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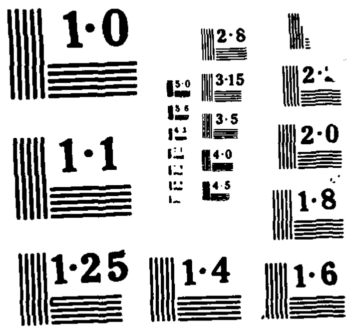
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U.S. ARMY INTELLIGENCE CENTER AND SCHOOL
SOFTWARE ANALYSIS AND MANAGEMENT SYSTEM

2-D VS 3-D: SINE OF SQUARED ANGULAR ERROR AND WEIGHTED PERPENDICULAR FIXING

TECHNICAL MEMORANDUM NO. 22

Mathematical Analysis Research Corporation

29 June 1987

National Aeronautics and
Space Administration

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JET PROPULSION LABORATORY
California Institute of Technology
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This report analyses the difference between the 3-D and 2-D approach to fixing. In the report the asymptotic difference between the two methods is found to be small, percentagewise. Assumptions concerning the gnomonic plane are examined. The mathematical derivations are appendicised.

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U.S. ARMY INTELLIGENCE CENTER AND SCHOOL
Software Analysis and Management System

2-D vs 3-D: Sine of Squared Angular Error and Weighted Perpendicular Fixing

Technical Memorandum No. 22

29 June 1987

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PREFACE

The work described in this publication was performed by the Mathematical Analysis Research Corporation (MARC) under contract to the Jet Propulsion Laboratory, an operating division of the California Institute of Technology. This activity is sponsored by the Jet Propulsion Laboratory under contract NAS7-918, RE182, A187 with the National Aeronautics and Space Administration, for the United States Army Intelligence Center and School.

This specific work was performed in accordance with the FY-87 statement of work (SOW #2).

2-D vs 3-D: Sine of Squared Angular Error and Weighted Perpendicular Fixing

INTRODUCTION

If

1) two dimensional algorithms work on variables from a gnomonic plane
2) gnomonic planes are based on the same point of tangency both within a single algorithm and between the two and three dimensional algorithms

then it is reasonable to ask the question:

How much difference does it make if the gnomonic projections are taken before or after fixing?

This question will be put into mathematical form and analyzed in this memo. The procedure used will be asymptotic expansion, on the assumption that error would go to zero as the radius of the Earth became infinite if the same gnomonic projected coordinates were involved. In this way the dependence of the difference of projections on the scale of the problem may be seen. This format is important because the scale restriction is a significant known parameter, best accounted for in this way.

For analysis purposes it is also relevant to identify how error models cross to the gnomonic plane. They also may or may not correspond well with two dimensional error models.

The two fix techniques being analyzed are the sine of squared angular error and its approximation, the weighted perpendicular. The weighted perpendicular technique is used by the Improved Guardrail V (IGRV) program, and differs from the sine of squared angular error technique in interesting ways that will be discussed later.

These topics are analyzed in detail in the sections that follow.

CONCLUSIONS

Analysis shows there is little difference between using the three dimensional model and using the two dimensional model, if the gnomonic plane is positioned close to the emitter. By little difference we mean that the distance between the fix determined by these two methods is small.

For small scale problems and relatively accurate lines of bearing, it can be shown that the distance between the fix estimates would be a small percentage of the maximum miss distance of an LOB. In particular, in the appendix it is shown that the percentage is less than the absolute value of the maximum of the miss distance to the radius of the Earth, ratio squared, times a factor to account for the size of the base relative to the size of the problem. Because of cancellation owing to terms with different signs, the percentage is usually less than the squared ratio.

The small difference that does exist between the fix methodologies only reflects a difference in the weighting of the different data. In line of sight applications, the weighting in the three dimensional method makes less sense than the weighting of the two dimensional method.

METHOD OF ANALYSIS

The fix methods and gnomonic projection are not affected by how latitude and longitude coordinates are assigned to the sphere (to within spherical rotations). Therefore to shorten expressions, we will start by assuming that the point of tangency of the plane with which the gnomonic projections are taken is at (0,0,R), where R is the radius of the Earth.

$z=R$ is the gnomonic Plane

(X_i,Y_i,R)=the i^{th} sensor location in the gnomonic plane, (X_i,Y_i) fixed.

θ_i=the angle of the i^{th} line of bearing in the gnomonic plane

(x,y,R)=the location estimate in the gnomonic plane.

The i^{th} plane of bearing corresponding to θ_i is spanned by (X_i,Y_i,R) and $(\sin\theta_i,\cos\theta_i,0)$, and goes through (0,0,0).

n_i = the unit normal corresponding to the i^{th} plane of bearing which is proportional to $(X_i,Y_i,R)x(\sin\theta_i,\cos\theta_i,0)$

or $(-R\cos\theta_i,R\sin\theta_i,X_i\cos\theta_i-Y_i\sin\theta_i)/\sqrt{R^2+(X_i\cos\theta_i-Y_i\sin\theta_i)^2}$

$(x,y,R)(R/\sqrt{x^2+y^2+R^2})$= point on the Earth with gnomonic projection (x,y,R)

$(X_i,Y_i,R)(R/\sqrt{X_i^2+Y_i^2+R^2})$= point on the Earth with gnomonic projection (X_i,Y_i,R)

The two dimensional likelihood function to be minimized for two dimensional sine of angular error is

$$L2 = \Sigma[(x-X_i,y-Y_i)(-\cos\theta_i,\sin\theta_i)]^2/[(x-X_i)^2+(y-Y_i)^2]$$

The three dimensional likelihood function to be minimized (based on the inverse image of the two dimensional sensor locations and bearings) is

$$L = \Sigma\{[(x,y,R)/\sqrt{x^2+y^2+R^2} -(X_i,Y_i,R)/\sqrt{X_i^2+Y_i^2+R^2}]\cdot\vec{n}_i\}^2/D_i$$

where

$C_i = (-R\cos\theta_i,\ R\sin\theta_i,\ X_i\cos\theta_i-Y_i\sin\theta_i)$

$d_i = -(x-X_i)\cos\theta_i+(y-Y_i)\sin\theta_i$ = signed perpendicular distance

$\vec{n}_i$ = normal unit vector = $C_i/||C_i||$

$F_i = ||C_i||^2 = R^2+(X_i\cos\theta_i-Y_i\sin\theta_i)^2$

$1/F_i= 1/||C_i||^2 = 1/R^2-1/R^4[(X_i\cos\theta_i-Y_i\sin\theta_i)^2] + O(1/R^6)$ (as $R\to\infty$)

$$D_i = ||[(x,y,R)/\sqrt{x^2+y^2+R^2} - (X_i,Y_i,R)/\sqrt{X_i^2+Y_i^2+R^2}]||^2$$

$$E_i = [(x,y,R)/\sqrt{x^2+y^2+R^2} - (X_i,Y_i,R)/\sqrt{X_i^2+Y_i^2+R^2}]\ C_i$$

CALCULATIONS

$$L = \Sigma E_i^2/(F_i D_i)$$

$$\lim_{R \to \infty} \min L = \Sigma[-(x-X_i)\cos\theta_i+(y-Y_i)\sin\theta_i]^2/[(x-X_i)^2+(y-Y_i)^2] = \min L2$$

This implies that the limit of the three dimensional fix (as the relative size of the Earth to the scale of the problem goes to infinity) is the two dimensional equal to the two dimensional fix.

To get an idea of the actual difference, expand the Taylor Series for the three dimensional fix about $R=\infty$. The constant term is the two dimensional fix. The first derivative term shows the dependence on R.

The two techniques mentioned above, solve the problem of minimizing L and L2 differently. The sine of squared angular error does so by taking the partial derivatives of the functions and solves them for zero. The weighted perpendicular technique approachs the problem by taking the partial derivative of only the numerator of the expression. In leaving the denominator alone, this technique weights the value of the derivative. The resulting expression is then set equal to zero and solved.

3-D FIX METHOD

We will need to use the correspondence between (x,y) in the gnomonic projection and $(x,y,R)/\sqrt{x^2+y^2+R^2}$ in the three dimensional representation. If $x=f(1/R)$ $y=g(1/R)$ denote the implicit functions defined above, then

$$L_x(f(1/R),g(1/R)) = L_y(f(1/R),g(1/R)) = 0$$

Hence with $U=1/R$

$$L_{xx}f'+L_{xy}g'+L_{xU} = L_{xy}f'+L_{yy}g'+L_{yU} = 0$$

Thus,

$$\begin{vmatrix} f' \\ g' \end{vmatrix} = - \begin{vmatrix} L_{xx} & L_{xy} \\ L_{xy} & L_{yy} \end{vmatrix}^{-1} \begin{vmatrix} L_{xU} \\ L_{yU} \end{vmatrix}$$

It turns out, however, that $L_{xU}=L_{yU}=0$ at $R=\infty$ thus, $f'(0)=g'(0)=0$. Therefore to get the first order term we need to compute f" and g". Note first that

$$L_{xx}f''+L_{xy}g''+L_{xxx}(f')^2+2L_{xxy}f'g'+L_{xyy}(g')^2+2L_{xxU}f'+2L_{xyU}g'+L_{xUU} = 0$$

$$L_{xy}f''+L_{yy}g''+L_{xxy}(f')^2+2L_{xyy}f'g'+L_{yyy}(g')^2+2L_{xyU}f'+2L_{yyU}g'+L_{yUU} = 0$$

However since $f'(0)=g'(0)$, this reduces to

$$L_{xx}f''+L_{xy}g''+L_{xUU} = 0$$

$$L_{xy}f''+L_{yy}g''+L_{yUU} = 0$$

$$\begin{vmatrix} f'' \\ g'' \end{vmatrix} = - \begin{vmatrix} L_{xx} & L_{xy} \\ L_{xy} & L_{yy} \end{vmatrix}^{-1} \begin{vmatrix} L_{xUU} \\ L_{yUU} \end{vmatrix}$$

at the point of interest ($U=0$ or $R=\infty$). Thus we need to evaluate this expression and take the limit as $U\to 0$ ($R\to\infty$). Since we are examining two techniques, this will be done for both. However, the calculations are similar for both techniques, so the majority of those calculations shown will be for the sine of squared angular error. Wherever a difference in the techniques is of interest, the calculations for both will be shown.

MATH APPENDIX

$$1/\sqrt{R^2+a} = \sum_{i=0}^{\infty} \binom{-1/2}{i} R^{2(-1/2-i)} a^i = (1/R)-(1/2)(1/R^3)a+(3/8)(1/R^5)a^2+ O(1/R^7)$$

Thus,

$$(x,y,R)/\sqrt{x^2+y^2+R^2}=(x,y,R)[(1/R)-(1/R^3)(x^2+y^2)/2+(3/8)(1/R^5)(x^2+y^2)^2+ O(1/R^7)]$$

$$(x,y,R)/\sqrt{x^2+y^2+R^2}-(X_i,Y_i,R)/\sqrt{X_i^2+Y_i^2+R^2}$$

$$=(x-X_i,y-Y_i,0)[1/R]-([(x,y,R)(x^2+y^2)-(X_i,Y_i,R)(X_i^2+Y_i^2)]/2)[1/R^3]$$

$$+ (O(1/R^5),O(1/R^5),O(1/R^4))$$

Thus $D_i=[(x-X_i)^2+(y-Y_i)^2][1/R^2]$

$$-[(x-X_i)(x(x^2+y^2)-X_i(X_i^2+Y_i^2))+(y-Y_i)(y(x^2+y^2)-Y_i(X_i^2+Y_i^2))][1/R^4]$$

$$+([(x^2+y^2)-(X_i^2+Y_i^2)]^2/4)[1/R^4] + O(1/R^6)$$

And so $1/D_i= [1/((x-X_i)^2+(y-Y_i)^2)][R^2] + O(R^4)$

Also,

$$E_i = [(x,y,R)/\sqrt{x^2+y^2+R^2}-(X_i,Y_i,R)/\sqrt{X_i^2+Y_i^2+R^2}]\, C_i$$

$$=[-(x-X_i)\cos\theta_i+(y-Y_i)\sin\theta_i]-[-x\cos\theta_i+y\sin\theta_i+X_i\cos\theta_i-Y_i\sin\theta_i](x^2+y^2)/2[1/R^2]$$

$$+ O(1/R^4)$$

$$=[-(x-X_i)\cos\theta_i+(y-Y_i)\sin\theta_i]+[(x-X_i)\cos\theta_i-(y-Y_i)\sin\theta_i](x^2+y^2)/2[1/R^2]+ O(1/R^4)$$

$$=d_i-d_i(x^2+y^2)/2[1/R^2] + O(1/R^4)$$

$$L_x=\Sigma[(2E_{i,x}D_i-E_iD_{i,x})E_i/(F_iD_i^2)] + O(R^2)$$

$$L_y=\Sigma[(2E_{i,y}D_i-E_iD_{i,y})E_i/(F_iD_i^2)] + O(R^2)$$

As was mentioned earlier, the weighted perpendicular technique (*) utilizes a different L_x and L_y. Specifically, L_x^* and L_y^* do not have $D_{i,x}$ and $D_{i,y}$ terms, respectively. Thus, the weighted perpendicular technique has

$$L_x^*= \Sigma[(2E_{i,x}E_i)/(F_iD_i)] + O(R^2)$$

$$L_y^*= \Sigma[(2E_{i,y}E_i)/(F_iD_i)] + O(R^2)$$

where

$$E_{i,x} = -\cos\theta_i+[\cos\theta_i(x^2+y^2)+2x((x-X_i)\cos\theta_i-(y-Y_i)\sin\theta_i)]/2[1/R^2] + O(1/R^4)$$

$$= -\cos\theta_i + [\cos\theta_i (x^2+y^2)/2 - xd_i][1/R^2] + O(1/R^4)$$

$$D_{i,x} = 2(x-X_i)[1/R^2] - [x(x^2+y^2) - X_i(X_i^2+Y_i^2) + (x-X_i)(3x^2+y^2) + (y-Y_i)2yx][1/R^4]$$
$$+[(x^2+y^2)-(X_i^2+Y_i^2)]x[1/R^4] + O(1/R^6)$$

$$= 2(x-X_i)[1/R^2] - [(x-X_i)(X_i^2+Y_i^2) + (x-X_i)(3x^2+y^2) + (y-Y_i)2yx][1/R^4] + O(1/R^6)$$

$$E_{i,y} = \sin\theta_i + [-\sin\theta_i (x^2+y^2)/2 - yd_i][1/R^2] + O(1/R^4)$$

$$D_{i,y} = 2(y-Y_i)[1/R^2] - [y(x^2+y^2) - Y_i(X_i^2+Y_i^2) + (y-Y_i)(x^2+3y^2) + (x-X_i)2yx][1/R^4]$$
$$+[(x^2+y^2)-(X_i^2+Y_i^2)]y[1/R^4] + O(1/R^6)$$

$$2E_{i,x}D_i - E_iD_{i,x} = [-2\cos\theta_i((x-X_i)^2+(y-Y_i)^2) - 2d_i(x-X_i)]\ [1/R^2]$$
$$+2(\cos\theta_i)[(x-X_i)(x(x^2+y^2)-X_i(X_i^2+Y_i^2)) + (y-Y_i)(y(x^2+y^2)-Y_i(X_i^2+Y_i^2))][1/R^4]$$
$$-2(\cos\theta_i)([(x^2+y^2)-(X_i^2+Y_i^2)]^2/4)[1/R^4]$$
$$+[\cos\theta_i(x^2+y^2) - 2xd_i][(x-X_i)^2+(y-Y_i)^2][1/R^4]$$
$$+d_i(x^2+y^2)(x-X_i)[1/R^4]$$
$$+d_i[(x-X_i)(3x^2+y^2+X_i^2+Y_i^2) + (y-Y_i)2yx][1/R^4] + O(1/R^6)$$

$$L_{xx} = \Sigma[(2E_{i,xx}D_i^2E_i - E_i^2D_{i,xx}D_i + 2E_{i,x}^2D_i^2) - 2(2E_{i,x}D_iE_i - E_i^2D_{i,x})D_{i,x}]/(F_iD_i^3)]$$

$$L_{yy} = \Sigma[(2E_{i,yy}D_i^2E_i - E_i^2D_{i,yy}D_i + 2E_{i,y}^2D_i^2) - 2(2E_{i,y}D_iE_i - E_i^2D_{i,y})D_{i,y}]/(F_iD_i^3)]$$

$$L_{xy} = \Sigma[(2E_{i,xy}D_i^2E_i - E_i^2D_{i,xy}D_i + 2E_{i,x}D_{i,y}D_iE_i - 2E_{i,y}D_{i,x}D_iE_i + 2E_{i,x}E_{i,y}D_i^2)$$
$$-2(2E_{i,x}D_iE_i - E_i^2D_{i,x})D_{i,y}]/(F_iD_i^3)]$$

$$= \Sigma[2E_{i,xy}D_i^2E_i - E_i^2D_{i,xy}D_i - 2E_{i,x}D_{i,y}D_iE_i - 2E_{i,y}D_{i,x}D_iE_i + 2E_i^2D_{i,x}D_{i,y}$$
$$+2E_{i,x}E_{i,y}D_i^2]/(F_iD_i^3)]$$

$$E_{i,xx} = E_{i,yy} = E_{i,xy} = 0 + O(1/R^2)$$

$$D_{i,xx} = 2[1/R^2] - [(X_i^2+Y_i^2) + (3x^2+y^2) + (x-X_i)6x + (y-Y_i)2y][1/R^4] + O(1/R^6)$$

$$D_{i,xy} = -[(x-X_i)2y + 2yx + (y-Y_i)2x][1/R^4] + O(1/R^6)$$

$$D_{i,yy} = 2[1/R^2] - [(X_i^2+Y_i^2) + (x^2+3y^2) + (y-Y_i)6y + (x-X_i)2x][1/R^4] + O(1/R^6)$$

Only the first order term matters as $R\to\infty$ for these terms. Thus at $R=\infty$,

$$E_{i,xx} = E_{i,yy} = E_{i,xy} = 0 \qquad R^2D_{i,xx} = R^2D_{i,yy} = 2 \qquad R^2D_{i,xy} = 0$$

$$E_{i,x} = -\cos\theta_i \qquad E_{i,y} = \sin\theta_i \qquad R^2D_{i,x} = 2(x-X_i) \qquad R^2D_{i,y} = 2(y-Y_i)$$

$E_i=[-(x-X_i)\cos\theta_i+(y-Y_i)\sin\theta_i]$ $\qquad$ $R^2D_i=[(x-X_i)^2+(y-Y_i)^2]$

$R^2/F_i=1$

$$L_{xx}=\Sigma[d_i^2(-D_{i,xx}D_i+2D_{i,x}^2)+4d_i\cos\theta_i(D_iD_{i,x})+2\cos^2\theta_i\ D_i^2]/[R^2D_i^3]$$
$$=\Sigma[d_i^2(6(x-X_i)^2-2(y-Y_i)^2)+8d_i(x-X_i)\cos\theta_i((x-X_i)^2+(y-Y_i)^2)$$
$$+2\cos^2\theta_i((x-X_i)^2+(y-Y_i)^2)^2]/[(x-X_i)^2+(y-Y_i)^2]3$$

$$L_{yy}=\Sigma(d_i^2(-D_{i,yy}D_i+2D_{i,y}^2)-4d_i\sin\theta_i(D_iD_{i,y})+2\sin^2\theta_iD_i^2)/(R^2D_i^3)$$
$$=\Sigma[d_i^2(6(y-Y_i)^2-2(x-X_i)^2)-8d_i(y-Y_i)\sin\theta_i((x-X_i)^2+(y-Y_i)^2)$$
$$+2\sin^2\theta_i((x-X_i)^2+(y-Y_i)^2)^2]/[(x-X_i)^2+(y-Y_i)^2]3$$

$$L_{xy} = \Sigma(2d_i^2D_{i,x}D_{i,y} + 2d_i(\cos\theta_iD_{i,y}D_i-\sin\theta_iD_{i,x}D_i) -2\sin\theta_i\cos\theta_iD_i^2)/(R^2D_i^3)$$
$$= \Sigma[8d_i^2(x-X_i)(y-Y_i) + 4d_i((y-Y_i)\cos\theta_i-(x-X_i)\sin\theta_i)((x-X_i)^2+(y-Y_i)^2)$$
$$- 2\sin\theta_i\cos\theta_i((x-X_i)^2+(y-Y_i)^2)^2] / ((x-X_i)^2+(y-Y_i)^2)3$$

Let

$x-X_i = \varepsilon_i$ $\qquad$ $y-Y_i = \eta_i$

Thus,

$$L_{xx} = \Sigma[(M_i+N_i+O_i)/(\varepsilon_i^2+\eta_i^2)^3]$$
$$L_{yy} = \Sigma[(P_i+Q_i+S_i)/(\varepsilon_i^2+\eta_i^2)^3]$$
$$L_{xy} = \Sigma[(T_i+U_i+V_i)/(\varepsilon_i^2+\eta_i^2)^3]$$
$$L_{yx} = L_{xy}$$

where

$$M_i = d_i^2(6\varepsilon_i^2-2\eta_i^2)$$
$$N_i = 8d_i\varepsilon_i\cos\theta_i(\varepsilon_i^2+\eta_i^2)$$
$$O_i = 2\cos^2\theta_i(\varepsilon_i^2+\eta_i^2)^2$$
$$P_i = d_i^2(6\eta_i^2-2\varepsilon_i^2)$$
$$Q_i = -8d_i\eta_i\sin\theta_i(\varepsilon_i^2+\eta_i^2)$$
$$S_i = 2\sin^2\theta_i(\varepsilon_i^2+\eta_i^2)^2$$
$$T_i = 8d_i^2\varepsilon_i\eta_i$$

$$U_i = 4d_i(\eta_i\cos\theta_i - \epsilon_i\sin\theta_i)(\epsilon_i^2+\eta_i^2)$$

$$V_i = -2\sin\theta_i\cos\theta_i(\epsilon_i^2+\eta_i^2)^2$$

Using similar calculations, the weighted perpendicular technique produces

$$L^*_{xx} = \Sigma(2(\epsilon_i^2+\eta_i^2)\cos^2\theta_i + 4\cos\theta_i d_i \epsilon_i)/(\epsilon_i^2+\eta_i^2)^2$$

$$L^*_{yy} = \Sigma(2(\epsilon_i^2+\eta_i^2)\sin^2\theta_i + 4\sin\theta_i d_i \eta_i)/(\epsilon_i^2+\eta_i^2)^2$$

$$L^*_{xy} = \Sigma(-2(\epsilon_i^2+\eta_i^2)\sin\theta_i\cos\theta_i + 4\cos\theta_i d_i \eta_i)/(\epsilon_i^2+\eta_i^2)^2$$

$$L^*_{yx} = \Sigma(-2(\epsilon_i^2+\eta_i^2)\sin\theta_i\cos\theta_i - 4\sin\theta_i d_i \epsilon_i)/(\epsilon_i^2+\eta_i^2)^2$$

Returning to the the sine of squared angular error technique, recall that

$$-\begin{vmatrix} L_{xx} & L_{yx} \\ L_{xy} & L_{yy} \end{vmatrix}^{-1} = \frac{-1}{\text{Determinant}} \begin{vmatrix} L_{yy} & -L_{xy} \\ -L_{yx} & L_{xx} \end{vmatrix}$$

where the Determinant $= L_{xx}L_{yy} - L_{xy}L_{yx}$

$$= \Sigma\Sigma(M_iP_j + M_iQ_j + M_iS_j + N_iP_j + N_iQ_j + N_iS_j + O_iP_j + O_iQ_j + O_iS_j)/[(\epsilon_i^2+\eta_i^2)^3(\epsilon_j^2+\eta_j^2)^3]$$

$$- \Sigma\Sigma(T_iT_j + 2{*}T_iU_j + 2{*}T_iV_j + 2{*}U_iV_j + V_iV_j + U_iU_j)/[(\epsilon_i^2+\eta_i^2)^3(\epsilon_j^2+\eta_j^2)^3]$$

Terms of order d_i may be ignored for relatively accurate LOBs. All of the above terms are at least order d_i except for O_iS_j and V_iV_j. Thus,

$$\text{Determinant} = [\Sigma\Sigma(O_iS_j - V_iV_j)/[(\epsilon_i^2+\eta_i^2)^3(\epsilon_j^2+\eta_j^2)^3]] + O(d_i)$$

$$= 4\Sigma\Sigma\cos\theta_i\sin\theta_j(\epsilon_i^2+\eta_i^2)^2(\epsilon_j^2+\eta_j^2)^2(\cos\theta_i\sin\theta_j - \sin\theta_i\cos\theta_j)/[(\epsilon_i^2+\eta_i^2)^3(\epsilon_j^2+\eta_j^2)^3] + O(d_i)$$

$$= 4\Sigma\Sigma\cos\theta_i\sin\theta_j\sin(\theta_j-\theta_i)/[(\epsilon_i^2+\eta_i^2)(\epsilon_j^2+\eta_j^2)] + O(d_i)$$

$$= 2\Sigma\Sigma\cos\theta_i\sin\theta_j\sin(\theta_j-\theta_i)/[(\epsilon_i^2+\eta_i^2)(\epsilon_j^2+\eta_j^2)]$$

$$+ 2\Sigma\Sigma\cos\theta_j\sin\theta_i\sin(\theta_i-\theta_j)/[(\epsilon_j^2+\eta_j^2)(\epsilon_i^2+\eta_i^2)] + O(d_i)$$

$$= 2\Sigma\Sigma\sin(\theta_j-\theta_i)(\cos\theta_i\sin\theta_j - \cos\theta_j\sin\theta_i)/[(\epsilon_i^2+\eta_i^2)(\epsilon_j^2+\eta_j^2)] + O(d_i)$$

Finally,

$$\text{Determinant} = 2\Sigma\Sigma\sin^2(\theta_j-\theta_i)/[(\epsilon_i^2+\eta_i^2)(\epsilon_j^2+\eta_j^2)] + O(d_i)$$

$$L_{xx} = [2\Sigma\cos^2\theta_i/(\epsilon_i^2+\eta_i^2) + O(d_i)] + O(1/R^2)$$

$$L_{yy} = [2\Sigma\sin^2\theta_i/(\epsilon_i^2+\eta_i^2) + O(d_i)] + O(1/R^2)$$

$$L_{xy} = [-2\Sigma\sin\theta_i\cos\theta_i/(\epsilon_i^2+\eta_i^2) + O(d_i)] + O(1/R^2) = L_{yx}$$

$$L_{i,x} = [-2\cos\theta_i d_i/(\epsilon_i^2+\eta_i^2) + O(d_i^2)] + O(1/R^2)$$

$L_{i,y}=[2\sin\theta_i d_i/(\varepsilon_i^2+\eta_i^2) + O(d_i^2)] + O(1/R^2)$

With these assumptions($(d_i \sim 0)$and$(R\to\infty)$), the IGRV values for L^*_{xx}, L^*_{yy}, L^*_{xy}, $L^*_{i,x}$, $L^*_{i,y}$, and the Determinant* coincide with those above.

Computing L_{xUU} at U=0 or $R=\infty$ is simplified by the fact that the terms making up L_x can be chosen so that $L_x=\Sigma a_k b_k \ldots c_k$ where $a_{k,U}=b_{k,U}=\ldots=c_{k,U}=0$ at U=0 ($R=\infty$). Thus mixed terms equal zero and one is left with

$L_{xUU}=(\Sigma a_k b_k \ldots d_k)(a_{k,UU}/a_k+b_{k,UU}/b_k+\ldots+c_{k,UU}/c_k)$

The terms used in the two kinds of products making up L_{xUU} are:

E_i , $D_{i,x}/U^2$, $E_{i,x}$, $F_i \cdot U^2$, and U^2/D_i .

At U=0 or $R=\infty$

$E_{i,UU}/E_i = -(x^2+y^2)$

$E_{i,xUU}/E_{i,x} = -(x^2+y^2)+2xd_i/\cos\theta_i$

$(D_{i,x}/U^2)_{UU}/(D_{i,x}/U^2) = -[(X_i^2+Y_i^2)+3x^2+y^2]+2[(y-Y_i)/(x-X_i)]yx$

$(F_i \cdot U^2)_{UU}/(F_i \cdot U^2) = 2(X_i\cos\theta_i-Y_i\sin\theta_i)^2$

$(U^2/D_i)_{UU}/(U^2/D_i) =$

$$2[(x-X_i)(x(x^2+y^2)-X_i(X_i^2+Y_i^2))+(y-Y_i)(y(x^2+y^2)-Y_i(X_i^2+Y_i^2))]/[(x-X_i)^2+(y-Y_i)^2]$$
$$-2([(x^2+y^2)-(X_i^2+Y_i^2)]^2/4)/[(x-X_i)^2+(y-Y_i)^2]$$

$E_{i,yUU}/E_{i,y} = -(x^2+y^2)+2yd_i/\sin\theta_i$

$(D_{i,y}/U^2)_{UU}/(D_{i,y}/U^2) = -[(X_i^2+Y_i^2)+3y^2+x^2]+2[(x-X_i)/(y-Y_i)]yx$

Terms which are dependent on i are proportional to L_x terms and hence add to zero. Therefore, at U=0 (or $R=\infty$) and $d_i \sim 0$

$\Sigma(\text{product})E_{i,UU}/E_i = 0$

$\Sigma(\text{product})\ E_{i,xUU}/E_{i,x} \sim 0$ $\qquad$ $\Sigma(\text{product})\ E_{i,yUU}/E_{i,y} \sim 0$

Thus,

$$L_{xUU}=\Sigma L_{i,x}[(D_{i,x}/U^2)_{UU}/(D_{i,x}/U^2)+(1/F\cdot U^2)_{UU}/(1/F\cdot U^2)+(U^2/D_i)_{UU}/(U^2/D_i)]$$

$$L_{yUU}=\Sigma L_{i,y}[(D_{i,y}/U^2)_{UU}/(D_{i,y}/U^2)+(1/F\cdot U^2)_{UU}/(1/F\cdot U^2)+(U^2/D_i)_{UU}/(U^2/D_i)]$$

In order to calculate f" and g", where

$$\begin{vmatrix} f'' \\ g'' \end{vmatrix} = - \begin{vmatrix} L_{xx} & L_{xy} \\ L_{xy} & L_{yy} \end{vmatrix}^{-1} \begin{vmatrix} L_{xUU} \\ L_{yUU} \end{vmatrix}$$

a special case will be examined. In particular, the emitter is assumed to be at $(x,y)=(0,0)$. This is, however, a logical place to put the plane.

Then,

$$(D_{i,x}/U^2)_{UU}/(D_{i,x}/U^2) = -(X_i^2+Y_i^2) = (D_{i,y}/U^2)_{UU}/(D_{i,y}/U^2)$$

$$(1/F_i \cdot U^2)_{UU}/(1/F_i \cdot U^2) = -2d_i^2$$

$$(U^2/D_i)_{UU}/(U^2/D_i) = 2[X_i^2+Y_i^2-(X_i^2+Y_i^2)/4] = 6(X_i^2+Y_i^2)/4$$

Therefore,

$$L_{xUU}=\Sigma[L_{i,x}(-2d_i^2 + (X_i^2+Y_i^2)/2)]$$

$$L_{yUU}=\Sigma[L_{i,y}(-2d_i^2 + (X_i^2+Y_i^2)/2)]$$

Since the weighted perpendicular technique utilizes a different L_x and L_y, it should produce a different L_{xUU} and L_{yUU}. In fact,

$$L^*_{xUU}= \Sigma[L_{i,x}(-2d_i^2 + 3/2(X_i^2+Y_i^2))]$$

$$L^*_{yUU}= \Sigma[L_{i,x}(-2d_i^2 + 3/2(X_i^2+Y_i^2))]$$

Notice that the only difference in these values and the ones above, is that the $(X_i^2+Y_i^2)$ term is multiplied by 3/2 rather than 1/2. This is due to the lack of a $D_{i,x}$ ($D_{i,y}$) term in the equation for L^*_x (L^*_y). {Remember that $(D_{i,x}/U^2)_{UU}/(D_{i,x}/U^2) = -(X_i^2+Y_i^2)$}

Finally,

$$f'' = [-1/\text{Determinant}][L_{yy}L_{xUU}-L_{xy}L_{yUU}]$$

$$g'' = [-1/\text{Determinant}][-L_{yx}L_{xUU}+L_{xx}L_{yUU}]$$

$$L_{yy}L_{xUU}-L_{xy}L_{yUU}=[-\Sigma\Sigma 2\sin^2\theta_i\cos\theta_j d_j/(\epsilon_i^2+\eta_i^2) +\Sigma\Sigma 2\sin\theta_i\cos\theta_i\sin\theta_j d_j/(\epsilon_i^2+\eta_i^2) +O(d_i^2)] + O(1/R^2)$$

$$=[2\Sigma\Sigma\sin\theta_i\sin(\theta_j-\theta_i)d_j/(\epsilon_i^2+\eta_i^2) + O(d_i^2)] + O(1/R^2)$$

$$-L_{yx}L_{xUU}+L_{xx}L_{yUU}=[-\Sigma\Sigma 2\sin\theta_i\cos\theta_i\cos\theta_j d_j/(\epsilon_i^2+\eta_i^2) +\Sigma\Sigma 2\cos^2\theta_i\sin\theta_j d_j/(\epsilon_i^2+\eta_i^2) +O(d_i^2)] + O(1/R^2)$$

$$=[2\Sigma\Sigma\cos\theta_i\sin(\theta_j-\theta_i)d_j/(\varepsilon_i^2+\eta_i^2) + O(d_i^2)] + O(1/R^2)$$

The only difference in $(-L_{yx}L_{xUU}+L_{xx}L_{yUU})$ and $(L_{yy}L_{xUU}-L_{xy}L_{yUU})$, and therefore in f" and g", resulting from the weighted perpendicular technique, is that both are increased by a factor of three. Thus,

$$L_{yy}L^*_{xUU}-L_{xy}L^*_{yUU}=[\Sigma\Sigma 6\sin\theta_i\sin(\theta_j-\theta_i)d_j/(\varepsilon_i^2+\eta_i^2) + O(d_i^2)] + O(1/R^2)$$

$$-L_{yx}L^*_{xUU}+L_{xx}L^*_{yUU}=[\Sigma\Sigma 6\cos\theta_i\sin(\theta_j-\theta_i)d_j/(\varepsilon_i^2+\eta_i^2) + O(d_i^2)] + O(1/R^2)$$

Returning to the sine of squared angular error technique, it follows that f" and g" can be expressed as

$$f"= -[2\Sigma\Sigma\sin\theta_i\sin(\theta_j-\theta_i)d_j/(\varepsilon_i^2+\eta_i^2)]/\{2\Sigma\Sigma\sin^2(\theta_j-\theta_i)/[(\varepsilon_i^2+\eta_i^2)(\varepsilon_j^2+\eta_j^2)]\} + O(d_i)$$

$$= -[\Sigma\Sigma\sin\theta_i\sin(\theta_j-\theta_i)d_j/(\varepsilon_i^2+\eta_i^2)]/\{\Sigma\Sigma\sin^2(\theta_j-\theta_i)/[(\varepsilon_i^2+\eta_i^2)(\varepsilon_j^2+\eta_j^2)]\} + O(d_i)$$

$$g"= -[2\Sigma\Sigma\cos\theta_i\sin(\theta_j-\theta_i)d_j/(\varepsilon_i^2+\eta_i^2)]/\{2\Sigma\Sigma\sin^2(\theta_j-\theta_i)/[(\varepsilon_i^2+\eta_i^2)(\varepsilon_j^2+\eta_j^2)]\} + O(d_i)$$

$$= -[\Sigma\Sigma\cos\theta_i\sin(\theta_j-\theta_i)d_j/(\varepsilon_i^2+\eta_i^2)]/\{\Sigma\Sigma\sin^2(\theta_j-\theta_i)/[(\varepsilon_i^2+\eta_i^2)(\varepsilon_j^2+\eta_j^2)]\} + O(d_i)$$

For the weighted perpendicular technique used in the IGRV code, a different f" and g" are produced.

$$f"= -3[\Sigma\Sigma\sin\theta_i\sin(\theta_j-\theta_i)d_j/(\varepsilon_i^2+\eta_i^2)]/\{\Sigma\Sigma\sin^2(\theta_j-\theta_i)/[(\varepsilon_i^2+\eta_i^2)(\varepsilon_j^2+\eta_j^2)]\} + O(d_i)$$

$$g"= -3[\Sigma\Sigma\cos\theta_i\sin(\theta_j-\theta_i)d_j/(\varepsilon_i^2+\eta_i^2)]/\{\Sigma\Sigma\sin^2(\theta_j-\theta_i)/[(\varepsilon_i^2+\eta_i^2)(\varepsilon_j^2+\eta_j^2)]\} + O(d_i)$$

Since the only difference in the terms for f" and g" between the two techniques is a constant multiplier, in simplifying the two terms, we need only concentrate on the sine of squared angular error technique.

Note that if $b_i \geq 0$ for all i, then

$$\min_i\{a_i/b_i\} \leq \Sigma a_i/\Sigma b_i \geq \max_i\{a_i/b_i\}$$

Additionally,

$$-\sin(\theta_j-\theta_i) = \sin(\theta_i-\theta_j)$$

$$||(\sin\theta_i,\cos\theta_i)|| = \sin^2\theta_i + \cos^2\theta_i = 1$$

In order to investigate the size of the f" and g" terms, we study the norm of (f",g"). Using the facts mentioned above, we can do the following.

$$||(f",g")|| \leq \sum_{i=1}^{n}||(\sin\theta_i,\cos\theta_i)|| \left| \frac{\sum_{j=1}^{n}[\sin(\theta_i-\theta_j)d_j/(\varepsilon_i^2+\eta_i^2)]}{\sum_{j=1}^{n}\{\sin^2(\theta_i-\theta_j)/[(\varepsilon_i^2+\eta_i^2)(\varepsilon_j^2+\eta_j^2)]\}} \right|$$

$$\leq \max_{i} \left| \sum_{j=1}^{n} \sin(\theta_i - \theta_j) d_j \;/\; \sum_{j=1}^{n} [\sin^2(\theta_i - \theta_j)/(\epsilon_j^2 + \eta_j^2)] \right|$$

Now, fix the i such that the expression is maximized. Thus,

$$||(f",g")|| \leq \left| \sum_{j=1}^{n} \sin(\theta_i - \theta_j) d_j \;/\; \sum_{j=1}^{n} [\sin^2(\theta_i - \theta_j)/(\epsilon_j^2 + \eta_j^2)] \right|$$

$$\leq \left| \max_{j}\{d_j\} \sum_{j=1}^{n} \sin(\theta_i - \theta_j) \;/\; \{[1/\max_{j}\{\epsilon_j^2 + \eta_j^2\}] \sum_{j=1}^{n} \sin^2(\theta_i - \theta_j)\} \right|$$

$$\leq \left| [\sum_{j=1}^{n} \sin(\theta_i - \theta_j) \;/ \sum_{j=1}^{n} \sin^2(\theta_i - \theta_j)] \max_{j}\{d_j\} \max_{j}\{\epsilon_j^2 + \eta_j^2\} \right|$$

Furthermore since $n\Sigma X_i^2 \geq [\Sigma X_i]^2$, calculations show that

$$\Sigma X_i / \Sigma X_i^2 \leq n / \Sigma X_i = 1/\overline{X}$$

Therefore,

$$||(f",g")|| \leq \left| \max_{j}\{d_j\} \max_{j}\{\epsilon_j^2 + \eta_j^2\} \;/\; \mathrm{avg}[\sin(\theta_i - \theta_j)] \right|$$

Finally, the distance between the fix produced by the two methodologies (two dimensional versus three dimensional) is

$$\text{distance} \leq ||(f",g")||^2 \, [1/R^2]$$

$$\leq \left| \max_{j}\{d_j\} \max_{j}\{\epsilon_j^2 + \eta_j^2\} \;/\; \mathrm{avg}[\sin(\theta_i - \theta_j)] \right|^2 [1/R^2]$$

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