	1.4421574E-02
7	1.5424208E-02
8	1.6907291E-02
9	2.9010303E-02
10	2.9186631E-02
11	3.2019943E-02
12	5.0999749E-02
13	5.5162951E-02
14	5.6429233E-02
15	6.2712573E-02

run 33

Sp	TMSEP	=	8.4313720E-02
Cp	TMSEP	=	8.4313720E-02
MSE	TMSEP	=	8.4313720E-02

TMSEPs FOR RUN 33

1	1.6472956E-02
2	1.9875426E-02
3	2.3888426E-02
4	2.4913736E-02
5	2.6807437E-02
6	3.0863002E-02
7	3.6406118E-02
8	4.1674461E-02
9	4.9072534E-02
10	5.0164599E-02
11	5.6148276E-02
12	7.2232231E-02
13	7.6316334E-02
14	8.4313720E-02
15	9.0484820E-02

run 34

Sp	TMSEP	=	1.9720267E-03
Cp	TMSEP	=	1.9720267E-03
MSE	TMSEP	=	1.9720267E-03

TMSEPs FOR RUN 34

1	1.4564074E-03
2	1.5348943E-03
3	1.5788405E-03
4	1.7205393E-03
5	1.9720267E-03
6	3.0269641E-03
7	3.1256014E-03
8	3.4055866E-03
9	3.6278572E-03
10	3.9648265E-03
11	4.2102486E-03
12	5.5269981E-03
13	6.0082953E-03
14	6.6690734E-03
15	1.7902035E-02

run 35

Sp	TMSEP	=	2.2126485E-02
Cp	TMSEP	=	2.2126485E-02
MSE	TMSEP	=	2.8070798E-02

TMSEPs FOR RUN 35

1	2.1942048E-03
2	3.8941863E-03
3	5.9093987E-03
4	5.9594219E-03
5	1.0993944E-02
6	1.1843625E-02
7	1.2036902E-02
8	1.3240814E-02
9	1.3524123E-02
10	1.6991993E-02
11	1.8741287E-02
12	1.9171823E-02
13	2.2126485E-02
14	2.3006901E-02
15	2.8070798E-02

run 36

Sp	TMSEP	=	1.5084335E-02
Cp	TMSEP	=	1.5084335E-02
MSE	TMSEP	=	2.3686530E-02

TMSEPs FOR RUN 36

	1	1.5084335E-02
	2	1.5954880E-02
	3	1.7274419E-02
	4	1.7573619E-02
	5	1.8410455E-02
	6	1.8481053E-02
	7	1.9330112E-02
	8	2.0087145E-02
	9	2.0443222E-02
	10	2.1288132E-02
	11	2.1304168E-02
	12	2.3686530E-02
	13	2.5245925E-02
	14	2.5533320E-02
	15	2.7481722E-02
run		37

Sp	TMSEP	=	3.8605933E-03
Cp	TMSEP	=	2.6786476E-02
MSE	TMSEP	=	2.6786476E-02

TMSEPs FOR RUN 37

	1	7.0788513E-04
	2	7.3154463E-04
	3	1.3002879E-03
	4	2.6231960E-03
	5	2.9667136E-03
	6	2.9975821E-03
	7	3.8605933E-03
	8	8.0252122E-03
	9	1.6348951E-02
	10	1.8301096E-02
	11	2.0436686E-02
	12	2.2819119E-02
	13	2.6190016E-02
	14	2.6786476E-02
	15	3.0853484E-02
run		38

Sp	TMSEP	=	1.0595867E-02
Cp	TMSEP	=	1.0595867E-02
MSE	TMSEP	=	2.3941863E-02

TMSEPs FOR RUN 38

1	2.9883636E-03
2	4.1764169E-03
3	4.8778988E-03
4	9.7571025E-03
5	9.9973157E-03
6	1.0465559E-02
7	1.0595867E-02
8	1.4187485E-02
9	1.7074063E-02
10	2.0584134E-02
11	2.3294877E-02
12	2.3941863E-02
13	2.5516720E-02
14	2.5926596E-02
15	2.7840657E-02

run 39

Sp	TMSEP	=	1.0622767E-02
Cp	TMSEP	=	1.0622767E-02
MSE	TMSEP	=	1.0622767E-02

TMSEPs FOR RUN 39

1	1.1929536E-03
2	1.3373954E-03
3	1.6708911E-03
4	3.4250987E-03
5	3.5063413E-03
6	4.6463534E-03
7	4.6803518E-03
8	8.6782407E-03
9	8.7229246E-03
10	8.8040661E-03
11	1.0622767E-02
12	1.2146195E-02
13	1.2587945E-02
14	1.3528628E-02
15	1.7254293E-02

run 40

Sp	TMSEP	=	3.5547487E-02
Cp	TMSEP	=	3.5547487E-02
MSE	TMSEP	=	4.1834652E-02

TMSEPs FOR RUN 40

1	7.5619790E-04
2	1.5781120E-03
3	2.6893581E-03
4	2.8271538E-03
5	1.0157598E-02
6	1.0704858E-02
7	1.1176147E-02
8	1.2656823E-02
9	2.0222088E-02
10	2.0667935E-02
11	2.1879649E-02
12	2.8531680E-02
13	3.0591035E-02
14	3.5547487E-02
15	4.1834652E-02

run 41

Sp	TMSEP	=	2.2413865E-02
Cp	TMSEP	=	2.2413865E-02
MSE	TMSEP	=	2.8072061E-02

TMSEPs FOR RUN 41

1	1.0048087E-02
2	1.5616417E-02
3	1.5880233E-02
4	1.6917937E-02
5	1.7490519E-02
6	1.7648792E-02
7	1.8472508E-02
8	1.9556798E-02
9	2.0597108E-02
10	2.1678284E-02
11	2.2413865E-02
12	2.4091654E-02
13	2.7307892E-02
14	2.8072061E-02
15	3.0794047E-02

run 42

Sp	TMSEP	=	4.7981236E-03
Cp	TMSEP	=	4.7981236E-03
MSE	TMSEP	=	4.7981236E-03

TMSEPs FOR RUN 42

1	3.4364881E-03
2	3.5259190E-03
3	3.6066370E-03
4	3.9199176E-03
5	4.0680161E-03
6	4.1261218E-03
7	4.7981236E-03
8	5.2037672E-03
9	5.3428058E-03
10	5.6819613E-03
11	5.7216412E-03
12	6.0622832E-03
13	6.0960101E-03
14	6.4953892E-03
15	1.5678339E-02

run 43

Sp	TMSEP	=	3.9310423E-03
Cp	TMSEP	=	3.9310423E-03
MSE	TMSEP	=	3.9310423E-03

TMSEPs FOR RUN 43

1	1.8738484E-03
2	2.3403340E-03
3	2.9570037E-03
4	3.1850252E-03
5	3.9310423E-03
6	4.7286372E-03
7	4.8334445E-03
8	4.8407987E-03
9	5.7194503E-03
10	6.7090066E-03
11	7.6004523E-03
12	9.2586232E-03
13	9.2892852E-03
14	1.2049303E-02
15	1.2300548E-02

run 44

Sp	TMSEP	=	1.2019108E-03
Cp	TMSEP	=	1.8140119E-02
MSE	TMSEP	=	1.8140119E-02

TMSEPs FOR RUN 44

1	3.9169341E-04
2	5.2826642E-04
3	1.2019108E-03
4	1.2748841E-03
5	3.4131568E-03
6	6.1522294E-03
7	6.9718626E-03
8	7.0441552E-03
9	7.1675945E-03
10	7.9703350E-03
11	8.1011681E-03
12	8.6136796E-03
13	1.8140119E-02
14	1.9227074E-02
15	2.1237271E-02

run 45

Sp	TMSEP	=	3.0700916E-03
Cp	TMSEP	=	3.0700916E-03
MSE	TMSEP	=	1.1097923E-02

TMSEPs FOR RUN 45

1	3.0700916E-03
2	3.3025786E-03
3	4.1044764E-03
4	5.5306656E-03
5	5.6966278E-03
6	5.9962580E-03
7	6.9032218E-03
8	7.6960446E-03
9	1.0958084E-02
10	1.1097923E-02
11	1.1505049E-02
12	1.1665934E-02
13	1.1997189E-02
14	1.2202757E-02
15	1.3101306E-02

run 46

Sp	TMSEP	=	2.2392911E-03
Cp	TMSEP	=	8.1968531E-03
MSE	TMSEP	=	1.1323700E-02

TMSEPs FOR RUN 46

1	2.2392911E-03
2	2.4520580E-03
3	2.9001222E-03
4	5.2466160E-03
5	5.2603823E-03
6	5.3152107E-03
7	6.3047521E-03
8	6.6063711E-03
9	7.9144575E-03
10	8.1968531E-03
11	8.2538687E-03
12	9.0868426E-03
13	1.1323700E-02
14	1.3579807E-02
15	1.5681304E-02

run 47

Sp	TMSEP	=	1.0096396E-02
Cp	TMSEP	=	1.0096396E-02
MSE	TMSEP	=	1.0096396E-02

TMSEPs FOR RUN 47

1	9.9929059E-03
2	1.0018086E-02
3	1.0096396E-02
4	1.0306062E-02
5	1.1321264E-02
6	1.1852885E-02
7	1.2029038E-02
8	1.2128158E-02
9	1.2128995E-02
10	1.2503473E-02
11	1.2757318E-02
12	1.4217354E-02
13	1.4767253E-02
14	1.6271524E-02
15	2.8789150E-02

run 48

Sp	TMSEP	=	7.3499735E-03
Cp	TMSEP	=	7.3499735E-03
MSE	TMSEP	=	1.9433314E-02

TMSEPs FOR RUN 48

1	7.3499735E-03
2	7.8419475E-03
3	1.0533275E-02
4	1.1174344E-02
5	1.1179664E-02
6	1.1932554E-02
7	1.5124455E-02
8	1.5476356E-02
9	1.5993901E-02
10	1.6502503E-02
11	1.7872537E-02
12	1.9433314E-02
13	2.0548729E-02
14	2.0941224E-02
15	2.2205124E-02

run 49

Sp	TMSEP	=	4.2170435E-02
Cp	TMSEP	=	4.2170435E-02
MSE	TMSEP	=	4.2170435E-02

TMSEPs FOR RUN 49

1	1.5701497E-02
2	1.6772849E-02
3	1.7934496E-02
4	1.9688854E-02
5	2.1170381E-02
6	2.2765471E-02
7	2.4873454E-02
8	2.5819845E-02
9	2.7326189E-02
10	2.9126355E-02
11	3.0813400E-02
12	3.1698938E-02
13	3.3878606E-02
14	4.2170435E-02
15	4.4911824E-02

run 50

Sp	TMSEP	=	2.2768520E-02
Cp	TMSEP	=	2.2768520E-02
MSE	TMSEP	=	3.6368068E-02

TMSEPs FOR RUN 50

1	1.5540421E-02
2	1.6995009E-02
3	1.7393902E-02
4	2.0387221E-02
5	2.1287283E-02
6	2.1346433E-02
7	2.2153363E-02
8	2.2768520E-02
9	2.3359068E-02
10	2.4774464E-02
11	2.5770180E-02
12	2.8919019E-02
13	3.0525209E-02
14	3.6368068E-02
15	3.8798217E-02

run 51

Sp	TMSEP	=	2.4604530E-03
Cp	TMSEP	=	2.4604530E-03
MSE	TMSEP	=	7.0150704E-03

TMSEPs FOR RUN 51

1	1.6393900E-03
2	1.8884840E-03
3	2.2606149E-03
4	2.4604530E-03
5	2.4627568E-03
6	2.6644068E-03
7	2.7719922E-03
8	3.1818897E-03
9	3.1976223E-03
10	3.4056555E-03
11	7.0150704E-03
12	7.1953204E-03
13	7.7710031E-03
14	7.9887751E-03
15	2.0532234E-02

run 52

Sp	TMSEP	=	1.5042799E-02
Cp	TMSEP	=	3.2231480E-02
MSE	TMSEP	=	3.2231480E-02

TMSEPs FOR RUN 52

1	1.3510313E-02
2	1.4268238E-02
3	1.5042799E-02
4	1.5315600E-02
5	1.6465578E-02
6	1.7620863E-02
7	1.8399971E-02
8	1.8756019E-02
9	2.1330154E-02
10	2.4062017E-02
11	2.4590410E-02
12	2.6639560E-02
13	2.7676962E-02
14	3.2231480E-02
15	3.5307933E-02

run 53

Sp	TMSEP	=	4.1374839E-03
Cp	TMSEP	=	4.1374839E-03
MSE	TMSEP	=	4.1374839E-03

TMSEPs FOR RUN 53

1	1.4167477E-03
2	1.8064859E-03
3	1.8127215E-03
4	4.1374839E-03
5	5.3663515E-03
6	5.3790277E-03
7	5.4524424E-03
8	5.7910713E-03
9	6.2028468E-03
10	6.9525018E-03
11	8.6443033E-03
12	8.7333312E-03
13	9.9164629E-03
14	1.0268040E-02
15	1.1267337E-02

run 54

Sp	TMSEP	=	1.4347644E-02
Cp	TMSEP	=	1.4347644E-02
MSE	TMSEP	=	1.4347644E-02

TMSEPs FOR RUN 54

1	5.8415225E-03
2	6.4000203E-03
3	7.0034480E-03
4	8.8544022E-03
5	9.1899643E-03
6	9.4587682E-03
7	9.9534485E-03
8	1.0385258E-02
9	1.0859730E-02
10	1.1279927E-02
11	1.4146212E-02
12	1.4347644E-02
13	1.5366105E-02
14	1.7616469E-02
15	1.8684378E-02

run

55

Sp	TMSEP	=	1.2993388E-02
Cp	TMSEP	=	1.2993388E-02
MSE	TMSEP	=	1.9165561E-02

TMSEPs FOR RUN

55

1	4.9696909E-03
2	7.7763824E-03
3	9.6101258E-03
4	1.1696715E-02
5	1.2170565E-02
6	1.2716134E-02
7	1.2899847E-02
8	1.2993388E-02
9	1.5015327E-02
10	1.5555417E-02
11	1.5641246E-02
12	1.6055508E-02
13	1.7838184E-02
14	1.9165561E-02
15	2.1346293E-02

run

56

Sp	TMSEP	=	6.3008070E-02
Cp	TMSEP	=	6.3008070E-02
MSE	TMSEP	=	7.3209882E-02

TMSEPs FOR RUN

56

1	6.1677363E-02
2	6.2864579E-02
3	6.3008070E-02
4	6.6310331E-02
5	6.6765495E-02
6	6.6853113E-02
7	7.1422204E-02
8	7.1454287E-02
9	7.3190719E-02
10	7.3209882E-02
11	7.5800024E-02
12	7.7444568E-02
13	7.8586839E-02
14	7.9587929E-02
15	8.5475355E-02

run 57

Sp	TMSEP	=	3.7371186E-03
Cp	TMSEP	=	3.7371186E-03
MSE	TMSEP	=	3.7371186E-03

TMSEPs FOR RUN 57

1	3.7371186E-03
2	3.8739347E-03
3	3.9046754E-03
4	4.0383120E-03
5	4.1861599E-03
6	4.4727097E-03
7	4.6059825E-03
8	4.7026719E-03
9	5.1347492E-03
10	5.8260681E-03
11	5.8471588E-03
12	6.2553254E-03
13	6.4490214E-03
14	7.0934095E-03
15	9.5695062E-03

run 58

Sp	TMSEP	=	1.2361378E-02
Cp	TMSEP	=	1.2361378E-02
MSE	TMSEP	=	1.2361378E-02

TMSEPs FOR RUN 58

1	2.4738992E-03
2	3.7456378E-03
3	3.8988995E-03
4	4.2070122E-03
5	4.5560552E-03
6	4.8246444E-03
7	6.3058720E-03
8	1.2361378E-02
9	1.3068646E-02
10	1.3421705E-02
11	1.3535345E-02
12	1.4164726E-02
13	1.6804852E-02
14	1.7755972E-02
15	2.3304023E-02

run 59

Sp	TMSEP	=	1.5110006E-02
Cp	TMSEP	=	1.5110006E-02
MSE	TMSEP	=	1.5110006E-02

TMSEPs FOR RUN 59

1	1.0106159E-02
2	1.1130731E-02
3	1.1358641E-02
4	1.1672366E-02
5	1.2471750E-02
6	1.3456321E-02
7	1.3687690E-02
8	1.5110006E-02
9	1.5243942E-02
10	1.7136147E-02
11	1.8399373E-02
12	1.8740477E-02
13	1.9748218E-02
14	2.0658856E-02
15	2.3248075E-02

run 60

Sp	TMSEP	=	1.8573448E-02
Cp	TMSEP	=	1.8573448E-02
MSE	TMSEP	=	1.8573448E-02

TMSEPs FOR RUN 60

1	1.1594909E-02
2	1.2438751E-02
3	1.2594829E-02
4	1.2842594E-02
5	1.3457467E-02
6	1.3940695E-02
7	1.5752792E-02
8	1.8573448E-02
9	2.0086005E-02
10	2.2414638E-02
11	2.3428341E-02
12	2.4330338E-02
13	2.5231192E-02
14	2.5254766E-02
15	2.7355636E-02

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ABSTRACT

The purpose of this thesis was to identify three promising least squares selection procedures discussed in the literature during the previous decade and then test them using simulation. The three criteria chosen for this study were minimum mean square error (Min MSE), minimum Sp , and minimum Cp .

Most of the previous simulations in this area are limited to investigating the usefulness of variable selection criteria when all relevant regressors and some noise variables are available. It is questionable whether all relevant variables will be included. This research has examined the effects of not including a significant variable in the variable pool.

In examining each criterion, emphasis was placed on the technique's performance under varying amounts of multicollinearity, variable variation, number of variables, and sample size. Response Surface Methodology was used to determine the effects of varying these factors. A comparison was then made using the results from the Response Surface.

To supplement the simulation research a comprehensive literature review of the most current journal articles dealing with several least squares criteria has been provided. This review includes a discussion of each technique's strengths and weaknesses. Since many of the least squares variable selection criteria are addressed, this thesis serves as a useful starting place for various regression questions.

Keywords: Factorial, statistical tests, etc.