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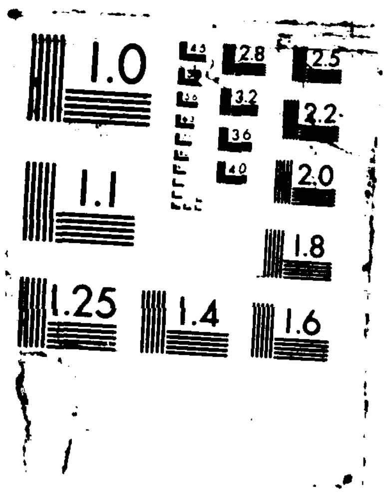
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# Incomplete Lipschitz-Hankel Integrals of MacDonald Functions


ALLEN R. MILLER

*Engineering Services Division*


March 15, 1988



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19 ABSTRACT (Continue on reverse if necessary and identify by block number)

Various representations for incomplete Lipschitz-Hankel integrals of MacDonald functions have been given in closed form using Kampé de Fériet double hypergeometric functions. In addition, reduction formulas for the Kampé de Fériet functions associated with these integrals have been derived in some cases.

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## CONTENTS

# INCOMPLETE LIPSCHITZ-HANKEL INTEGRALS OF MACDONALD FUNCTIONS

## INTRODUCTION

An incomplete Lipschitz-Hankel integral of MacDonald functions $K_\nu(z)$ may be defined as the following function of two complex variables:

$$Ke_\nu(a, z) \equiv \int_0^z e^{at} t^\nu K_\nu(t)\,dt\,. \tag{1}$$

Here the symbol $e$ denotes the presence of the exponential function and $\nu$ may be complex. Analogously, we may define integrals which contain the functions sin $(at)$ and cos $(at)$ in place of exp $(at)$:

$$Ks_\nu(a, z) \equiv \int_0^z \sin(at) t^\nu K_\nu(t)\,dt$$

$$Kc_\nu(a, z) \equiv \int_0^z \cos(at) t^\nu K_\nu(t)\,dt\,. \tag{2}$$

To assure convergence of the integrals in Eqs. (1) and (2), it is necessary that Re $\nu > -1/2$. $Ks_\nu(a, z)$ converges for Re $\nu > -1$.

Agrest and Maksimov [1] have found representations for $Ke_\nu(a, z)$ by using incomplete cylindrical functions. In this report we derive representations for $Ke_\nu(a, z)$, $Ks_\nu(a, z)$, and $Kc_\nu(a, z)$ by using Kampé de Fériet double hypergeometric functions. We then give a representation for a generalization of $Ke_\nu(a, z)$.

## PRELIMINARY RESULTS AND DEFINITIONS

To begin, we state some well-known results that are found in Refs. 2 or 3:

$$\int^z t^\mu K_\nu(t)dt = (\mu + \nu - 1)zK_\nu(z)s_{\mu-1,\nu-1}(z) - zK_{\nu-1}(z)s_{\mu,\nu}(z) \tag{3}$$

$$\int^z t^\mu K_\nu(t)dt = (\mu + \nu - 1)zK_\nu(z)S_{\mu-1,\nu-1}(z) - zK_{\nu-1}(z)S_{\mu,\nu}(z) \tag{4}$$

where the $s_{\mu,\nu}$, $S_{\mu,\nu}$ are Lommel functions:

$$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu+\nu+1)(\mu-\nu+1)}\,{}_1F_2\left[1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; \frac{-z^2}{4}\right]. \tag{5}$$

When either of the numbers $\mu \pm \nu$ is an odd positive integer,

$$S_{\mu,\nu}(z) = z^{\mu-1}{}_3F_0\left[1, \frac{\nu-\mu+1}{2}, -\frac{\nu+\mu-1}{2}; —; \frac{-4}{z^2}\right]. \tag{6}$$

Manuscript approved December 8, 1987.

We shall also need the Wronskian

$$K_{\nu+1}(z)I_\nu(z) + K_\nu(z)I_{\nu+1}(z) = 1/z. \tag{7}$$

By using Eqs. (4) and (6) we obtain for $n$ an odd positive integer

$$\int_0^z t^{\nu+n}K_\nu(t)dt = 2^{\nu+n-1}\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{2\nu+n+1}{2}\right)$$

$$-(2\nu+n-1)z^{\nu+n-1}K_\nu(z)\,{}_3F_0\left[1, \frac{1-n}{2}, \frac{3-2\nu-n}{2}; —; \frac{4}{z^2}\right]$$

$$-z^{\nu+n}K_{\nu-1}(z)\,{}_3F_0\left[1, \frac{1-n}{2}, \frac{1-2\nu-n}{2}; —; \frac{4}{z^2}\right]. \tag{8}$$

However, for all nonnegative integers $n$ we find by using Eqs. (3) and (5)

$$\int_0^z t^{\nu+n}K_\nu(t)dt = \frac{z^{\nu+n+1}}{n+1}K_\nu(z)\,{}_1F_2\left[1; \frac{n+3}{2}, \frac{2\nu+n+1}{2}; \frac{z^2}{4}\right]$$

$$+\frac{z^{\nu+n+2}}{(n+1)(2\nu+n+1)}K_{\nu-1}(z)\,{}_1F_2\left[1; \frac{n+3}{2}, \frac{2\nu+n+3}{2}; \frac{z^2}{4}\right]. \tag{9}$$

Equations (8) and (9) are given in Ref. 2 for $\nu = 0$.

We define the following Kampé de Fériet double hypergeometric functions and give associated generating relations [4,5]:

$$L[\alpha, \beta; \gamma, \delta; x, y] \equiv F^{0:1;1}_{2:0;0}\left[\begin{matrix} — : & \alpha; & \beta; \\ \gamma, \delta : & —; & —; \end{matrix} x, y\right] \qquad |x| < \infty,\ |y| < \infty$$

$$M[\alpha, \beta; \gamma, \delta; x, y] \equiv F^{1:0;1}_{1:1;0}\left[\begin{matrix} \alpha : & —; & \beta; \\ \gamma : & \delta; & —; \end{matrix} x, y\right] \qquad |x| < \infty,\ |y| < 1$$

$$Q[\alpha, \beta, \gamma; \mu, \nu, \lambda; x, y] \equiv F^{0:2;1}_{2:1;0}\left[\begin{matrix} — : & \alpha, \beta; & \gamma; \\ \mu, \nu : & \lambda; & —; \end{matrix} x, y\right] \qquad |x| < \infty,\ |y| < \infty$$

$$L[\alpha, \beta; \gamma, \delta; x, y] = \sum_{m=0}^{\infty} \frac{(\alpha)_m}{(\gamma)_m(\delta)_m}\frac{x^m}{m!}\,{}_1F_2[\beta; m+\gamma, m+\delta; y] \tag{10}$$

$$M[\alpha, \beta; \gamma, \delta; x, tx] = \sum_{m=0}^{\infty} \frac{(\alpha)_m}{(\gamma)_m(\delta)_m}\frac{x^m}{m!}\,{}_3F_0[\beta, -m, 1-\delta-m; —; t] \tag{11}$$

$$Q[\alpha, \beta, \gamma; \mu, \nu, \lambda; x, y] = \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{(\mu)_m (\nu)_m (\lambda)_m} \frac{x^m}{m!} \, {}_1F_2[\gamma; m + \mu, m + \nu; y]. \tag{12}$$

It is easy to see that the function $L$ is a special case of $Q$:

$$Q[\alpha, \lambda, \beta; \gamma, \delta, \lambda; x, y] = L[\alpha, \beta; \gamma, \delta; x, y].$$

## REPRESENTATION FOR $K_{e_\nu}(a, z)$, $|a| < 1$

Integrating the right-hand side of Eq. (1) term by term, we have

$$K_{e_\nu}(a, z) = \sum_{n=0}^{\infty} \frac{a^{2n}}{(2n)!} \int_0^z t^{\nu+2n} K_\nu(t)\, dt + \sum_{n=0}^{\infty} \frac{a^{2n+1}}{(2n+1)!} \int_0^z t^{\nu+2n+1} K_\nu(t)\, dt. \tag{13}$$

By using Eqs. (8) and (9), observing that

$$\sum_{n=0}^{\infty} 2^{\nu+2n} \Gamma(n+1)\Gamma(\nu+n+1) \frac{a^{2n+1}}{(2n+1)!} = 2^\nu \, \Gamma(1+\nu)\, a \; {}_2F_1 \left[ 1, 1+\nu; \frac{3}{2}; a^2 \right],$$

and taking note of Eqs. (10) and (11), we obtain

$$K_{e_\nu}(a, z) = 2^\nu \, \Gamma(1+\nu)\, a \; {}_2F_1 \left[ 1+\nu, 1; \frac{3}{2}; a^2 \right]$$

$$+ z^\nu K_\nu(z) \left\{ zL \left[ \frac{1}{2} + \nu, 1; \frac{3}{2}, \frac{1}{2} + \nu; \frac{a^2 z^2}{4}, \frac{z^2}{4} \right] - 2a\nu M \left[ 1+\nu, 1; \frac{3}{2}, \nu; \frac{a^2 z^2}{4}, a^2 \right] \right\}$$

$$+ z^{\nu+1} K_{\nu-1}(z) \left\{ \frac{z}{1+2\nu} L \left[ \frac{1}{2} + \nu, 1; \frac{3}{2}, \frac{3}{2} + \nu; \frac{a^2 z^2}{4}, \frac{z^2}{4} \right] - aM \left[ 1+\nu, 1; \frac{3}{2}, 1+\nu; \frac{a^2 z^2}{4}, a^2 \right] \right\}. \tag{14}$$

We readily show that

$$\lim_{\delta \to 0} \delta M[\alpha, \beta; \gamma, \delta; x, y] = \frac{\alpha x}{\gamma} M[\alpha+1, \beta; \gamma+1, 2; x, y]$$

from which it follows that

$$\lim_{\nu \to 0} 2\nu M \left[ \nu+1, 1; \frac{3}{2}, \nu; \frac{a^2 z^2}{4}, a^2 \right] = \frac{1}{3} a^2 z^2 M \left[ 2, 1; \frac{5}{2}, 2; \frac{a^2 z^2}{4}, a^2 \right].$$

Since $K_\nu(z) = K_{-\nu}(z)$, ${}_2F_1\left[1, 1; \frac{3}{2}; a^2\right] = \sin^{-1}a/a(1 - a^2)^{1/2}$, Eq. (14) for $\nu = 0$ gives the result

$$K_{e_0}(a, z) = (1 - a^2)^{-1/2}\sin^{-1}a$$

$$+ zK_0(z)\left\{L\left[\frac{1}{2}, 1; \frac{3}{2}, \frac{1}{2}; \frac{a^2z^2}{4}, \frac{z^2}{4}\right] - \frac{a^3z}{3}M\left[2, 1; \frac{5}{2}, 2; \frac{a^2z^2}{4}, a^2\right]\right\}$$

$$+ zK_1(z)\left\{zL\left[\frac{1}{2}, 1; \frac{3}{2}, \frac{3}{2}; \frac{a^2z^2}{4}, \frac{z^2}{4}\right] - aM\left[1, 1; \frac{3}{2}, 1; \frac{a^2z^2}{4}, a^2\right]\right\}. \qquad (15)$$

Equation (14) is valid for Re $\nu > -1/2$ and $|a| < 1$. It is shown in Ref. 4 that

$$M[\alpha, 1; \gamma, \delta; tx, t] = 1 + {}_0F_1[-; \delta; x]\{{}_2F_1[\alpha, 1; \gamma; t] - 1\}$$

$$- \frac{\alpha t x^2}{2\gamma\delta(\delta + 1)}Q[1 + \alpha, 1, 1; 2 + \delta, 3, 1 + \gamma; tx, x]. \qquad (16)$$

This equation provides the corollary that $M[\alpha, 1; \gamma, \delta; tx, t]$ converges on the unit circle $|t| = 1$ if and only if ${}_2F_1[\alpha, 1; \gamma; t]$ does. We then deduce that Eq. (14) is conditionally convergent on $|a| = 1$, $a \neq \pm 1$ provided that $|\mathrm{Re}\,\nu| < 1/2$.

## REPRESENTATIONS FOR $K_{e_\nu}(a, z)$, $|a| < \infty$

We may, however, use Eq. (16) to better advantage. We find

$$M\left[1 + \nu, 1; \frac{3}{2}, \nu; \frac{a^2z^2}{4}, a^2\right] = 1 + \left(\frac{2}{z}\right)^{\nu - 1}\Gamma(\nu)I_{\nu-1}(z)\left\{{}_2F_1\left[1 + \nu, 1; \frac{3}{2}; a^2\right] - 1\right\}$$

$$- \frac{a^2(z^2/4)^2}{3\nu}Q\left[2 + \nu, 1, 1; 2 + \nu, 3, \frac{5}{2}; \frac{a^2z^2}{4}, \frac{z^2}{4}\right]$$

$$M\left[1 + \nu, 1; \frac{3}{2}, 1 + \nu; \frac{a^2z^2}{4}, a^2\right] = 1 + \left(\frac{2}{z}\right)^{\nu}\Gamma(1 + \nu)I_\nu(z)$$

$$\cdot\left\{{}_2F_1\left[1 + \nu, 1; \frac{3}{2}; a^2\right] - 1\right\}$$

$$- \frac{a^2(z^2/4)^2}{3(2 + \nu)}Q\left[2 + \nu, 1, 1; 3 + \nu, 3, \frac{5}{2}; \frac{a^2z^2}{4}, \frac{z^2}{4}\right].$$

By using these equations with Eq. (14) and taking note of Eq. (7), we deduce

$$K_{e_\nu}(a, z) = 2^\nu \Gamma(1 + \nu) a + z^\nu K_\nu(z) A_\nu(a, z) + z^{\nu+1} K_{\nu-1}(z) B_\nu(a, z) \tag{17}$$

where

$$A_\nu(a, z) \equiv zL\left[\frac{1}{2} + \nu, 1; \frac{3}{2}, \frac{1}{2} + \nu; \frac{a^2z^2}{4}, \frac{z^2}{4}\right]$$

$$+ \frac{a^3z^4}{24} Q\left[2 + \nu, 1, 1; 2 + \nu, 3, \frac{5}{2}; \frac{a^2z^2}{4}, \frac{z^2}{4}\right] - 2\nu a$$

$$B_\nu(a, z) \equiv \frac{z}{1 + 2\nu} L\left[\frac{1}{2} + \nu, 1; \frac{3}{2}, \frac{3}{2} + \nu; \frac{a^2z^2}{4}, \frac{z^2}{4}\right]$$

$$+ \frac{a^3z^4}{48(2 + \nu)} Q\left[2 + \nu, 1, 1; 3 + \nu, 3, \frac{5}{2}; \frac{a^2z^2}{4}, \frac{z^2}{4}\right] - a .$$

This equation is valid everywhere in the complex $a$-plane.

We obtain another somewhat simpler representation for $K_{e_\nu}(a, z)$, $|a| < \infty$ by using Eq. (13) together with Eqs. (9), (10), and (12):

$$K_{e_\nu}(a, z) = z^{\nu+1} K_\nu(z) \left\{L\left[\frac{1}{2} + \nu, 1; \frac{3}{2}, \frac{1}{2} + \nu; \frac{a^2z^2}{4}, \frac{z^2}{4}\right]\right.$$

$$\left. + \frac{az}{2} Q\left[1 + \nu, 1, 1; 1 + \nu, 2, \frac{3}{2}; \frac{a^2z^2}{4}, \frac{z^2}{4}\right]\right\}$$

$$+ z^{\nu+2} K_{\nu-1}(z) \left\{\frac{1}{1 + 2\nu} L\left[\frac{1}{2} + \nu, 1; \frac{3}{2}, \frac{3}{2} + \nu; \frac{a^2z^2}{4}, \frac{z^2}{4}\right]\right.$$

$$\left. + \frac{az}{4(1 + \nu)} Q\left[1 + \nu, 1, 1; 2 + \nu, 2, \frac{3}{2}; \frac{a^2z^2}{4}, \frac{z^2}{4}\right]\right\} . \tag{18}$$

**REPRESENTATIONS FOR $K_{s_\nu}(a, z)$, $K_{c_\nu}(a, z)$**

By using Eqs. (14), (17), and (18) we easily obtain the following:

$$K_{s_\nu}(a, z) = 2^\nu \Gamma(1+\nu)\, a \; {}_2F_1\left[1+\nu, 1; \frac{3}{2}; -a^2\right]$$

$$-az^\nu \left\{2\nu K_\nu(z) M\left[1+\nu, 1; \frac{3}{2}, \nu; \frac{-a^2z^2}{4}, -a^2\right]\right.$$

$$\left. + zK_{\nu-1}(z) M\left[1+\nu, 1; \frac{3}{2}, 1+\nu; \frac{-a^2z^2}{4}, -a^2\right]\right\} \quad (19)$$

$$K_{s_\nu}(a, z) = 2^\nu \Gamma(1+\nu)\, a$$

$$- z^\nu K_\nu(z) \left\{\frac{a^3z^4}{24} Q\left[2+\nu, 1, 1; 2+\nu, 3, \frac{5}{2}; \frac{-a^2z^2}{4}, \frac{z^2}{4}\right] + 2\nu a\right\}$$

$$- z^{\nu+1} K_{\nu-1}(z) \left\{\frac{a^3z^4}{48(2+\nu)} Q\left[2+\nu, 1, 1; 3+\nu, 3, \frac{5}{2}; \frac{-a^2z^2}{4}, \frac{z^2}{4}\right] + a\right\}$$

$$= \frac{az^{2+\nu}}{2} \left\{K_\nu(z) Q\left[1+\nu, 1, 1; 1+\nu, 2, \frac{3}{2}; \frac{-a^2z^2}{4}, \frac{z^2}{4}\right]\right.$$

$$\left. + \frac{zK_{\nu-1}(z)}{2(1+\nu)} Q\left[1+\nu, 1, 1; 2+\nu, 2, \frac{3}{2}; \frac{-a^2z^2}{4}, \frac{z^2}{4}\right]\right\}$$

$$K_{c_\nu}(a, z) = z^{1+\nu} K_\nu(z) L\left[\frac{1}{2}+\nu, 1; \frac{3}{2}, \frac{1}{2}+\nu; \frac{-a^2z^2}{4}, \frac{z^2}{4}\right]$$

$$+ \frac{z^{2+\nu}}{1+2\nu} K_{\nu-1}(z) L\left[\frac{1}{2}+\nu, 1; \frac{3}{2}, \frac{3}{2}+\nu; \frac{-a^2z^2}{4}, \frac{z^2}{4}\right].$$

These equations are valid for $|a| < \infty$ except Eq. (19) which is valid for $|a| < 1$. For $\nu = 0$, Eq. (19) may be written

$$K_{s_0}(a, z) = (1+a^2)^{-1/2} \sinh^{-1} a + az \left\{\frac{a^2z}{3} K_0(z) M\left[2, 1; \frac{5}{2}, 2; \frac{-a^2z^2}{4}, -a^2\right]\right.$$

$$\left. - K_1(z) M\left[1, 1; \frac{3}{2}, 1; \frac{-a^2z^2}{4}, -a^2\right]\right\} \quad |a| \leq 1, \quad a \neq \pm i$$

## REDUCTION FORMULAS

It is shown in Ref. 5 that

$$Q[\alpha, 1, 1; \gamma, \delta, \beta; x, x] = \frac{1-\beta}{\alpha-\beta+1}\, {}_1F_2[1; \gamma, \delta; x]$$

$$+ \frac{\alpha}{\alpha-\beta+1}\, {}_2F_3[1, \alpha+1; \gamma, \delta, \beta; x] \tag{20}$$

where, when $\gamma$ or $\delta$ is a positive integer, we find the following useful: for $n = 1, 2, \ldots$

$${}_1F_2[1; 1+n, 1+\nu; y] = \frac{n!}{y^n} \prod_{k=0}^{n-1} (\nu - k) \left\{ {}_0F_1[-; 1+\nu-n; y] \right.$$

$$\left. - \sum_{k=0}^{n-1} \frac{y^k}{(1+\nu-n)_k\, k!} \right\}. \tag{21}$$

We may show by using Eq. (20) that

$$L\left[\frac{1}{2}+\nu, 1; \frac{3}{2}, \frac{3}{2}+\nu; \frac{z^2}{4}, \frac{z^2}{4}\right] = \frac{\sinh z}{z} \tag{22}$$

$$L\left[\frac{1}{2}+\nu, 1; \frac{3}{2}, \frac{1}{2}+\nu; \frac{z^2}{4}, \frac{z^2}{4}\right] = \frac{2\nu}{1+2\nu}\frac{\sinh z}{z} + \frac{1}{1+2\nu}\cosh z\,. \tag{23}$$

By using Eqs. (20) and (21) we arrive at

$$Q\left[2+\nu, 1, 1; 3+\nu, 3, \frac{5}{2}; \frac{x^2}{4}, \frac{x^2}{4}\right]$$

$$= \frac{2+\nu}{1+2\nu}\frac{48}{x^4}\left\{\cosh x - 2\,\Gamma(2+\nu)\left(\frac{2}{x}\right)^{\nu} I_\nu(x) + 1 + 2\nu\right\}. \tag{24}$$

It is not too difficult to verify that

$${}_1F_2\left[2; 4, \frac{7}{2}; \frac{x^2}{4}\right] = \frac{360}{x^6}\left\{x \sinh x - 4\cosh x + x^2 + 4\right\}.$$

Then by using Eqs. (20), (21), and this result, we find

$$Q\left[2+\nu,\,1,\,1;\,2+\nu,\,3,\,\frac{5}{2};\,\frac{x^2}{4},\,\frac{x^2}{4}\right] \tag{25}$$

$$=\frac{1}{1+2\nu}\,\frac{24}{x^4}\left\{x\sinh x+2\nu\cosh x-4\,\Gamma(2+\nu)\left(\frac{2}{x}\right)^{\nu-1}I_{\nu-1}(x)+2\nu(1+2\nu)\right\}.$$

We may show directly from the definition of $Q$ that

$$Q[\alpha,\,1,\,1;\,\beta,\,\gamma,\,\delta;\,x,\,y]={}_1F_2[1;\,\beta,\,\gamma;\,y]$$

$$+\frac{\alpha x}{\beta\gamma\delta}\,Q[\alpha+1,\,1,\,1;\,\beta+1,\,\gamma+1,\,\delta+1;\,x,\,y].$$

In particular, we find by using Eq. (21) that

$$Q\left[\alpha+1,\,1,\,1;\,\beta+1,\,2,\,\delta;\,\frac{x^2}{4},\,\frac{y^2}{4}\right]=\frac{4\beta}{y^2}\left\{\left(\frac{2}{y}\right)^{\beta-1}\Gamma(\beta)I_{\beta-1}(y)-1\right\}$$

$$+\frac{\alpha+1}{\delta(\beta+1)}\,\frac{x^2}{8}\,Q\left[\alpha+2,\,1,\,1;\,\beta+2,\,3,\,\delta+1;\,\frac{x^2}{4},\,\frac{y^2}{4}\right]. \tag{26}$$

Then, using this result and Eq. (24) we find

$$Q\left[1+\nu,\,1,\,1;\,2+\nu,\,2,\,\frac{3}{2};\,\frac{x^2}{4},\,\frac{x^2}{4}\right]$$

$$=\frac{1+\nu}{1+2\nu}\,\frac{4}{x^2}\left\{\cosh x-\left(\frac{2}{x}\right)^{\nu}\Gamma(1+\nu)I_{\nu}(x)\right\}. \tag{27}$$

This may also be obtained directly from Eqs. (20) and (21). Now by using Eqs. (25) and (26) we deduce

$$Q\left[1+\nu,\,1,\,1;\,1+\nu,\,2,\,\frac{3}{2};\,\frac{x^2}{4},\,\frac{x^2}{4}\right]$$

$$=\frac{2}{1+2\nu}\,\frac{1}{x}\left\{\frac{2\nu\cosh x}{x}+\sinh x-\left(\frac{2}{x}\right)^{\nu}\Gamma(1+\nu)I_{\nu-1}(x)\right\}. \tag{28}$$

Equations (22)-(25), (27), and (28) may be used together with Eqs. (17) or (18), the identity [3]

$$zK_{\nu-1}(z) - z\,K_{\nu+1}(z) + 2\nu K_{\nu}(z) = 0$$

and Eq. (7) to obtain

$$K_{e_\nu}(1, z) = \left\{K_{\nu}(z) + K_{\nu+1}(z)\right\} \frac{e^{z} z^{\nu+1}}{1 + 2\nu} - \frac{2^{\nu}\,\Gamma(1 + \nu)}{1 + 2\nu} \tag{29}$$

$$K_{e_\nu}(-1, z) = \left\{K_{\nu}(z) - K_{\nu+1}(z)\right\} \frac{e^{-z} z^{\nu+1}}{1 + 2\nu} + \frac{2^{\nu}\,\Gamma(1 + \nu)}{1 + 2\nu} .$$

Equation (29) was first derived by Luke [6] in 1950 for integral $\nu$. Another derivation for these results is given in Ref. 1.

## GENERALIZATION OF $\mathbf{K_{e_\nu}(a, z)}$

We define for complex numbers $a$ and $z$

$$K_{e_{\mu,\nu}}(a, z) \equiv \int_0^z e^{at} t^{\mu} K_{\nu}(t)\,dt . \tag{30}$$

Analogously, we may define $K_{s_{\mu,\nu}}(a, z)$ and $K_{c_{\mu,\nu}}(a, z)$ by replacing exp $(at)$ by sin $(at)$ and cos $(at)$ respectively in the integrand of Eq. (30). For convergence of the integral in Eq. (30), it is necessary that Re $(\mu + 1) > |\mathrm{Re}\,\nu|$.

A computation similar to the one used in obtaining Eq. (9) gives for Re $(\mu + 1) > |\mathrm{Re}\,\nu|$

$$\int_0^z t^{\mu} K_{\nu}(t)\,dt = \frac{z^{1+\mu}}{\mu - \nu + 1} \left\{ K_{\nu}(z)\,{}_1F_2\left[1;\ \frac{\mu - \nu + 3}{2},\ \frac{\mu + \nu + 1}{2};\ \frac{z^2}{4}\right] \right.$$

$$\left. + \frac{zK_{\nu-1}(z)}{\mu + \nu + 1}\,{}_1F_2\left[1;\ \frac{\mu - \nu + 3}{2},\ \frac{\mu + \nu + 3}{2};\ \frac{z^2}{4}\right] \right\} . \tag{31}$$

Eq. (9), of course, is just the special case $\mu = \nu + n$.

In the same manner we obtained Eq. (18), but now using Eqs. (12) and (31), we deduce

$$K_{e_{\mu,\nu}}(a, z) = \tag{32}$$

$$z^{1+\mu}K_\nu(z) \left\{ \frac{1}{\mu - \nu + 1} Q\left[\frac{\mu+\nu+1}{2}, \frac{\mu-\nu+1}{2}, 1; \frac{\mu+\nu+1}{2}, \frac{\mu-\nu+3}{2}, \frac{1}{2}; \frac{a^2z^2}{4}, \frac{z^2}{4}\right]\right.$$

$$\left. + \frac{az}{\mu-\nu+2} Q\left[\frac{\mu+\nu+2}{2}, \frac{\mu-\nu+2}{2}, 1; \frac{\mu+\nu+2}{2}, \frac{\mu-\nu+4}{2}, \frac{3}{2}; \frac{a^2z^2}{4}, \frac{z^2}{4}\right]\right\}$$

$$+ z^{2+\mu}K_{\nu-1}(z) \left\{ \frac{1}{(\mu+\nu+1)(\mu-\nu+1)} Q\left[\frac{\mu+\nu+1}{2}, \frac{\mu-\nu+1}{2}, 1;\right.\right.$$

$$\left.\frac{\mu+\nu+3}{2}, \frac{\mu-\nu+3}{2}, \frac{1}{2}; \frac{a^2z^2}{4}, \frac{z^2}{4}\right]$$

$$\left. + \frac{az}{(\mu+\nu+2)(\mu-\nu+2)} Q\left[\frac{\mu+\nu+2}{2}, \frac{\mu-\nu+2}{2}, 1; \frac{\mu+\nu+4}{2}, \frac{\mu-\nu+4}{2}, \frac{3}{2}; \frac{a^2z^2}{4}, \frac{z^2}{4}\right]\right\}.$$

For $\mu = \nu$, we obtain Eq. (18), i.e., $K_{e_\nu}(a, z) = K_{e_{\nu,\nu}}(a, z)$.

By using Eq. (32) we may write

$$K_{s_{\mu,\nu}}(a, z) =$$

$$\frac{az^{2+\mu}K_\nu(z)}{\mu-\nu+2} Q\left[\frac{\mu+\nu+2}{2}, \frac{\mu-\nu+2}{2}, 1; \frac{\mu+\nu+2}{2}, \frac{\mu-\nu+4}{2}, \frac{3}{2}; \frac{-a^2z^2}{4}, \frac{z^2}{4}\right]$$

$$+ \frac{az^{3+\mu}K_{\nu-1}(z)}{(\mu+\nu+2)(\mu-\nu+2)} Q\left[\frac{\mu+\nu+2}{2}, \frac{\mu-\nu+2}{2}, 1;\right.$$

$$\left.\frac{\mu+\nu+4}{2}, \frac{\mu-\nu+4}{2}, \frac{3}{2}; \frac{-a^2z^2}{4}, \frac{z^2}{4}\right]$$

$$K_{c_{\mu,\nu}}(a,z) =$$

$$\frac{z^{1+\mu}K_\nu(z)}{\mu-\nu+1}\, Q\left[\frac{\mu+\nu+1}{2}, \frac{\mu-\nu+1}{2}, 1; \frac{\mu+\nu+1}{2}, \frac{\mu-\nu+3}{2}, \frac{1}{2}; \frac{-a^2z^2}{4}, \frac{z^2}{4}\right]$$

$$+ \frac{z^{2+\mu}K_{\nu-1}(z)}{(\mu+\nu+1)(\mu-\nu+1)}\, Q\left[\frac{\mu+\nu+1}{2}, \frac{\mu-\nu+1}{2}, 1;\right.$$

$$\left.\frac{\mu+\nu+3}{2}, \frac{\mu-\nu+3}{2}, \frac{1}{2}; \frac{-a^2z^2}{4}, \frac{z^2}{4}\right].$$

From the latter two equations, the representations for $K_{s_\nu}(a,z)$ and $K_{c_\nu}(a,z)$ obtained earlier may be deduced by setting $\mu = \nu$.

## CONCLUSION

Various representations for the incomplete Lipschitz-Hankel integral $K_{e_{\mu,\nu}}(a,z)$ and related integrals have been given in closed form by using Kampé de Fériet functions in two variables. These representations should prove useful in numerical computation. In addition, reduction formulas for the Kampé de Fériet functions associated with $K_{e_\nu}(a,z)$ have been given.

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