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A BROWNIAN BRIDGE CONNECTED WITH EXTREME VALUES

by

L. de Haan

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A Brownian bridge connected with extreme values

L. de Haan
Erasmus University Rotterdam
and
Center for Stochastic Processes
University of North Carolina

Abstract

A stochastic process formed from the intermediate order statistic is shown to converge to a Brownian bridge under conditions that strenghten the domain of attraction conditions for extreme-value distributions.

Keywords: random variables, distribution functions, (KR)

1. Introduction

Suppose X_1, X_2, ... are i.i.d. random variables from some distribution function F. Let $X_{(1,n)} \leq \cdots \leq X_{(n,n)}$ be the n-th order statistic.
It is well-known (cf. e.g. Resnick 1987) that, if F is the domain of attraction of an extreme-value distribution G_γ, the point process represented by the points

$$\left\{\left(\frac{i}{n}, \frac{X_i - b_n}{a_n}\right)\right\}_{i=1}^{\infty}$$

(with $a_n > 0$ and b_n chosen appropriately, n = 1, 2, ...) converges ($n\to\infty$) in distribution to a Poisson process with mean measure $dt.\frac{dG_\gamma(x)}{G_\gamma(x)}$. From this convergence one can infer the joint limiting distribution of extreme order statistics $(X_{(n,n)}, \ldots, X_{(n-k,n)})$ with k fixed ($n\to\infty$). However limiting distributions for intermediate order statistics $X_{(n-k,n)}$ where $k = k(n) \to \infty$ $k(n)/n \to 0$ ($n\to\infty$) cannot be inferred from the above-mentioned point process convergence.
If $F(x) = 1 - e^{-x}$ ($x > 0$), then by Rényi's representation $X_{(n-k,n)}$ and the stochastic process $\{X_{(n-k+[ks],n)} - X_{(n-k,n)}\}_{0\leq s\leq 1}$ are independent. Since the latter is equal in distribution to $\{X_{([ks],\, k)}\}_{0\leq s\leq 1}$, it converges – if

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properly normalized – to a Brownian bridge, provided $k = k(n) \to \infty$ $(n\to\infty)$. We are going to extend this behaviour to any distribution function satisfying a natural strengthening of the conditions for the domain of attraction of an extreme-value distribution.

So, in contrast with the situation for extreme order statistics, the process convergence underlying the limit behaviour of intermediate order statistics, is Gaussian. The same phenomenon by the way is present in the neighborhood of a quantile of the distribution (compare the asymptotic normality of sample quantiles with the point process convergence of de Haan and Resnick (1981)).

2. The results

Theorem 1

Define $U := (\frac{1}{1-F})^{\leftarrow}$ (inverse function) and suppose U has a positive derivative U'. Suppose further that $U \in RV_\gamma$ and there exists a positive function a such that for some $\rho \geq 0$ and all $x > 0$ (with either choice of sign)

$$(2.1) \qquad \lim_{t\to\infty} = \frac{(tx)^{1-\gamma}U'(tx) - t^{1-\gamma}U'(t)}{a(t)} = \pm \frac{x^{-\rho} - 1}{-\rho}$$

(read $\pm \log x$ for the right-hand side if $\rho = 0$), then for $k = k(n) \to \infty$, $k(n) = o(n/g^{\leftarrow}(n))$ where $g(t) := t^{3-2\gamma}\{U'(t)/a(t)\}^2$, $n\to\infty$

1. there exists a sequence of Brownian bridges $B_n(s)$ such that for all $\varepsilon > 0$

$$(2.2) \qquad \sup_{0<s<1} \sqrt{k}.s^{\gamma+1}\left\{\frac{X_{(n-[ks],n)} - X_{(n-k,n)}}{\{1 - F(X_{(n-k,n)})\}/F'(X_{n-k,n})} + \frac{1 - s^{-\gamma}}{\gamma}\right\} - B_n(s) \to 0$$

in probability $(n\to\infty)$.

2.

$$(2.3) \qquad \sqrt{k}\, \frac{X_{(n-k,n)} - U(\frac{n}{k})}{\frac{n}{k} \cdot U'(\frac{n}{k})} \text{ is asymptotically standard normal.}$$

3. the process under 1. and the random variable under 2. are asymptotically independent.

Before proceeding to the proof of the theorem we first formulate condition (2.1) in terms of the distribution function and its density (for the proof of the equivalence see Dekkers and de Haan 1988, section 2 and appendix).

<u>Proposition 2.1</u>

Relation (2.1) with $\rho = 0$ is equivalent to:

for $\gamma > 0$: $\pm t^{1+1/\gamma}F'(t) \in \Pi(b)$,

for $\gamma < 0$: $U(\infty) := \lim_{t\to\infty} U(t) < \infty$ and $\mp t^{-1-1/\gamma}F'(U(\infty)-t^{-1}) \in \Pi(b)$,

for $\gamma = 0$: let $f_0 := (1-F)/F'$ and $x^* := \sup\{x|F(x) < 1\}$. There exists a positive function α with $\alpha(t) \to 0$ $(t\uparrow x^*)$ such that for $x > 0$ locally uniformly

$$\lim_{t\uparrow x} \frac{\frac{1-F(t+xf_0(t))}{1-F(t)} - e^{-x}}{\alpha(t)} = \pm\frac{x^2}{2}e^{-x}.$$

<u>Proposition 2.2</u>

Relation (2.1) with $\rho > 0$ is equivalent to:

for $\gamma > 0$: for some $c > 0$ the function $t^{1+1/\gamma}F'(t) - c$ is of constant sign and

$$\lim_{t\to\infty} \frac{(xt)^{1+1/\gamma}F'(tx) - c}{t^{1+1/\gamma}F'(t) - c} = x^{-\rho}$$

(regular variation with exponent $-\rho$, notation $\pm\{t^{1+1/\gamma}F'(t) - c\} \in RV_{-\rho}$).

for $\gamma < 0$: for some $c > 0$ the function $\pm\{t^{-1-1/\gamma}F'(U(\infty) - t^{-1}) - c\} \in RV_{-\rho}$.

The proof of the theorem will be broken up in a series of lemma's.

<u>Lemma 1</u>

Relation (2.1) implies

(2.4) $\quad U' \in RV_{\gamma-1}$

(i.e. U' is regularly varying at infinity with index $\gamma-1$) and

$$\text{(2.5)} \qquad \lim_{t\to\infty} \left\{ \frac{U(tx) - U(t)}{a_0(t)} - \frac{x^\gamma - 1}{\gamma} \right\} / a_1(t) = \Psi_1(x)$$

for $x > 0$ locally uniformly, where a_0 and a_1 are positive functions and

$$\text{(2.6)} \qquad \pm\, \Psi_1(x) = \begin{cases} x^\gamma \{\frac{x^{-\rho} - 1}{-\rho}\} & \gamma \neq 0 \\ (\log x)^2/2 & \gamma = 0. \end{cases}$$

Furthermore

$$\text{(2.7)} \qquad a_0(t) = \begin{cases} \gamma\, U(t) & \gamma > 0 \\ t\, U'(t) & \gamma = 0 \\ -\gamma\{U(\infty) - U(t)\} & \gamma < 0 \end{cases}$$

and

$$\text{(2.8)} \qquad a_1(t) = \begin{cases} \gamma^{-1} a(t)\; t^\gamma / U(t) & \gamma > 0 \\ a(t)/\{t.U'(t)\} & \gamma = 0 \\ (-\gamma)^{-1} a(t)\; t^\gamma / \{U(\infty) - U(t)\} & \gamma < 0. \end{cases}$$

<u>Corollary 1</u>

$a_0 \in RV_\gamma$ and $a_1 \in RV_0$. Moreover $\lim_{t\to\infty} a_1(t) = 0$.

<u>Corollary 2</u>

Relation (2.5) also holds with $a_0(t) = t\, U'(t)$ for all γ (cf. Dekkers and de Haan, th. A1 and A3).

<u>Proof</u>

<u>$\gamma > 0$</u>: Relation (2.1) implies (cf. Dekkers and de Haan 1988, th. A.1)

$$(2.9)\qquad \lim_{t\to\infty} \frac{(tx)^{-\gamma}\, U(tx) - t^{-\gamma}\, U(t)}{\gamma^{-1}.a(t)} = \pm \frac{x^{-\rho} - 1}{-\rho}$$

locally uniformly for $x > 0$, i.e.

$$(2.10)\qquad \lim_{t\to\infty} \left\{ \frac{U(tx) - U(t)}{\gamma\, U(t)} - \frac{x^{\gamma} - 1}{\gamma} \right\} / \{\gamma^{-1}\, a(t)\, t^{\gamma}/U(t)\} = \pm\, x^{\gamma}.\frac{x^{-\rho} - 1}{-\rho}.$$

$\underline{\gamma = 0}$: Relation (2.1) implies (cf. Dekkers and de Haan 1988, th. A.5)

$$(2.11)\qquad \lim_{t\to\infty} \left\{ \frac{U(tx) - U(t)}{t\, U'(t)} - \log x \right\} / \{a(t)/t.U'(t)\} = \pm\, (\log x)^{2}/2$$

for $x > 0$.

$\underline{\gamma < 0}$: Relation (2.1) implies (cf. Dekkers and de Haan 1988, th. A.3)

$$(2.12)\qquad \lim_{t\to\infty} \frac{(tx)^{-\gamma}\{U(\infty) - U(tx)\} - t^{-\gamma}\{U(\infty) - U(t)\}}{-\gamma^{-1}.a(t)} = \pm \frac{x^{-\rho} - 1}{-\rho}$$

for $x > 0$ locally uniformly, i.e.

$$(2.13)\qquad \lim_{t\to\infty} \left\{ \frac{U(tx) - U(t)}{-\gamma\{U(\infty) - U(t)\}} - \frac{x^{\gamma} - 1}{\gamma} \right\} / \{(-\gamma)^{-1} a(t) t^{\gamma}/(U(\infty) - U(t))\} =$$

$$= \mp\, x^{-\gamma}\, \frac{x^{-\rho} - 1}{-\rho}$$

for $x > 0$.

<u>Lemma 2</u>

If (2.1) holds with a + sign for $\gamma \geq 0$ and a − sign for $\gamma < 0$, then given $\varepsilon > 0$ there exists t_0 such that for $t \geq t_0$ and $x \geq 1$

1. $\gamma > 0$

$$(1-\varepsilon)\, x^{\gamma}\, \frac{x^{-\rho-\varepsilon} - 1}{-\rho-\varepsilon} - \varepsilon x^{\gamma} < \left\{ \frac{U(tx) - U(t)}{a_0(t)} - \frac{x^{\gamma} - 1}{\gamma} \right\} / a_1(t) <$$

$$(1+\varepsilon)\, x^{\gamma}\, \frac{x^{-\rho+\varepsilon}-1}{-\rho+\varepsilon} + \varepsilon x^{\gamma}$$

2. $\gamma = 0$

$$(1-2\varepsilon)(\log x)^2/2 - 2\varepsilon \log x - \varepsilon < \left\{ \frac{U(tx)-U(t)}{a_0(t)} - \log x \right\} / a_1(t) <$$

$$(1+\varepsilon)^2\, x^{\varepsilon}\, (\log x)^2/2 + 2\,\varepsilon \log x + \varepsilon.$$

3. $\gamma < 0$

$$(1-\varepsilon)\, x^{\gamma}\, \frac{x^{-\rho-\varepsilon}-1}{-\rho-\varepsilon} - \varepsilon . x^{\gamma} < \left\{ \frac{U(tx)-U(t)}{a_0(t)} - \frac{x^{\gamma}-1}{\gamma} \right\} / a_1(t) <$$

$$(1+\varepsilon)\, x^{\gamma}\, \frac{x^{-\rho+\varepsilon}-1}{-\rho+\varepsilon} + \varepsilon x^{\gamma}.$$

Here a_0 and a_1 are as in Lemma 1.

Proof

For 1) and 3) see Geluk and de Haan 1987, p. 10 and p. 27. For 2) adapt the proof of Lemma 3 in Dekkers, Einmahl and de Haan (1988).

Lemma 3

Let $Y_{(1,n)} \le \dots \le Y_{(n,n)}$ be the n-th order statistics from the distribution function $1 - x^{-1}$ $(x > 1)$. If $k(n) \to \infty$ and (2.1) holds with a + sign for $\gamma \ge 0$ and a − sign for $\gamma < 0$, then given $\varepsilon > 0$ there exist n_0 such that for $n \ge n_0$, $i \le k$ almost surely $(n\to\infty)$

1. $\gamma \neq 0$

$$\sqrt{k}\, a_1(Y_{(n-k,n)})\{Y_{(n-i,n)}/Y_{(n-k,n)}\}^{\gamma}\, [(1-\varepsilon)\, \frac{\{Y_{(n-i,n)}/Y_{(n-k,n)}\}^{-\rho-\varepsilon}-1}{-\rho-\varepsilon} - \varepsilon]$$

$$< \sqrt{k}\left\{\frac{U(Y_{(n-i,n)}) - U(Y_{(n-k,n)})}{a_0(Y_{(n-k,n)})} - \frac{(Y_{(n-i,n)}/Y_{(n-k,n)})^{\gamma} - 1}{\gamma}\right\}$$

$$\sqrt{k}\, a_1(Y_{(n-k,n)})\, \{Y_{(n-i,n)}/Y_{(n-k,n)}\}^{\gamma}\, [(1+\varepsilon)\frac{\{Y_{(n-i,n)}/Y_{(n-k,n)}\}^{-\rho-\varepsilon} - 1}{-\rho-\varepsilon} + \varepsilon]$$

2. $\gamma = 0$

$$\sqrt{k}\, a_1(Y_{(n-k,n)})[(1-2\varepsilon)\{\log(Y_{n-i,n}/Y_{n-k,n})\}^2/2$$

$$- 2\,\varepsilon\, \log\{Y_{(n-i,n)}/Y_{(n-k,n)}\} - \varepsilon]$$

$$< \sqrt{k}\left\{\frac{U(Y_{n-i,n}) - U(Y_{(n-k,n)})}{a_0(Y_{(n-k,n)})} - \log Y_{n-i,n} - \log Y_{n-k,n}\right\} <$$

$$\sqrt{k}\, a_1(Y_{n-k,n})[(1+\varepsilon)^2\{Y_{(n-i,n)}/Y_{(n-k,n)}\}^{\varepsilon}\{\log(Y_{(n-i,n)}/Y_{(n-k,n)})\}^2/2$$

$$+ 2\,\varepsilon\, \log\{Y_{(n-i,n)}/Y_{(n-k,n)}\} + \varepsilon].$$

Proof

This is a straightforward application of Lemma 2, taking into account that $Y_{(n-k,n)} \to \infty$ $(n\to\infty)$.

Corollary 3

Under the conditions of theorem 1 for each $\varepsilon > 0$

$$\sup_{0<s<1} s^{\gamma+\varepsilon} \sqrt{k}\left\{\frac{U(Y_{n-[ks],n}) - U(Y_{n-k,n})}{a_0(Y_{n-k,n})} - \frac{\{Y_{(n-[ks],n)}/Y_{(n-k,n)}\}^{\gamma} - 1}{\gamma}\right\} \to 0$$

$(n\to\infty)$ in probability.

Proof

The conditions on $\{k(n)\}$ imply (cf. Dekkers and de Haan 1988, proof th. 2.3)

$$\lim_{n\to\infty} \sqrt{k}(n)\, a_1\left(\frac{n}{k(n)}\right) = 0.$$

Since $\lim_{n\to\infty} Y_{(n-k(n))} \cdot \frac{k(n)}{n} = 1$ in probability (cf. Smirnov (1949)) and $a_1 \in RV_0$, also

$$\lim_{n\to\infty} \sqrt{k}(n)\, a_1\left(Y_{(n-k(n))}\right) = 0.$$

Further by D. Mason ((1982) theorem 3)

$$\sup_{0<s<1} s^{\gamma+2\varepsilon} \{Y_{(n-[ks],n)}/Y_{(n-k,n)}\}^{\gamma} \left[(1+\varepsilon)\frac{\{Y_{(n-[ks],n)}/Y_{(n-k,n)}\}^{-\rho+\varepsilon} - 1}{-\rho+\varepsilon}\right]$$

is bounded $(n\to\infty)$ in probability and so is

$$\sup_{0<s<1} s^{1+2\varepsilon}[(1+\varepsilon)^2\{Y_{(n-[ks],n)}/Y_{(n-k,n)}\}^{\varepsilon}\{\log(Y_{(n-[ks],n)})(Y_{(n-k,n)})\}^2/2 +$$

$$+ 2\varepsilon \log\{Y_{m-[ks],n}/Y_{(n-k,n)}\} + \varepsilon].$$

The left hand bounds are treated similarly.

Lemma 4

There exists a sequence of Brownian bridges $B_n(y)$ such that for $y < 1$

$$\sup_{0<y<1} |\sqrt{k}\, \tilde{h}_{\gamma}(y)\left\{\frac{Y^{\gamma}_{([ky],k)} - 1}{\gamma} - \frac{(1-y)^{-\gamma} - 1}{\gamma}\right\} - B_k(y)| \to 0 \ (k\to\infty)$$

in probability with $\tilde{h}_{\gamma}(y) = (1-y)^{\gamma+1}$.

Proof

Cor. 3.2.1, p. 27, M. Csörgö (1983).

Proof of theorem 1

1. Combine Corollary 3 and Lemma 3, further note the distributional equality

$$\{Y_{(n-i,n)}/Y_{(n-k,n)}\}_{i=1}^{k-1} = \{Y_{(k-i,k)}\}_{i=1}^{k-1}.$$

It remains to show that

$$\left\{\frac{X_{(n-[ks],n)} - X_{(n-k,n)}}{(1-F(X_{n-k,n}))/F'(X_{n-k,n})}\right\} \stackrel{d}{=} \left\{\frac{U(Y_{(n-[ks],n)}) - U(Y_{n-k,n})}{Y_{(n-k,n)}\cdot U'(Y_{n-k,n})}\right\}$$

with $\{Y_{(i,n)}\}_{i=1}^{n}$ as in Lemma 4. For the numerator this is obvious. For the denominator note that

$$\{1-F(U(y))\}/F'(U(y)) = y\, U'(y).$$

2. It is well-known (Smirnov 1949) that

$$R_n := \sqrt{k}\,\{\tfrac{k}{n}\, Y_{(n-k,n)} - 1\}$$

converges in distribution to a standard normal random variable R, say ($n\to\infty$, $k = k(n)/n \to 0$). Hence

$$\sqrt{k}\,.\,\frac{U(Y_{n-k,n}) - U(\frac{n}{k})}{\frac{n}{k}\cdot U'(\frac{n}{k})} = \sqrt{k}\int_1^{1+R_n/\sqrt{k}} \frac{U'(\frac{n}{k}.s)}{U'(\frac{n}{k})}\,ds \to R$$

in distribution.

3. Follows immediately from Corollary 3 and the reasoning under 2.

Remark

It is possible to formulate the convergence towards the Brownian bridge in one of the other forms given on p. 26/27 of M. Csörgö (1983). This then allows us to infer the asymptotic normality of Hill's estimate for the index of regular variation (Hill (1975)) via an invariance principle.

Theorem 2

Under the conditions of Theorem 1

$$\sup_{0<s<1} \sqrt{k}\, s^{\gamma+1}\left\{\frac{X_{(n-[ks],n)} - U(\frac{n}{k})}{\frac{n}{k}U'(\frac{n}{k})} + \frac{1-s^{-\gamma}}{\gamma}\right\} - \{B_n(s) + sR_n\} \to 0$$

in probability ($n\to\infty$), where $\{B_n(s)\}$ and R_n are as in Theorem 1, $R_n \to R$, standard normal, and R_n and $\{B_n(s)\}$ are asymptotically independent.

Proof

$$\sqrt{k}\, s^{\gamma+1}\left\{\frac{X_{(n-[ks],n)} - U(\frac{n}{k})}{\frac{n}{k}U'(\frac{n}{k})} + \frac{1-s^{-\gamma}}{\gamma}\right\} =$$

$$\stackrel{d}{=} \sqrt{k}\, s^{\gamma+1}\left\{\frac{U(Y_{(n-[ks],n)}) - U(\frac{n}{k})}{\frac{n}{k}U'(\frac{n}{k})} + \frac{1-s^{-\gamma}}{\gamma}\right\} =$$

$$\sqrt{k}\, s^{\gamma+1}\left[\frac{Y_{(n-k,n)}\,U'(Y_{(n-k,n)})}{\frac{n}{k}U'(\frac{n}{k})}\left\{\frac{U(Y_{(n-[ks],n)}) - U(Y_{(n-k,n)})}{Y_{(n-k,n)}U'(Y_{n-k,n})} + \frac{1-s^{-\gamma}}{\gamma}\right.\right.$$

$$\left.\left. - \frac{1-s^{-\gamma}}{\gamma}\left[1 - \frac{\frac{n}{k}U'(\frac{n}{k})\cdot(\frac{k}{n}\cdot Y_{(n-k,n)})^{\gamma}}{Y_{(n-k,n)}U'(Y_{(n-k,n)})}\right]\right\} - \frac{1-s^{-\gamma}}{\gamma}\left\{(\frac{k}{n}Y_{n-k,n})^{\gamma} - 1\right\}\right.$$

$$\left. - \frac{U(\frac{k}{n}) - U(Y_{n-k,n})}{\frac{n}{k}\cdot U'(\frac{n}{k})}\right].$$

We now investigate the asymptotics of the various terms.

$$\sqrt{k}\left\{1 - \frac{\frac{n}{k}U'(\frac{n}{k})\cdot(\frac{k}{n}\cdot Y_{(n-k,n)})^{\gamma}}{Y_{(n-k,n)}U'(Y_{(n-k,n)})}\right\} = \sqrt{k}\, a_1(\tfrac{n}{k})\frac{Y^{1-\gamma}_{(n-k,n)}U'(Y_{n-k,n}) - (\frac{n}{k})^{1-\gamma}U'(\frac{n}{k})}{a_1(\frac{n}{k})\cdot Y^{1-\gamma}_{(n-k,n)}\,U'(Y_{(n-k,n)})}$$

$$\sim o(1) \cdot \frac{Y^{1-\gamma}_{(n-k,n)} U'(Y_{n-k,n}) - (\frac{n}{k})^{1-\gamma} U'(\frac{n}{k})}{a(\frac{n}{k})} \to 0 \ (n\to\infty)$$

in probability by (2.1), since $\frac{n}{k} Y_{(n-k,n)} \to 1$ $(n\to\infty)$ in probability.
In particular

$$\frac{Y_{(n-k,n)} \cdot U'(Y_{(n-k,n)})}{\frac{n}{k} \cdot U'(\frac{n}{k})} \to 1 \ (n\to\infty)$$

in probability. Next (note that this term is zero for $\gamma = 0$)

$$\sqrt{k} \, \{(\frac{k}{n} Y_{(n-k,n)})^{\gamma} - 1\} \sim \gamma \sqrt{k} \, \{(\frac{k}{n} Y_{(n-k,n)}) - 1\} \to \gamma \cdot R,$$

$n\to\infty$ (cf. proof of Theorem 1). Finally as in the proof of Theorem 1

$$\sqrt{k} \frac{U(Y_{n-k,n}) - U(\frac{n}{k})}{\frac{n}{k} \cdot U'(\frac{n}{k})} \to R$$

$(n\to\infty)$ in probability.

Remark

The differentiability of F is not required for Theorem 2 and part 1 of Theorem 1: condition (2.5) suffices.

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