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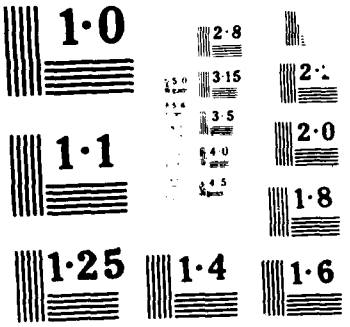
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FINAL REPORT ON
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We summarize work done under AFOSR Contract F49620-83-C-0064. The major results include: development of renormalization group techniques for large-eddy simulations of turbulent flows, the first direct numerical simulation of turbulent spots in channel and boundary-layer flows, the further development of spectral methods for turbulence simulations, the identification of secondary instability modes in free shear layers, the development of an efficient multi-grid marching method for solution of the parabolized Navier-Stokes equations, a mathematical analysis of boundary conditions for the parabolized compressible Navier-Stokes equations, the further development of a method to improve numerical solution of singular perturbation problems by use of asymptotic approximations.

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MARCHING ITERATIVE METHODS FOR THE PARABOLIZED AND THIN LAYER NAVIER-STOKES EQUATIONS

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MARCHING ITERATIVE METHODS
FOR THE PARABOLIZED AND THIN LAYER NAVIER-STOKES EQUATIONS

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Abstract

Downstream marching iterative schemes for the solution of the Parabolized or Thin Layer (PNS or TL) Navier-Stokes equations are described. Modifications of the primitive equation global relaxation sweep procedure result in efficient second-order marching schemes. These schemes take full account of the reduced order of the approximate equations as they behave like the SLOR for a single elliptic equation. The improved smoothing properties permit the introduction of Multi-Grid acceleration. The proposed algorithm is essentially Reynolds number independent and therefore can be applied to the solution of the subsonic Euler equations. The convergence rates are similar to those obtained by the Multi-Grid solution of a single elliptic equation; the storage is also comparable as only the pressure has to be stored on all levels. Extensions to three-dimensional and compressible subsonic flows are discussed. Numerical results are presented.

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1. INTRODUCTION

Considerable evidence accumulated recently about the applicability of the Parabolized Navier-Stokes equations for high Reynolds number flows with a principal flow direction; see Rubin [1]. The PNS equations are obtained by neglecting the streamwise viscous terms in the Navier-Stokes (NS) equations. When the viscous terms in the circumferential direction are also neglected, one gets the Thin Layer approximation.

The steady PNS equations still have an elliptic nature, and therefore the initial value problem in the marching direction is not well posed [2]. A well posed initial-boundary value problem can be formulated by specifying (for example) upstream and side conditions for the velocities and one downstream condition for the pressure. This coupled system of partial differential equations behaves like a single elliptic equation for the pressure. Therefore the PNS equations must be solved globally and cannot be solved by a single sweep marching. The reduced order of the PNS equation can be exploited by constructing an iterative marching method for updating the pressure field only. Such a multiple sweep iteration method has the advantage that the velocity fields are generated during the marching process and only the pressure field has to be stored from sweep to sweep. A considerable saving in storage results. However, simple minded marching does not result in good convergence properties and sometimes diverges. For the two-dimensional incompressible case, Israeli and Lin [3] devised a stable marching scheme that behaves like the Successive Line Over Relaxation (SLOR) method for a single elliptic equation. The good smoothing properties of the above mentioned scheme can be used in a Multi-Grid (MG) framework in order to accelerate the convergence of the solution of the PNS (or TL) equations. The marching scheme is implemented using a new stable algorithm which is second order also in the marching direction. The same method can be used without modification for the subsonic Euler equations as the effect of the Reynolds number on the convergence rate is insignificant. In two dimensions the PNS and TL equations are identical and therefore the same analysis applies to both.

It turns out that the extension to three-dimensions is conceptually simple; but the resulting algorithm, a successive plane over relaxation, is complicated by the requirement of the simultaneous solution of the equations in planes perpendicular to the marching direction. This problem can be alleviated by splitting of the equation of continuity from the momentum equations.

The extension of the method to compressible flows is conceptually non trivial. The original iterative method is based on the concept that the convergence relies on the implicit relaxation of a single quantity, the pressure, which approximately satisfies a single elliptic equation. In the compressible case a viable approach is to eliminate the pressure and to derive an equation for $\tilde{p}$, the logarithm of the density. It can be shown that $\tilde{p}$ satisfies approximately equation (1.1),

$$(1 - M^2)\tilde{p}_{ss} + \tilde{p}_{nn} = 0, \qquad (1.1)$$

where M is the Mach number and s and n are coordinates along and perpendicular to the flow direction. Although this equation is never derived or used in the algorithm, it reveals the fact that for $M < 1$ the upstream influence is transmitted through the quantity $\tilde{p}$, and therefore only this quantity should be stored or updated. The flow of information should be downstream for the velocity and temperature and upstream for the density, and the difference scheme must be built accordingly. For supersonic flows the flow of information should be only downstream and the marching method is non-iterative. For supersonic flows with imbedded subsonic regions, the iterative method should be used, combined with an appropriate switching at shock waves and sonic lines.

It should be pointed out here that the present approach is very different conceptually from that of Reddy and Rubin [4]. Although they used our idea (Israeli [6]) of back shifting the pressure, one full mesh distance, with respect to the velocity for incompressible flows, their generalization to compressible flows is a Mach number dependent shift

which vanishes for $M > 1$. This smooth transition from subsonic to supersonic flows is questionable since the change of type of equation (1.1) is sudden, at $M = 1$. Indeed, only our full shift is used in their papers and properly results in a conservative scheme across a shock.

Another question raised by the above mentioned paper is that of the distinction between the pressure which uses downstream data and the density which uses upstream data. This obscures the issue of the direction of flow of information and proper location of boundary conditions. This approach should result in inconsistency of boundary data and may eventually lead to ill posedness and divergence.

In the next sections we will summarize our previous theoretical results, present some new numerical results and the extensions to 3-D and compressible flows.

2. FORMULATION FOR THE INCOMPRESSIBLE CASE

For simplicity we will consider initially the case of the steady, incompressible, and two-dimensional PNS (or TL) equations in cartesian coordinates [x;y]:

$$U_x + V_y = 0 \tag{2.1}$$

$$(U^2)_x + (UV)_y = -P_x + U_{yy}/Re \tag{2.2}$$

$$(UV)_x + (V^2)_y = -P_y + V_{yy}/Re \tag{2.3}$$

where x is the mainstream direction, Re is the Reynolds number. U and V are the nondimensional velocity components in the x and y direction, respectively. P is the nondimensional pressure.

The two-dimensional NS equations are elliptic of order four - Brandt and Dinar [5]. The PNS are elliptic only of order two like the Poisson equation (the mathematical nature of several two-dimensional and three-dimensional approximations to the Navier-Stokes equations was analyzed in [7]). This ellipticity is due to the pressure gradient terms via the continuity equation. A well posed problem can be formulated by defining the boundary conditions as described in Fig. 2. The following Dirichlet conditions may be specified:

* upstream boundary (AB): $U = U_{in}$; $V = V_{in}$ (2.4)

* at a solid wall (AD): $U = U_{wall}$; $V = V_{wall}$ (2.5)

* at the outer boundary (BC): $U = U_{out}$; $V = V_{out}$ (2.6)

* at the downstream boundary (CD): $P = P_{down}$. (2.7)

Other boundary conditions can be used, but the same number of conditions on each boundary must be kept.

In order to separate linear and non linear effects, some of the convergence tests were performed with the following linear version of equations (2.1)-(2.3):

$$U_x + V_y = 0 \tag{2.8}$$

$$(aU)_x + (bU)_y = -P_x + U_{yy}/Re \tag{2.9}$$

$$(aV)_x + (bV)_y = -P_y + V_{yy}/Re \tag{2.10}$$

where a and b are known functions of x and y.

3. DISCRETIZATION AND MARCHING

Numerical solutions of Eqs. (2.1)-(2.3) are obtained by spreading a grid over the computational domain. Let us assume that the grid points are distributed evenly along the x and y coordinates with the spacing Δx and Δy respectively. When differencing these equations it should be remembered that their nature should be reflected [1,8] in the finite difference approximation. In order to be consistent with the boundary layer (parabolic) nature of the flow, the axial gradients of the velocities should be computed using only upstream values, while the elliptic nature is preserved by forward differencing the axial pressure gradient [1,8,9]. Consequently, it was assumed that a stable marching scheme must be of the first order in the marching direction. It turns out that this effect can be achieved by a judicious choice of the placement of the variables to be solved at each station. The choice can be explained most easily by taking $V = 0$ and $\frac{1}{Re} = 0$ in Eq. (2.2) for U, yielding

$$u_x^2 = - p_x .$$

A first order difference scheme then becomes

$$u_{m,j}^2 - u_{m-1,j}^2 = p_{m,j} - p_{m+1,j} ;$$

the unknowns are $u_{m,j}$ and $p_{m,j}$. The scheme first suggested by Israeli [9,10] is:

$$u_{m+1,j}^2 - u_{m,j}^2 = p_{m,j} - p_{m+1,j}$$

with the unknowns $u_{m+1,j}$ and $p_{m,j}$. The scheme is centered about $m+\frac{1}{2}$ and is second order. This approach was subsequently used by Rubin and Reddy [8] and Reddy and Rubin [4].

In addition, one may stagger the velocity V with respect to the other variables as shown in Fig. 3, where the centering points of the different difference equations are also plotted. The differential equations are approximated by central second-order approximations whenever needed averaging was used as is usually done for staggered grids.

Numerical experiments with a first order computer code show that the solution after one marching sweep is not close to the final solution of the PNS equations when the initial pressure field is constructed using the boundary layer assumption $p_y = 0$. Since the p_x term is forward differenced, some global iterations over the whole solution domain should be performed in order to converge the explicit contribution to this pressure term. The simplest global iterative technique to solve the equations is by multiple marching sweeps with the primitive equations where only the pressure field is kept from iteration to iteration [1]. Numerical experiments also show that for certain nets this procedure diverges. The divergence occurs also for the linearized version of Eqs. (2.1) - (2.3). Figure 1 presents the residual of the pressure field as a function of the global iteration's sweep number for a 21×11 field. A jump is encountered every 10 iterations (probably related to the arrival of the boundary pressure pulse traveling at the numerical

scheme speed) leading to ultimate divergence. However convergence was reported with different mesh and boundary conditions and also when combining the above procedure with a multigrid technique [4]. It was thought that the replacement of one of the momentum equations by the Poisson equation for the pressure will improve the convergence rate, but the solution did not satisfy the replaced momentum equation. A successful implementation of the marching technique is derived in the next section. A short and reduced version of the analysis was presented first in [3].

4. A MULTI-GRID ALGORITHM

The Multi-Grid technique is a numerical strategy for substantially improving the convergence rate of an iterative procedure. In order to facilitate comparison with theory, the accomodative C-cycle MG algorithm was chosen.*

Each MG process consists of three basic parts: relaxation, restriction, and interpolation [5].

The Relaxation Scheme

The overall convergence rate of any MG process is greatly influenced by the smoothing properties of the relaxation scheme. It can be shown analytically and experimentally that the usual multiple sweep marching [1] does not have good convergence and smoothing properties because short wave errors are not efficiently smoothed. Israeli and Lin [3] showed that certain modifications in the streamwise momentum equation, which vanish upon convergence, give rise to an iterative scheme which is equivalent, in the linear case, to the SLOR method for one Poisson equation. In the general nonlinear case the modified iterative process is essentially equivalent to the relaxation of a single nonlinear Poisson-like equation for the pressure. The velocities can be viewed as auxiliary variables needed during the marching since they have no "memory" by themselves.

Furthermore, we have automatically gained the good smoothing properties of the line relaxation scheme of a single Poisson equation. The problems associated with the loss of ellipticity of the difference

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* Some of the elements of the present approach were used independently by Rubin and Reddy [8]. Detailed comparisons cannot be made because convergence rates and storage estimates were not presented there.

approximation for the Navier-Stokes equations at high Reynolds number [5] are thus avoided and no upstream-weighting or artificial viscosity are required. There results a considerable saving in storage, as well as a simpler relaxation scheme (compare to the distributive relaxation [5]) where the convergence rate is essentially independent of the Reynolds number. We note that the same marching algorithm can thus be used for the (subsonic) Euler equation with the same favorable convergence rate. (For supersonic flows the marching method is non iterative.)

A part of the analysis of [3] is repeated here to motivate the later extensions to three-dimensional and compressible flows. We start with the PNS equations (2.1)-(2.3) and linearize them about a constant state. We also introduce

$$\bar{L}(f) = \frac{\partial}{\partial x}(\bar{U}f) + \frac{\partial}{\partial y}(\bar{V}f) - \frac{1}{Re}\frac{\partial^2}{\partial y^2} f \tag{4.1}$$

where $\bar{U}$ and $\bar{V}$ are constant reference velocities. The next step is to discretize the equations only in the x direction to obtain:

$$-\frac{U_{m-1}-U_m}{\Delta x} + (V_y)_m = 0 \tag{4.2}$$

$$D(U_m) = -\frac{P_{m+1}-P_m}{\Delta x} \tag{4.3}$$

$$D(V_m) = -(P_y)_m \tag{4.4}$$

where U_m, V_m and P_m are functions of y. Here $D(f_m)$ is the semi discretized form of $\bar{L}(f)$ at the marching station m. The semi - discretized system should be discretized also in the y direction before solution is attempted, but since the specific form of this discretization is not important for the following argument, we postpone this step for the sake of transparency.

The marching iterative procedure assumes that $U_1^k(y)$, $V_1^k(y)$ are known as well as P_m^{k-1} for $m = 2,3,4,\ldots M$, where k is the current iteration index. Therefore, the marching scheme for $m \geq 2$ is:

$$-\frac{U_{m-1}^k - U_m^k}{\Delta x} + (V_y)_m = 0 \tag{4.5}$$

$$D(U_m^k) = -\frac{P_{m+1}^{k-1} - P_m^k}{\Delta x} \tag{4.6}$$

$$D(V_m^k) = -(P_y^k)_m \quad . \tag{4.7}$$

We now apply D to Eq. (4.5) and differentiate Eq. (4.7) with respect to y. Elimination of the V terms between Eqs. (4.5) and (4.7) gives:

$$D(U_{m-1}^k - U_m^k) = -(P_{yy}^k)_m \, \Delta x \quad . \tag{4.8}$$

Now by substitution of Eq. (4.6) into Eq. (4.8) we get:

$$P_{m+1}^{k-1} - P_m^k - P_m^{k-1} + P_{m-1}^k + \Delta x^2 (P_{yy}^k)_m = 0 \, . \tag{4.9}$$

It follows that the marching scheme for the primitive system (4.2)-(4.4) can be viewed as a line iterative scheme for the semi-discretized Laplace equation; indeed upon convergence Eq. (4.9) will become:

$$\frac{P_{m+1} - 2P_m + P_{m-1}}{\Delta x^2} + (P_{yy})_m = 0 \, . \tag{4.10}$$

In order to find out the rate of convergence of Eq. (4.9) to the final state (4.10), we Fourier transform Eq. (4.9) in y assuming appropriate boundary conditions in that direction:

$$P_m^k = \bar{Z}_m e^{I\,ny} \quad , \quad P_m^{k-1} = Z_m e^{I\,ny} \tag{4.11}$$

where $I^2 = -1$ and n is the Fourier wave number. After substituting these definitions into Eq. (4.9) we get:

$$Z_{m+1} - \bar{Z}_m - Z_m + \bar{Z}_{m-1} - \Delta x^2 n^2 \bar{Z}_m = 0. \tag{4.12}$$

Transforming in the x direction we define again

$$Z = A e^{I\,\theta x} \tag{4.13}$$

where θ is the wave number in the x direction. By substitution of this definition into Eq. (4.12) we get:

$$\left|\frac{\bar{A}}{A}\right| = \left|\frac{1 - e^{-I\theta}}{1 + \Delta x^2 n^2 - e^{-I\theta}}\right| \, . \tag{4.14}$$

This means that all the long waves (with small $\Delta x^2 n^2$) in the cross flow

direction are only weakly damped irrespective of their structure in the marching direction. In particular the $n = 0$ modes which exist for derivative boundary conditions in the cross flow direction are not affected at all by the relaxation, i.e.,

$$\left|\frac{\bar{A}}{A}\right| = 1.$$

On the other hand, the well known SLR scheme for the Poisson equation gives (after the same Fourier transformations):

$$Z_{m+1} - (2 + \Delta x^2 n^2)\bar{Z}_m + \bar{Z}_{m-1} = 0 \tag{4.15}$$

and

$$\left|\frac{\bar{A}}{A}\right| = \left|\frac{1}{q - e^{-I\theta}}\right|; \quad q = 2 + \Delta x^2 n^2 \geq 2 \tag{4.16}$$

and also

$$\left|\frac{\bar{A}}{A}\right|^2 = \frac{1}{q^2 + 1 - 2q\cos\theta}\,. \tag{4.17}$$

This quantity is less than 1 for all acceptable q's and $\cos\theta < 1$. Most waves are strongly damped, and only the longest waves in both directions are weakly damped by the iteration. This behavior was used to accelerate the convergence as is done by the SLOR technique, Chebychev acceleration, or Multi-Grid method.*

The question is how to generate an equivalent relaxation scheme for the primitive system in the marching form. This means that we may add terms which can be evaluated during the marching process but should vanish upon convergence.

A rational approach to the construction of the relaxation scheme is to retrace backwards the steps of the derivation of the discrete Laplace equation from the discrete primitive equations. We start from the SLOR equation (4.15):

$$Z_{m+1} - 2\bar{Z}_m + \bar{Z}_{m-1} = \Delta x^2 n^2 \bar{Z}_m$$

which we inverse Fourier transform with respect to y to get:

$$P^{k-1}_{m+1} - 2P^k_m + P^k_{m-1} = -\Delta x^2 (P^k_{yy})_m\,.$$

*It was pointed out by J. South that (4.3) can be viewed as an over-relaxed version of (4.10) with an over-relaxation factor: $\Omega = 2$ which is not a good choice for Ω.

Now, we substitute Eq. (4.8) for the right hand side of the last equation to get:

$$P_{m+1}^{k-1} - 2P_m^k + P_{m-1}^k = \Delta x D(U_{m-1}^k - U_m^k)$$

which can be written as:

$$(P_{m+1}^{k-1} - P_m^k) + (P_m^{k-1} - P_m^k) - (P_m^{k-1} - P_{m-1}^k) = \Delta x D(U_{m-1} - U_m);$$

adding the equations from $m = 2$ and using the linear form of D we get:

$$-(P_{m+1}^{k-1} - P_m^k) - \sum_{i=2}^{m} (P_i^{k-1} - P_i^k) + (P_2^{k-1} - P_1^k) = \Delta x D(U_m - U_1),$$

but from Eq. (4.3):

$$P_2^{k-1} - P_1^k = -\Delta x D(U_1^k); \tag{4.18}$$

therefore, we get for $m \geq 2$

$$D(\bar{U}_m) = -\frac{P_{m+1}^{k-1} - P_m^k}{\Delta x} - \frac{1}{\Delta x} \sum_{i=2}^{m} (P^{k-1} - P^k). \tag{4.19}$$

Eq. (4.19) contains all the modifications required in order to convert the iteration scheme of (4.5)-(4.7) into a scheme equivalent to the SLOR scheme for one Laplace equation with "over-relaxation" factor $\Omega = 1$. We see that in this approach only the x momentum equation is modified. The new added term can be generated easily during the marching process and is inexpensive in storage (one extra line vector) and computation (one substruction per grid point). In what follows we will derive Eq. (4.19) in a more general way and introduce the over-relaxation parameter $\Omega > 1$.

In practice we will use difference approximations and boundary conditions also in the y direction, and the resulting scheme may not be amenable to the discrete analogue of the Fourier transform. It is therefore worthwhile to generalize the previous approach by using the matrix finite difference formulation.

Let the vectors U_m, V_m, P_m contain the N values of the corresponding variables on the m-th line (x = constant) of the marching sweep (including the specified boundary values). The y-momentum

equation (1.5) can be written in the form:

$$P_{m+1} - P_m = -\Delta x D(U_m) \equiv R_m \,. \tag{4.20}$$

On the other hand elimination of V_m between the continuity and the V-momentum equations will result in:

$$FP_m = R_m - R_{m-1} \tag{4.21}$$

where $F = \Delta x^2 I \frac{\partial^2}{\partial y^2}$. Substructing successively u-momentum equations (4.20) and using Eq. (4.21) gives:

$$P_{m+1} - (2I + F)P_m + P_{m-1} = 0, \quad m = 3,4,... \tag{4.22}$$

which is Laplace's equation. The first equation of (4.20) can be used as a derivative condition at the left (inlet) boundary, namely:

$$P_3 - P_2 = R_2 \,. \tag{4.23}$$

We now apply the SLOR scheme to the last two equations (ignoring temporarily the downstream boundary condition) to get the downstream marching form:

$$-P_2^* + P_3^{(k-1)} = R_2 \tag{4.24}$$

$$P_{m-1}^{(k)} - (2I + F)P_m^* + P_{m+1}^{(k-1)} = 0, \quad m = 3,4,... \tag{4.25}$$

where $P_m^{(k)} = \Omega P_m^* + (1-\Omega)P_m^{(k-1)}$; Ω is the overrelaxation factor, and the superscript denotes the iteration sweep number. In order to recover the primitive variable formulation, we relate the velocity field in Eq. (4.21) to the starred pressure field, i.e.,

$$FP_m^* = R_m - R_{m-1} \,. \tag{4.26}$$

Substitution in Eq. (4.25) gives:

$$P_{m-1}^{(k)} - 2P_m^* + P_{m+1}^{(k-1)} = R_m - R_{m-1}, \quad m = 3,4,... \tag{4.27}$$

Successive summations of Eqs. (4.24) and (4.25) give:

$$P_{m+1}^{(k-1)} - P_m^* = R_m + S_m, \qquad m = 2,3,4,... \tag{4.28}$$

which is the primitive variable marching form of the u-momentum equation. The source term S_m in Eq. (4.28) satisfies:

$$S_m = S_{m-1} + (P_m^* - P_m^{(k-1)}) + (P_{m-1}^* - P_{m-1}^{(k)}), \quad m = 3,4,... \tag{4.29}$$

with $S_2 = 0$. It can be seen that S_m vanishes upon convergence The computational form of (4.28) for $m = 3,4,...$ is:

$$-2P_m^* = R_m + \bar{S}_m \tag{4.30}$$

$$\bar{S}_m = \bar{S}_{m-1} - P_{m+1}^{(k-1)} + 2P_{m-1}^* - P_{m-1}^{(k)}; \quad \bar{S}_2 = -P_2^* - P_3^* . \tag{4.31}$$

Thus, the theory of overrelaxation can be applied exactly to the constant coefficient case of system (2.8)-(2.10). For the non-linear case this theory can serve as a guide to the choice of Ω. Alternately, one can choose $\Omega = 1$ and apply the Multi-Grid procedure.

Restriction and Storage Requirements

Let the finite difference approximation of equations (2.1)-(2.3) on the finest grid M be represented as in [5]:

$$L_j^M \tilde{w}^M(\tilde{x}) = F_j^M(\tilde{x}) \tag{4.32}$$

where $\tilde{x} = (x,y)$, $\tilde{w}^M = [U^M, V^M, P^M]^T$ is the exact solution of the difference equations, and j is the number of the differential equation, $j = 1,2,3$.

The problem is transferred from the current level k to a coarser level $k-1$, see Fig. 4, by correcting the right hand side of (4.32)

$$F_j^{k-1}(\tilde{x}) = L_j^{k-1}(\tilde{I}_{j,k}^{k-1} \tilde{w}^k(\tilde{x})) + I_{j,k}^{k-1}[F_j^k(\tilde{x}) - L_j^k \tilde{w}^k(\tilde{x})] \tag{4.33}$$

in the Full Approximation Storage (FAS) mode. $\tilde{w}^k(\tilde{x})$ is an approximation to $\tilde{w}^k(\tilde{x})$ in the finer level. $I_{j,k}^{k-1}$ and $\tilde{I}_{j,k}^{k-1}$ are proper restriction operators for equation j.

The term in square bracket in equation (4.33) is the residual of the j-th equation. For the present marching scheme there is no residual in the continuity and in the y-momentum equations since they are solved exactly in each step. The residual of the x-momentum equation results

only from the streamwise pressure gradient term and its computation needs only one substraction. $\tilde{I}^{k-1}_{j,k}$ was chosen to be linear interpolation, which yields for the continuity equation: $L^{k-1}_1(\tilde{I}^{k-1}_{1,k}\tilde{w}^k(\tilde{x})) = 0$. $I^{k-1}_{j,k}$, $j = 1,2$ is computed by averaging in both the x and y directions. $I^{k-1}_{3,k}$ is a simple injection.

In summary, equation (4.33) takes the following terms:

$$F^{k-1}_1(\tilde{x}) = 0$$

$$F^{k-1}_2(\tilde{x}) = L^{k-1}_2(\tilde{I}^{k-1}_{2,k}\tilde{w}^k(\tilde{x})) + I^{k-1}_{2,k}(F^k_2 - L^{k}_2\tilde{w}^k(\tilde{x}))$$

$$F^{k-1}_3(\tilde{x}) = L^{k-1}_3(\tilde{I}^{k-1}_{3,k}\tilde{w}^k(\tilde{x})).$$

Two consequences should be emphasized:

(a) Only two corrections ($F^{k-1}_2(\tilde{x})$, $F^{k-1}_3(\tilde{x})$) have to be computed and stored.

(b) All the dependent variables must be transferred in order to compute the corrections ($L^{k-1}_j(\tilde{I}^{k-1}_{j,k}\tilde{w}^k(\tilde{x}))$, $j = 2,3$). Since only the pressure is stored, these corrections must be computed during the marching process.

It follows that in addition to the pressure on all grids, one has to save one correction term for each momentum equation on the coarser grids. Assuming N computational points on the finest grid, a simple-minded estimate gives 35N/7 storage locations for the three-dimensional NS Multi-Grid solution, and 11N/7 for the PNS marching MG solution. For the two-dimensional case the corresponding figures are 14N/3 and 6N/3.

Interpolation

Since the present marching scheme generates the velocity field from the pressure, only the correction to the pressure must be interpolated back to the fine grid.

5. GENERALIZATIONS

In order to generalize the preceding approach we note that the essence of the relaxation procedure is the replacement of the term $\Delta x \partial P/\partial x$ by the marching difference form:

$$\Delta x \frac{\partial P}{\partial x} \cong -(2P_m^* + \bar{S}_m) \tag{5.1}$$

where $\bar{S}_m$ is already known. If (5.1) is differenced it will (using the definition of $\bar{S}_m$) give rise to

$$2(P_m^* - P_{m-1}^*) + \bar{S}_m - \bar{S}_{m-1} = -(P_{m+1}^{k-1} - 2P_m^{*\prime} + P_{m-1}^k) .$$

Thus, the correct successive line over relaxation form is implicitly obtained for the second derivative of the pressure in the marching direction. (It should be emphasized again that the second order elliptic equation for the pressure is neither derived nor used in the algorithm itself.)

The implication of the present technique is: if it is known that the equations can be manipulated so that some variable will satisfy approximately a second order elliptic equation, we should use the replacement (5.1) for the derivative of that variable in the main flow direction. An efficient marching scheme will thus be generated.

The present version of the algorithm will be applied to the subsonic compressible multi-dimensional Navier-Stoke's equations. Several particular cases will be examined.

The first step is the derivation of an elliptic equation starting with:

$$\vec{V}\cdot\nabla\vec{V} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{V} + \frac{\nu}{3}\nabla(\nabla\cdot\vec{V}) \tag{5.2}$$

where $\vec{V}$ is the velocity vector. In addition we will require the equation of state of a perfect gas

$$p = \rho RT \tag{5.3}$$

and the continuity equation in the form

$$\nabla\cdot\vec{V} = -\vec{V}\cdot\nabla \ell n \rho. \tag{5.4}$$

It follows from (5.3) that

$$\frac{1}{\rho}\nabla p = RT\nabla \ell n\rho + R\nabla T \tag{5.5}$$

and therefore

$$\nabla\cdot\frac{1}{\rho}\nabla p = RT\nabla^2 \ell n\rho + R\nabla^2 T + \text{l.o.t.} \tag{5.6}$$

(l.o.t. stands for lower order terms).

Also

$$\nabla\cdot(\vec{V}\cdot\nabla\vec{V}) = (\vec{V}\cdot\nabla)(\nabla\cdot\vec{V}) + \text{l.o.t.} = -(\vec{V}\cdot\nabla)^2 \ell n\rho + \text{l.o.t.} \tag{5.7}$$

Taking the divergence of (5.2) and using (5.6) and (5.7) we get

$$(\vec{V}\cdot\nabla)^2 \ell n\rho = RT\nabla^2 \ell n\rho + R\nabla^2 T + \frac{4\nu}{3}\nabla^2(\nabla\cdot V) + \text{l.o.t.} \tag{5.8}$$

Several special cases follow:

1) Incompressible case. Here $\nabla\cdot\vec{V} = 0$, and we get directly (taking the divergence of (5.2)):

$$\nabla^2 p = \text{l.o.t.}; \tag{5.9}$$

thus the gradient term was differentiated once and the replacement (5.1) should apply in two or three dimensions. In the later case we have to compute simultaneously all the variables in the marching plane, m, and so we get a successive plane overrelaxation scheme. It is possible that an alternating direction scheme can be used to solve the coupled system in the mth plane, but a multi-grid approach seems to be preferable. At the present time numerical results for the three dimensional case are not available.

2) Isothermal case. We get

$$(\vec{V}\cdot\nabla)^2 \ell n\rho = \frac{a^2}{\gamma}\nabla^2 \ell n\rho + \text{l.o.t.} + \text{v.t.}$$

where $a^2 = \gamma RT$ is the adiabatic speed of sound, γ is the ratio of specific heats, and v.t. is the viscous term to be discussed later.

3) Isentropic case. Here, $(\gamma-1)\nabla \ell n\rho = \nabla \ell nT$ and therefore:

$$(V\cdot\nabla)^2 \ell n\rho = a^2\nabla^2 \ell n\rho + \text{l.o.t.} + \text{v.t.}$$

4) Constant stagnation enthalpy. Here, $a_c^2 = a^2 + \frac{\gamma-1}{2}\vec{V}^2$ and we get

$$(V\cdot\nabla)^2 \ell n\rho + (\gamma-1)V\cdot(\nabla\cdot V)\cdot\vec{V} = a^2\nabla^2 \ell n\rho + \text{l.o.t.} + \gamma\text{v.t.}$$

In all the compressible cases considered the prominent balance is:

$$(\vec{V}\cdot\nabla)^2 \tilde{p}_{xx} \cong a^2 \nabla^2 \tilde{p}_{xx} .$$

Using local stream aligned coordinates s and n, we find

$$(\vec{V}\cdot\nabla)^2 = V^2 \frac{\partial^2}{\partial s^2}$$

$$\nabla^2 = \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial n^2} ;$$

therefore $\tilde{p}$ in p_{xx} appears in the form:

$$(1-M^2) \frac{\partial^2 \tilde{p}}{\partial s^2} + \frac{\partial^2 \tilde{p}}{\partial n^2} = \text{other terms}.$$

After the parabolization of the viscous term, only the left hand side has a second derivative of $\tilde{p}$ in the streamwise direction. Specifically $\nu\nabla^2\nabla\cdot V$ is replaced by

$$\nu \frac{\partial^2}{\partial n^2} \nabla\cdot V = -\nu \frac{\partial^2}{\partial n^2} \vec{V}\cdot\vec{\nabla} p_{xx} .$$

This term cannot become large since the pressure does not have large gradients in the boundary layer.

We argue that if our iteration is appropriate for the $\tilde{p}$ equation it will be a good scheme overall.

To get a successive line (or plane) over relaxation scheme, all we have to do is replace all the occurrences of $\frac{\partial \tilde{p}}{\partial x}$ with the marching form (5.1). All the properties of Section 4 will be the same as long as $M^2 < 1$.

In fact, better convergence can be expected as M^2 approaches 1 since the quantity q of (4.16) will become now $2 + (\Delta x^2 n^2/1-M^2)$. Only $\tilde{p}$ will have memory and must be globally saved and updated by the iteration procedure. $\tilde{p}$ will also transmit the downstream information and must be specified there.

For transonic flows a conservation form is preferred and it may be more convenient to work with ρ rather than p_{xx}. An elliptic equation can be derived for ρ, but care must be taken to transmit the downstream information via ρ. Upstream information should not be transmitted by ρ and ρ should not be specified at the inflow otherwise, the problem will be overspecified. Consider, for example, the term $\rho_x u^2_x$ as it is discretized as

$$(\rho_m U_m^2 - \rho_{m-1} U_{m-1}^2)$$

however, at the station m we should compute the U_m velocities coupled with the ρ_{m-1} densities. The approach of Reddy and Rubin [12] where the pressure is specified both at inflow and at outflow is inconsistent unless one happens to know the right pressures before the computation. The inconsistency and consequent error can be easily demonstrated by one-dimensional examples.

6. RESULTS

In order to check the MG algorithm, we choose the following analytical solution. It satisfies the continuity equation but gives rise to source terms in the momentum equations:

$$U = A + (x+y)^m; \quad V = -(x+y)^m; \quad P = -(E1+E2)(x+y)^m \tag{6.1}$$

where a and b from equations (3) are defined by:

$$a = E1 + F(x+y)^n; \quad b = E2 - F(x+y)^n \tag{6.2}$$

and $E1 = 1$; $E2 = .2$; $F = .2$; $A = 5$; $Re = 1000$; $m = 4$; $n = 2$. The coarsest grid consists of 4×4 intervals.

Figure 5 compares the MG convergence history of different relaxation schemes. In the MG solutions three levels were involved (M=3). The horizontal coordinate gives the number of Work Units (WU), where each work unit is equivalent to one global iteration on the finest grid. The vertical coordinate gives the logarithm of the dynamic residual ε. The dots show the solution of the equivalent Poisson equation (with the same solution for the pressure but with Dirichlet condition over all the boundaries). The linearized PNS equations were solved with and without the streamwise pressure gradient correction of [3]. The corresponding (17×17 points) single grid convergence history is plotted for comparison (for the case of $\Omega = 1$). The corrected discrete equations and the Poisson equation exhibit very similar convergence whereas the convergence of the unmodified equations is much worse. Upon increasing the number of grids in the unmodified equations, the convergence deteriorates.

The Reynolds number independence of the scheme is demonstrated in Figure 6, where the convergence history is presented for Reynolds numbers 1, 10^3 and infinity.

In order to check the non-linear version of the code, several test

cases were run; the incompressible flow over a flat plate, the flow along an axisymmetric cylinder, entrance flow between two flat plates, and the flow behind the trailing edge of a flat plate. In all cases good agreement was obtained with known solutions. The details will be presented elsewhere. Here we show (Figure 7) the convergence history for a flow over a flat plate with uniform upstream profile and Neumann condition for the pressure at the exit. While the number of levels is varied, the finest grid remains the same and consists of 65×65 points. In Figure 8 there is a comparison between the present results for the flow near the trailing edge of a flat plate and the results of reference [11]. The skin friction coefficient CF is shown for $z < 1$ while the center line velocity UC is shown for $z > 1$. The trailing edge is at $z = 1$.

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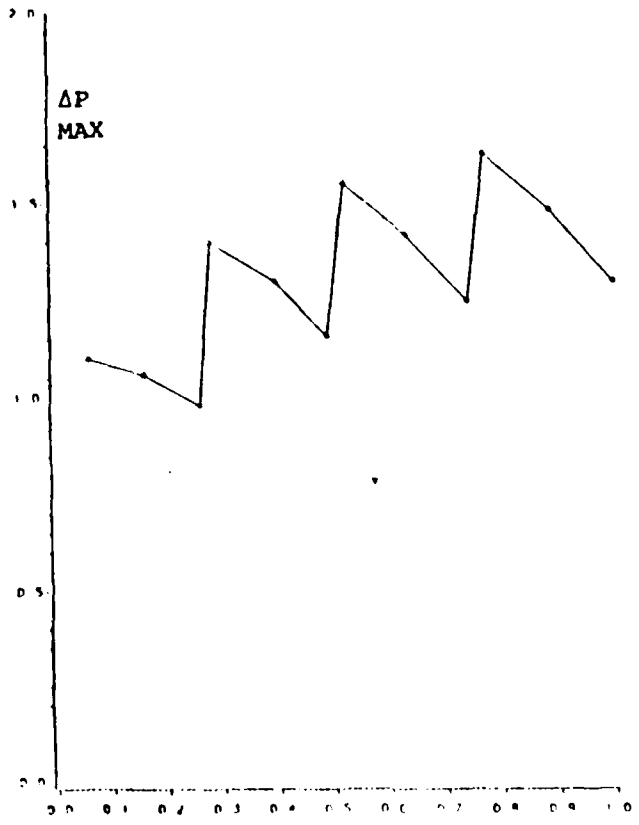

Figure 1: Pressure Residue vs. the Global Iteration Number.

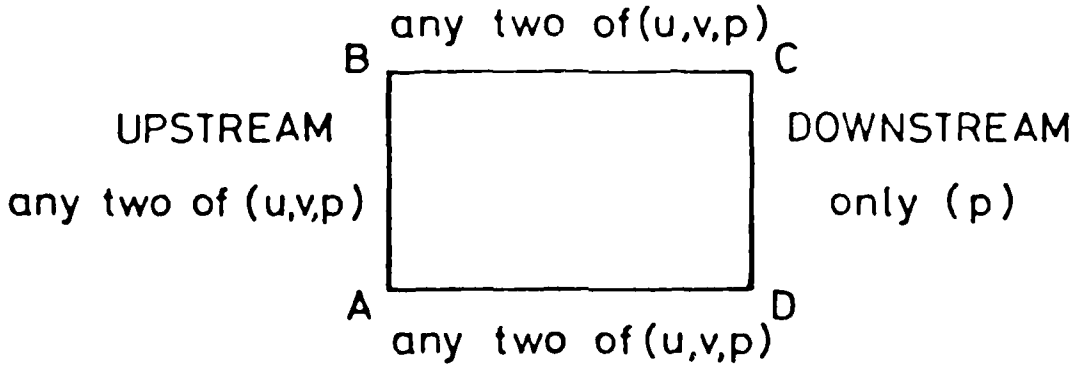

Figure 2: Example of Permissible Boundary Conditons.

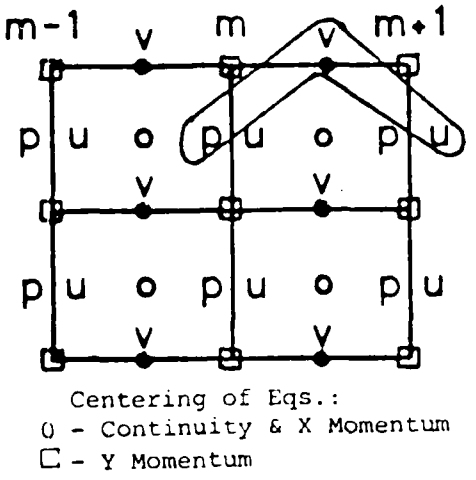

Figure 3: The staggered grid.

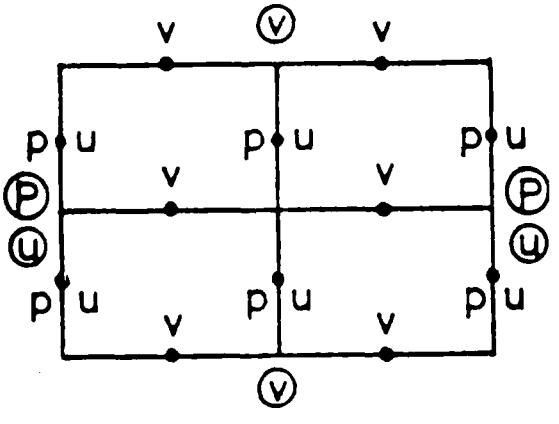

Figure 4: Relative placement of variables on two successive grids.

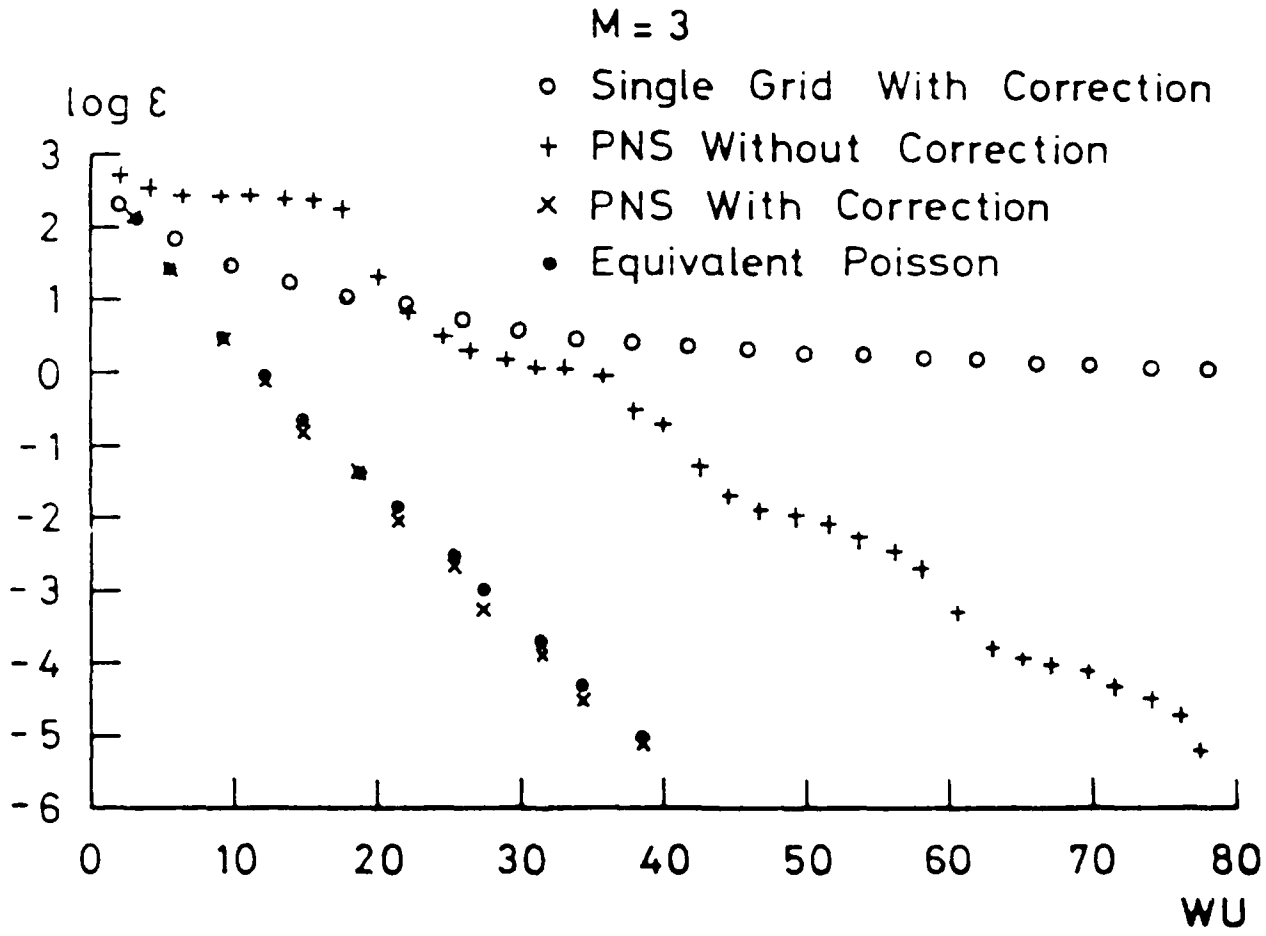

Figure 5: Convergence history for different relaxation schemes (M = 3).

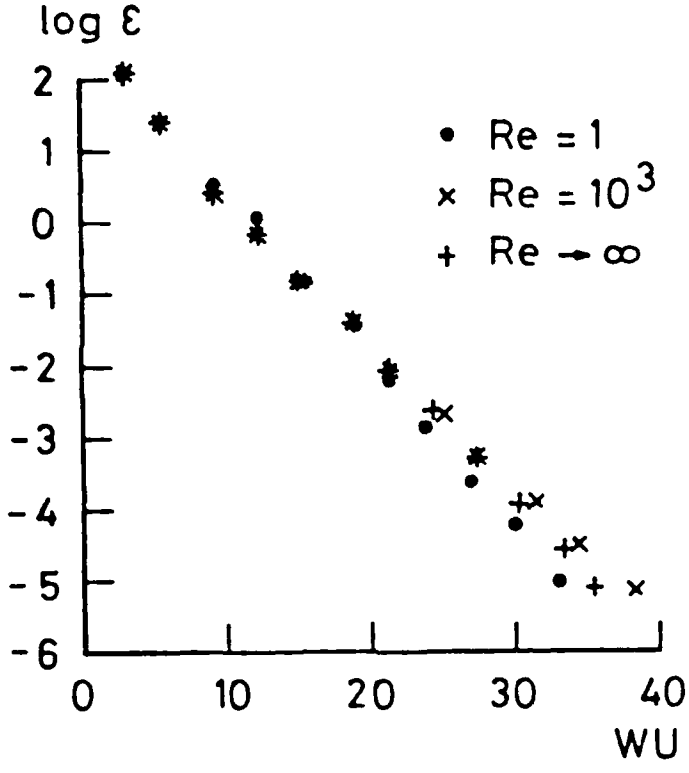

Figure 6: Convergence history for several Reynolds numbers (M = 3).

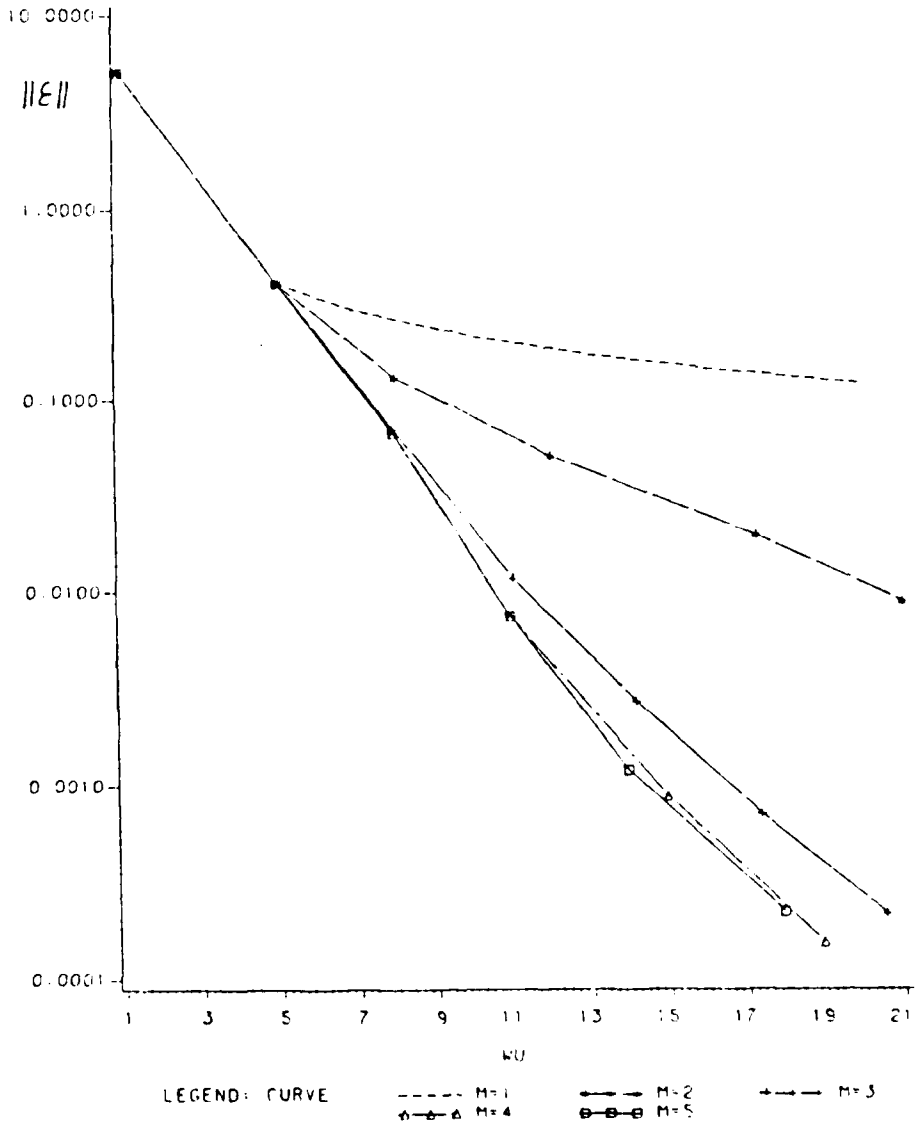

Figure 7: Convergence history, semi-infinite flat plate.
RE = 10000

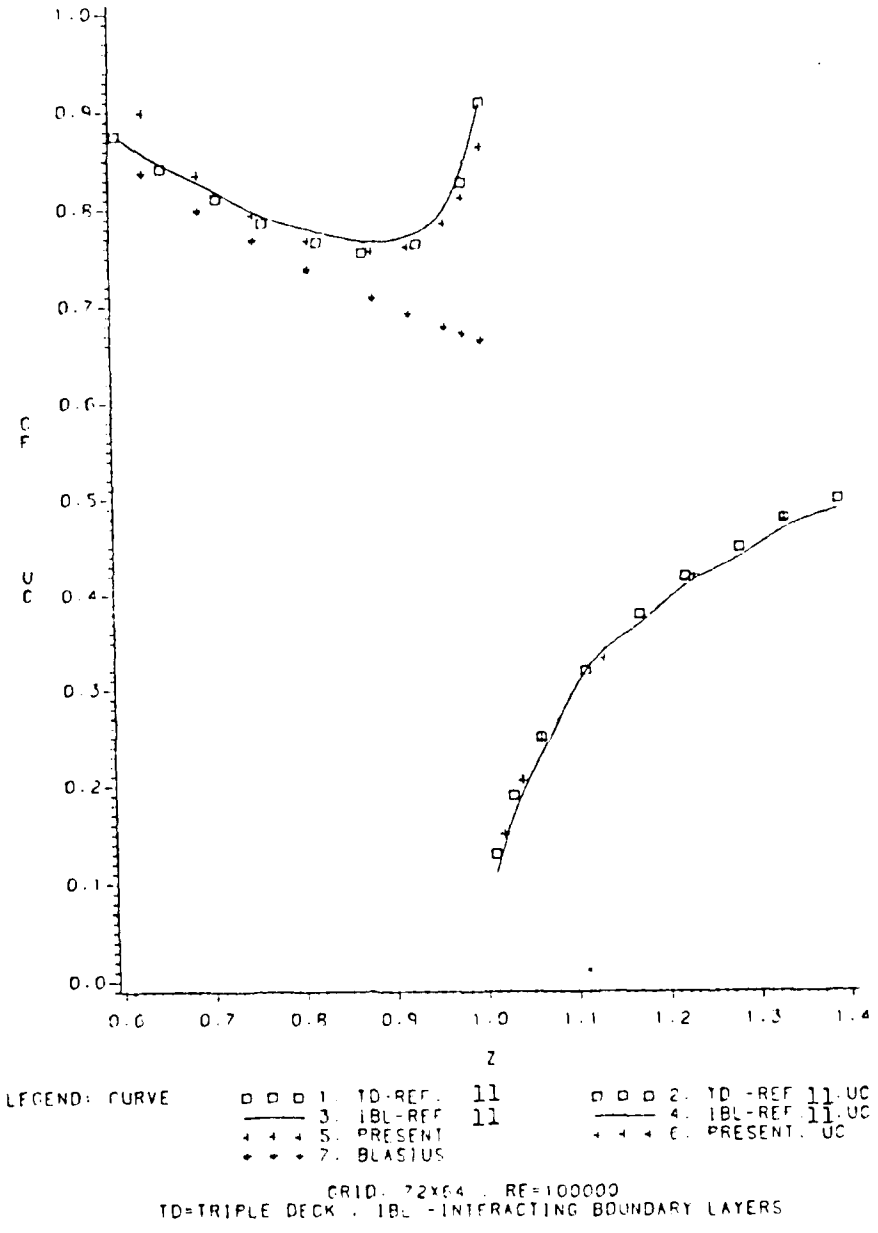

Figure 8: Flow near the trailing edge of a flat plate.

CF - skin friction

UC - Centerline velocity

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