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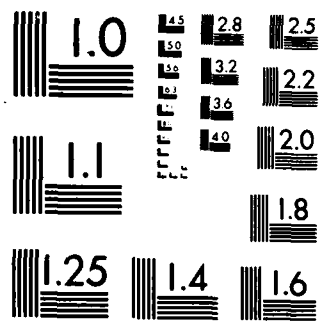

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JOINTLY CONSTRAINED BILINEAR PROGRAMMING: THE LINEAR COMPLEMENTARITY PROBLEM

by

Faiz A. Al-Khayyal

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-2-

Abstract

Jointly Constrained Bilinear Programming: The Linear Complementarity Problem

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~~We~~ investigates refinements to an existing nonconvex programming algorithm that exploit the special structure of linear complementarity problems. ~~We~~ prove that the working bases in the linear programming subproblems can be reduced from 3n×3n to n×n. In addition, ~~we~~ show that the procedure (in general, infinitely convergent) is finite under a nondegeneracy assumption. The procedure compares favorably with two recently proposed algorithms and is competitive with a third related method.

Key Words: Bilinear Programming, Linear Complementarity Problem, Quadratic Programming.

Abbreviated Title: Jointly Constrained BLP: The LCP

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Jointly Constrained Bilinear Programming: The Linear Complementarity Problem

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In a recent paper, Al-Khayyal and Falk [1] describe a branch-and-bound algorithm for finding a global solution to the nonconvex program

$$\begin{aligned} &\text{minimize} \quad f(x) + x^T y + g(y) \\ &\text{subject to} \quad (x,y) \in S \cap \Omega \end{aligned} \tag{1}$$

where f and g are convex over the feasible region, which is defined by the intersection of a nonempty, closed, convex set S and a compact hyper-rectangle Ω. For an appropriate choice of Ω the above problem contains as a special case the well-known linear complementarity problem (LCP) of finding a real n-vector x such that

$$Mx + q \geq 0, \quad x \geq 0, \quad x^T(Mx+q) = 0, \tag{2}$$

where M is a given real square matrix and q is a given real n-vector. The relationship of (2) to (1) is apparent from the following indefinite quadratic programming formulation of (2):

$$\text{minimize} \quad x^T y$$

$$\text{subject to} \quad -Mx+y = q, \quad x \geq 0, \quad y \geq 0. \tag{3}$$

To solve (3) by the branch-and-bound method in [1] we need a compact hyperrectangle Ω that contains at least one complementary solution. Such a set is easy to construct in this case because, when the LCP has a solution, at least one complementary solution is a basic feasible solution of the system $-Mx+y = q$. Hence an appropriate Ω may be constructed by bounding all basic solutions using the following result [6, Lemma 2.1].

<u>Lemma 1</u> Let x be a basic solution of the $m \times n$ system $Ax = b$. Then

$$|x_j| \leq m! \; \alpha^{m-1} \beta$$

where

$$\alpha = \max_{i,j}\{|a_{ij}|\} \text{ and } \beta = \max_{j}\{|b_j|\}.$$

Tighter bounds can be obtained, in the case when the set $D = \{x: Mx+q \geq 0, x \geq 0\}$ is bounded, by solving the n linear programs $\max\{x_j : x \in D\}$ for $j = 1,\ldots,n$. Once upper bounds on x are determined, similar bounds for $y = Mx+q$ are easily computed.

In this paper we investigate refinements to the method in [1] that exploit the structure of problem (3). In particular, we show that the size of the working bases in the linear programming subproblems can be reduced from $3n \times 3n$ to $n \times n$, and that convergence is finite for linear complementarity problems with nondegenerate complementary solutions. We also compare the procedure to three recent algorithms. The preliminary findings are that the proposed procedure is faster than the cutting plane method of Ramarao and Shetty [7], and can solve more problems than the piecewise linear equation approach of Solow and Sengupta [8]. Only the enumeration procedure of Al-Khayyal [3] is potentially faster, but the two methods appear to give comparable execution times on some problems. For brevity, we assume knowledge of the algorithm in [1] and will make reference only to components of the latter procedure that are provably refined here.

1. Algorithmic Refinements

For the general problem, the algorithm in [1] branches into four subproblems at each stage. These problems are defined by partitioning the hyperrectangle Ω into four subsets in the following way. Given a point $(\bar{x},\bar{y})\in\Omega = \{(x,y):\ \ell_x<x<u_x,\ \ell_y<y<u_y\}$, we choose an index I to specify the partition of Ω into the four sets $\Omega^1,\ldots,\Omega^4$, where

$$\Omega^1 = \{(x,y):\ \ell_{x_I}<x_I<\bar{x}_I,\ \ell_{y_I}<y_I<\bar{y}_I,\ \ell_{x_i}<x_i<u_{x_i},\ \ell_{y_i}<y_i<u_{y_i},\ i\neq I\},$$

$$\Omega^2 = \{(x,y):\ \bar{x}_I<x_I<u_{x_I},\ \ell_{y_I}<y_I<\bar{y}_I,\ \ell_{x_i}<x_i<u_{x_i},\ \ell_{y_i}<y_i<u_{y_i},\ i\neq I\},$$

$$\Omega^3 = \{(x,y):\ \bar{x}_I<x_I<u_{x_I},\ \bar{y}_I<y_I<u_{y_I},\ \ell_{x_i}<x_i<u_{x_i},\ \ell_{y_i}<y_i<u_{y_i},\ i\neq I\},$$

$$\Omega^4 = \{(x,y): \ell_{x_I} \le x_I \le \bar{x}_I,\ \bar{y}_I \le y_I \le u_{y_I},$$
$$\ell_{x_i} \le x_i \le u_{x_i},\ \ell_{y_i} \le y_i \le u_{y_i},\ i \ne I\}.$$

Each of the four subsets is a candidate for future partition in the search for an optimal solution. Without loss of generality, we may assume that $u_x > \ell_x \geq 0$ and $u_y > \ell_y \geq 0$.

A solution to the linear complementarity problem (2) must satisfy $x_i y_i = 0$ for all i. Hence, the index I is chosen such that $\bar{x}_I \bar{y}_I > 0$ at a feasible noncomplementary solution $(\bar{x},\bar{y})$. The set Ω chosen for partition is the one among all candidate subsets that produces the lowest value of $\bar{x}^T\bar{y}$, where each subset has a point $(\bar{x},\bar{y})$ associated with it. Clearly, the subset Ω^3 can be eliminated from further search (fathomed) since it cannot contain a complementary solution. We show in the next section that the set Ω can be fathomed if either $\ell_{x_I} > 0$ or $\ell_{y_I} > 0$ together with $\bar{x}_I \bar{y}_I > 0$. Thus, I is a partitioning index for Ω only if $\ell_{x_I} = \ell_{y_I} = 0$. In this way, the active nodes (corresponding to subsets that potentially contain complementary solutions) are guaranteed to satisfy $\ell_{x_i} \ell_{y_i} = 0$ for all i.

The greatest computational savings are realized from the refinement of the subproblem solved at each node. Let Ω be the active partition associated with an arbitrary node. The subproblem solved at that node is defined as

$$\text{minimize} \quad \phi(x,y)$$
$$\text{subject to} \quad (x,y) \in S \cap \Omega \tag{4}$$

where $S = \{(x,y): -Mx+y = q\}$ and

$$\phi(x,y) = \sum_{i=1}^{n} \max\{\ell_{y_i} x_i + \ell_{x_i} y_i,\ u_{y_i} x_i + u_{x_i} y_i - u_{x_i} u_{y_i}\}$$

is the convex envelope of $x^T y$ over Ω (see [1]). Note that $\phi(x,y) \geq 0$ on the nonnegative orthant since, by assumption, $\ell_x \geq 0$ and $\ell_y \geq 0$. The general procedure calls for solving this problem using an equivalent linear program involving 3n decision variables (2n are lower and upper bounded, and n are unrestricted) and 3n constraints (n are equality and 2n are inequality). For the LCP, however, we can improve on the linear programming subproblems. To simplify the presentation, assume that $S \ \Omega \neq \emptyset$, thereby guaranteeing the existence of an optimal solution to problem (4). Let $(\bar{x},\bar{y})$ denote such a solution and let $\bar{v} = \phi(\bar{x},\bar{y})$.

Consider the linear program

$$\text{minimize} \quad \psi(x,y)$$
$$\text{subject to} \quad (x,y) \in S \cap C \tag{5}$$

where $$\psi(x,y) = \ell_y^T x + \ell_x^T y,$$

$$C = \{(x,y): x \geq \ell_x,\ y \geq \ell_y,\ (u_{y_i} - \ell_{y_i}) x_i + (u_{x_i} - \ell_{x_i}) y_i \leq u_{x_i} u_{y_i},\ i \in N\},$$

and $N = \{1,2,\ldots,n\}$. When $S \cap C \neq \emptyset$, let $(\hat{x},\hat{y})$ denote an optimal solution to problem (5) and let $\hat{v} = \psi(\hat{x},\hat{y})$. Notice that $C \subset \Omega$ and that $\phi(x,y) \geq \psi(x,y)$ for all (x,y). In addition, $\psi(x,y) \geq 0$ for $(x,y) \geq (0,0)$. We show the equivalence of problems (4) and (5) for active partitions Ω, but first we prove the following simple Lemma.

<u>Lemma 2</u> If $S \cap \Omega$ contains a complementary solution, then $\bar{v} = \hat{v} = 0$.

<u>Proof</u> Suppose $(x^*,y^*) \varepsilon S \cap \Omega$ is a complementary solution. By construction, we have $\ell_x \geq 0$ and $\ell_y \geq 0$ such that $\ell_x^T \ell_y = 0$. It follows that $(x^*,y^*) \varepsilon C$. Because ϕ is the convex envelope of $x^T y$ over Ω, we have

$$0 = (x^*)^T y^* \geq \phi(x^*,y^*) \geq \phi(\bar{x},\bar{y}) \geq \psi(\bar{x},\bar{y}) \geq \psi(\hat{x},\hat{y}) \geq 0. \qquad \blacksquare$$

<u>Theorem 3</u> Problems (4) and (5) are equivalent in the sense that either the optimal solution of one solves the other, or $S \cap \Omega$ does not contain a complementary solution.

<u>Proof</u> We first show that either the optimal solutions of problem (4) solve problem (5) or $S \cap \Omega$ does not contain a complementary solution. If $(\bar{x},\bar{y}) \varepsilon C$, then $\phi(\bar{x},\bar{y}) = \psi(\bar{x},\bar{y})$. Hence,

$$\begin{aligned} \bar{v} &= \min\{\phi(x,y)\colon (x,y) \varepsilon S \cap C\} \\ &\geq \min\{\psi(x,y)\colon (x,y) \varepsilon S \cap C\} \\ &\geq \min\{\psi(x,y)\colon (x,y) \varepsilon S \cap \Omega\} = \bar{v} \end{aligned}$$

and therefore $(\bar{x},\bar{y})$ solves problem (5). Now suppose that

$(\bar{x},\bar{y}) \notin C$. Then $\bar{v} > 0$ and, by Lemma 2, $S \cap \Omega$ does not contain a complementary solution.

To complete the proof, we now show that either the optimal solutions of problem (5) solve problem (4) or $S \cap \Omega$ does not contain a complementary solution. The latter holds when $S \cap C = \emptyset$, since every complementary solution in $S \cap \Omega$ must necessarily be in C. Now assume that $S \cap C \neq \emptyset$. We consider two cases: when $\hat{v} = 0$ and when $\hat{v} > 0$. First assume that $\hat{v} = 0$ and recall that $(\hat{x},\hat{y})$ is feasible to problem (4). We have

$$0 = \hat{v} = \phi(\hat{x},\hat{y}) \geq \phi(\bar{x},\bar{y}) = \bar{v} \geq 0.$$

Hence $\phi(\hat{x},\hat{y}) = 0$ and $(\hat{x},\hat{y})$ solves problem (4). Finally, assume that $\hat{v} > 0$. We now prove that $\bar{v} > 0$ and invoke Lemma 2 to claim that $S \cap \Omega$ does not contain a complementary solution. By contradiction, assume that $\hat{v} > 0$ and $\bar{v} = 0$. Then

$$0 = \phi(\bar{x},\bar{y}) \geq \psi(\bar{x},\bar{y}) \geq 0$$

and $(\bar{x},\bar{y})$ is feasible to problem (5); otherwise, $(\bar{x},\bar{y}) \notin C$ and $\phi(\bar{x},\bar{y}) > 0$. Hence,

$$0 = \psi(\bar{x},\bar{y}) \geq \psi(\hat{x},\hat{y}) = \hat{v} > 0$$

which is a contradiction. Thus we conclude that $\bar{v} > 0$. ∎

By exploiting the foregoing theorem, the branch-and-bound algorithm [1] involves linear programming subproblems with working bases of order n. This is achieved by scaling the variables in problem (5). Specifically, for each $i \in N$, let

$$x_i' = \left(\frac{u_{y_i} - \ell_{y_i}}{u_{x_i} u_{y_i}}\right) x_i \quad \text{and} \quad y_i' = \left(\frac{u_{x_i} - \ell_{x_i}}{u_{x_i} u_{y_i}}\right) y_i \quad . \tag{6}$$

In terms of the new variables, problem (5) becomes

$$\begin{aligned} \min \quad & c^T x' + d^T y' \\ \text{subject to} \quad & Ax' + By' = q \\ & x' + y' \leq b \\ & x' \geq \ell_{x'} \\ & y' \geq \ell_{y'} \end{aligned} \tag{7}$$

where the parameters $c, d, A, B, b, \ell_{x'}$ and $\ell_{y'}$ are easily derived from the coordinate transformation (6). Problem (7) need not be solved to completion if a complementary solution is uncovered. It is easy to recognize when such a solution is encountered because $x^T y = 0$ if and only if $(x')^T y' = 0$.

The linear program (7) has n equality costraints, n generalized upper bounding (GUB) constraints, and 2n lower bound (LB) restrictions on the decision variables. Recall that both GUB and LB constraints can be handled separately (see, e.g., Chapter 6 in Lasdon [4]). Therefore, all pivot computations can be performed using only an $n \times n$ basis matrix, which is a considerable improvement on the $3n \times 3n$ basis that arises in the general case.

2. Finite Convergence

For the general problem (1), the branch-and-bound algorithm [1] can only be guaranteed to converge in the limit. In this section we prove that the process is finite for linear complementarity problems whose solutions are nondegenerate extreme points of $S \cap \Omega$. We first show that the procedure always branches into three subproblems and then argue that all paths in the branch-and-bound tree are fathomed after a finite number of branches under the nondegeneracy assumption.

Let Ω represent the initial hyperrectangle and let C be the associated set in problem (5). Thus, $\ell_x = \ell_y = 0$ and $\psi(x,y) = 0$ for all (x,y). Suppose $(\bar{x},\bar{y}) \in S \cap C$ such that $\bar{x}_I \bar{y}_I > 0$. Note that $(\bar{x},\bar{y})$ solves problem (5). Let $\Omega^1,\ldots,\Omega^4$ be the partition of Ω defined above, and let $C^1,\ldots,C^4$ and $\psi^1,\ldots,\psi^4$ be the associated sets and objective functions, respectively, in problem (5). For $j = 1,\ldots,4$, let (ℓ_x^j,ℓ_y^j) and (u_x^j,u_y^j) denote the lower bound and upper bound vectors in Ω^j.

Since $(\ell^3_{x_I},\ell^3_{y_I}) > (0,0)$, the set Ω^3 cannot contain a complementary solution; therefore, we fathom that subset. For $j = 1,2,4$, if $S \cap C^j = \emptyset$ then by Theorem 3 the set Ω^j also does not contain a complementary solution and is consequently fathomed. Otherwise, let (x^j,y^j) denote an optimal solution to problem (5) with $\psi = \psi^j$ and $C = C^j$. For convenience, we shall henceforth designate such problems as (5-j). Now, for $j = 2,4$, if $\psi^j(x^j,y^j) > 0$ then the subset Ω^j is fathomed by Lemma 2. Hence, set Ω^j is a candidate for further partition only if $\psi^j(x^j, y^j) = 0$. In particular, for j = 2 and j = 4 we must have $y_I^2 = 0$ and $x_I^4 = 0$, respectively. Consequently, all subsequent subdivisions of Ω^2 and Ω^4 must involve an index different from I. Otherwise, an optimal solution to (5) with $\Omega \subset \Omega^j$ (j=2,4) would have both x_I and y_I components positive, and by Lemma 2 this means that Ω is fathomed. Therefore, any path in the branch-and-bound tree that only involves subsets of the form Ω^2 and Ω^4 can have at most n branches.

So long as $\ell_{x_i} = \ell_{y_i} = 0$, the index i can conceivably arise repeatedly for partitioning a sequence of nested subsets. We argue below that the sequence is finite under the nondegeneracy assumption. Suppose the index I is used, in accordance with the algorithm, to repeatedly partition the set Ω^j. For k=1,2,..., let Ω^{j_k} denote the subsets of Ω^j that have lower bounds of zero for both x_I and y_I; that is, $\ell_{x_I}^{j_k} = \ell_{y_I}^{j_k} = 0$. Similarly, let C^{j_k} and ψ^{j_k} be the associated sets and objective functions in problem ($5-j_k$). Further assume that a nondegenerate complementary solution, say (x^*, y^*), is in C^{j_k} for all k. Thus, we assume that the algorithm is converging to the point (x^*, y^*) by partitioning Ω^{j_k}, for each k, into four subsets using the same index I. With (x^{j_k}, y^{j_k}) denoting an optimal solution to problem ($5-j_k$), we have

$$C^j \supset C^{j_1} \supset C^{j_2} \supset \ldots,$$

$$(x^*, y^*) \in \bigcap_{k=1}^{\infty} C^{j_k},$$

and

$$(x^{j_k}, y^{j_k}) \to (x^*, y^*),$$

where we assume for the moment that the sequence $\{(x^{j_k}, y^{j_k})\}$ converges to (x^*, y^*) in the limit. We also have, by construction, that (x^*, y^*) is a nondegenerate extreme point of $S \cap C^{j_k}$ for every k whenever (x^*, y^*) is a nondegenerate extreme point of $S \cap \Omega$.

Under the nondegeneracy assumption, (x^*, y^*) is a locally unique global solution to the problem

$$\begin{aligned} &\text{minimize} \quad x^T y \\ &\text{subject to} \quad (x,y) \varepsilon S \cap C^{j_k} \end{aligned}$$

for every k. Since $\{(x^{j_k}, y^{j_k})\}$ converges to (x^*, y^*), then for sufficiently large k the point (x^*, y^*) uniquely solves the linear program

$$\begin{aligned} &\text{minimize} \quad (y^{j_k})^T x + (x^{j_k})^T y \\ &\text{subject to} \quad (x,y) \varepsilon S \cap C^{j_k}. \end{aligned} \qquad (8)$$

Thus, convergence is achieved finitely when a linear program is solved at a point in the converging sequence which is sufficiently close to the nondegenerate limit point (x^*, y^*). This, in fact, is already a feature of the general algorithm in [1].

The following scheme is implemented in the algorithm and was originally intended to accelerate convergence to nonextreme point global solutions to jointly constrained bilinear programs. Before partitioning Ω^{j_k} using the solution (x^{j_k}, y^{j_k}) of problem $(5\text{-}j_k)$, we solve the linear program

$$\begin{aligned} &\text{minimize} \quad (y^{j_k})^T x + (x^{j_k})^T y \\ &\text{subject to} \quad (x,y) \in S \cap \Omega^{j_k} \end{aligned} \tag{9}$$

and let $(\tilde{x}^{j_k}, \tilde{y}^{j_k})$ denote an optimal solution. Finally, we minimize $x^T y$ over the line segment connecting (x^{j_k}, y^{j_k}) and $(\tilde{x}^{j_k}, \tilde{y}^{j_k})$, and use the minimizing point to define the partiton of Ω^{j_k}. Notice that the only difference between problems (8) and (9) is in the sets C^{j_k} and Ω^{j_k} which intersect with S. In general, we would expect the optimal objective value of (9) to be less than or equal to that of (8) because $C^{j_k} \subset \Omega^{j_k}$. However, because every complementary solution in Ω^{j_k} must also be in C^{j_k}, for k sufficiently large and assuming nondegeneracy of the limit point (x^*, y^*), we have that $(\tilde{x}^{j_k}, \tilde{y}^{j_k})$ also solves problem (8). Therefore, our algorithm converges finitely for linear complementarity problems that have at least one complementary solution, each one being a nondegenerate extreme point of the associated polyhedral set.

3. Concluding Remarks

A related enumeration procedure [3] always branches into two nodes and solves the subproblems of minimizing x_I and minimizing y_I over a face (specified by the parent node) of the set $S^+ = \{(x,y) \in S: x > 0, y > 0\}$, where the index I and the branching node are selected in the same way as in our algorithm above. If the minimum value of x_I or y_I is zero, then the associated node represents a face of S^+ having a lower dimension than the face of the parent node; otherwise, the corresponding node is fathomed.

A comparison between the bilinear programming approach above and the enumeration procedure in [3] reveals that minimizing x_I over the face defined by the branching node is roughly equivalent to minimizing $\psi^4(x,y)$ over $S \cap (C^1 \cup C^4)$. Analogously, minimizing y_I over the branching node's face in [3] corresponds to minimizing $\psi^2(x,y)$ over $S \cap (C^1 \cup C^2)$ in our procedure. Another difference between the two strategies is that the branching variable is fixed to zero in descendant nodes (thus defining the faces of the offspring) in [3], whereas the branching variable in our procedure can conceivably take on nonzero optimal values in descendant subproblems, although this is discouraged by our construction which maintains a positive objective function coefficient for the branching variables in all offspring subproblems.

An implementation of the bilinear programming procedure, which does not exploit the refinements described herein, was tested against the enumeration procedure in a preliminary experiment reported in [2]. The enumeration procedure solved all seven problems faster, but only two problems ran faster by one order of magnitude and the methods had run times within 25% of each other on another two problems. This suggests that the two methods are comparable on some problems.

Despite the poor implementation of the bilinear procedure, it nevertheless outperformed the cutting-plane method of Ramarao and Shetty [7] on the only two problems in common among the test problems solved; specifically, problems 5 and 6 in [2] correspond to problems PB-20-02 and PB-20-05 in [7], respectively. These problems were solved on the same machine (a CDC Cyber 170/730) and produced run times favoring our procedure of 6.7 seconds versus 27.57 seconds for problem 5 (PB-20-02) and 12.7 seconds versus 25.85 seconds for problem 6 (PB-20-05). Our code was unable to solve problems with more than twenty variables because of core memory limitations. However, the memory size burden can be considerably reduced by the refinements to the procedure presented above.

Another novel approach to solving the linear complementarity problem is the method proposed by Solow and Sengupta [8] which involves finding roots of a piecewise linear convex function. They prove their algorithm to be finite when all of the principal minors of M are positive (i.e., when M is a P-matrix). While their approach is interesting, the computational results they reported using positive definite matrices still give Lemke's method [5] an edge. By contrast, our approach is finite for arbitrary matrices and can solve problems on which Lemke's method fails.

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