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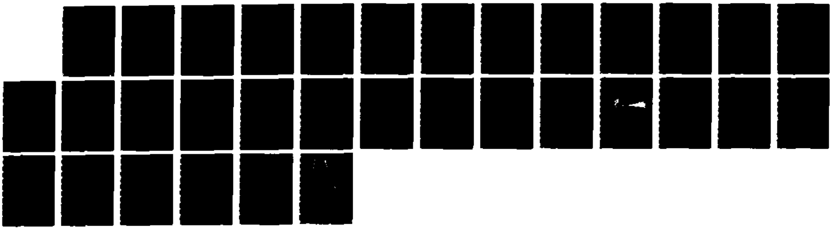

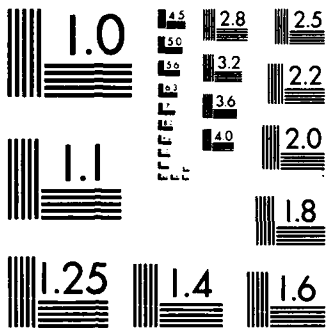
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A PROBABILISTIC MODEL OF BRITTLE CRACK FORMATION

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Probability of a brittle crack formation in an elastic solid with fluctuating strength is considered. A set Ω of all possible crack trajectories reflecting the fluctuation of the strength field is introduced. The probability $P(X)$ that crack penetration depth exceeds X is expressed as a functional integral over Ω of a conditional probability of the same event taking place along a particular path. Various techniques are considered to evaluate the integral. Under rather non-

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20. ABSTRACT (Cont.)

restrictive assumptions we reduce the integral to solving a diffusion type equation. Here a new characteristic of fracture process, 'crack diffusion coefficient', is introduced. Then an illustrative example is considered where the integration is reduced to solving an ordinary differential equation. Effect of the crack diffusion coefficient and of the magnitude of strength fluctuations (ratio of minimal and mean values of the strength field) on probability density of crack penetration depth is presented. Practical implications of the proposed model are discussed.

I. Introduction

The significant role played by microdefects (damage) in the process of crack formation and growth is commonly recognized. We distinguish two extreme cases of the influence of defects on the fracture process; modeling of these two cases requires essentially different formalisms.

Case 1 - The intensity of damage formed as a response to the stress concentration at the tip of a propagating crack is much greater than the intensity of the pre-existing damage. The crack propagation is then inseparable from the evolution of the damage accompanying the crack. This strongly cooperative phenomenon is modeled, for example, by Crack Layer Theory[1] based on thermodynamics of irreversible processes.

Case 2 - A crack propagates through a pre-existing field of defects causing negligible changes to the field. The fluctuation of the microdefect field is directly reflected in stochastic features of fracture surfaces and also leads to the scatter of experimentally observed fracture parameters, such as critical crack length, critical load, etc. A probabilistic approach seems most adequate under these circumstances.

Below, we present a probabilistic model of crack formation. Our consideration is based on the approach outlined earlier[2].

II. Probabilistic Model of Crack Formation

We address the following problem: what is the probability of crack formation beyond a particular depth in an elastic medium with fluctuating strength properties under prescribed loading conditions.

Let us consider an ensemble of macroscopically identical specimens subjected to identical loading conditions. Let the resulting fracture profiles be superimposed on the same plot as shown in Fig. 1 (such experiments are reported[3,4,5] for simple tension, and[6] for four point bending). Apparently, each crack trajectory is unique (no two coincide). More importantly, in a given test from the above sequence of identical tests, any of the plotted trajector for each specimen ies must be viewed as a priori possible. Statistical analysis of the observed trajectories allows one to formulate reasonable assumptions about the nature of the set Ω of all possible crack trajectories for each specimen under given test conditions. Thus Fig. 1 presents a sample subset of Ω.

If we assume that only one crack is formed in each specimen (single path fracture), then the probability $P(X)$ that a crack is formed beginning at the notch and extending to or beyond the depth X can be written as

$$P(X) = \sum_{\Omega} P\{X|\omega\}P\{\omega\} \quad , \tag{1}$$

where $P\{\omega\}$ is the probability that the crack 'chooses' a path ω among all possible paths, and $P\{X|\omega\}$ is the conditional probability that the depth X is reached by a crack formed along ω. In a continuum-based model, the space Ω is uncountable, so the sum in (1) has to be substituted by an integral

$$P(X) = \int_{\Omega} P\{X|\omega\} \, d\ (\omega) \quad . \tag{2}$$

This brings about three tasks: to propose an adequate crack path space Ω, to choose a probability measure $d\mu(\omega)$ on Ω, and to determine the probability $P\{X|\omega\}$.

The first two tasks are closely related. Their solution, in principle, is based on a statistical analysis of fracture surfaces, and preliminary data has been obtained recently. Here we consider a 'diffusion' model, i.e., we model crack trajectories by Brownian paths $y = \omega(x)$, $0 \leq x \leq X$, $\omega(0) = 0$, ($\Omega_X = \{\omega\}$ will denote the space) and choose $d\mu(\omega)$ to be a Wiener measure[7]

$$d\mu_X^{(D)}(\omega) = \text{const} \cdot \exp\left\{-\int_0^X \frac{1}{2D} [\omega'(x)]^2 \, dx\right\} \prod_{x=0}^{X} d\omega(x) \qquad (3)$$

Below we will meet the parameter $D > 0$ from (3) in a diffusion-type equation (13) for a probability of crack formation. By analogy, we refer to D as 'crack diffusion coefficient'. D reflects the tendency of crack trajectories to deviate from the X-axis and is experimentally measurable[5]. Evaluation of D uniquely determines, in the 'diffusion' model, the set of all possible crack trajectories as well, as the measure in (2).

The third task, determination of the conditional probability $P\{X|\omega\}$, is based on the formulation of a fracture criterion.

III. Probability of Crack Formation Along Given Path

In the case of brittle fracture of a homogeneous material, the Griffith criterion of uncontrolled crack propagation is commonly written as

$$J(\ell)\Big|_{\ell = \ell_c} = 2\gamma \quad , \quad \frac{\partial J}{\partial \ell}\Big|_{\ell = \ell_c} > 0 \; , \qquad (4)$$

where J is the energy release rate, γ is the specific surface energy, and ℓ_c is the critical crack length for the crack-loading configuration under consideration. Strictly speaking, (4) is a condition of global instability, if $\partial J/\partial \ell > 0$ for all $\ell > \ell_c$. Namely, (4) then assures that, as the crack advances, there is sufficient amount of potential energy released to compensate for the energy required for new surface formation

$$J(\ell) > 2\gamma \tag{5}$$

for $\ell > \ell_c$

In a heterogeneous material, γ becomes a random field which we assume to be statistically homogeneous. Now we take (5) as the criterion of local crack propagation. Thus, for a heterogeneous strength field γ, the global instability criterion should be written as an integral requirement that (5) is met everywhere along a fracture path. We further assume that, at distances exceeding certain r, values of the field γ are independent. This correlation distance r will be assumed to be much smaller than the crack size.

Subdividing the crack trajectory into small portions and assuming that the crack arrest at each step is an unlikely event with the probability proportional to the length of the step, we arrive at the following expression for the probability of crack penetration to a depth X along a particular path ω (see Appendix):

$$P\{X|\omega\} = \exp\left\{-\int_0^X P\{2\gamma \geq J_\omega(x)\}\,\frac{dx}{r}\right\}, \tag{6}$$

where $P\{2\gamma \geq J_\omega(X)\}$ stands for the probability that the value of the random field γ exceeds the value of the energy release rate at a current point $(x, \omega(x))$ of the path ω.

To describe $P\{2\gamma \geq J_\omega(x)\}$, we assume that locally a crack chooses the 'easiest' path, i.e., it minimizes the energy barrier $2\gamma - J$. This is equivalent to the assumption that γ along crack trajectories represents minimal values of the γ-field, since fluctuations of the γ-field occur at a scale small in comparison to crack increments, considered for energy release rate evaluation. According to the theory of statistics of extremes,[8] we employ a Weibull distribution for γ at every point of crack trajectory

$$F(\gamma) = \begin{cases} 1 - \exp\left[-\left(\dfrac{\gamma - \gamma_{min}}{\gamma_o}\right)^{\alpha}\right] & (\gamma \geq \gamma_{min}) \\ 0 & (\gamma < \gamma_{min}) \end{cases} \quad , \tag{7}$$

where $\alpha > 0$, γ_{min} and γ_o (commonly known as shape, minimal value and scale parameters) are empirical constants. Then

$$P\{2\gamma \geq J_\omega(x)\} = 1 - F(J_\omega(x)/2) = \exp\left[-\left(\frac{J_\omega(x)/2 - \gamma_{min}}{\gamma_o}\right)^{\alpha}\right] . \tag{8}$$

Putting Eqs. (2), (6) and (8) together we get the following expression for the probability that a crack is formed to a depth greater than X:

$$P(X) = \int_{\Omega_X} \exp\left\{-\int_0^X \exp\left[-\left(\frac{J_\omega(x)/2 - \gamma_{min}}{\gamma_o}\right)^{\alpha}\right]\frac{dx}{r}\right\} d\mu_X^{(D)}(\omega) \tag{9}$$

where $d\mu_X^{(D)}(\omega)$ is given by (3).

The functional integral in (9) can be evaluated by various techniques[7], some of which we use below.

IV. Crack Diffusion Equation

In this section we reduce integration in (9) to solving a conventional diffusion type equation.[9] This is achieved by representing $P(X)$ as

$$P(X) = \int_{-\infty}^{\infty} P(X,Y)\,dY \quad , \tag{10}$$

where the new function $P(X,Y)$ is introduced by

$$P(X,Y) = \int_{\Omega_{X,Y}} \exp\left\{-\int_0^X P\{2\gamma \geq J_\omega(x)\} \frac{dx}{r}\right\} d\mu_{X,Y}^{(D)}(\omega) . \tag{11}$$

Here $P\{2\gamma > J(x)\}$ is given by (8), $\Omega_{X,Y}$ is the set of all possible crack trajectories beginning at the origin and crossing a point (X,Y), and $d\mu^{(D)}_{X,Y}(\omega)$ is the corresponding conditional Wiener measure. The meaning of $P(X,Y)dY$ is the probability that a crack is formed beginning at the notch and passing through the 'window' $Y \leq y \leq Y + dY$ at the depth X.

The probability $P\{2\gamma \geq J_\omega(x)\}$ given by (8) is a functional of ω, since the energy release rate $J_\omega(x)$ is a functional of the crack trajectory. A special feature of this functional is that it is insensitive to small perturbations away from the crack tip, i.e., functional derivatives such as $\delta J_\omega(x)/\delta\omega$ can be approximated as being proportional to δ-functions concentriated at the crack tip. It follows by means of a functional Taylor expansion near a conditional average

$$\omega_o(x) = \int_{\Omega_{X,Y}} \omega(x)\, d\mu^{(D)}_{X,Y}(\omega)$$

that $P\{2\gamma \geq J_\omega(x)\}$ can be substituted by a function V of current crack tip coordinates x and $\omega(x)$ thus allowing to rewrite (11) as

$$P(X,Y) = \int_{\Omega_{X,Y}} \exp\left\{-\int_0^X V(x,\omega(x))dx\right\} d\mu^{(D)}_{X,Y}(\omega) \quad . \tag{12}$$

Exploiting results of M. Kac[9] concerning integrals of the form (12), we conclude that P(X,Y) is the solution of

$$\frac{\partial P(X,Y)}{\partial X} = \frac{D}{2}\frac{\partial^2 P(X,Y)}{\partial Y^2} - V(X,Y)\,P(X,Y) \tag{13}$$

which satisfies

$$\begin{cases} P(X,Y) \rightarrow 0, & \text{as} \quad Y \rightarrow \pm\infty \quad (X > 0) \\ P(0,Y) = \delta(Y) \end{cases} \tag{14}$$

We refer to (13) as 'crack diffusion equation'.

We remind that $P(X,Y)$ gives rise to the function $P(X)$ via (10), and consequently determines the probability density $p(X)$ of a crack penetrating e.. .ly to a particular depth X

$$p(X) = -\frac{dP(X)}{dX} \tag{15}$$

(by its definition, $1-P(X)$ is the distribution function for the X-coordinate of the crack tip). Apparently, $P(X)$ contains less information than $P(X,Y)$. However, it allows one to reconstruct the γ-field parameters on the basis of experimental data, as discussed in section VI.

V. Illustrative Example

In this section, we consider the effect of the γ-field parameters and of the crack diffusion coefficient on the crack depth distribution $p(X)$ for a particular example of specimen-loading configuration.

Let us consider a random crack formed in a semi-infinite plane, under the action of a pair of forces, as shown in Fig. 2.

In Section III the evaluation of the probability of crack penetration to a depth greater than X was reduced to the evaluation of the energy release rate $J_\omega(x)$ as a functional of ω (see (9)). The energy release rate dependency on a crack trajectory is quite complex; no general solution is available. There are 'long wave' and 'short wave' ω-contributions to $J_\omega(x)$. The 'long wave' effect is accounted for by perturbation methods or other means.[10,11,12,13] The 'short wave' ω-contribution can be approximated by existing kinked crack solutions,[11,14] since the functional J_ω is highly sensitive to the direction of the crack at the very tip and is practically not affected by small perturbations away from the tip. It follows from an analysis of the existing solutions that the 'short wave' effect dominates. Considering the nature of the kinks in the 'crack diffusion model', we take the following approximation for the energy release rate associated with the crack formed along ω to a depth x:

$$J_\omega(x) = J_o(x) - k\, J_o(x)\, (\omega/r)^2 \tag{16}$$

where $J_o(x)$ stands for the conventional energy release rate for a rectilinear crack of the length x formed along the X-axis

$$J_o(x) = \frac{2.2\,\mathcal{F}^2}{E \cdot x} \quad ; \tag{17}$$

the factor $k = 0.5$ is evaluated on the basis of a numerical solution for a kinked crack.[13]

Before proceeding to the evaluation of the probability $P(X)$ (see (9)), it is convenient to introduce the mean value γ^* of the γ-field instead of γ_o in the Weibull distribution (7), obtaining

$$F(\gamma) = \begin{cases} 1 - \exp\left[- \left(c(\alpha) \dfrac{\gamma - \gamma_{min}}{\gamma^* - \gamma_{min}} \right)^\alpha \right] & (\gamma \geq \gamma_{min}) \\ 0 & (\gamma < \gamma_{min}) \end{cases} \quad . \tag{18}$$

We used (a straightforward calculation)

$$\gamma^* = \gamma_{min} + c(\alpha)\gamma_o\ , \qquad c(\alpha) = \int_0^\infty \exp\left(- z^\alpha \right) dz \tag{19}$$

The corresponding changes in (8) and (9) are immediate.

Zeroth Approximation (indicated below by the subscript 0). Substituting $J_\omega(x)$ in (9) by $J_0(x)$ (see (16,17)), and performing the functional integration (which is trivial since the integrand does not depend on ω), we obtain

$$P_0(X) = \exp\left\{-\int_0^X \exp\left[-\left(c(\alpha)\frac{J_0(x)/2 - \gamma_{min}}{\gamma^* - \gamma_{min}}\right)^\alpha\right]\frac{dx}{r}\right\} \tag{20}$$

It is convenient to rewrite (20) in a dimensionless form.

Let us introduce a 'Griffith length' ℓ_G determined from $J_0(\ell_G) = 2\gamma^*$ and representing the length of an equilibrial rectilinear crack along the X-axis in an ideally homogeneous material with the fracture energy γ^*. This yields dimensionless crack penetration depth and correlation distance

$$z = x/\ell_G \quad , \qquad \rho = r/\ell_G \quad . \tag{21}$$

We also have from (17)

$$\frac{J_0(x)}{2\gamma^*} = \frac{\ell_G}{x} = \frac{1}{z} \quad . \tag{22}$$

Passing to the dimensionless quantities in (20), we get

$$P_0(Z) = \exp\left\{-\int_0^Z \exp\left[-\left(c(\alpha)\frac{1/z - q}{1 - q}\right)^\alpha\right]\frac{dz}{\rho}\right\} \quad , \tag{23}$$

where

$$q = \gamma_{min}/\gamma^* \qquad (0 \le q \le 1) \quad . \tag{24}$$

The probability density $p_0(Z) = -dP_0(Z)/dZ$ of the dimensionless crack penetration depth Z (cf. (15)) is shown in Fig. 3 for various q and $\alpha = 3$, $\rho = 10^{-2}$. 15

Notice first that the mean crack depth $\langle X \rangle_o = \ell_G \langle Z \rangle_o$ does not coincide with the depth ℓ_G of the 'Griffith crack' corresponding to the mean value of the fracture energy (as should be expected).

Secondly, as q increases, the mean crack depth increases, the variance of the crack depth decreases (though non-monotonically in general), and, when q approaches 1, $p_o(Z)$ tends to $\delta(Z-1)$. The latter is a consequence of the fact that $\gamma_{min}/\gamma^* = 1$ corresponds to the distribution $F(\gamma)$ being a step-function, i.e., the field γ being a homogeneous non-random field with the value γ^*.

Our final remark on the zeroth approximation concerns a straightforward way to arrive at (20) without functional integrals. Namely, let us consider a rectilinear crack propagating along the X-axis through the same random field γ, as above. Then adopting the Poisson process type assumption (see Appendix, specially, (A7)) one arrives, in a standard way, at the differential equation

$$\begin{cases} \dfrac{dP_o(X)}{dX} = \dfrac{1}{r} P\{2\gamma \geq J_o(x)\} \cdot P_o(X) \\ P_o(X)\Big|_{X=0} = 1 \end{cases} \tag{25}$$

whose solution is exactly (20) (in view of (8), (17)).

Next Approximation. Now we consider $J_\omega(x)$ given by (16). As in the zeroth approximation, it is convenient here to pass to the dimensionless quantities $z = x/\ell_G$, $Z = X/\ell_G$, $\rho = r/\ G$, $q = \gamma_{min}/\gamma^*$, and, in addition, $\varphi(z) = \omega(z)/r$. Substituting (16) into (9), using (22), and taking into consideration a simple relationship for functional integrals

$$\int_{\Omega_X} (\ldots)\ d\mu_X^{(D)}(\omega) = \int_{\Omega_Z} (\ldots)\ d\mu_Z^{(\Delta)}(\varphi) \quad , \quad \Delta = \frac{\ell_G \cdot D}{r^2} \quad , \tag{26}$$

we get the following expression for the probability of crack penetration to a depth greater than Z:

$$P(Z) = \int_{\Omega_Z} \exp\left\{ - \int_0^Z \exp\left[- \left(c(\alpha)\ \frac{(1-k\varphi^2)/z-q}{1-q} \right)^{\alpha} \right] \frac{dz}{\rho} \right\} d\mu_Z^{(\Delta)}(\varphi) \tag{27}$$

Maintaining in the inner integrand the terms of the degree up to two (in φ), we obtain

$$P(Z) = P_o(Z) \cdot \int_{\Omega_Z} \exp\left\{ - \int_0^Z u(z)\varphi^2(z)dz \right\} d\mu_Z^{(\Delta)}(\varphi) \quad , \tag{28}$$

where $P_o(Z)$ is the probability of crack pentration to a depth grater than Z evaluated in the zeroth approximation (23);

$$u(z) = \frac{\alpha k}{\rho} \left(\frac{c(\alpha)}{1-q} \right)^{\alpha} \frac{(1/z-q)^{\alpha-1}}{z} \exp\left[- \left(c(\alpha)\ \frac{1/z-q}{1-q} \right)^{\alpha} \right] \tag{29}$$

Evaluation of the functional integral in (33) can be reduced to solving an ordinary differential equation,[16] namely,

$$\int_{\Omega_Z} \exp\left\{ - \int_0^Z u(z)\varphi^2(z)dz \right\} d\mu_Z^{(\Delta)}(\varphi) = [Q(z)]^{-1/2} \Big|_{z=0} \quad , \tag{30}$$

where the function $Q(z)$ satsfies

$$\begin{cases} Q''(z) - \Delta \cdot u(z) \cdot Q(z) = 0 \\ Q(z)\Big|_{z=Z} = 1 \quad , \qquad Q'(z)\Big|_{z=Z} = 0 \quad . \end{cases} \tag{31}$$

The probability density $p(Z) = -dP(Z)/dZ$ of the dimensionless crack penetration depth Z (cf. (15)) is shown in Fig. 4 for various ., $q = 0$ and α, ρ as above.

Notice that, as D increases, the mean crack depth decreases. This can be understood in the following way. In terms of energy, one finds lower average energy release rate for trajectories associated with higher D, since those have steeper kinks. In addition, the statistical weight of such trajectories increases with D.

Due to this effect, evaluation of γ^* on the basis of measured average crack depths $(\gamma^* = 1.1 \mathcal{F}^2/(E \cdot \langle X \rangle))$ ignoring 'crack diffusion' would result in overestimation of the strength. More generally, reconstruction of the γ-field parameters requires crack diffusion analysis. The latter can be based on a straightforward statistical analysis of fracture surfaces in an ensemble of identical specimens[5] or on a more sophisticated analysis of a single fracture surface from a sufficiently large specimen.

VI. Conclusions

In brittle solids, specific fracture energy evaluated on the basis of fracture experiments displays large scatter and specimen geometry dependence. The probabilistic model outlined above treats specific fracture energy γ as a random field and suggests certain experimental procedure for evaluation of the γ-field parameters.

Two types of parameters enter the expression for the probability distribution of crack penetration depth: Weibull distribution parameters (pointwise characteristics of the γ-field) and the correlation distance together with the 'crack diffusion coefficient', both reflecting non-local properties related to material structure such as grain size.[6] The latter ones can be extracted from fractographic analysis of a sample set of crack trajectories. Then a comparison of the mathematical expectation, variance and other moments of the crack penetration depth, as rendered by the model, with those found experimentally yields the values of the Weibull distribution parameters.

Thus determined γ-field parameters can be used (a) to rank brittle materials with regard to toughness; (b) to predict structural reliability.

The dimensionless 'crack diffusion coefficient' $\Delta = \ell_G \cdot D/r^2$ introduced above gives a new similarity criterion for single path fracture (to be used in addition to similitude in geometry and stresses). It involves a mixture of macroscopical and microscopical parameters, thus constraining the choice of materials for small scale modelling of brittle fracture.

APPENDIX - CALCULATION OF $P\{X|\omega\}$

We subdivide the interval $0 \le x \le X$ into m equal portions Δx_k whose length Δx is small enough so that $J_\omega(x)$ can be considered constant over each Δx_k. At the same time, we assume that Δx is much greater than the correlation distance of the γ-field, $\Delta x \gg r$. Let us further split each Δx_k into $n = \Delta x/r$ subintervals δx_j $((k-1)n < j \le kn)$ of the length r (see Fig. 5) and consider γ to be constant over each δx_j. Thus the interval $0 \le x \le X$ is subdivided into $m \cdot n$ subintervals δx_j.

Crack formation between the notch and a depth X means failure over all δx_j's. According to the energy criterion of local failure, the failure over δx_j occurs, if the released potential energy $\Delta\Pi_j$ is greater than the energy required for new surface formation $\Delta\Pi_j > 2\gamma_j \cdot \delta x_j$.[18] Thus

$$P\{X|\omega\} = P\left\{ \bigcap_{j=1}^{mn} \left(\Delta\Pi_j > 2\gamma_j \cdot \delta x_j \right) \right\} \tag{A1}$$

To evaluate the probability, we consider crack formation as an ordered sequence of local failures. Therefore it is convenient to represent $P\{X|\omega\}$ as the following product of conditional probabilities:

$$P\{X|\omega\} = \prod_{j=1}^{mn} P\left\{ \left(\Delta\Pi_j > 2\gamma_j \cdot \delta x_j \right) \middle| \bigcap_{i=1}^{j-1} \left(\Delta\Pi_i > 2\gamma_i \cdot \delta x_i \right) \right\} . \tag{A2}$$

The potential energy released at the j-th step under the condition that all preceeding elements failed can be evaluated through the conventional, in fracture mechanics, energy release rate $J_\omega(x)$

$$\Delta\Pi_j = J_\omega(x_j) \cdot \delta x_j \quad . \tag{A3}$$

Thus (A2) becomes

$$P\{X|\omega\} = \prod_{j=1}^{mn} P\{J_\omega(x_j) > 2\gamma_j\}$$

$$= \prod_{k=1}^{m} \prod_{j=(k-1)n+1}^{kn} P\{J_\omega(x_j) > 2\gamma_j\} \qquad (A4)$$

where the inner product represents the (conditional) probability of failure over Δx_k. According to the choice of Δx, $J_\omega(x)$ is (approximately) constant over each Δx_k, i.e., for a given k, all the values $J_\omega(x_j)$ in the inner product in (A4) can be substituted by the one and same value $J_\omega(x_{kn})$. Also γ_j's are independent random variables, which have the same (Weibull) distribution (7). We conclude that the probabilities in (A4) are all equal, for a given k. Passing to the opposite event in each of the $n = \Delta x/r$ (equal) factors in (A4), assuming that its probability is small, and using

$$(1 - \varepsilon)^n \approx \exp(-n\varepsilon) \quad (\text{small } \varepsilon) \quad , \qquad (A5)$$

we rewrite (A4), as follows:

$$P\{X|\omega\} = \prod_{k=1}^{m} \left(1 - P\{2\gamma \geq J_\omega(x_{kn})\}\right)^{\Delta x/r}$$

$$\approx \exp\left[-\sum_{k=1}^{m} P\{2\gamma \geq J_\omega(x_{kn})\frac{\Delta x}{r}\}\right] . \qquad (A6)$$

Substituting the above summation by integration, one arrives at (6).

It is worth noting that one can arrive at (6) by a standard Poisson process type argument, if one postulates that the probability of the crack arrest over a segment $x \leq x' \leq x + \Delta x$ is proportional to Δx

$$P\left\{ 2\gamma(x',\omega(x')) \geq J_\omega(x') \quad \text{for some} \quad x \leq x' \leq x + \Delta x \right\}$$

$$= P\{2\gamma > J_\omega(x)\} \; \frac{\Delta x}{r} \quad . \tag{A7}$$

[1]A. Chudnovsky, J. Appl. Mech., in print.

[2]A. Chudnovsky, in Studies on Elasticity and Plasticity, No. 9 (L. M. Kachanov, ed.), Leningrad, 3-41 (1973) (in Russian).

[3]A. Chudnovsky, in Proc. Workshop Continuum Mech. Approach to Damage and Life Prediction (D. C. Stouffer, ed.), May 1980, Carrolton, Kentucky, 79-87.

[4]A. Chudnovsky, N. Loginova, Yu Shraiber, Bull. Am Phys. Soc., Vol. 25, No. (1984).

[5]M. A. Mull, A Chudnovsky, A. Moet, Philos. Mag., to appear.

[6]A. Chudnovsky, Ph. C. Perdikaris, Fourth Int. Conf. on Appl. of Stat. and Prob. in Soil and Structural Engineering, Univ. of Florence, Italy (1983).

[7]I. M. Gelfand, A. M. Jaglom, Russian Math. Surveys, XI, No. 1 (67) 48-79 (1956).

[8]M. R. Leadbetter, G. Lindgren, H. Rootzen, Extremes and Related Properties of Random Sequences and Processes (Springer-Verlag, New York, 1983).

[9]M. Kac, Proc. 2nd Berkeley Sympos. Math Stat. and Probab., Berkeley, 189-215 (1951).

[10]N. V. Banichuk, Izv. AN SSSR, MTT 7, No. 2, 130-137 (1970) (in Russian).

[11]B. Cotterell, J. R. Rice, MRL E-110 Report, Brown Univ., Providence, R. I. (October 1978).

[12]V. Khandogin, A. Chudnovsky, in 'Dynamics and Strength of Aircraft Structures' (L. M. Kurshin, ed.), Novosibirsk, 1978 (in Russian).

[13]R. V. Goldstein, R. L. Salganik, Izv. AN SSSR, MTT 7, No. 3, 69-82 (1970) (in Russian).

[14]C. H. Wu, J. Elasticity, Vol. 8, No. 3, 235-257 (1978).

[15]The choice of $\alpha = 3$ in the illustrative example is motivated by previous experimental work (see footnote 5).

[16] R. H. Cameron, W. T. Martin, Bull. Amer. Math. Soc. 51, No. 2, 73-90 (1945), see also footnote 8, p. 81.

[17] The selection of the values of Δ has purely illustrative purpose. The values 0, 10^2 and 10^3 correspond to the standard deviation of the crack trajectories at the depth ℓ_G being equal to 0, $0.1 \cdot \ell_G$ and $0.3 \cdot \ell_G$, respectively. (The variance of the ordinate of a crack trajectory at a depth X equals $D \cdot X = \Delta \cdot r^2 \cdot X/\ell_G$.)

[18] Here as well, as in (A3), it would be more appropriate to use, instead of δx_j, a length of the corresponding $\delta \omega_j$. In the 'diffusion approximation,' the conventional length of a crack trajectory is meaningless. However, there exists a Hausdorf measure (closely related to the dimension 3/2 of the Brownian paths) such that the length of $\delta\omega$ is almost surely finite and proportional to the corresponding δx (see footnote 19).

[19] B. Fristedt, in 'Advances in Probability and Related Topics' (P. Ney, S. Port, eds) Marcel Dekker, Inc., New York (1974), Vol. 3.

Figure 1. A ensemble of macroscopically identical specimens cracked under identical fatigue load and the superimposition of the profilograms of the crack trajectories (see footnote 3).

Figure 2. The specimen-load configuration and a sample crack.

Figure 3. The probability density of the crack penetration depth for various $q = \gamma_{min}/\gamma^*$ in the zeroth approximation. Parameters α and $\rho = r/\ell_G$ are 3 and 10^{-2}, respectively.

Figure 4. The probability density of the crack penetration depth for various Δ. Parameters α, $\rho = r/\ell_G$, and $q = \gamma_{min}/\gamma^*$ are 3, 10^{-2} and 0, respectively (see footnote 17).

Figure 5. Discretization of a crack trajectory for the purpose of the probability calculation.

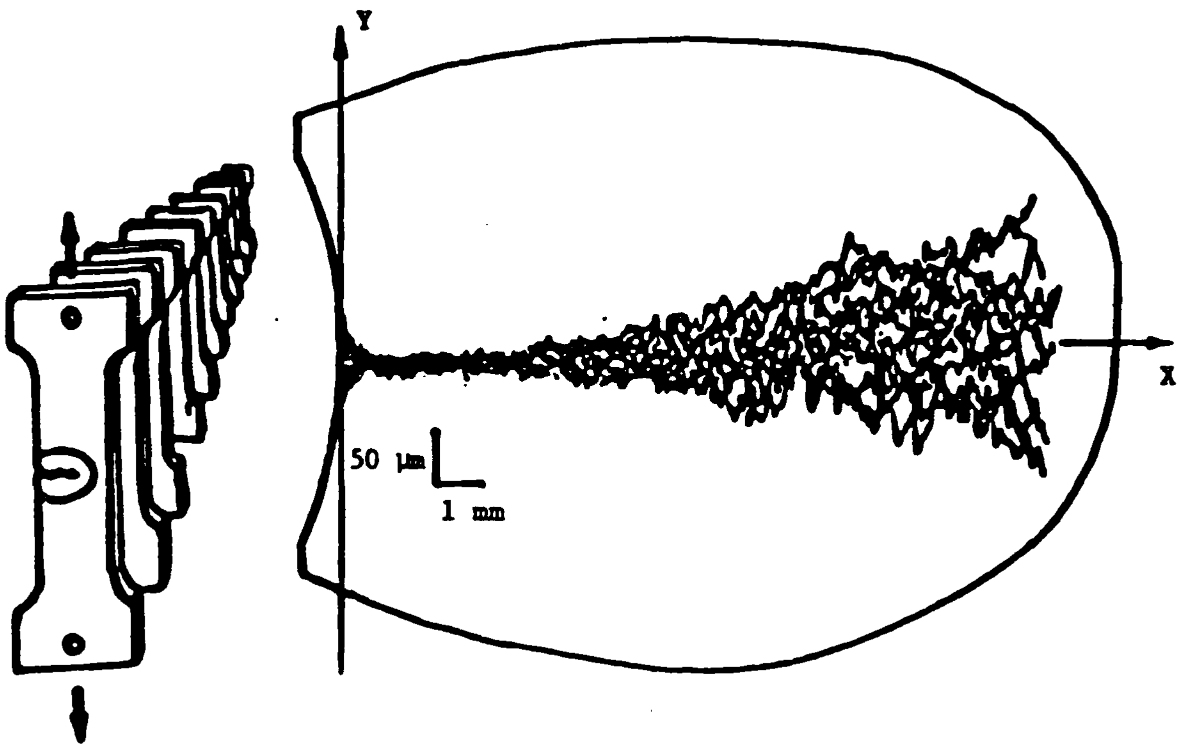
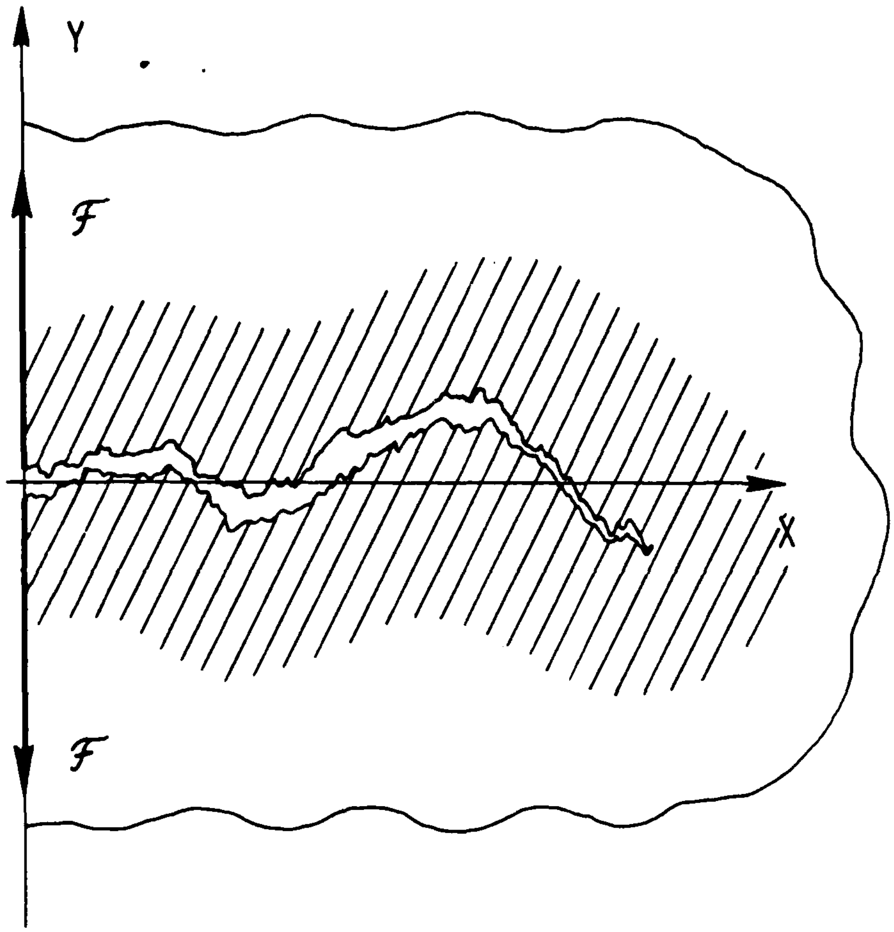
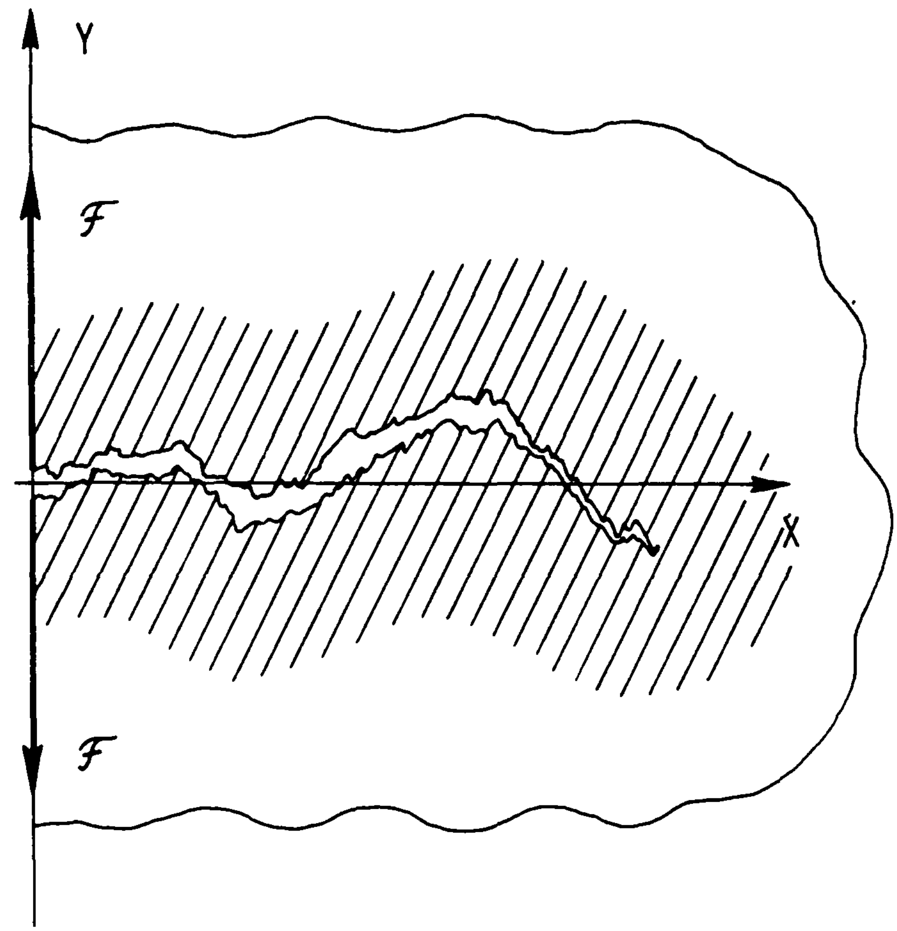


FIG. 1 Superimposition of the profilograms of crack trajectories from identical Ti-alloy samples[3].



Please notice that Figures 2 thru 5 have been duplicated.
We did not remove the duplicate to avoid confusion that
discontinuity of pagination would have caused.

Fig 2


Y
𝓕
X
𝓕

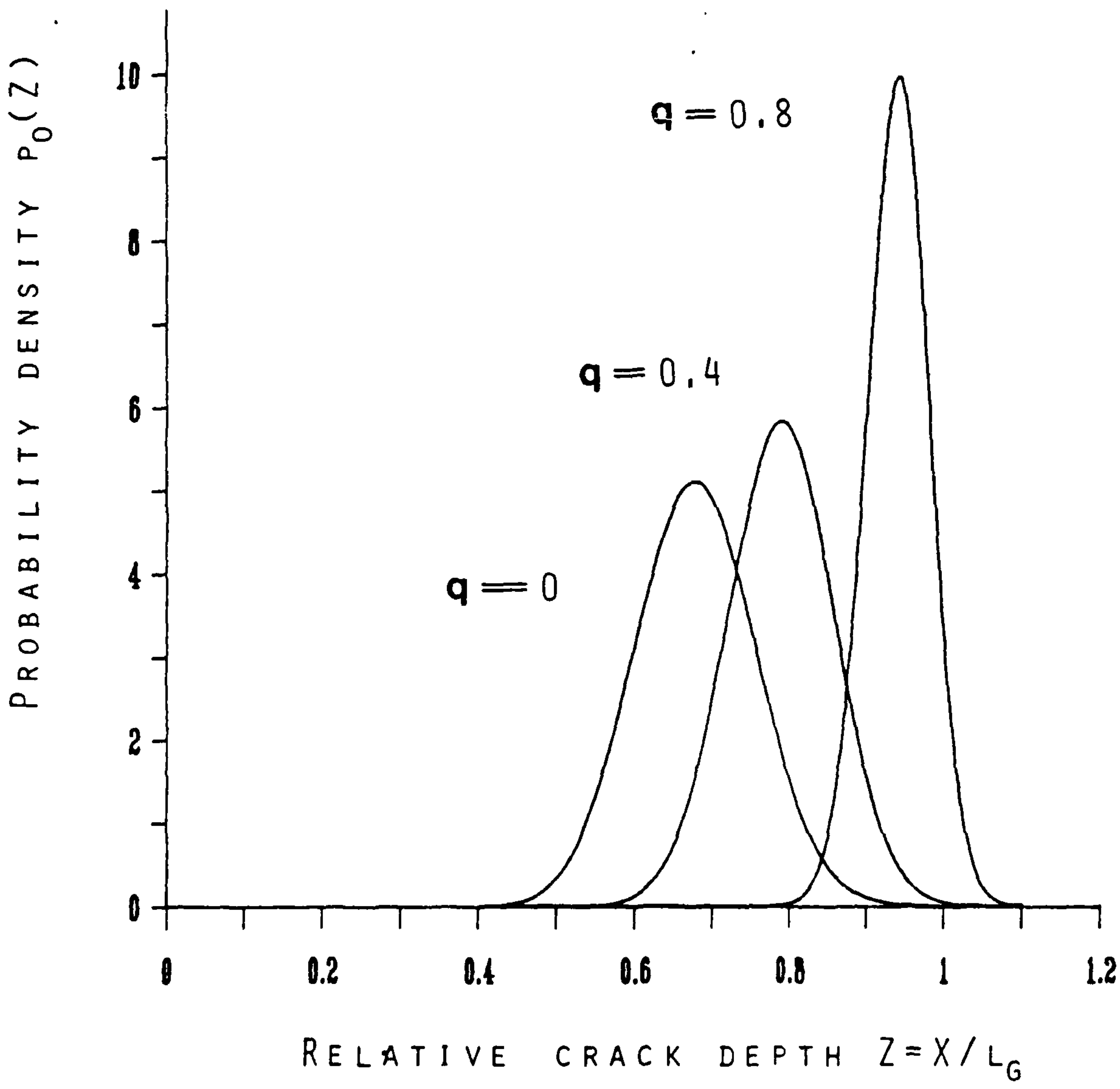
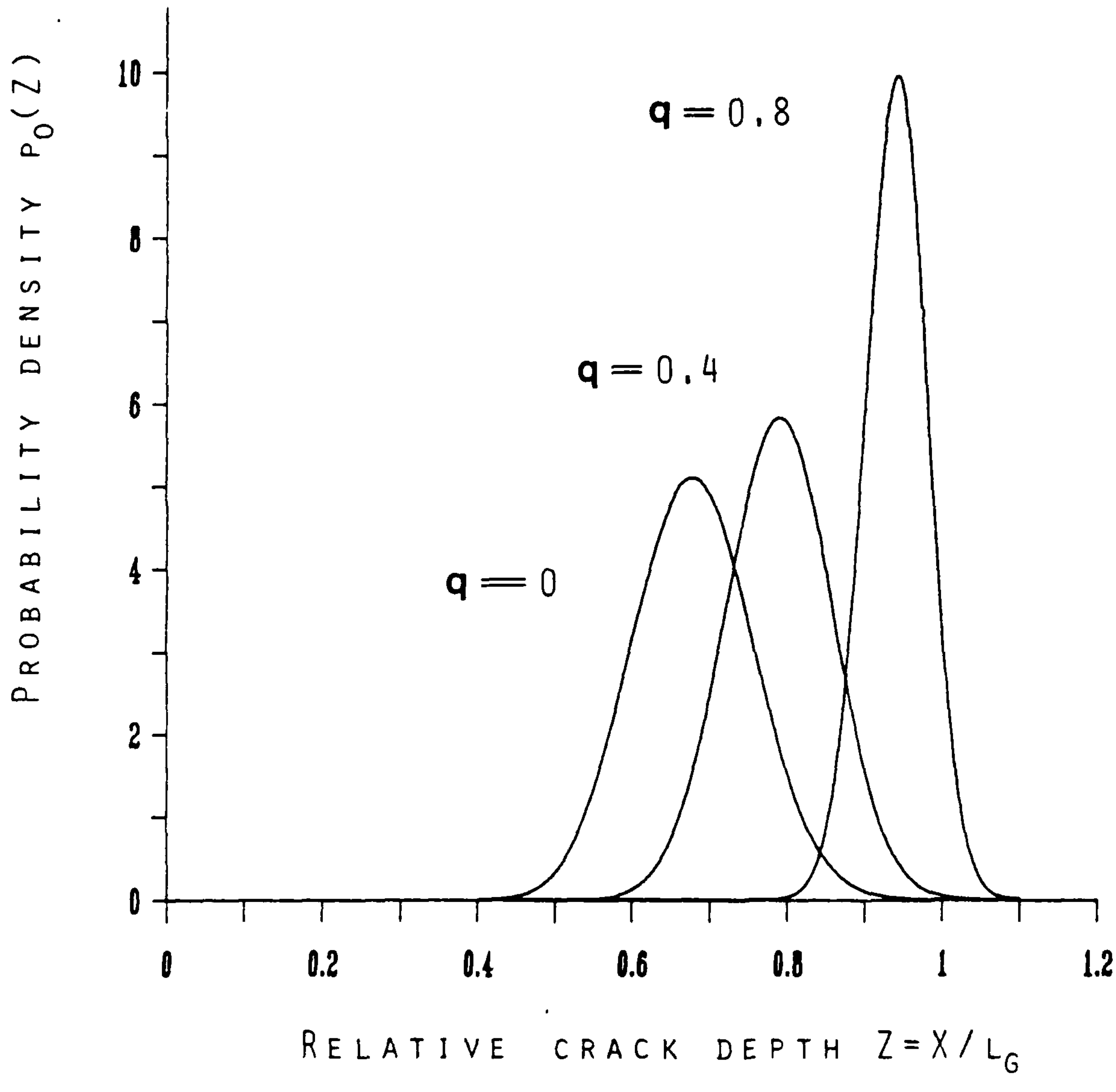
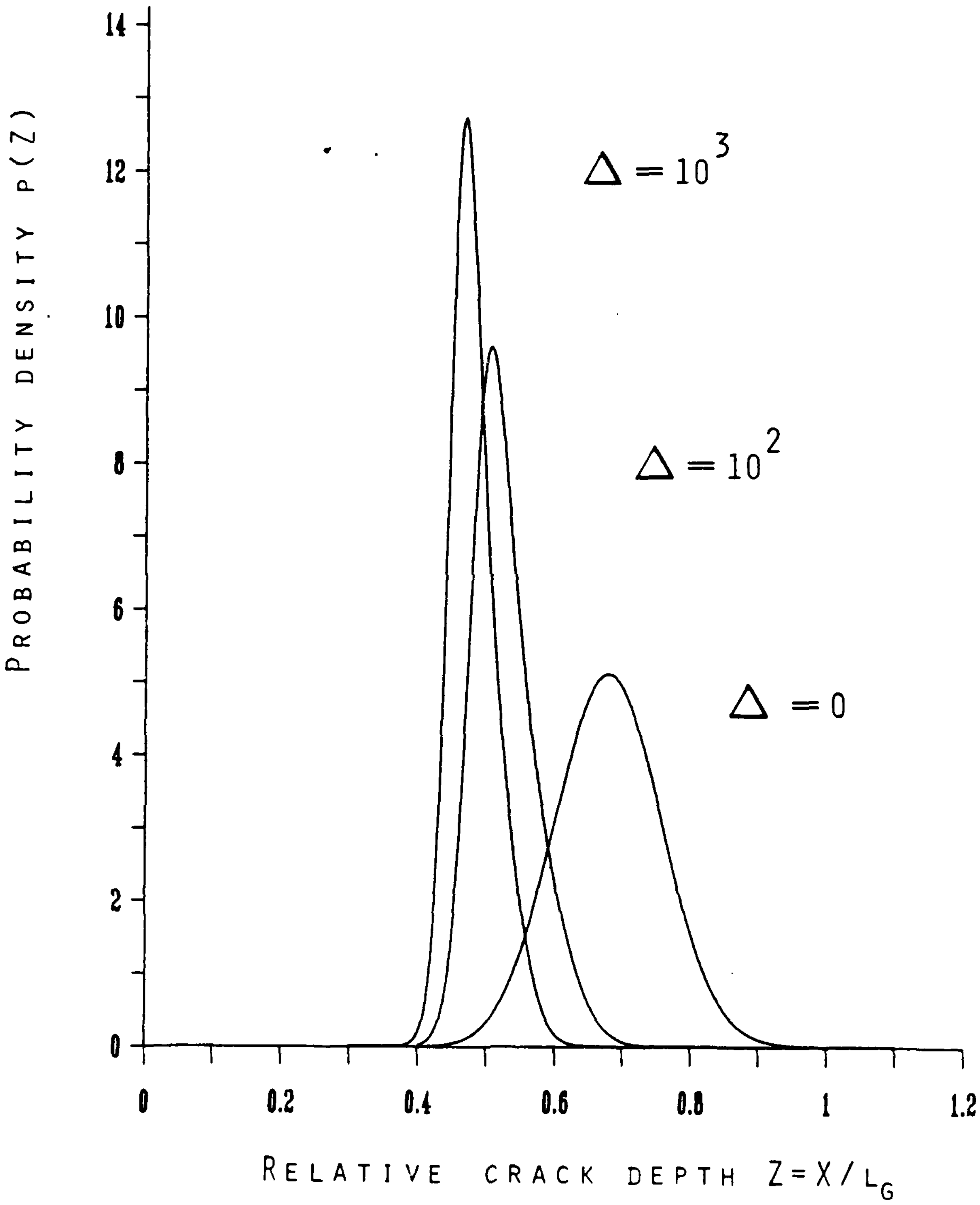

PROBABILITY DENSITY $P_0(Z)$
10
8
6
4
2
0
q = 0.8
q = 0.4
q = 0
0
0.2
0.4
0.6
0.8
1
1.2
RELATIVE CRACK DEPTH $Z=X/L_G$

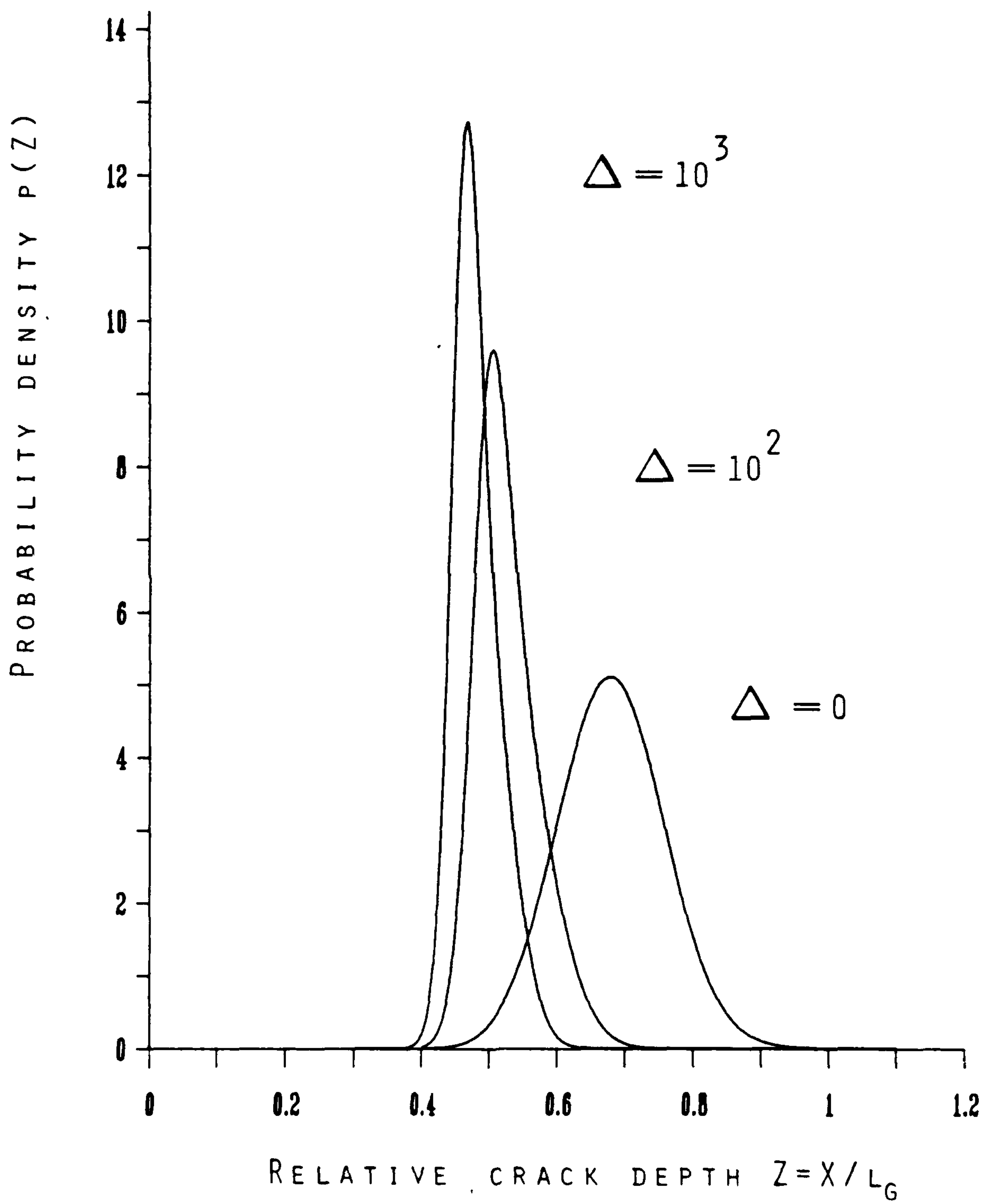
Fig 3

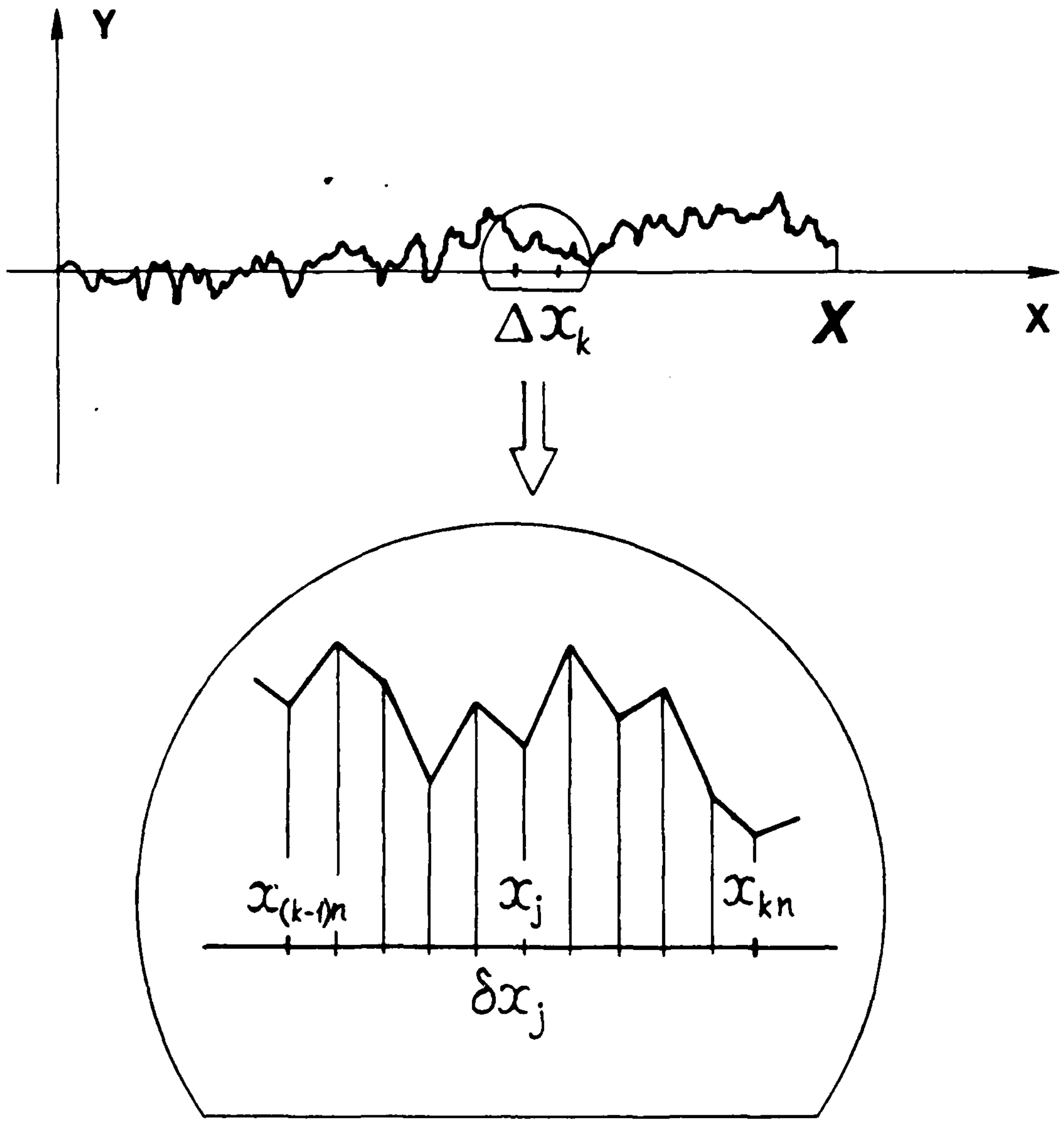

PROBABILITY DENSITY $P_0(Z)$
10
8
6
4
2
0
q = 0.8
q = 0.4
q = 0
0
0.2
0.4
0.6
0.8
1
1.2
RELATIVE CRACK DEPTH $Z = X/L_G$

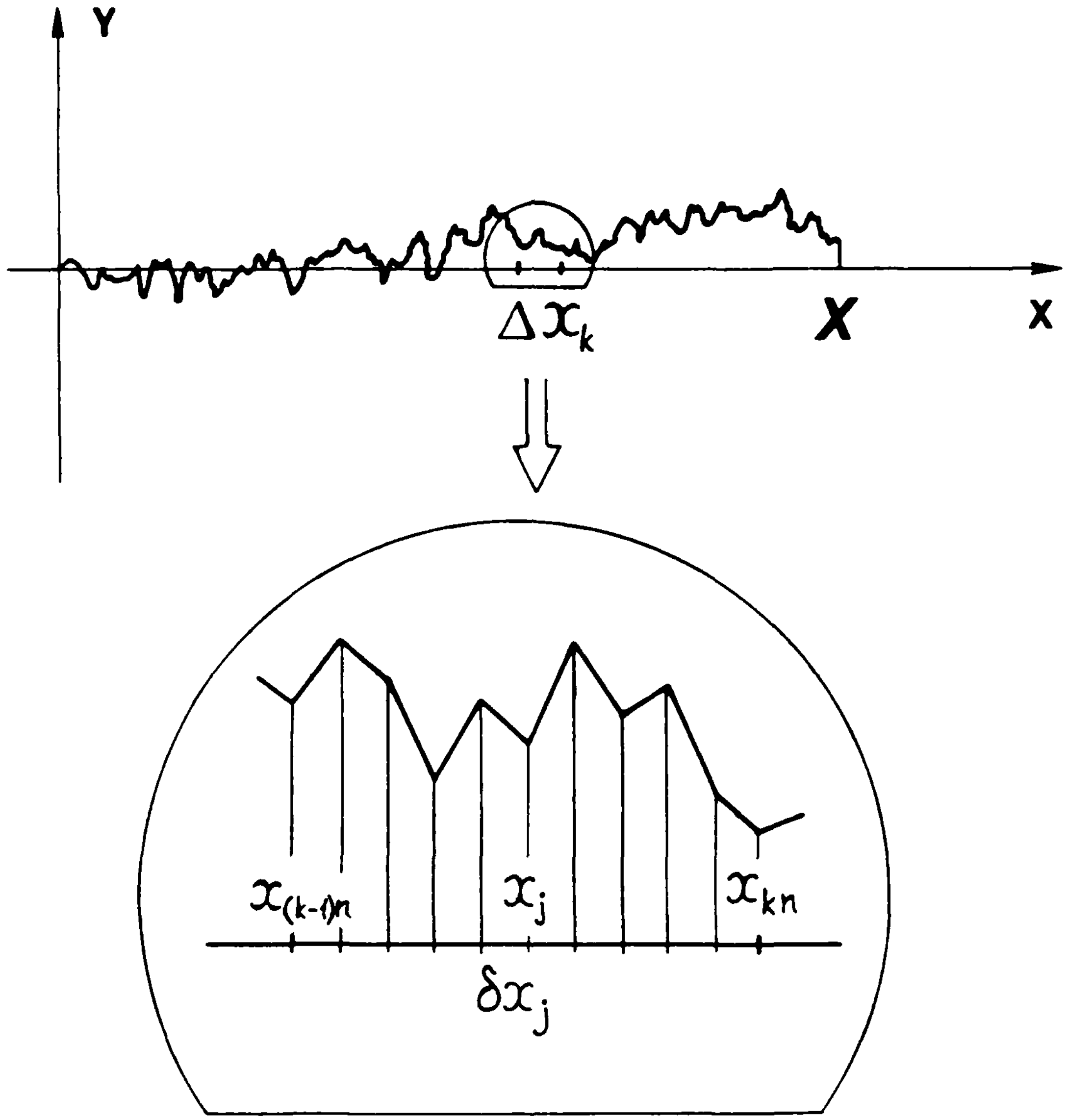


14
12
10
8
6
4
2
0
PROBABILITY DENSITY p(Z)
Δ = 10^3
Δ = 10^2
Δ = 0
0
0.2
0.4
0.6
0.8
1
1.2
RELATIVE CRACK DEPTH Z=X/L_G

Fig. 4


PROBABILITY DENSITY P(Z)
14
12
10
8
6
4
2
0
$\Delta = 10^3$
$\Delta = 10^2$
$\Delta = 0$
0
0.2
0.4
0.6
0.8
1
1.2
RELATIVE CRACK DEPTH $Z = X/L_G$


Y
Δx_k
X
x
$x_{(k-1)n}$
x_j
x_{kn}
δx_j


Y
Δx_k
X
x
$x_{(k-1)n}$
x_j
x_{kn}
δx_j

END

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