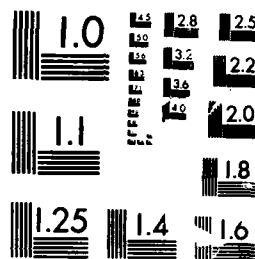


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BALLOONING MODES OR FOURIER MODES IN A TOROIDAL PLASMA?

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Abstract

The relationship between two different descriptions of eigenmodes in a torus is investigated. In one the eigenmodes are similar to Fourier modes in a cylinder and are highly localised near a particular rational surface. In the other they are the ~~so-called~~ ballooning modes which extend over many rational surfaces. Using a model which represents both drift waves and resistive interchanges we investigate the transition from one of these structures to the other. In this simplified model the transition depends on a single parameter which embodies the competition between toroidal coupling of Fourier modes (which enhances ballooning) and variation in frequency of Fourier modes from one rational surface to another (which diminishes ballooning). As the coupling is increased each Fourier mode acquires a sideband on an adjacent rational surface and these sidebands then expand across the radius to form the extended mode described by the conventional ballooning mode approximation. This analysis shows that the ballooning approximation is appropriate for drift waves in a tokamak but not for resistive interchanges in a pinch. In the latter the conventional ballooning effect is negligible but they may nevertheless show a ballooning feature. This is localised near the same rational surface as the primary Fourier mode and so does not lead to a radially extended structure.

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1. INTRODUCTION

Two essentially distinct descriptions have been used for short wave length perturbations, such as drift waves and resistive instabilities, in an axisymmetric toroidal plasma. The first regards toroidal curvature as a small effect and expresses the plasma perturbation in terms of coupled cylindrical Fourier modes with toroidal and poloidal mode number n and m . In the absence of toroidal effects each Fourier mode is centered on its appropriate rational surface where $m = nq(r)$ and its properties are determined by the magnetic shear, density gradient etc. at the rational surface; derivatives of the shear and density gradient are neglected. In this approximation modes with different m are almost degenerate. In the case of drift waves an "outgoing wave" boundary condition applied to each Fourier mode leads to the so-called 'shear damping' effect.¹

On the other hand, in the second description it is precisely the degeneracy of the Fourier modes which is exploited. This leads to the ballooning representation² in which disturbances centered on neighboring rational surfaces are strongly coupled to produce a radially extended structure - similar to the quasi-mode of Roberts and Taylor³ - which "balloons" on one side of the torus. The coupling reduces the shear damping⁴ of drift waves - and it may disappear altogether if the coupling is sufficiently strong.

In this paper we discuss the relationship between these two opposing views of mode structure in toroidal systems. In Section 2 we introduce a simple model for drift waves in which Fourier modes are coupled but not degenerate. In Section 3 we show that a similar model can describe

resistive interchange instabilities. In Sections 4 and 5 we use these models to explore the transition from a Fourier mode description to a ballooning mode description as the toroidal curvature is increased.

This investigation shows that as the transition occurs, each Fourier mode first acquires a ballooning component centered on neighboring rational surfaces and this structure then expands across the radius to form the conventional ballooning mode. However, for resistive interchanges the overlap between Fourier modes on adjacent rational surfaces is exponentially small and this extended ballooning structure does not arise. Instead an alternative ballooning effect, discussed in Section 6, can occur. In this the ballooning contribution is centered on the same rational surface as the primary Fourier mode. Consequently although the mode "balloons" it does not have the quasi-mode character. In Section 7 we discuss the relationship of these results to the conventional ballooning mode approximation.

2. DRIFT WAVES

A simple model for drift waves in a large aspect ratio tokamak has been described in Refs. 5, 6 and 7. The perturbed potential is written in the form:

$$\phi = \phi(\theta, \rho) \exp[i(n\zeta - m_0\theta) - i\omega t] \quad (1)$$

where ρ is the radial distance from the rational surface $m_0 = nq(r)$ and θ is the poloidal angle. Then $\phi(\theta, \rho)$ satisfies

$$(L_0 + L_1 + L_2)\phi = 0 \quad (2)$$

where

$$L_0 = a_i^2 \frac{\partial^2}{\partial \rho^2} - b - \frac{(\omega - \omega_* - i\omega\delta)}{(\omega\tau + \omega_*)} - \left(\frac{\omega_*}{\omega} \cdot \frac{\epsilon_c}{ka_i}\right)^2 \left(\frac{\partial}{\partial \theta} + iks\rho\right)^2, \quad (3)$$

$$L_1 = -2\epsilon_n \frac{\omega_*}{\omega} \left(\cos\theta + \frac{i\sin\theta}{k} \frac{\partial}{\partial \rho}\right), \quad (4)$$

$$L_2 = -\frac{\rho^2}{L_*^2} \frac{\omega_*}{(\omega\tau + \omega_*)}. \quad (5)$$

In Eqs. (3)-(5), $k = nq/r$, $s = (r/q)(dq/dr)$, $a_i^2 = c^2 m_i T_i / e^2 B^2$, $b = k^2 a_i^2$, $\epsilon_n = q\epsilon_c = r_n/R\tau$ where r_n is the density scale length and $\tau = T_e/T_i$. The diamagnetic frequency $\omega_* = (k\tau_e/eBr_n)$ is expanded about a local maximum with $L_*^{-2} = (1/\omega_*)(d^2\omega_*/dr^2)$ and the parameter $i\delta$ represents the destabilising effect of electron Landau resonance and trapped electrons. This model can be obtained from the general equation for drift waves in a tokamak, given by Tang,⁸ by taking the long wavelength limit $ka_i \ll 1$ and neglecting temperature gradients.

The operator L_0 describes drift waves in a cylinder (e.g. as discussed by Pearlstein & Berk¹) and depends only on the shear and density gradient at the rational surface itself; L_1 describes the toroidal effects and L_2 introduces the radial variation of ω_* .

To simplify the problem further we consider the regime $\epsilon_c s$,

$b, \delta < 1$, in which $\omega \approx \omega_*$. Then Eq. (2) becomes

$$\left[\frac{\partial^2}{\partial x^2} - \sigma^2 \left(\frac{\partial}{\partial \theta} + ix \right)^2 - \epsilon (\cos \theta + is \sin \theta \frac{\partial}{\partial x}) - \kappa x^2 - \lambda \right] \phi = 0 \quad (6)$$

where $x = kps$. The three parameters in the model are then

$$\sigma = \frac{\epsilon_c}{bs}, \quad \epsilon = \frac{2\epsilon_n}{bs^2}, \quad \kappa = \frac{1}{k^2 L_*^2 bs^4 (1 + \tau)} \quad (7)$$

and the eigenvalue λ is related to the mode frequency ω by

$$\lambda = \frac{1}{bs^2(1 + \tau)} \left(\frac{\omega}{\omega_*} - 1 + b(1 + \tau) - i\delta \right). \quad (8)$$

In the approximation $\omega \approx \omega_*$ the parameter σ^2 is real and there is no explicit dissipation in the model. However, in the usual way, we suppose that ω has a small positive imaginary part in order to justify neglect of initial value terms. Correspondingly σ^2 acquires a small negative imaginary part.

3. RESISTIVE INSTABILITIES

Another instability which can be represented by a model similar to Eq. (6) is the localised electrostatic resistive interchange^{3,9} in a large aspect ratio toroidal pinch. In this case the model can be derived from the high mode number resistive ballooning equation discussed by Bateman and Nelson.¹⁰ In the electrostatic limit $\gamma < \eta^2 |V_S|^2$ [where γ is the

growth rate, η the resistivity and the mode has an eikonal representation $\phi \sim \phi \exp(inS)$] this equation takes the form

$$\frac{\gamma}{\eta^2} \frac{B \cdot \nabla}{B^2} \frac{1}{B^2} B \cdot \nabla \phi - \rho_0 \gamma^2 \frac{|VS|^2}{B^2} \phi + 2p' \frac{B \times VS \cdot \tilde{K}}{B^2} \phi = 0 \quad (9)$$

where ρ_0 is the density and $\tilde{K}$ is the field line curvature.

In a large aspect ratio pinch, with magnetic field

$B = B_0(r)[1 - \Delta(r) \cos \theta]$, Eq. (9) can be further simplified.

Reverting to the form (1) for the perturbations and taking the long wavelength limit it becomes

$$\left[\frac{\partial^2}{\partial x^2} + \sigma_R^2 \left(\frac{\partial}{\partial \theta} + ix \right)^2 + \alpha (\rho_1 \cos \theta + i \rho_2 \sin \theta \frac{\partial}{\partial x}) - \kappa_R x^2 + \alpha \right] \phi = 0 \quad (10)$$

where the three principal parameters are

$$\sigma_R^2 = \frac{S}{\gamma \tau_A n^2 q^2 s^2}, \quad \alpha = \frac{r^2 \beta_\theta}{r_p R_c s^2 (\gamma \tau_A)^2}, \quad \kappa_R = \frac{\alpha^2}{(nqsL)^2} \quad (11)$$

Here τ_A is the Alfvén time, S is the magnetic Reynolds number and r_p is the pressure scale length. The other parameters ρ_1 , ρ_2 , R_c are related to the magnetic field strength by

$$\frac{1}{R_c} = - \frac{d(\ln B_0)}{dr}, \quad \rho_1 = R_c \frac{d\Delta}{dr} - \Delta, \quad \rho_2 = \frac{\Delta R_c}{r} \left(\frac{B_\theta}{B_0} \right)^2$$

and $L^{-2} = - (1/\alpha)(d^2\alpha/dr^2)$ represents the radial variation of the cylindrical curvature term α .

4. DEGENERACY AND ITS REMOVAL

We have noted in Sections 2 and 3 that drift waves and resistive instabilities in a large aspect ratio torus can both be described by the model equation

$$\left[\frac{\partial^2}{\partial x^2} - \sigma^2 \left(\frac{\partial}{\partial \theta} + ix \right)^2 - \epsilon \left(\cos \theta + is \sin \theta \frac{\partial}{\partial x} \right) - \kappa x^2 - \lambda \right] \phi = 0 \quad (12)$$

(where we set $\alpha p_1 = \alpha p_2 = \epsilon$ in the resistive case). If Fourier modes $\sim \exp(-im\theta)$ are introduced into Eq. (12) it can be seen that σ determines their radial width, κ describes the difference between modes on neighboring rational surfaces and ϵ describes the coupling between modes. Thus, despite the many simplifications, the model retains the features necessary for both Fourier mode and ballooning mode descriptions. The distinction between drift and resistive instabilities lies in the sign of σ^2 . For drift waves $\sigma^2 > 0$ and for resistive instabilities $\sigma^2 < 0$. In this section we concentrate on drift waves.

We treat both ϵ and κ as first order small quantities and seek a solution of Eq. (12) in the form

$$\phi = \sum_{m=-\infty}^{\infty} c_m u_m(x) \exp(-im\theta) \quad (13)$$

where

$$u_m(x) = u(x-m) = \exp[-i\sigma(x-m)^2/2] \quad (14)$$

satisfies the zero-order equation

$$\left[\frac{d^2}{dx^2} + \sigma^2(m-x)^2 - \lambda_0 \right] u_m = 0, \quad (15)$$

with $\lambda_0 = -i\sigma$.

Each $u_m(x)$ represents a separate wave propagating from its rational surface $x \approx m$. When σ has the small imaginary part referred to in Section 2 the amplitude of this wave decreases with distance from the rational surface. The eigenvalue λ_0 corresponds to a drift wave frequency

$$\frac{\omega}{\omega_*} = 1 - b(1 + \tau) + i \left[\delta - \frac{r_n s}{Rq} \left(\frac{1 + \tau}{\tau} \right) \right] \quad (16)$$

which exhibits the competition between the destabilising term δ and the 'shear damping' $\sim (r_n s/Rq)$ associated with the 'outgoing wave' feature⁽¹⁾ of the $u_m(x)$.

So far, the modes $u_m(x)$ are degenerate and the coefficients c_m

are completely arbitrary. The appropriate combination of c_m for the toroidal problem is determined when the additional terms of Eq. (12) are introduced and the degeneracy is removed. Then we have

$$\hat{L}_0 \phi_1 - (\epsilon \hat{L}_1 + \kappa \hat{L}_2) \phi_0 - \lambda_0 \phi_1 - \lambda_1 \phi_0 = 0 \quad (17)$$

where $\hat{L}_0$ is the differential operator which appears in Eq. (15) and $\epsilon \hat{L}_1$ and $\kappa \hat{L}_2$ are the remaining operators of Eq. (12). The quantity ϕ_1 can be annihilated by the operator

$$\int_{-\infty}^{\infty} dx \, \phi \, d\theta \, u_m(x) \exp(im\theta) \quad (18)$$

This leads to a recurrence relation for the c_m :

$$\begin{aligned} (\lambda_1 \langle u_m^2 \rangle + \kappa \langle x^2 u_m^2 \rangle) c_m + \frac{\epsilon}{2} (\langle u_m u_{m+1} \rangle + s \langle u_m D u_{m+1} \rangle) c_{m+1} \\ + \frac{\epsilon}{2} (\langle u_m u_{m-1} \rangle - s \langle u_m D u_{m-1} \rangle) c_{m-1} = 0 \end{aligned} \quad (19)$$

where

$$\langle \dots \rangle = \int_{-\infty}^{\infty} \dots dx \quad \text{and} \quad D \equiv \frac{d}{dx} \quad (20)$$

Using the explicit expression (14) for the $u_m(x)$, Eq. (19) can be reduced to a form well-known in the theory of Mathieu functions:

$$q(c_{m+1} + c_{m-1}) = (a - m^2) c_m \quad (21)$$

where

$$q = \frac{\epsilon}{2\kappa} \left(1 + \frac{i\sigma}{2}\right) \exp\left(\frac{-i\sigma}{4}\right) \quad \text{and} \quad a = \left(\frac{i}{2\sigma} - \frac{\lambda_1}{\kappa}\right) . \quad (22)$$

Note that the symbol q is used here to conform with standard notation for Mathieu functions: it should not be confused with the toroidal 'safety factor' in Section 1. The boundary conditions on Eq. (21) are that $c_m \rightarrow 0$ as $|m| \rightarrow \infty$ and the required solutions are related to the 'period- π ' Mathieu functions ce_{2n} and se_{2n} .

5. STRUCTURE OF TOROIDAL EIGENMODES

The preceding section shows that the structure of the toroidal eigenmodes is determined, through the recurrence relation (21), by the parameter q . This parameter incorporates the influence of toroidal curvature, the width of a basic Fourier mode and the variation of diamagnetic frequency with the location of its resonant surface.

When $|q|$ is small the fundamental eigenvalue of Eq. (21) is¹¹

$$a = -2q^2 + \frac{7}{2}q^4 + \dots \quad (23)$$

and the associated coefficients c_m decrease rapidly with m ,

$$c_0 = 1, \quad c_1 = -2q, \quad c_2 = q^2/2, \quad \dots \quad (24)$$

Consequently, when $|q|$ is small only c_0 is significant. Recalling that a factor $\sim \exp(-im_0\theta)$ was extracted from the perturbation (Eq. (1)) this shows that the toroidal eigenmode comprises a single, well-localised Fourier mode on the rational surface $nq(r) = m_0$ with weak and rapidly decreasing sidebands on adjacent rational surfaces. These weak sidebands provide the only ballooning effect. There is a small reduction in shear damping (compared to Eq. (16)) but this is due to the variation in ω_* across the plasma profile rather than to toroidal coupling.

On the other hand, when $|q|$ is large the toroidal eigenmodes take on an entirely different structure. In one of them c_m varies slowly with m and Eq. (21) may be replaced by a differential equation, treating m as a continuous variable⁽⁴⁾:

$$q \frac{d^2 c_m}{dm^2} + (m^2 - a + 2q) c_m = 0. \quad (25)$$

Then the fundamental eigenvalue is

$$a = 2q - iq^{1/2} \quad (26)$$

and the corresponding c_m are

$$c_m = \exp(-im^2/2q^{1/2}) \quad (27)$$

(where we take $0 < \text{Arg } q < 2\pi$). Thus the c_m are essentially constant

from $m = 0$ out to $m \sim \pm |q|^{1/4}$ and fall off exponentially beyond this point.

There is also another eigenmode when q is large. In this $c_m \sim \exp(im\pi)$ - which corresponds to replacing the poloidal angle θ by $\theta + \pi$ in a mode with slowly varying c_m . This signifies that this second mode is centered on the inside edge of the torus ($\theta = \pi$) rather than the outside ($\theta = 0$). Writing $c_m = e^{im\pi} c'_m$, where c'_m varies slowly, one finds for this eigenmode

$$a = -2q + q^{1/2} \quad (28)$$

and

$$c_m = \exp(im\pi - m^2/2q^{1/2}) \quad (29)$$

with $-\pi < \text{Arg } q < \pi$.

At large q , therefore, the toroidal eigenmodes extend over many rational surfaces, involve many poloidal m numbers and have a strong ballooning character. This is the situation described by the conventional ballooning mode approximation. If $\text{Re}(q) > 0$ the more strongly ballooning mode is centered at $\theta = \pi$, (on the inside edge of the torus) and the shear damping of this mode is reduced by toroidal coupling if $s > 1/2$ - in agreement with the detailed computations of toroidal drift modes by Hastie, Hesketh and Taylor.⁷

In a typical tokamak ($B \sim 3T$, $T_e \sim T_i \sim 2 \text{ keV}$, $a \sim 20 \text{ cm}$, $R \sim 100 \text{ cm}$) q is $\sim 10^2 - 10^3$ for drift waves with $ka_i \sim 1$ so that such drift waves lie well within the ballooning regime. They are therefore well-described by the conventional ballooning mode approximation (see Section 7).

6. ANOTHER BALLOONING EFFECT

For resistive interchanges we replace σ by $-i\sigma_R$ and the coupling coefficient q becomes

$$q_R = \frac{\epsilon_R}{2\kappa_R} \left(1 + \frac{\sigma_R^s}{2} \right) \exp\left(-\frac{\sigma_R}{4}\right) \quad (30)$$

where we have again set $\rho_1 = \rho_2$ and $\epsilon_R = \alpha\rho_2$.

In a typical pinch experiment σ_R is large and q_R is exponentially small. Consequently resistive interchanges do not show the ballooning effect discussed so far and are not described by the conventional ballooning mode approximation. However, they may show a different ballooning effect which, although small when σ_R is large, is not exponentially small.

To investigate this alternative ballooning effect we return to the model equation (10)

$$\left[\frac{\partial^2}{\partial x^2} + \sigma^2 \left(\frac{\partial}{\partial \theta} + i\kappa \right)^2 + \alpha(1 + \rho_1 \cos \theta + i\sigma \rho_2 \sin \theta \frac{\partial}{\partial x}) - \kappa x^2 \right] \phi = 0 \quad (31)$$

where we drop the subscripts on σ and κ but explicitly indicate the normal and geodesic curvatures since the geodesic curvature ρ_2 plays a dominant role when $\sigma \gg 1$.

We now observe that when $\sigma \gg 1$ the basic Fourier mode is very narrow and therefore introduce a new length scale $y = \sigma^{1/2}x$; then

$$\left(\frac{\partial^2}{\partial \theta^2} + \frac{i}{\sigma^{1/2}} \left(2y \frac{\partial}{\partial \theta} + \frac{\alpha \sigma \rho_2}{\sigma} \sin \theta \frac{\partial}{\partial y} \right) + \frac{1}{\sigma} \left[\frac{\partial^2}{\partial y^2} - y^2 \left(1 + \frac{\kappa}{\sigma^2} \right) + \frac{\alpha}{\sigma} (1 + \rho_1 \cos \theta) \right] \right) \phi = 0. \quad (32)$$

We now seek a solution

$$\phi = \phi_0(y, \theta) + \frac{1}{\sigma^{1/2}} \phi_1(y, \theta) + \dots \quad (33)$$

and, to ensure maximal ordering, we take $\alpha \sim \sigma$, $\kappa \sim \sigma^2$. Then in lowest order $\phi_0 = \phi_0(y)$ and in first order

$$\phi_1 = \frac{i \alpha \sigma \rho_2}{\sigma} \sin \theta \frac{d\phi_0}{dy} + \bar{\phi}_1(y). \quad (34)$$

In second order, after annihilating ϕ_1 by integration over θ , we have

$$\left[\left(1 - \frac{1}{2} \left(\frac{\alpha \rho_2}{\sigma} \right)^2 \right) \frac{\partial^2}{\partial y^2} - y^2 \left(1 + \frac{\kappa}{\sigma^2} \right) + \frac{\alpha}{\sigma} \right] \phi_0 = 0. \quad (35)$$

Thus at large σ , ϕ_0 differs slightly from the basic Fourier mode and there is a small shift in the eigenvalue. However, the interesting feature is the appearance of the contribution ϕ_1 , proportional to $\sin \theta$. This is a "ballooning" component located near the same rational surface as the primary Fourier mode ϕ_0 and proportional to $\sigma^{-3/2}$. It is thus quite distinct from the ballooning components considered hitherto which are located on rational surfaces adjacent to that of the primary mode and are exponentially small in σ . It is, however, the effect referred to as "ballooning" in discussions of interchange modes by some Soviet authors (see, for example, Ref. 12).

This 'centered' ballooning effect can be regarded as a result of coupling between the harmonics of the basic Fourier modes u_m which are usually neglected. These harmonics are given by Hermite functions

$$u_m^k(x) = \left(\frac{\sqrt{\sigma}}{2^k k! \sqrt{\pi}} \right)^{1/2} \exp \left[-\frac{\sigma}{2} (x - m)^2 \right] H_k \left[\sqrt{\sigma} (x - m) \right] \quad (36)$$

with $k > 0$. If we return to Section 4 and include these harmonics in the expansion of ϕ_0 , so that

$$\phi_0 = \sum_k \sum_m c_m^k u_m^k(x) \exp(-im\theta) \quad (37)$$

then in place of the recurrence relation (21) we obtain (retaining only the dominant geodesic curvature ρ_2 in Eq. (31) and ignoring κ),

$$(\alpha_k - \alpha)c_m^k = \frac{\alpha_s \rho_2}{2} \sum_l (\langle u_m^k D u_{m+1}^l \rangle c_{m+1}^l - \langle u_m^k D u_{m-1}^l \rangle c_{m-1}^l), \quad (38)$$

with $\alpha_k = 2k + 1$.

Bearing in mind that c_m^k for $k \neq 0$ is small compared to c_m^0 and that $\alpha \sim \alpha_0$, we have for $k = 0$

$$(\alpha_0 - \alpha)c_m^0 = \frac{\alpha_0 s \rho_2}{2} \sum_l (\langle u_m^0 D u_{m+1}^l \rangle c_{m+1}^l - \langle u_m^0 D u_{m-1}^l \rangle c_{m-1}^l) \quad (39)$$

and for $k \neq 0$

$$(\alpha_k - \alpha_0)c_m^k = \frac{\alpha_0 s \rho_2}{2} (\langle u_m^k D u_{m+1}^0 \rangle c_{m+1}^0 - \langle u_m^k D u_{m-1}^0 \rangle c_{m-1}^0) \quad (40)$$

so that

$$(\alpha_0 - \alpha)c_m^0 = \frac{\alpha_0 s \rho_2}{2} \{ \langle u_m^0 D u_{m+1}^0 \rangle c_{m+1}^0 - \langle u_m^0 D u_{m-1}^0 \rangle c_{m-1}^0 \} +$$

$$\frac{\alpha_0^2 s^2 \rho_2^2}{4} \sum_{l \neq 0} \frac{1}{(\alpha_l - \alpha_0)} \{ \langle u_m^0 D u_{m+1}^l \rangle [\langle u_{m+1}^l D u_{m+2}^0 \rangle c_{m+2}^0 - \langle u_{m+1}^l D u_m^0 \rangle c_m^0]$$

$$- \langle u_m^0 D u_{m-1}^l \rangle [\langle u_{m-1}^l D u_m^0 \rangle c_m^0 - \langle u_{m-1}^l D u_{m-2}^0 \rangle c_{m-2}^0] \}.$$

(41)

Compared to the earlier calculation of Section 4, there is a change in the coefficient of c_m^0 equivalent to a shift in the eigenvalue α by an amount

$$\frac{\alpha_0^2 s^2 \rho_2^2}{4} \sum_{l \neq 0} \frac{1}{(\alpha_l - \alpha_0)} \{ \langle u_m^0 D u_{m+1}^l \rangle^2 + \langle u_m^0 D u_{m-1}^l \rangle^2 \} . \quad (42)$$

Eq. (40) shows that there is also an additional component of the perturbation, given by

$$\frac{\alpha_0 s \rho_2}{2} c_m^0 e^{-i(m+1)\theta} \sum_{l \neq 0} \frac{1}{(\alpha_l - \alpha)} \langle u_{m+1}^l D u_m^0 \rangle u_{m+1}^l . \quad (43)$$

The individual matrix elements in (42) and (43) involve the overlap between Hermite functions centered on different rational surfaces and are consequently exponentially small when $\sigma \gg 1$. However, the expressions (42) and (43) are not themselves exponentially small. It can be shown, using the Hermite generating functions, that for large σ the expression

(42) is $\sim \alpha_0^2 s^2 \rho_2^2 / \sigma^2$ and agrees with the shift in eigenvalue given by Eq. (35). Similarly the expression (43) is $\sim \alpha_0 s \rho_2 / \sigma$ and agrees with Eq. (34) at large σ . Note again, that although this contribution to the perturbation is proportional to $\exp(-i(m+1)\theta)$ it is localised near the m^{th} rational surface, (i.e. near the same surface as the primary component c_m^0).

7. THE BALLOONING REPRESENTATION

In the usual theory of high n ballooning modes^{2,7} one maps the poloidal angle θ on to an extended coordinate η with $-\infty < \eta < \infty$ and writes the perturbation in the form

$$\phi(x, \eta) = A(x) e^{-ix(\eta + k)} f(\eta, x) . \quad (44)$$

Then $A(x)$ is taken to be a slowly varying 'envelope' (on the scale $x\sqrt{n}$) and to lowest order in $(1/\sqrt{n})$, $f(\eta, x)$ satisfies a 'local' one-dimensional eigenvalue equation in the extended η coordinate (with x fixed). For the present drift wave model this one-dimensional equation is

$$\left[\sigma^2 \frac{d^2}{d\eta^2} + (\eta + k)^2 + \epsilon(\cos \eta + s(\eta + k)\sin \eta) - \kappa x^2 + \lambda \right] f(\eta, x) = 0 , \quad (45)$$

where $\lambda(k, x)$ is the local eigenvalue.

The parameter k , representing an as yet undetermined radial wave-number, and an equation for the radial envelope function $A(x)$ are determined in higher orders of the expansion in $(1/\sqrt{n})$. Thus k is obtained from the equation

$$\frac{\partial \lambda}{\partial k}(x, k) = 0 \quad (46)$$

and $A(x)$ satisfies the equation

$$\frac{1}{2} \frac{\partial^2 \lambda}{\partial k^2} \frac{d^2 A}{dx^2} + \left(\lambda - \lambda_0 - \frac{x^2}{2} \frac{\partial^2 \lambda}{\partial x^2} \right) A = 0 \quad (47)$$

The condition for self-consistency of the theory, that $A(x)$ vary slowly, is satisfied therefore if

$$\left(\frac{\partial^2 \lambda}{\partial k^2} / \frac{\partial^2 \lambda}{\partial x^2} \right)^{1/4} \gg 1 \quad (48)$$

To assess this criterion in the present model we can determine $\lambda(k, x)$ from Eq. (45) by perturbation in ϵ and κ . [This expansion is quite distinct from the ballooning expansion in $1/\sqrt{n}$.] In leading order the outgoing wave condition⁷ implies

$$f_0 = \exp[i(\eta + \kappa)^2 / 2\sigma] \quad (49)$$

with eigenvalue $\lambda_0 = -i\sigma$. In next order

$$\left[\sigma^2 \frac{d^2}{d\eta^2} + (\eta + k)^2 + \lambda_0 \right] f_1$$

$$+ \left[\varepsilon(\cos \eta + s(\eta + k)\sin \eta) - \kappa x^2 + \lambda_1 \right] f_0 = 0$$

Annihilating f_1 by the operation $\int_{-\infty}^{\infty} d\eta f_0$ (recalling that σ has a small positive imaginary part) we find

$$\lambda_1 = \kappa x^2 - 2\kappa q \cos k \quad (51)$$

where q is the parameter introduced in Section 4.

The requirement of Eq. (46) can therefore be satisfied by $k = 0$ or $k = \pi$, corresponding to ballooning modes centered on the inside or the outside of the torus, consistent with the discussion of Section 5. It then follows from Eq. (51) that

$$\frac{\partial^2 \lambda}{\partial k^2} = \pm 2\kappa q \quad \text{and} \quad \frac{\partial^2 \lambda}{\partial x^2} = 2\kappa \quad (52)$$

and the self-consistency condition (48) reduces to $|q| \gg 1$. This is again precisely what we expect in the light of the discussion given in Section 5 and confirms that the ballooning approximation is valid when $|q| \gg 1$.

When $|q|$ is small the ballooning approximation is invalid; $A(x)$

does not vary slowly and the consistency condition (48) is not satisfied. Nevertheless the ballooning formalism may sometimes be useful even when $|q| < 1$. This is because the Ansatz

$$\phi(x, \eta) = e^{-ix\eta} f(\eta) \quad (53)$$

satisfies the model equation exactly if κ/ϵ can be neglected. For drift waves this is never possible when $|q| < 1$ since κ/ϵ must then be large. However, for resistive interchanges, when q is given by Eq. (30) and σ_R is large, we may have both $|q|$ and κ/ϵ small. In such an event the lowest order approximation of ballooning theory (analogous to Eq. (45)) provides a good estimate for the correct eigenvalue (and $f(\eta)$ incorporates the alternative ballooning effect discussed in Section 6) - even though the formal development of the ballooning approximation breaks down.

8. SUMMARY AND CONCLUSIONS

We have investigated the connection between two seemingly opposed views of toroidal eigenmodes. In one the eigenmodes are basically cylinder Fourier modes $\sim u_m(x)\exp(-im\theta)$ localised near a particular rational surface. In the other, the eigenmodes are ballooning structures which extend over many rational surfaces. Using a model which represents both drift waves and electrostatic resistive interchange instabilities, we have determined the parameter which determines whether the eigenmodes are Fourier-like or ballooning-like. This parameter q , defined in Eq. (22),

embodies the competing physical effects of coupling between Fourier modes (which enhances the ballooning tendency) and variation in the frequency of Fourier modes from one rational surface to another (which reduces the ballooning tendency).

As q increases, the conventional ballooning effect first manifests itself through coupling of the fundamental Fourier modes $u_m^0(x)$ on adjacent surfaces. This produces a toroidal eigenfunction consisting of a primary Fourier mode with weak 'sidebands' on neighboring rational surfaces. These 'sidebands' then increase in amplitude and extend across the radius until the 'quasi-mode'-like structure of the conventional toroidal ballooning mode is produced. When $|q|$ is large the ballooning mode extends over $\sim q^{1/4}$ rational surfaces.

This analysis supports the criterion given earlier² for the validity of the conventional ballooning approximation in a toroidal system and shows that this approximation is appropriate for drift waves in a tokamak but not for resistive interchanges in a pinch. For resistive interchanges the conventional ballooning effect is negligible but they may show another ballooning effect. This involves a sideband centered on the same rational surface as the primary Fourier mode and does not lead to a radially extended quasi-mode structure. It can be considered as the result of indirect coupling between harmonics of the Fourier modes. Surprisingly, although the conventional ballooning theory then breaks down, its lowest order approximation may still be useful.

References

- ¹ L. D. Pearlstein and H. L. Berk. Phys. Rev. Letts. 23, 220 (1969).
- ² J. W. Connor, R. J. Hastie and J. B. Taylor. Proc. Roy. Soc. 365, 1 (1979).
- ³ K. V. Roberts and J. B. Taylor. Phys. Fluids 8, 315 (1965).
- ⁴ J. B. Taylor in Plasma Physics and Controlled Thermonuclear Research (Proc. 6th Int. Conf, Berchtesgaden, 1976) I.A.E.A. Vienna, Vol. 2, 323 (1977).
- ⁵ J. G. Cordey and R. J. Hastie. Nucl. Fusion 17, 533 (1977).
- ⁶ W. Horton Jr., R. Estes, H. Kwak and Duk-in Choi. Phys. Fluids 21, 1367 (1978).
- ⁷ R. J. Hastie, K. W. Hesketh and J. B. Taylor. Nucl. Fusion 19, 1223 (1979).
- ⁸ W. M. Tang. Nucl. Fusion 18, 1089 (1978).
- ⁹ J. L. Johnson and J. M. Greene. Plasma Physics 9, 611 (1967).
- ¹⁰ G. Bateman and D. B. Nelson. Phys. Rev. Letts. 41, 1804 (1978).

¹¹ Gertrude Blanch in Handbook of Mathematical Functions, edited by M. Abramowitz and I. A. Stegun, National Bureau of Standards Applied Mathematical Series no. 55 (U.S. Government Printing Office, D.C. 1964) Chap. 20, p. 721.

¹² A. B. Mikhailovski. Nucl. Fusion 15, 95 (1975).

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