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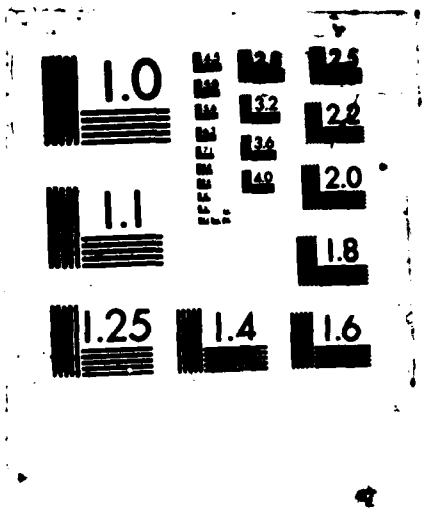

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SOME RECENT RESULTS ON GRAPH MATCHING

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1. Introduction

A **matching** in a graph G is a set of lines, no two of which share a common point. A matching is **perfect** if it spans $V(G)$. The problem of finding a matching of maximum cardinality in a graph models a number of significant real-world problems and, in addition, is of considerable mathematical interest in its own right. Matchings are in a sense among the best understood graph-theoretic objects: there exist efficient algorithms to find and good characterizations for the existence of perfect matchings and for the maximum weight of a matching; there are nice descriptions of polyhedra associated with matchings; good bounds and, for a few special classes, exact formulas for the number of perfect matchings in a graph. But there are many important questions that remain unanswered. What is the number of perfect matchings in a general graph? Which graphs can be written as the disjoint union of perfect matchings (i.e., which r-regular graphs are r-line-colorable)? How does one generate a random perfect matching? Matching theory has often been in the front lines of research in graph theory and many results in matching theory have served as pilot results for new branches of study in combinatorics (e.g., minimax theorems, good characterizations and polyhedral descriptions).

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While some of the open "matching" problems mentioned above are NP-hard (for example, the enumeration of perfect matchings [V] and

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the line-coloring problem [H]), there are others the polynomial-time solvability of which have been established only recently and not without considerable difficulty.

In this paper the continuing development of a canonical decomposition theory for graphs in terms of their matchings plays a central role. Using results of Gallai [G1, G2] and, independently, Edmonds [E1] which describe how to canonically decompose any graph in terms of the maximum matchings it contains, one can, in a sense, reduce the decomposition theory to the special case when the graph has a perfect matching.

Unfortunately, if the graph G in question has a perfect matching, the Gallai–Edmonds procedure gives no information.

Kotzig [K1, K2, K3] began to develop a method of decomposing graphs with perfect matchings in the late 1950's. This procedure was extended by Lovász [L1] and by Lovász and Plummer [LP1]. The building blocks which emerged from this procedure are the 3-connected bicritical graphs — or **bricks**. (A graph G is **bicritical** if $G - u - v$ has a perfect matching for all choices of distinct points $u, v \in V(G)$.) This procedure will be called the **Brick Decomposition Procedure**, or simply **BDP**.

In Section 2 of the present paper, the main result presented is a description of the matching lattice; i.e., the lattice generated by the incidence vectors of perfect matchings of a graph. This in turn yields a good characterization of the number of perfect matchings linearly independent over various fields. The BDP will be seen to play a vital role in the determination of the matching lattice.

The BDP, however, comes to an abrupt halt when the graph G is itself a brick. A method for decomposing bricks is currently unknown.

In Section 3 we describe another approach motivated by the desire to better understand the structure of bricks. If $|V(G)| \geq 2n+2 \geq 4$, we say that graph G is **n-extendable** if every matching of cardinality n is a subset of a perfect matching. In [P1] it was shown that, in particular, a 2-extendable graph which is not bipartite must be a brick. This fact, coupled with the fact that as long as $n \geq 2$, an n-extendable graph G must also be $(n-1)$-extendable, gives us an interesting nested family of subsets of bricks to study. Some results obtained relating n-extendability and such other graphical invariants as connectivity, toughness and genus are presented.

For any terminology not defined here, as well as more detailed information on the Gallai–Edmonds decomposition and the BDP, the reader is referred to [LP3].

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2. The BDP and the Matching Lattice

We begin by sketching the BDP. It is best understood if we recall the classical theorem of Tutte [T].

2.1. THEOREM. *A graph G has a perfect matching if and only if for every $X \subseteq V(G)$, the number of odd components of $G-X$ is at most $|X|$.*

Tutte's Theorem is a very powerful tool, for example, in proving the sufficiency of certain conditions for the existence of a perfect matching. However, it does not answer questions about how many perfect matchings a given graph has, how these are distributed, which lines occur in perfect matchings, etc. The present decomposition theory starts with the observation that if we delete all lines from a graph which do not occur in perfect matchings, the set of perfect matchings clearly is not changed. Moreover, if the graph has several connected components, then its perfect matchings are composed of perfect matchings of these components. Hence from now on, we shall restrict our attention to connected graphs with the property that every line is contained in some perfect matching. Such graphs have been variously called **matching-covered graphs** [L2], **U-graphs** [N] and **1-extendable graphs** [P1]. We shall adopt the last of these three names in the present paper.

Let G be a 1-extendable graph. A subset $B \subseteq V(G)$ is called a **barrier** if it gives equality in Tutte's condition; i.e., if $G-B$ has exactly $|B|$ odd components. (Note that trivially $G-B$ has *at most* $|B|$ odd components since G has a perfect matching.) From the hypothesis that G is 1-extendable, it also follows that if $G-B$ has exactly $|B|$ odd components then it has no even ones, and furthermore no line of G joins two points of B.

Define two points x and y in G to be **equivalent** if $G-x-y$ has no perfect matching. Kotzig [K1, K2, K3] and Lovász [L1] proved the following result concerning this relation.

2.2. THEOREM. *If G is 1-extendable then the equivalence of points is an equivalence relation. The equivalence classes of this relation are barriers and every barrier of G is contained in one of these equivalence classes.*

We call the partition defined by this equivalence relation the **canonical partition** of the graph G. Note that the canonical partition is the discrete partition if and only if $G-x-y$ has a perfect matching for every two points x and y. This is equivalent to saying that no barrier has more

than one element. We recognize such graphs to be precisely the so-called *bicritical* graphs defined in the Introduction.

If B is a barrier then we can decompose G into $|B|$ pieces as follows. For each (odd) component H of $G-B$, let H' denote the graph obtained from G by deleting all the other components of $G-B$ and shrinking B to a single point. It is easy to see that the graphs H' are 1-extendable and that each perfect matching in G yields a unique perfect matching in each of the graphs H'. For the problems mentioned above (the matching lattice, etc.) it is usually a routine step to prove the results for G if they are already established for each of the pieces H'.

If the pieces H' are not bicritical (i.e., if they contain non-trivial barriers) then we can continue to decompose the H''s recursively until we end up with a list of bicritical graphs. (We delete the trivial bicritical graphs on just two points from this list.)

Presently we can go one small step further. It is trivial to show that any bicritical graph with at least four points must be 2-connected. Suppose that G is a bicritical graph which is not 3-connected; that is, suppose we can write $G = G_1 \cup G_2$ where $V(G_1) \cap V(G_2) = \{u, v\}$. Now we separate G_1 and G_2 and add the line uv to each of them (regardless of whether or not this line was present in G). It is not difficult to show that this results in two new smaller bicritical graphs. We can continue this process on each non-3-connected bicritical graph until we end up with a list of 3-connected bicritical graphs, the so-called *bricks*. Moreover, we denote the final list of bricks of G by $\mathcal{L}(G)$.

We illustrate the BDP in Figure 2.1. In this example, $S_1 = \{v_1, v_2\}$ and $S_2 = \{v_3, v_4\}$ are two different equivalence classes (i.e., maximal barriers) of G.

Presently we do not know how to decompose bricks. We shall see, however, that several matching properties of graphs can be read off from their lists of bricks. Note also that the BDP is not unique; that is, one is free to choose barriers and cutsets in any order. However, the final list of bricks $\mathcal{L}(G)$ turns out to be independent of these choices [L5]:

2.3. THEOREM. *The list $\mathcal{L}(G)$ of bricks for any 1-extendable graph G is uniquely determined by G.*

To understand the internal structure of the set of perfect matchings, one often represents them by their incidence vectors. Let $\mathcal{M} = \mathcal{M}(G) \subseteq \Re^{E(G)}$ denote the set of incidence vectors of perfect matchings of the graph G. Much important information is to be found by studying the *convex hull*, $\mathrm{conv}(\mathcal{M})$, of this set of $|E(G)|$-dimensional vectors. This

FIGURE 2.1.

polytope was characterized as the solution set of a system of linear inequalities by Edmonds [E2]:

2.4. THEOREM. *The convex hull of incidence vectors of the perfect matchings in a graph G is the solution set of the following system of linear equations and inequalities:*

(i)	$x_e \geq 0$	*for every line e,*
(ii)	$\sum_{e \in \nabla(v)} x_e = 1$	*for every point v,*
(iii)	$\sum_{e \in \nabla(S)} x_e \geq 1$	*for every set* $S \subseteq V(G)$*, with* $\lvert S \rvert$ *odd.*

Here for every $S \subseteq V(G)$, $\nabla(S)$ denotes the set of lines of G connecting S to $V(G) - S$.

It seems perhaps a more immediate question to describe the *linear* hull $\mathrm{lin}_F(\mathcal{M})$ over various fields F. In the case of the rational (or, equivalently, the real or the complex) field, $\mathrm{lin}_F(\mathcal{M})$ was characterized by Naddef [N]. Note that the difference of any two equations in group (ii) gives a linear relation which is satisfied by every vector in $\mathrm{lin}_{\mathcal{Q}}(\mathcal{M})$. It is easy to see that if G is bipartite then this yields $n-2$ independent linear relations, while in the non-bipartite case, $n-1$ independent linear relations result. The difficulty is that some inequalities in group (iii) (possibly even exponentially many) also yield (hidden) linear constraints. Naddef described how these can be recognized and a maximum independent family characterized. We do not formulate his result here because our approach is different, being based upon the work of Edmonds, Lovász and Pulleyblank [ELP], who gave a polynomial-time algorithm to compute $\mathrm{lin}_{\mathcal{Q}}(\mathcal{M})$. In that paper the following formula was also derived:

2.5. THEOREM. *Let G be a* 1*-extendable graph. Then:*
$\dim \mathrm{lin}_{\mathcal{Q}}(\mathcal{M}(G)) = |E(G)| - |V(G)| + 2 - |\mathcal{L}(G)|$.

The main step in the proof occurs when G itself is a brick. For these the formula above shows that hidden linear constraints from group (iii) do not arise; that is, the following is true.

2.6. THEOREM. *Let G be a brick. Then a vector* $x \in \mathcal{Q}^{E(G)}$ *belongs to* $\mathrm{lin}_{\mathcal{Q}}(\mathcal{M}(G))$ *if and only if*

$$x(\nabla(u)) = x(\nabla(v))$$

for every pair of points u and v in $V(G)$.

The proof of these two results depends heavily on Edmonds' Theorem 2.4, that is, on the ordered structure of $\mathcal{Q}$, and therefore it does not remain valid if the field $\mathcal{Q}$ is replaced by any field of characteristic different from 0. A different approach [L5], which we shall sketch below, shows that the two preceding results are still valid if we replace $\mathcal{Q}$ by any field with characteristic different from 2. For fields with characteristic 2, however, a different formula holds:

2.7. THEOREM. *Let G be a 1-extendable graph and F a field of characteristic 2. Then if $\mathcal{P}(G)$ denotes the subset of $\mathcal{L}(G)$ which are Petersen graphs, we have:*

$$\dim \operatorname{lin}_F(\mathcal{M}(G)) = |E(G)| - |V(G)| + 2 - |\mathcal{L}(G)| - |\mathcal{P}(G)|.$$

(Roughly speaking, we lose one dimension for each Petersen graph found among the bricks.)

These results follow from a description of the (*perfect*) *matching lattice*, rather than from the perfect matching polytope. The (perfect) **matching lattice** of G, $\operatorname{lat}(\mathcal{M}(G)) = \operatorname{lat}(\mathcal{M})$, is the lattice generated by the vectors in $\mathcal{M}$; that is, the set of all linear combinations of vectors in $\mathcal{M}$ with integral coefficients.

This lattice arises in a very natural way as a relaxation of the chromatic index problem. Let G be an r-regular graph. Then clearly G is r-line-colorable if and only if the vector $\mathbf{1} \in \Re^{E(G)}$ consisting of all 1's can be written as a linear combination of vectors in $\mathcal{M}$ with non-negative integral coefficients. If we drop the integrality assumption here, we obtain a necessary condition for r-line-colorability, which, as pointed out by Edmonds [E2] *is equivalent to checking if* $\frac{1}{2}\mathbf{1} \in \operatorname{conv}(\mathcal{M})$. On the other hand, if we drop the non-negativity of the coefficients then we obtain another necessary condition for r-line-colorability equivalent to demanding that $\mathbf{1} \in \operatorname{lat}(\mathcal{M})$. The Petersen graph shows that this condition is non-trivial: for the Petersen graph, the first necessary condition above is satisfied, but not the second.

In describing the matching lattice, once again it turns out that the main case to consider is the case when the graph is a brick. We can then prove the following [L5]:

2.8. THEOREM. *Let G be a brick different from the Petersen graph. Then a vector $x \in \Re^{E(G)}$ belongs to* $\operatorname{lat}(\mathcal{M}(G))$ *if and only if*

(*i*) *x is integral, and*

(*ii*) *$x(\nabla(u)) = x(\nabla(v))$ for every pair of points u and v in $V(G)$.*

Using the BDP described above, we obtain a good characterization for membership in the matching lattice. In fact, the condition of Theorem 2.8 can be checked in polynomial time. From this it is not difficult to compute a basis in the matching lattice and solve other fundamental questions concerning $\operatorname{lat}(\mathcal{M})$ and $\operatorname{lin}_F(\mathcal{M})$. We shall not go into the details of this, however, but instead we shall formulate some consequences of these results.

2.9. COROLLARY. *Let G be an r-regular brick different from the Petersen graph. Then* $\mathbf{1} \in \mathrm{lat}(\mathcal{M}(G))$.

2.10. COROLLARY. *Let G be a graph which is not contractible to the Petersen graph and suppose $x \in \Re^{E(G)}$. Assume that x is integral and $x \in \mathrm{lin}_{\Re}(\mathcal{M}(G))$. Then $x \in \mathrm{lat}(\mathcal{M}(G))$.*

(Note that in particular every planar graph satisfies the hypothesis of this Corollary.)

2.11. COROLLARY. *Let G be any graph and suppose $x \in \Re^{E(G)}$. Assume that x is even integral and that $x \in \mathrm{lin}_{\Re}(\mathcal{M}(G))$. Then $x \in \mathrm{lat}(\mathcal{M}(G))$.*

In a paper on line-coloration of cubic graphs, Seymour [S] studied properties of so-called *r-graphs*. A graph G is an **r-graph** if it is regular of degree r and for each $X \subseteq V(G)$ with $|X|$ odd, $|\nabla(X)| \geq r$. It follows easily from Tutte's Theorem that every connected r-graph is 1-extendable. The next result is a special case of Corollary 2.10.

2.12. COROLLARY. *Let G be an r-graph which is not contractible to the Petersen graph. Then* $\mathbf{1} \in \mathrm{lat}(\mathcal{M}(G))$.

Finally, from Corollary 2.11 we have:

2.13. COROLLARY. *Let G be an r-graph. Then the vector* $\mathbf{2} = (2, 2, \ldots, 2) \in \mathrm{lat}(\mathcal{M}(G))$.

Seymour proved both of these final two results for the case $r = 3$ and conjectured their validity for general r.

3. Bricks and n-extendable graphs

In Section 2 of this paper we saw how the brick decomposition procedure can be carried out on an arbitrary 1-extendable graph and that, in fact, the procedure is "canonical" in the sense that the final list of bricks so obtained is an invariant of the graph. Furthermore, we saw how this procedure can be used to obtain the matching ranks of any 1-extendable graph over the field $GF(2)$ as well as over $\Re$. However, there are bricks having the same numbers of points, lines and ranks (both real and $GF(2)$) which have *different* numbers of perfect matchings. For example, the two graphs G_1 and G_2 in Figure 3.1 each have 12 points and 19 lines. Each is an example of a Halin graph and hence is bicritical by Theorem 2.2 of [LP2]. (A graph G is a **Halin Graph** if it can be

FIGURE 3.1.

expressed as $T \cup C$ where T is a tree having no points of degree 0 or 2 and C is a cycle passing through the endpoints of T so that $T \cup C$ is planar.) Thus $r_{\Re}(G_i) = r_{GF(2)}(G_i)$ for $i = 1, 2$ by a theorem of Naddef and Pulleyblank [NP]. But since each is a brick, the rank of each is $q - p + 1 = 8$. However, G_1 has 15 perfect matchings while G_2 has only 14.

The brick decomposition procedure tells us nothing in such a case and hence it is natural to seek some further decomposition procedure — this time of the bricks themselves.

This appears to be a very hard problem. We begin this section by mentioning two properties of bricks derived in [ELP] and [L5] respectively. The proof of Theorem 2.3 of the present paper makes essential use of the concept of a *tight cut*. A cutset L of lines in a 1-extendable graph G is called **tight** if every perfect matching of G contains exactly one line of L. A cut is **trivial** if all its lines meet at one common point.

3.1. THEOREM. *A brick contains no non-trivial tight cut.*

It is interesting to note that although this result sounds as if its proof should be one involving only elementary graph-theoretical ideas, no such proof is known. The proof of the preceding theorem presented in [ELP] makes essential use of the linear description of $PM(G)$.

We mention next a second property of bricks derived in [L5]. As usual, denote the complete graph on four points by K_4 and notice that the complement of a six-cycle, $\overline{C_6}$, is another way to describe the trian-

gular prism.

A few words are in order to try to place this next result in context. Let G' be a subgraph of graph G and let P be a path in G. Path P is called an **ear** of G' if the endpoints of P lie in $V(G')$, but no other points of P belong to $V(G')$. (In particular, a line $e = ab$ is an ear of G' if $e \in E(G) - E(G')$ and $\{a, b\} \subseteq V(G')$.) An ear is **odd** if it contains an odd number of lines.

The next result was proved in [LP1].

3.2. THEOREM. *Let G be a 1-extendable graph. Then G contains a 1-extendable subgraph G' such that G is the union of G' and one or two odd ears of G'. If two odd ears, then they are point-disjoint.*

Note that both K_4 and $\overline{C_6}$ suffice to show that the simultaneous addition of two ears may indeed be necessary.

Given a 1-extendable graph G, a chain of proper subgraphs $K_2 \subset G_1 \subset \cdots \subset G_{k-1} \subset G$ is an **ear decomposition** of G if each G_i in the sequence is 1-extendable and G_i is obtained from G_{i-1} by adding a single odd ear or perhaps two point-disjoint odd ears. In addition, we shall assume that the ear decomposition is as "fine" as possible in the sense that two odd ears are added only when neither odd ear added by itself would result in a 1-extendable graph. Since bricks are 3-connected, the final step in any ear decomposition for G must consist of adding one or two single lines. We are now prepared to state our next result.

3.3. THEOREM. *If G is a brick $\neq K_4$ or $\overline{C_6}$, then the last step in any ear decomposition of G must consist of adding a single line.*

Restated slightly, this theorem says that every brick different from K_4 or $\overline{C_6}$ must contain a line e such that $G - e$ is 1-extendable. Perhaps this result will turn out to be helpful in induction proofs yet to come concerning the structure of bricks.

Another approach was begun in [P1]. Let $p \geq 4$ and n be positive integers with $1 \leq n \leq (p-2)/2$. Call a graph G **n-extendable** if every set of n independent lines in G is a subset of a perfect matching. The family of 1-extendable graphs was the focus of Section 1 of this paper. The study of n-extendable graphs for values of $n > 1$ was motivated by the following results found in [P1].

3.4. THEOREM. *If G is 2-extendable then either G is bipartite or G is bicritical.*

3.5. THEOREM. *If $n \geq 1$ and G is n-extendable, then G is $(n+1)$-connected.*

We see immediately therefore that every non-bipartite 2-extendable graph is a brick. This fact, coupled with the next theorem, shows that in the case of non-bipartite graphs, as n increases, we obtain a nested family of subsets of bricks.

3.6. THEOREM. *If $n \geq 2$ and G is n-extendable, then G is $(n-1)$-extendable.*

In order to learn more about their structure, some initial studies of n-extendable graphs vis-à-vis other well-known graphical invariants have been carried out. The first results we mention might be thought of as "fine-tuning" the connectivity of an n-extendable graph somewhat. The proofs may be found in [P3].

3.7. THEOREM. *If $n \geq 1$ and G is n-extendable and has a point cutset S with $|S| = n+1$, then:*

(a) S is independent and

(b) if $n \geq 2$, $G-S$ has at most $n+1$ components. Moreover, equality holds iff $G = K_{n+1,n+1}$.

Recall that a graph G is **locally connected** if the subgraph induced by the neighborhood $\Gamma(u)$ of point u is connected for every $u \in V(G)$.

3.8. COROLLARY. *If $n \geq 1$ and G is both n-extendable and locally connected, then G is $(n+2)$-connected.*

Now let us recall that if $G \neq K_p$, the **toughness** of G, tough(G), is defined as $\min \frac{|S|}{c(G-S)}$, where $c(G-S)$ denotes the number of components of $G-S$ and the minimum is taken over all cutsets $S \subseteq V(G)$. Theorem 3.7 might lead one to conjecture that an n-extendable graph must have its toughness bounded below (perhaps even by 1), but this is not true. In [P4] for each $n \geq 1$ we constructed n-extendable graphs with arbitrarily small toughness. Such graphs have large numbers of points as one might expect, so it is reasonable to amend the question to ask the following. If $p = |V(G)|$ is there a function $f(p)$ such that if G is $f(p)$-extendable, then G is, say, 1-tough? The answer to this question is "yes" as the next theorem illustrates.

3.9. THEOREM. *Let G be a graph with p points and n a positive integer. Suppose that G is n-extendable, but that* tough$(G) < 1$. *Then $n \leq \lfloor \frac{p-2}{6} \rfloor$ and this bound is sharp for all n.*

On the other hand, it seems reasonable to expect that sufficiently large toughness will guarantee n-extendability. In connection with this, we should mention that in his introductory paper on toughness [C], Chvátal observed that if a graph G has an even number of points and has $\text{tough}(G) \geq 1$, then by Tutte's Theorem, G contains a perfect matching. If $\text{tough}(G) > 1$, we have the following:

3.10. THEOREM. *If G has p points with p even and if n is an integer such that $1 \leq n \leq (p-2)/2$, then if* $\text{tough}(G) > n$, *graph G is n-extendable. Moreover this lower bound on* $\text{tough}(G)$ *is sharp for all n.*

It is interesting to compare this result with the next theorem due to Enomoto, Jackson, Katerinis and A. Saito [EJKS]. Recall that for any integer $n \geq 1$ an **n-factor** of graph G is a spanning subgraph of G in which the degree of each point is n.

THEOREM. *Suppose G is a graph with at least $n+1$ points and suppose that* $\text{tough}(G) \geq n$. *Then if $n|V(G)|$ is even, G has an n-factor.*

The proofs of Theorems 3.9 and 3.10 may be found in [P4] where we also showed that Theorem 3.10 and the above result are independent in that neither implies the other.

The last graphical invariant we mention here in relation to n-extendability is the genus. In [P2] we obtained the next result.

3.11. THEOREM. *No planar graph is 3-extendable.*

More recently we proved the following extension to graphs of positive genus. (See [P5].) Let $\gamma = \gamma(G)$ denote the genus of graph G.

3.12. THEOREM. *If $\gamma > 0$, then G is not* $\left\lfloor \frac{9}{2} + \frac{18(\gamma-1)}{7+\sqrt{48\gamma-47}} \right\rfloor$*- extendable.*

Moreover we exhibited a family of 3-extendable toroidal graphs thus showing that the bound on extendability of Theorem 3.12 is sharp, at least when $\gamma = 1$. For $\gamma \geq 2$, it is not known if the bound of Theorem 3.12 is best possible.

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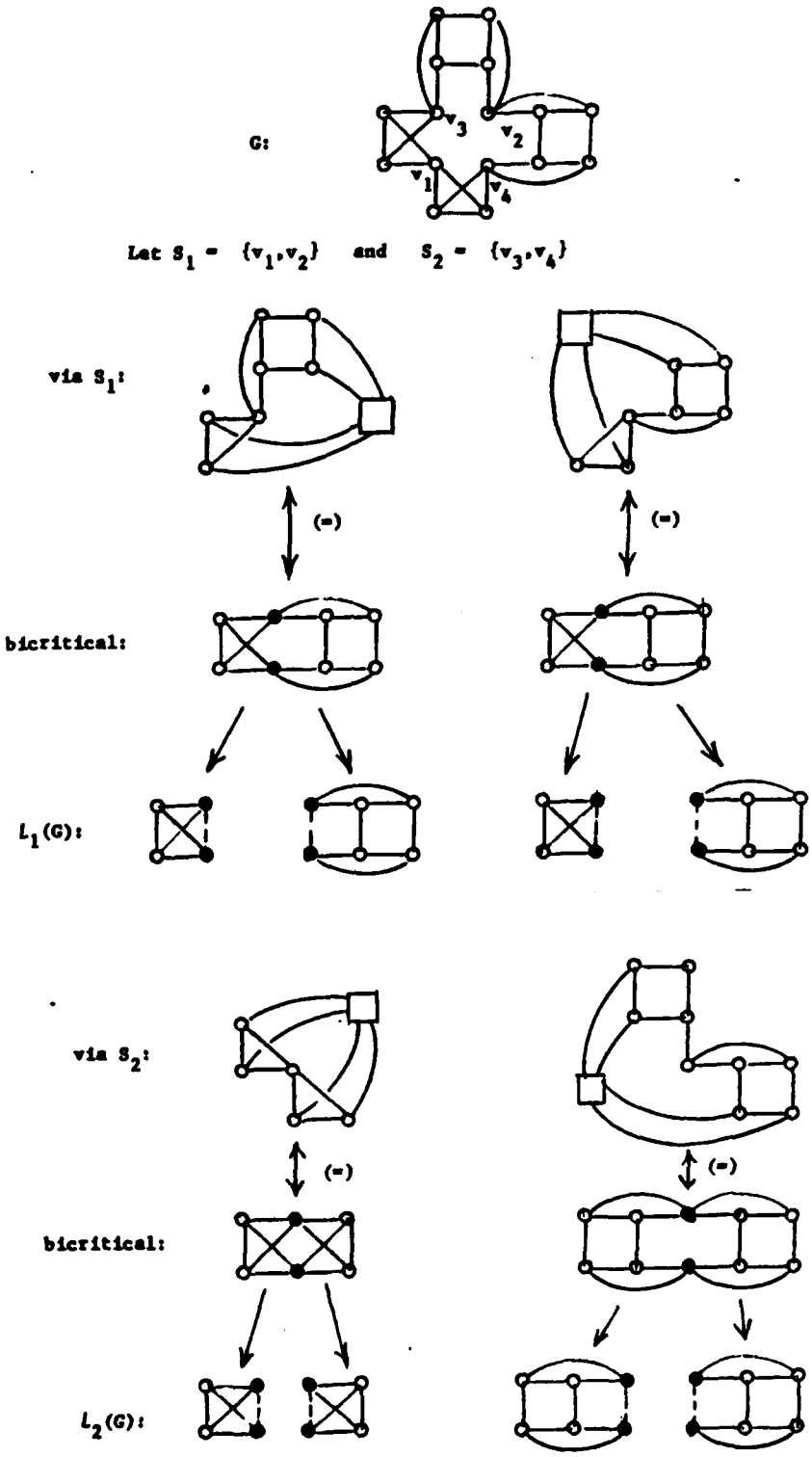
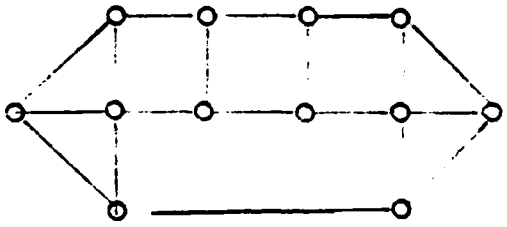

G:
v_3
v_2
v_1
v_4
Let $S_1 = \{v_1, v_2\}$ and $S_2 = \{v_3, v_4\}$
via S_1:
(=)
(=)
bicritical:
$L_1(G)$:
via S_2:
(=)
(=)
bicritical:
$L_2(G)$:

Figure 2.1

G_1:



G_2:

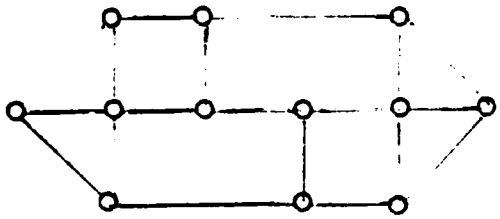


Figure 3.1

END
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