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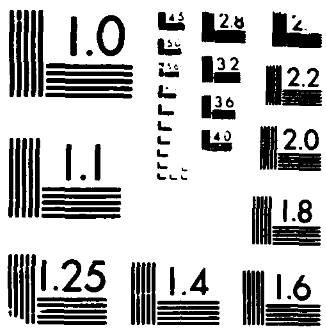

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19. ABSTRACT (Continue on reverse if necessary and identify by block number)

This research addresses several substantial problems in the theory and numerical analysis of ill-posed problems and integral equations: (i) Collocation, and Galerkin methods for Volterra and Abel equations of the first kind; (ii) Galerkin and collocation methods for nonlinear Abel-Volterra integral equations on the half-line and on a finite interval; (iii) a new approach to classification and regularization of ill-posed operator equations, and quantification of ill-posedness; (iv) operator extremal problems in the theory of compensation and representation of control systems; (v) constrained least-squares solutions of linear inclusions and singular control problems in Hilbert space.

New notions of bivariational and singular variational derivatives for functionals are also studied. They will be applied to extend the von Mises calculus for statistical functionals and its applications to robustness and approximation theorems.

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ILL-POSED PROBLEMS AND INTEGRAL EQUATIONS

Final Report

M. Zuhair Nashed

Paul P. Eggermont

February 22, 1988

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Ill–Posed Problems and Integral Equations

The research of Dr. **Eggermont** was mostly concentrated on the analysis of numerical methods for the approximate solution of Volterra integral equations of the first kind and (monotone) nonlinear Abel–Volterra equations of the second kind.

A. Collocation and Galerkin Methods for Volterra and Abel Equations of the First Kind

In [1] and [2] the superconvergence of collocation methods for Volterra equations of the first kind is investigated. It was known that collocation using piecewise polynomials of degree at most p produces convergence of order $p+2$ at selected points (depending on the collocation points) but that higher order convergence is in general impossible, despite the freedom of choosing collocation points. In [1] and [2] a way around this constraint is provided. A suitable "averaging" interpolation in the collocation solution was developed to obtain convergence of order $p+3$ in the mesh size h. Moreover, this was experimentally verified for $p = 2$ and 3. These results should be useful even with noisy data, since the increased order of convergence for the noiseless case would justify a smaller "optimal" mesh width h.

In [3] collocation methods similar to the ones used in [1] and [2] are applied to Abel–type integral equations of the first kind. It is still unknown how to choose the collocation points so that the collocation methods would yield stable projectional schemes. It is shown in [3] that this type of stability pertains if and only if the collocation methods are robust, in the sense that the amplification of noise in the data is asymptotically as small as possible. This is in effect a constraint on the condition number of the matrices arising from the collocation method. This constraint can be easily checked in practice. Some choices of collocation points are then shown experimentally to yield robust methods.

Whereas the stability of collocation methods cannot be proven, the stability of Galerkin methods *can* be proven. This is done in [4], together with the demonstration that suitable quadratures for the Galerkin method do not alter the order of convergence.

B. Galerkin and Collocation Methods for Nonlinear Volterra and Abel Equations of the Second Kind

The application of Galerkin methods to ***monotone*** Abel–type equations of the second kind is investigated in [5]. We reanalyzed the trapezoidal method for the solution of the nonlinear Abel–Volterra integral equation on the half–line

$$x(t) + \lambda \int_0^t (t-s)^{\alpha-1} g(s, x(s))ds = y(t),\ t > 0,$$

where $0 < \alpha < 1$. We proved the convergence of the method in the uniform norm, provided the nonlinear integral operator is Lipschitz–continuous and strictly monotone. The results depend on the "condition number" of the nonlinearity; there is no restriction on the size of λ.

In [7] we considered Galerkin methods for monotone Abel–Volterra integral equations on the half–line. The L^2 theory follows from Kolodoner's theory of monotone Hammerstein equations. We derived the L^∞ theory (i.e., uniform norm estimates) from the L^2 theory by relating the L^2– and L^∞–spectra of the operator

$x \to b \star (ax)$ to one another. Here $\star$ denotes convolution and $b \in L^1$ and $a \in L^\infty$. An extra order condition is imposed, namely $b(t) = O(t^{-\alpha-1})$, with $\alpha > 0$. We also proved the discrete analogue. In particular, we verified that the Galerkin matrix satisfies the "discrete" conditions.

In [6] we proposed a new method for the approximation of nonsmooth solutions of the nonlinear Abel–Volterra integral equations of the second kind on a finite interval. The method consists of the standard collocation method using piecewise polynomials on a uniform grid applied to a modified Abel–Volterra equation. The modification essentially subtracts the nonsmooth part of the integrand in the Abel–Volterra equation. The modification itself is computed by (another) collocation method on a small interval near zero.

C. On a Problem of S. Mazur

Together with Dr. Y. J. Leung, also of the University of Delaware, Dr. Eggermont investigated two problems of S. Mazur, dating from the 1930's (Problems 8 and 88 in the Scottish Book (D. Mauldin, editor), Birkhäuser, Boston 1980). The intimate relation between these two problems apparently had not been noticed before, cfr. Erdös (loc. cit.). These problems deal with the question whether every convergent sequence $\{z(n)\}$ can be written as

$$z(n) = (n+1)^{-1} \sum_{k=1}^{n} x(n-k)y(k)$$

with convergent sequences $\{x(n)\}$ and $\{y(n)\}$. The answer, as shown in [8], turned out to be no. This may find application to convolution type integral equations since the above summation is the discrete convolution of x and y.

• • •

The research of Dr. **Nashed** was concentrated on the following topics [(i) (iii) and parts of (iv) and (v) are interrelated]:

(i) Numerical analysis of integral equations of the first kind, with emphasis on classes of kernel functions and their implications for approximation and regularization methods [10], [14].

(ii) A new approach to classification and regularization of ill posed operator equations, and quantification of ill-posedness [10], [12].

(iii) A study of several mathematical models leading to nonlinear ill-posed problems [11].

(iv) Operator extremal problems in the theory of compensation and representation of control systems [13].

(v) Theory of normed linear relations; constrained least-squares solutions of linear inclusions and singular control problems in Hilbert space; and Tikhonov regularization when the adjoint is a multivalued operator as in the case of differential equations with nonstandard boundary conditions. This is joint research with Dr. S. J. Lee, [17], [18], [19], [20].

(vi) Bivariational and singular variational derivatives, and their applications to statistical functionals and robustness [24], [25], [26].

D. Ill-Posed Linear Operator Equations

The three classes of ill-posed problems considered are:

$$\int_a^b K(s,t)f(t)dt = g(s),$$

$$\mathcal{K}f = g,$$

$$Af = g,$$

where $\mathcal{K}$ is a compact linear operator and A is a bounded linear operator with nonclosed range. Nonlinear analogues of these problems are also considered. In least squares problems for the matrix equation $Mx = y$, where M is an $m \times n$ matrix, one distinguishes between the full rank case, where rank $M = \min(m, n)$, and the deficient rank case; the former being easier to treat theoretically and computationally. For linear ill-posed problems, the operator has infinite rank and, moreover, its range is nonclosed. We developed in [10] analogues of the full rank case and the deficient rank case for ill-posed integral and operator equations. This provided a delineation of subclasses among all operators with infinite rank that quantifies the ill-posedness in terms of maximum range resolution as described in [10].

We introduced new concepts of *regularizing families for the* ill-posed problem $Af = g$, where A is a bounded linear operator on a Banach space, based on bounded outer inverses of A, and classified ill-posed problems according to the type of regularizing families they admit. This analysis also led to new results in generalized inverse operator theory and interesting deeper results in the geometry of Banach spaces and operator ranges [9], [10].

We introduced and studied regularizers in the form of bounded outer inverses with infinite dimensional range (within this class, convergent regularizers can be selected to provide "optimal" resolution), approximate outer inverses and approximate right inverses in scales of norms.

The set of all operators in $\mathbf{L}(X, Y)$ that have bounded outer inverses with infinite dimensional range and the set of full-rank matrices of order $m \times n$ share several common properties: each of them is both *open* and *dense*, and all elements of each of the sets have outer inverses with the maximal possible rank (namely, the same as the rank of A). These properties and other results we established indicate that in Hilbert spaces, an equation involving a bounded *noncompact* operator with nonclosed range is *"less" ill-posed* than an equation with a *compact* operator with infinite dimensional range. In comparison with least-squares problems for $m \times n$ matrices, one may say that for operators with nonclosed range, the case of a *noncompact* operator corresponds to the *full-rank case* for matrices, while the case of a (nondegenerate) *compact* operator is the infinite dimensional analogue of the *rank-deficient case* for matrices. For the new notions of regularizers introduced we established convergence for specific families of regularizers and a choice of an "optimal" or "suitable" regularizer within a specific family in the presence of a contamination (i.e., what corresponds to a "regularizing parameter" and its choice in the present setting).

E. Classes of Kernel Functions and Their Implications in the Numerical Analysis of Integral Equations of the First Kind ([11], [10], [14], [22]).

The severeness or mildness of ill-posedness of integral equations of the first kind $\mathcal{K}f = g$ depends upon properties of the kernel and the regularity of the data g. We formulated our quantitative measures of ill-posedness directly in terms of kernel properties. In particular, properties of the following classes of kernels are delineated:

i) kernels with a finite smoothing index;

ii) infinitely smooth kernels inducing compact operators;

iii) kernels inducing operators of the Schatten classes;

iv) kernels inducing noncompact operators with nonclosed range;

v) Laplace transform kernel;

vi) kernels that make the range of $\mathcal{K}$ a reproducing kernel Hilbert space;

vii) kernels inducing operator equations of the first kind which are well-posed in certain Sobolev spaces or scales of Sobolev spaces.

Quantification of ill-posedness has often been too general or too vague and not related to the specific features of the problem at hand. In our approach, the emphasis has been on sharp results for specific problems and classes of kernels. We also distinguish kernels that correspond to the full-rank case and deficient-rank case as discussed above. The Laplace transform, convolution on an infinite interval, and other kernels inducing noncompact operators with nonclosed range are all examples of the full-rank case. The operator-theoretic distinctions among these classes also manifest themselves in the *numerical analysis* of these integral equations. Implications for discretization, moment discretization, projection and regularization methods, and for iterative schemes for the approximate solutions of these classes of integral equations are analyzed. Properties of matrices arising from these methods corresponding to the various classes of kernels and integral operators need to be further developed, particularly results relating the singular values of the "discretization" matrices to the singular values of the integral operator, and lower and upper bounds for the condition numbers of these matrices.

F. Nonlinear Ill–Posed Problems

In [11] we showed that there are several distinct notions of well–posedness for nonlinear extremal problems and nonlinear operator equations. We also analyzed certain nonlinear ill–posed problems arising from integral and operator equations, minimization of functionals, and variational inequalities. The theory of regularization of ill–posed *nonlinear* operator equations still needs considerable development. The paper [11] is an initial step we are taking toward the study and applications of such equations. An overview of some regularization–approximation methods for these problems will be given in a sequel paper. An extensive bibliography (100 references) is included in [11].

G. Operator Extremal Theory of Compensation and Representation of Systems and Control Problems

In [13] we addressed several problems in systems and control theory and related operator–theoretic extremal problems. The problems can be said to arise from compensation, representation, stabilization and approximation of singular and/or unstable systems.

In many problems of systems and control theory, (and more generally for problems in operator theory), one often needs a notion of a "metric" for measuring the distance between two systems and/or a notion of "ordering" or "partial ordering" between two systems. For stable systems the operator norm of the input–output mapping is often used as a metric. The same is true for bounded linear operators on a Banach space. However, this norm cannot be used for comparison or approximation of unstable systems or unbounded linear operators since the domains may be a different and the norm is unbounded. For such problems several notions are useful: the gap between closed subspaces, partial ordering in terms of Hermitian forms (also called Loewner ordering), a class of Schatten norms (including the Hilbert–Schmidt and the trace norms), and variants of these concepts. We defined notions of system pseudoinverses and system partial inverses which extend well known notions (e.g., Zadeh and Desoer) and analyze several compensation problems in systems and control theory in terms of new operator extremal characterizations of generalized inverses. We also considered generalizations of an operator inequality (relative to Hermitian ordering) which states that the inverse function is "convex" on the convex set of positive definite symmetric matrices. The generalizations hold for not necessarily bounded operators and do not require equality of the ranges of the operators. Finally, we addressed the "covariance condition" for the generalized inverse, namely $(T^{-1}AT)^{\dagger} = T^{-1}A^{\dagger}T$, and showed that this problem is related to the representation of a singular system relative to a basis. We provided a new outlook on the "covariance condition" and showed that the condition always holds relative to a transformed pair of projectors.

H. Normed Linear Relations and Constrained Least–Squares Solutions for Linear Inclusions and Singular Control Problems in Hilbert Spaces ([15]–[21])

The study of multi–valued operator equations (inclusions) and optimization problems involving multi–valued mappings has gained considerable importance in recent years due to the increasing occurrence of such problems in various areas of analysis, differential equations, continuum mechanics, control theory, and mathematical economics.

Let H_1 and H_2 be Hilbert spaces and let N be an algebraic subspace of H_1. The least-squares problem for a linear relation $L \subset H_1 \oplus H_2$ restricted to an algebraic coset $S := g + N$, $g \in H_1$, is considered in [16]. Various characterizations of a minimizer are derived in the form of *inclusion relations* that incorporate the constraints, and conditions under which a minimizer exists are developed. In particular, it is shown that generalized forms of the "normal equations" for constrained least-squares problems become *"normal inclusions"* that involve a multi-valued adjoint. The setting includes large classes of constrained least-squares minimization problems and optimal control problems subject to generalized boundary conditions.

An application of the abstract theory is given to a singular control problem involving ordinary differential equations with generalized boundary conditions, where the control may generate multiresponses. Characterizations of an optimal solution are developed in the form of inclusions that involve either an integrodifferential equation or a differential equation, and adjoint subspaces and/or solutions of certain linear equations.

In [19] we developed a theory for Tikhonov's regularization of an ill-posed problem $Af = g$, where A is an arbitrary closed linear operator, and the regularizing *closed* linear operator L is chosen so that either the domain of L is contained in the domain of A or vice versa. Error estimates were also obtained. We view Tikhonov's regularization as a least-squares problem in a *product space*, an approach that was first considered by Nashed in 1974. In the general setting described above, there is a subtle difficulty in that the adjoint of A and/or L is multivalued and hence the usual form of the regularized equation $(A^*A + \alpha L^*L)f = A^*g$ has no meaning. We circumvent this difficulty using adjoint subspaces and generalized graph–topology adjoints. We show that our approach applies to classes of problems to which the existing methodology and theory of Tikhonov's regularization do not apply, e.g. when the "smoothing" differential operator L is subjected to Stieltjes–type or nonstandard boundary conditions.

I. Bivariational and Singular Variational Derivatives in Function Spaces. Mathematical Foundations of the Von Mises Calculus of Statistical Functionals

The rigorous theory and extension of Volterra's variational derivatives for functionals on function spaces (the space of continuous functions, the space of functions with continuous n^{th} derivative) was developed by Dr. Nashed, in collaboration with Dr. E. P. Hamilton [24], [25]. See also the earlier paper:

E. P. Hamilton and M. Z. Nashed, Global and local variational derivatives and integral representations of Gâteaux differentials, *J. Functional Analysis* 49 (1982), 128–144.

A new notion of **bivariational derivative** and notions of **singular** variational derivatives were introduced [25]. These notions enabled us to handle an outstanding problem in this area, namely what corresponds to the variational derivative in the singular case and how to obtain a representation theorem when singular measures are involved. We established representation theorems for the Gâteaux differential in terms of the regular and singular bivariational derivatives [25].

Dr. Nashed has recently extended these notions and results to statistical functionals. These extensions provide the mathematical foundations for von Mises calculus for statistical functionals. We quote from a recent monograph by William J. J. Rey, ***Robust Statistical Methods***, Springer–Verlag, New York, 1978. From Section 5 entitled "Open Avenues":

"5.1. Estimators seen as functional of distributions

Section 2.4 is at the root of most derivations presented in this work however, and the fact must be stressed, it has been hardly possible to state precisely under what conditions a functional $T(\cdot)$, evaluated on distribution g, can be expanded with respect to distribution f according to

$$T(g) = T(f) + \int \psi(X)g(x)dx + \frac{1}{2}\iint \psi(x,y)g(x)g(y)dx\,dy + \dots$$

This state of affair is ***unsatisfactory*** although the high plausibility of the expansion has been demonstrated for a large class of functionals and distributions. As already noted the attempts by many groups of experts to state precise conditions have failed. This failure can partly be attributed to the complexity of the problem but, possibly, also to the shortage of motivated analysts with top–qualification in topologies."

This state of affair is further emphasized in Rey's more recent book entitled "***Introduction to Robust and Quasi–Robust Statistical Methods***," Springer–Verlag, New York, 1983.

We are pleased to report that our research last summer has led to a satisfactory resolution of this state of affairs [26]. Dr. Nashed has made in the past several worthwhile excursions into the area of differential calculus in abstract analysis; this area has become a mathematical hobby of his and the papers he published on this topic have been very rewarding. This most recent contribution to this area, which we discussed with several experts in the theory of statistical functionals and estimation is rather exciting. Applications of these results to robustness (both in statistics and ill-posed problems) are currently being investigated. It is a happy ending!

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