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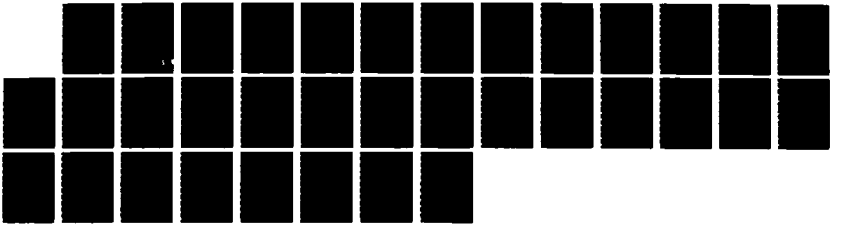

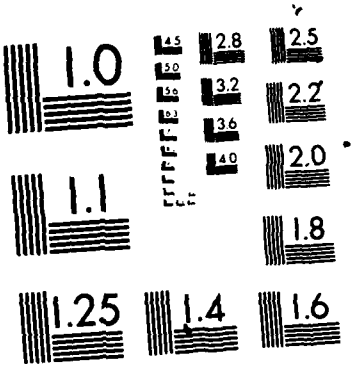

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A Two Compartment Storage Model with an Underlying Semi-Markov Process

by

Eric S. Tollar

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A two compartment storage model with an underlying semi-Markov process

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ABSTRACT

A storage model with an underlying semi-Markov process is proposed to model the behavior of a two compartment storage system with one way flow. It is shown that based on first moment assumptions, the divergence or convergence of each compartment is determined. For the eight separate cases in which at least one compartment does not converge, the bivariate asymptotic behavior of the compartments, when appropriately normalized, is determined under second moment conditions.

1. INTRODUCTION

In this paper, a two compartment storage model based on an underlying semi-Markov process is examined. This model is an extension of the popular single compartment model considered initially by Senturia and Puri (1973), with subsequent research by Senturia and Puri (1974), Puri and Senturia (1975), Puri (1978), Balagopal (1979), Puri and Woolford (1981), and Puri and Tollar (1985), among others. The extension to a model with an arbitrary amount of compartments based on an underlying Markov chain has been considered in Tollar (1985a, 1985b). For the sake of simplicity, we have reduced the number of compartments to two in this paper. This retains the flavor of the multiple compartment model's variety and difficulties, while removing much of the intractability of certain expressions (for some understanding of both of these features, see Tollar (1985a, 1985b)). Of course, all of the difficulties which typically arise from considering continuous time instead of discrete time are still present.

All of the random variables are considered to be defined on a given underlying probability (Ω, A, P). Let J be some denumerable set, and $\{(X_n, T_n), n = 0, 1, 2, \ldots\}$ a Markov renewal process with semi-Markov matrix $\underset{\sim}{A}(t) = (A_{ij}(t))$, (see Çinlar (1969), where for all $i, j \in J$, $t \geq 0$,

$$P(X_n = j, T_n - T_{n-1} \leq t | T_0, X_0, T_1, X_1, \ldots T_{n-1}, X_{n-1} = i) = A_{ij}(t). \quad (1.1)$$

We assume $\{X_n, n = 0, 1, 2, \ldots\}$ is a positive recurrent, aperiodic, irreducible Markov chain with transition matrix $\underset{\sim}{P} = (p_{ij}) = (A_{ij}(\infty))$, and stationary measure $\underset{\sim}{\pi} = (\pi_i)$. The moments of the sojourn times in a state $i \in J$ are then given by

$$m_i^{(k)} = \int_0^\infty t^k \sum_{j \in J} A_{ij}(dt), \quad k = 1, 2, \ldots,$$

where for simplicity, we write $m_i \equiv m_i^{(1)}$.

With each $i \in J$, we associate a triplet sequence $\{(U_n(i), V_n(i), W_n(i)), n = 0, 1, \ldots\}$ where the triplets are i.i.d. random triplets, independent of $\{(X_n, T_n), n = 0, 1, \ldots\}$ and of $\{(U_n(j), V_n(j), W_n(j), n = 1, \ldots\}$ for $j \neq i$. In addition, we assume for all $i \in J$ that $E|U_1(i)| < \infty$, $E|V_1(i)| < \infty$, $E|W_1(i)| < \infty$,

$$P(U_n(i) \neq 0, V_n(i) \neq 0, W_n(i) \neq 0) = 0 \quad \forall\, n \tag{1.2}$$

and

$$P(U_n(i) < 0) = P(V_n(i) < 0) = P(W_n(i) < 0) = 0. \tag{1.3}$$

We then define random variables $Z_{1,n}$ and $Z_{2,n}$ by

$$\begin{aligned} Z_{1,n} &= \max(0, Z_{1,n-1} + U_n(X_n)) \\ Z_{2,n} &= \max(0, Z_{2,n-1} + \min(V_n(X_n), Z_{1,n-1} + U_n(X_n)) - W_n(X_n)). \end{aligned} \tag{1.4}$$

From (1.4) it follows that

$$\begin{aligned} Z_{1,n} &= \max(Z_{1,0} + S_n, \max_{1 \le j \le n} (S_n - S_j)) \\ Z_{2,n} &= \max[Z_{1,0} + Z_{2,0} + R_n, Z_{1,0} + \max_{1 \le k \le n} (S_k + R_n - R_k), \\ &\qquad \max_{1 \le j \le k \le n} (S_k - S_j + R_n - R_k)] - Z_{1,n}, \end{aligned} \tag{1.5}$$

where $S_n = \sum_{i=1}^{n} [U_i(X_i) - V_i(X_i)]$ and $R_n = \sum_{i=1}^{n} [U_i(X_i) - W_i(X_i)]$. For details of the derivation of (1.5), see Tollar (1985a).

Finally, to introduce continuous time, we define these quantities for an arbitrary time t by

$$(Z_1(t),\ Z_2(t)) = (Z_{1,M(t)},\ Z_{2,M(t)}), \tag{1.6}$$

where

$$M(t) = \sup\{n: T_n \le t\}. \tag{1.7}$$

Close examination of (1.4) yields that $(Z_1(t),\ Z_2(t))$ represent the amount of material stored in the first and second compartments of a two compartment storage system with one-way flow. The sequence $\{U_n(X_n),\ n = 1,\ 2,\ \ldots\}$ represents the flow of material into the first compartment. The sequence $\{V_n(X_n),\ n = 1,\ 2,\ \ldots\}$ represents the amount of transfer demanded between the first and second compartment. Finally, $\{W_n(X_n),\ n = 1,\ 2,\ \ldots\}$ represents the amount of output demanded from the system, via the second compartment. As such at time T_n, if $U_n(X_n) > 0$, $U_n(X_n)$ is added to compartment one. If $V_n(X_n) > 0$, then either $V_n(X_n)$ is transferred from the first compartment to the second, or all the material present in compartment one, whichever is smaller. Finally, if $W_n(X_n) > 0$, then we have an output of material via compartment two of either $W_n(X_n)$ or all the material in compartment two, whichever is smaller. It should be noted that the assumptions (1.2) and (1.3) are not mathematically

necessary, but based on the physical model outlined above. These assumptions are made so that simultaneous transfers and negative input/transfer/outputs are not allowed. The subsequent results in this paper are identical with or without these assumptions.

Finally, we let

$$\beta = \sum_{i \in J} \pi_i m_i, \tag{1.8}$$

and

$$E_\pi U = \sum_{i \in J} \pi_i E(U_1(i)), \tag{1.9}$$

with equivalent definitions for $E_\pi V$ and $E_\pi W$. In 1985, Tollar established that in the discrete time case the moments $E_\pi U$, $E_\pi V$ and $E_\pi W$ determine the convergence properties of the amount contained in the compartments as time increases to infinity. The convergent compartments will be called subcritical, while the divergent ones will be called critical or supercritical, depending upon the rate of divergence. It was also shown that when the critical or supercritical compartments are appropriately normalized they converge in distribution, independent of the subcritical compartments.

In this paper it will be shown that all of the results in Tollar (1985a,b) still hold true in the continuous time case, as long as at least one compartment requires some normalization. Section 2 is devoted to the four cases in which neither compartment is subcritical, while section 3 is devoted to the four cases where exactly one compartment is critical. The remaining case, where both compartments are subcritical, is left to a subsequent paper.

2. NEITHER COMPARTMENT SUBCRITICAL

As established in Tollar (1985a,b), the values $Z_1(t)$ and $Z_2(t)$ diverge as t approaches infinity if $E_\pi U \geq E_\pi V \geq E_\pi W$, for $E_\pi U$, $E_\pi V$, and $E_\pi W$ defined as in (1.9). Depending upon whether or not the equality is strict, there are four distinct cases for the behavior of the process. It is shown that the limit behavior of all the cases can be represented as functionals of Brownian motion when the contents are appropriately normalized.

For the proof in this and the subsequent section, the following definitions are needed. For $i_0 \in J$, an arbitrary element, let us define $t_1, t_2, \ldots t_n$ recursively by $t_0 \equiv 0$, and

$$t_n = \min\{i > t_{n-1} : X_i = i_0\}. \tag{2.1}$$

Let
$$M_{i_0}(t) = \sup\{n : T_{t_n} \leq t\}. \tag{2.2}$$

Further, let us define $Y^{(i)}_{u-v}$ by

$$Y^{(i)}_{u-v} = \sum_{j=t_{i-1}+1}^{t_i} [U_i(X_i) - V_i(X_i)] - (T_{t_i} - T_{t_{i-1}})\beta^{-1}(E_\pi U - E_\pi V), \tag{2.3}$$

and define $Y^{(i)}_{u-w}$ and $Y^{(i)}_{v-w}$ analagously. The properties of $Y^{(i)}_{u-v}$ can be found in Balagopal (1978), but essentially they are that $\{Y^{(n)}_{u-v} : n = 2, 3, \ldots\}$ is and i.i.d. sequence with $E\, Y^{(i)}_{u-v} = 0$. Let us also define $\tilde{Y}^{(i)}_{u-v}$ by

$$\tilde{Y}^{(i)}_{u-v} = \sum_{j=t_{i-1}+1}^{t_i} |U_i(X_i) - V_i(X_i)|. \tag{2.4}$$

As in most proofs for normalized processes defined on Markov renewal processes, we will show that the process at hand behaves asymptotically like a functional of i.i.d. random variables by focusing on the return times to the state i_0. The limit results will then follow from the behavior of the i.i.d. random variables.

In all four cases, since $Z_1(t)$ and $Z_2(t)$ will be appropriately normalized, the initial values $Z_{1,0}$ and $Z_{2,0}$ are immaterial, and will be set to zero. Then (1.5) reduces to the simpler form of

$$[Z_1(t),\ Z_2(t)] = [\max_{0\le j\le M(t)} (S_{M(t)} - S_j),$$

$$\max_{0\le j\le k\le M(t)} (S_k - S_j + P_{M(t)} - R_k) - \max_{0\le j\le M(t)} (S_{M(t)} - S_j)]. \tag{2.5}$$

THEOREM 2.1. If $E_\pi U > E_\pi V > E_\pi W$, $\beta < \infty$,

$\sigma_1^2 = E[(Y^{(1)}_{u-v})^2 | X_0 = i_0] < \infty$, $\sigma_2^2 = E[(Y^{(1)}_{v-w})^2 | X_0 = i_0) < \infty$, then as $t \to \infty$,

$$P(Z_1(t) - t\beta^{-1}(E_\pi U - E_\pi V) \le x(t\pi_{i_0}\sigma_1^2\beta^{-1})^{-\frac{1}{2}},$$

$$Z_2(t) - t\beta^{-1}(E_\pi V - E_\pi W) \le y(t\pi_{i_0}\sigma_2^2\beta^{-1})^{-\frac{1}{2}}) \to P(X \le x,\ Y \le y)$$

where $(X,Y) \sim N(0,\Sigma)$, with

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \text{ and } \rho = (\sigma_1\sigma_2)^{-1}E(Y^{(1)}_{u-v}Y^{(1)}_{v-w} | X_0 = i_0).$$

PROOF. From (2.5), it can be shown that

$$Z_1(t) = S_{M(t)} - \max_{0\le j\le M(t)} (-S_j)$$

and

$$Z_2(t) = R_{M(t)} - S_{M(t)} + \max_{0\le j\le k\le M(t)} (-S_j + S_k - R_k) - \max_{0\le j\le M(t)} (-S_j).$$

Since $E_\pi U > E_\pi V > E_\pi W$, it follows from Tollar (1985a) that $\lim_{n\to\infty} \max_{0\le j\le n} (-S_j) = Z$,

where $P(Z < \infty) = 1$, and that

$$P(\lim_{n\to\infty} \max_{0\le j\le k\le n} (-S_j + S_k - R_k) < \infty) = 1.$$

Since both are monotone functions in n, we see that

$$\lim_{t\to\infty} \max_{0\le j\le M(t)} (-S_j) < \infty \text{ a.s.}, \quad \lim_{t\to\infty} \max_{0\le j\le k\le M(t)} (-S_j + S_k - R_n) < \infty \text{ a.s.}.$$

As such,

$$\lim_{t\to\infty} t^{-1/2}[Z_1(t) - S_{M(t)}] = 0 \text{ a.s.},$$

$$\lim_{t\to\infty} t^{-1/2}[Z_2(t) - (R_{M(t)} - S_{M(t)})] = 0 \text{ a.s.}.$$

It is well-known (see Puri and Woolford (1981)) that for sums on semi-Markov processes, $\lim_{t\to\infty} P(S_{M(t)} - t\beta^{-1}(E_\pi U - E_\pi V)) \le x(t\pi_{i_0}\beta^{-1}\sigma_1^2)^{\frac{1}{2}})$ converges to the standard normal cumulative distribution function. A simple extension to bivariate processes yields the theorem. □

THEOREM 2.2. If $E_\pi U > E_\pi V = E_\pi W$, $\beta < \infty$, $\sigma_1^2 = E[(Y_{u-v}^{(1)})^2 | X_0 = i_0] < \infty$, and $E[(\tilde{Y}_{v-w}^{(1)})^2 | X_0 = i_0] < \infty$, then as $t \to \infty$

$$P(Z_1(t) - t\beta^{-1}(E_\pi U - E_\pi V) \le x(t\pi_{i_0}\beta^{-1}\sigma_1^2)^{\frac{1}{2}},$$

$$Z_2(t) \le y(t\pi_{i_0}\beta^{-1}\sigma_2^2)^{\frac{1}{2}})$$

$$\to P(\rho Z_2(1) + \sqrt{1-\rho^2}\, Z_1(1) \le x, \sup_{0\le t\le 1}(Z_2(t)) \le y),$$

where $Z_1(\cdot)$, $Z_2(\cdot)$ are independent standard Brownian motion processes, $\sigma_2^2 = E[(Y_{v-w}^{(1)})^2 | X_0 = i_0)$, and $\rho = (\sigma_1\sigma_2)^{-1}E[Y_{u-v}^{(1)}Y_{v-w}^{(1)} | X_0 = i_0]$.

PROOF. As in theorem 2.1, it is clear that $\lim_{t\to\infty} t^{-\frac{1}{2}}|Z_1(t) - S_{M(t)}| = 0$. From (2.5), it is clear that

$$\max_{0\le k\le M(t)}((R_{M(t)} - S_{M(t)}) - (R_k - S_k)) - \max_{0\le j\le M(t)}(-S_j)$$

$$\le Z_2(t) \le \max_{0\le k\le M(t)}((R_{M(t)} - S_{M(t)}) - (R_k - S_k)).$$

Since $t^{-\frac{1}{2}} \max_{0\le j\le M(t)} (-S_j) \to 0$, we have that

$$\lim_{t\to\infty} t^{-\frac{1}{2}}[Z_2(t) - \max_{0\le k\le M(t)} [(R_{M(t)} - S_{M(t)}) - (R_k - S_k)] = 0. \qquad (2.6)$$

From (2.2) and (2.3), we have that

$$S_{M(t)} - t\beta^{-1}(E_\pi U - E_\pi V)$$

$$= Y^{(1)}_{u-v} + \sum_{i=2}^{M_{i_0}(t)} Y^{(i)}_{u-v} + \sum_{t_{M_{i_0}(t)+1}}^{M(t)} [U_i(X_i) - V_i(X_i)] - (t - T_{t_{M_{i_0}(t)}})\beta^{-1}(E_\pi U - E_\pi V).$$

By appealing to renewal theory, as in Puri and Woolford (1981), it is easily seen that

$$\lim_{t\to\infty} P\Big(\sum_{t_{M_{i_0}(t)+1}}^{M(t)} [U_i(X_i) - V_i(X_i)] \le x\Big) \to F(x), \text{ for some cumulative}$$

distribution function F. As such,

$$t^{-\frac{1}{2}} \sum_{t_{M_{i_0}(t)+1}}^{M(t)} [U_i(X_i) - V_i(X_i)] \longrightarrow 0.$$

By a simple renewal result (see Karlin and Taylor (1975)), it also follows that

$$t^{-\frac{1}{2}}(t - T_{t_{M_{i_0}(t)}}) \xrightarrow{p} 0.$$

As such,

$$t^{-\frac{1}{2}}[Z_1(t) - \sum_{i=2}^{M_{i_0}(t)} Y_{u-v}^{(i)}] \xrightarrow{p} 0.$$

Also,

$$\left| \max_{0\le k\le M(t)} [(R_{M(t)} - S_{M(t)}) - (R_k - S_k)] \right.$$

$$\left. - \max_{0\le k\le t_{M_{i_0}(t)}} [(R_{t_{M_{i_0}(t)}} - S_{t_{M_{i_0}(t)}}) - (R_k - S_k)] \right|$$

$$\le \sum_{t_{M_{i_0}(t)}+1}^{M(t)} |V_i(X_i) - W_i(X_i)|,$$

and a similar technique yields that

$$t^{-\frac{1}{2}} \sum_{t_{M_{i_0}(t)}+1}^{M(t)} |V_i(X_i) - W_i(X_i)| \xrightarrow{p} 0.$$

Let

$$U_j = \max_{t_{j-1}\le \ell\le t_j} [(R_{t_j} - S_{t_j}) - (R_\ell - S_\ell)].$$

It follows that

$$\max_{1\le j\le M_{i_0}(t)}\left(\sum_{j+1}^{M_{i_0}(t)} Y_{v-w}^{(i)}\right)$$

$$\le \max_{0\le k\le t_{M_{i_0}(t)}} [(R_{t_{M_{i_0}(t)}} - S_{t_{M_{i_0}(t)}}) - (R_k - S_k)] \tag{2.7}$$

$$\le |Y_{v-w}^{(1)}| + \max_{1\le j\le M_{i_0}(t)}\left(\sum_{j+1}^{M_{i_0}(t)} Y_{v-w}^{(i)}\right) + \max_{1\le j\le M_{i_0}(t)} (U_j).$$

From Karlin and Taylor (1975), there is a net $\{\varepsilon_t : t \ge 0\}$ where $\varepsilon_t \downarrow 0$, and for all $t > 0$, letting

$$\begin{aligned} \ell_1(t) &= [(\pi_{i_0}\beta^{-1} - \varepsilon_t)t], \\ \ell(t) &= [\pi_{i_0}\beta^{-1}t], \\ \ell_2(t) &= [(\pi_{i_0}\beta^{-1} + \varepsilon_t)t], \end{aligned} \tag{2.8}$$

we have that

$$P(\ell_1(t) \le M_{i_0}(t) \le \ell_2(t)) > 1 - \varepsilon_t. \tag{2.9}$$

Therefore,

$$P\Big(\max_{1\le j\le M_{i_0}(t)}(U_j) > \epsilon t^{\frac{1}{2}}\Big) \le \sum_{i=1}^{\ell_2(t)} P(U_i > \epsilon t^{\frac{1}{2}}) + \epsilon_t$$

$$= \ell_2(t)P(U_1 > \epsilon t^{\frac{1}{2}}) + \epsilon_t.$$

Since $E\,[(\tilde{Y}^{(1)}_{v-w})^2 | X_0 = i_0) < \infty$, $\lim_{n\to\infty} n^2 P(U_1 \ge \epsilon n) = 0$,

which implies

$$\lim_{t\to\infty} P\Big(\max_{1\le j\le M_{i_0}(t)}(U_j) > \epsilon t^{\frac{1}{2}}\Big) = 0.$$

Clearly $t^{-\frac{1}{2}}|Y^{(1)}_{v-w}| \xrightarrow{P} 0$. Therfore, it follows from (2.6) and (2.7) that

$$t^{-\frac{1}{2}}\Big[Z_2(t) - \max_{1\le j\le M_{i_0}(t)}\Big(\sum_{i=j+1}^{M_{i_0}(t)} Y^{(i)}_{v-w}\Big)\Big] \xrightarrow{P} 0. \tag{2.10}$$

Clearly,

$$\Big[\sum_{i=2}^{M_{i_0}(t)} Y^{(i)}_{u-v},\ \max_{1\le j\le M_{i_0}(t)}\Big(\sum_{j+1}^{M_{i_0}(t)} Y^{(i)}_{v-w}\Big)\Big]$$

$$\stackrel{d}{=}\Big[\sum_{i=2}^{M_{i_0}(t)} Y^{(i)}_{u-v},\ \max_{1\le j\le M_{i_0}(t)}\Big(\sum_{i=2}^{j} Y^{(i)}_{v-w}\Big)\Big].$$

From Kolmogorov's inequality and (2.9) we have that

$$P(|\sum_{2}^{M_{i_0}(t)} Y_{u-v}^{(i)} - \sum_{2}^{\ell(t)} Y_{u-v}^{(i)}| > \epsilon t^{\frac{1}{2}})$$

$$\leq P(\max_{\ell_1(t)\leq j\leq \ell(t)} |\sum_{\ell_1(t)+1}^{\ell(t)} Y_{u-v}^{(i)}| > \epsilon t^{\frac{1}{2}}) + P(\max_{\ell(t)\leq j\leq \ell_2(t)} |\sum_{\ell(t)+1}^{\ell_2(t)} Y_{u-v}^{(i)}| > \epsilon t^{\frac{1}{2}}) + \epsilon_t$$

$$\leq (\ell(t) - \ell_1(t)\sigma_1^2(\epsilon^2 t)^{-1} + (\ell_2(t) - \ell(t))\sigma_1^2(\epsilon^2 t)^{-1} + \epsilon_t.$$

$$= [\ell_2(t) - \ell_1(t)]\sigma_1^2(\epsilon^2 t)^{-1} + \epsilon_t.$$

From (2.8) we see that

$$\lim_{t\to\infty} P(|\sum_{2}^{M_{i_0}(t)} Y_{u-v}^{(i)} - \sum_{2}^{\ell(t)} Y_{u-v}^{(i)}| > \epsilon t^{\frac{1}{2}}) = 0. \tag{2.11}$$

Similarly,

$$P(|\max_{1\leq j\leq M_{i_0}(t)} (\sum_{2}^{j} Y_{v-w}^{(i)}) - \max_{1\leq j\leq \ell(t)} (\sum_{2}^{j} Y_{v-w}^{(i)})| > \epsilon t^{\frac{1}{2}})$$

$$\leq (\ell(t) - \ell_1(t))\sigma_2^2(\epsilon^2 t)^{-1} + (\ell_2(t) - \ell(t))\sigma_2^2(\epsilon^2 t)^{-1} + \epsilon_t. \tag{2.12}$$

From (2.6), (2.10), (2.11) and (2.12) it follows that

$$\lim_{t\to\infty} P(Z_1(t) - t\beta^{-1}(E_\pi U - E_\pi V) \le x(t\pi_{i_0}\beta^{-1}\sigma_1^2)^{\frac{1}{2}},\ Z_2(t) \le y(t\pi_{i_0}\beta^{-1}\sigma_2^2)^{\frac{1}{2}})$$

$$= \lim_{t\to\infty} P(\sum_{2}^{\ell(t)} Y_{u-v}^{(i)} \le x(t\pi_{i_0}\beta^{-1}\sigma_1^2)^{\frac{1}{2}},\ \max_{1\le j\le \ell(t)} (\sum_{i=2}^{j} Y_{v-w}^{(i)}) \le y(t\pi_{i_0}\beta^{-1}\sigma_2^2)^{\frac{1}{2}}). \tag{2.13}$$

Since $(n^{-\frac{1}{2}}\sum_{2}^{[ns]} Y_{u-v}^{(i)},\ n^{-\frac{1}{2}}\sum_{2}^{[nt]} Y_{v-w}^{(i)})$ converges uniformly to bivariate Brownian motion, and the maximum is a continuous functional on $C[0,1]$, it follows from results of Donsker (1951) and Billingsley (1968) that

$$\lim_{t\to\infty} P(\sum_{2}^{\ell(t)} Y_{u-v}^{(i)} \le x(t\pi_{i_0}\beta^{-1}\sigma_1^2)^{\frac{1}{2}},\ \max_{1\le j\le \ell(t)} (\sum_{i=2}^{j} Y_{v-w}^{(i)}) \le y(t\pi_{i_0}\beta^{-1}\sigma_2^2)^{\frac{1}{2}})$$

$$= P(\rho Z_2(1) + \sqrt{1-\rho^2}\, Z_1(1) \le x,\ \sup_{0\le t\le 1} (Z_2(t)) \le y),$$

for $Z_1(\cdot)$ and $Z_2(\cdot)$ independent standard Brownian motion, which from (2.13) completes the proof. □

It should be noted that the above theorem is just the continuous time analog to the theorems proven in Tollar (1985b) for the discrete time case. The techniques used to reduce from continuous time to a fixed number of i.i.d. random couplets are similar to those used in the above and analogous to those in Tollar (1985b). As such, the following proofs will be less detailed in this reduction.

THEOREM 2.3. If $E_\pi U = E_\pi V > E_\pi W$, $\beta < \infty$, $E[(\tilde{Y}^{(1)}_{u-v})^2 | X_0 = i_0) < \infty$, and

$\sigma_2^2 = E[(Y^{(1)}_{u-w})^2 | X_0 = i_0) < \infty$, then as $t \to \infty$

$$P(Z_1(t) \le x(t\pi_{i_0}\beta^{-1})^{\frac{1}{2}},\ Z_2(t) - t\beta^{-1}(E_\pi U - E_\pi W) \le y(t\pi_{i_0}\beta^{-1})^{\frac{1}{2}})$$

$$\to P(\sigma_1 \sup_{0\le t\le 1}(Z_1(t)) \le x,\ \sigma_2(\rho Z_1(1) + \sqrt{1-\rho^2}\, Z_2(1)) - \sigma_1 \sup_{0\le t\le 1}(Z_1(t)) \le y),$$

where $Z_1(\cdot)$, $Z_2(\cdot)$ are independent standard Brownian motion processes,

$\sigma_1^2 = E[(Y^{(1)}_{u-v})^2 | X_0 = i_0)$, and $\rho = (\sigma_1\sigma_2)^{-1}E[Y^{(1)}_{u-v}Y^{(1)}_{u-w} | X_0 = i_0)$.

PROOF. From (3.5), we can show that

$$Z_2(t) = R_{M(t)} + \max_{0\le j\le k\le M(t)}(-S_j + S_k - R_k) - \max_{0\le j\le M(t)}(S_{M(t)} - S_j)).$$

Again, from Tollar (1985b), we have that $t^{-\frac{1}{2}} \max_{0\le j\le k\le M(t)}(-S_j + S_k - R_k) \to 0$.

By arguments similar to those of theorem 2.2, it can be shown that for

$\ell(t) = [\pi_{i_0}\beta^{-1}t]$

$$t^{-\frac{1}{2}}|Z_1(t) - \max_{1\le j\le \ell(t)}(\sum_{i=j+1}^{\ell(t)} Y^{(i)}_{u-v})| \xrightarrow{P} 0,$$

and

$$t^{-\frac{1}{2}}|Z_2(t) - t\pi_{i_0}\beta^{-1}(E_\pi U - E_\pi W) - \sum_{2}^{\ell(t)} Y^{(i)}_{u-w} + \max_{1\le j\le \ell(t)} \Big(\sum_{i=j+1}^{\ell(t)} Y^{(i)}_{u-v}\Big)| \xrightarrow{P} 0.$$

Again, by uniform convergence, we can establish that

$$\lim_{t\to\infty} P\Big[\max_{1\le j\le \ell(t)} \Big(\sum_{i=j+1}^{\ell(t)} Y^{(i)}_{u-v}\Big) \le x(t\pi_{i_0}\beta^{-1})^{\frac{1}{2}},$$

$$\sum_{2}^{\ell(t)} Y^{(i)}_{u-w} - \max_{1\le j\le \ell(t)} \Big(\sum_{i=j+1}^{\ell(t)} Y^{(i)}_{u-v}\Big) \le y(t\pi_{i_0}\beta^{-1})^{\frac{1}{2}}\Big]$$

$$= P(\sigma_1 \sup_{0\le t\le 1} (Z_1(t) \le x,\ \sigma_2(\rho Z_1(1) + \sqrt{1-\rho^2}\, Z_2(1)) - \sigma_1 \sup_{0\le t\le 1} (Z_1(t)) \le y). \quad \square$$

THEOREM 2.4. If $E_\pi U = E_\pi V = E_\pi W$, $\beta < \infty$, $E[(\tilde{Y}^{(1)}_{u-v})^2 | X_0 = i_0] < \infty$, and

$E[(\tilde{Y}^{(1)}_{v-w})^2 | X_0 = i_0] < \infty$, then as $t \to \infty$

$$P(Z_1(t) \le x(t\pi_{i_0}\beta^{-1})^{\frac{1}{2}},\ Z_2(t) \le y(t\pi_{i_0}\beta^{-1})^{\frac{1}{2}}) \longrightarrow P(\sigma_1 \sup_{0\le t\le 1} (Z_1(t)) \le x,$$

$$\Big[\sup_{0\le t\le s\le 1} (\sigma_2(\rho Z_1(t) + \sqrt{1-\rho^2}\, Z_2(t)) + \sigma_1 Z_1(s)) - \sigma_1 \sup_{0\le t\le 1} (Z_1(t))\Big] \le y),$$

where $Z_1(\cdot)$, $Z_2(\cdot)$, σ_1^2, σ_2^2, and ρ are as in theorem 3.2.

PROOF. As in theorem 2.3, it can be established that

$$t^{-\frac{1}{2}}(Z_1(t) - \max_{1\le j\le \ell(t)} (\sum_{i=j+1}^{\ell(t)} Y_{u-v}^{(i)})) \xrightarrow{P} 0,$$

and $t^{-\frac{1}{2}}(Z_2(t) - \max_{1\le j\le k\le \ell(t)} (\sum_{i=j+1}^{k} Y_{u-v}^{(i)} + \sum_{i=k+1}^{\ell(t)} Y_{v-w}^{(i)}) + \max_{1\le j\le \ell(t)} (\sum_{i=j+1}^{\ell(t)} Y_{u-v}^{(i)})) \xrightarrow{P} 0.$

Again by the i.i.d. nature of $\{(Y_{u-v}^{(i)}, Y_{v-w}^{(i)})\}$, we have that

$$\lim_{t\to\infty} P(\max_{1\le j\le \ell(t)} (\sum_{i=2}^{\ell(t)} Y_{u-v}^{(i)}) \le x(\ell(t))^{-\frac{1}{2}},$$

$$\max_{1\le j\le k\le \ell(t)} (\sum_{i=2}^{j} Y_{v-w}^{(i)} + \sum_{i=j+1}^{k} Y_{u-v}^{(i)}) - \max_{1\le j\le \ell(t)} (\sum_{i=2}^{\ell(t)} Y_{u-v}^{(i)}) \le y(\ell(t))^{\frac{1}{2}})$$

$$= P(\sigma_1 \sup_{0\le t\le 1} (Z_1(t) \le x,$$

$$[\sup_{0\le t\le s\le 1} (\sigma_2(\rho Z_1(t) + \sqrt{1-\rho^2}\, Z_2(t)) + \sigma_1 Z_1(s)) - \sigma_1 \sup_{0\le t\le 1} (Z_1(t))] \le y). \quad \square$$

It should be pointed out that the Brownian motion function expressions of theorems 2.2 and 2.3 can be expressed in an integral form instead of the form given. However, the expression of theorem 2.4 seems not to have a form simpler than given.

In the next section, we examine the four cases in which one compartment is subcritical, and the other is either critical or supercritical.

3. ONE COMPARTMENT SUBCRITICAL.

Again, there are four separate cases to be considered. In all four cases, it will be shown that the subcritical compartment is asymptotically independent of the suitably normalized critical or supercritical compartment. The marginal behavior of the two compartments then completes the results.

The independence is established by appealing to theorem 5.2 of Puri and Tollar (1985), which essentially states that for two processes Z(t) and Y(t) where

$$\begin{aligned}
&\text{a)} \quad \text{for each } \varepsilon > 0 \text{ there is a } \tau > 0, \text{ a } T > 0, \text{ and a process} \\
&\qquad Z_\tau(t) \text{ where for } t > T,\ P(|Z_\tau(t) - Z(t)| > \varepsilon) < \varepsilon, \\
&\text{b)} \quad t^{-\frac{1}{2}}(Y(t-\tau) - Y(t)) \xrightarrow{p} 0, \qquad (3.1) \\
&\text{c)} \quad Y(t-\tau) \text{ and } Z_\tau(t) \text{ are conditionally independent} \\
&\qquad \text{given } X_{M(t-\tau)+1},\ T_{M(t-\tau)+1},
\end{aligned}$$

then

$$\lim_{t\to\infty} P(Z(t) \le x,\ Y(t) \le yt^{\frac{1}{2}}) = \lim_{t\to\infty} P(Z(t) \le x) \lim_{t\to\infty} P(Y(t) \le yt^{\frac{1}{2}}) \qquad (3.2)$$

Lemma 5.1 of Puri and Tollar (1985) observes that for Z(t) of the form,

$$Z(t) = \max\left[Z_0 + \sum_{i=1}^{M(t)} U_i(X_i),\ \max_{1\le j\le M(t)} \left(\sum_{i=j+1}^{M(t)} U_i(X_i)\right)\right] \qquad (3.3)$$

where $E_\pi U < 0$, then

$$Z_\tau(t) = \max_{M(t-\tau)\le j\le M(t)} \left(\sum_{i=j+1}^{M(t)} U_i(X_i) \right) \tag{3.4}$$

will have the property of (3.1a).

In the following theorems we define for $-\infty < y < \infty$,

$$\Phi(y) = \int_{-\infty}^{y} (2\pi)^{-\frac{1}{2}} \exp(-u^2/2)du,$$

and

$$|\Phi|(y) = [2\Phi(y) - 1]I(y \ge 0).$$

THEOREM 3.1. If $E_\pi U < E_\pi V$, $E_\pi U > E_\pi W$, $\beta < \infty$, $\sigma_1^2 = E[(Y_{u-w}^{(1)})^2 | X_0 = i_0) < \infty$, then for all continuity points x of $P(Z_1 \le x)$,

$$\lim_{t\to\infty} P(Z_1(t) \le x,\ Z_2(t) - t(E_\pi U - E_\pi W)\beta^{-1} \le y[\sigma_1^2 t \pi_{i_0} \beta^{-1}]^{\frac{1}{2}})$$

$$= P(Z_1 \le x)\Phi(y),$$

where Z_1 is a random variable such that $P(|Z_1| < \infty) = 1$.

PROOF. From (1.5), since $E_\pi(U - V) < 0$, it is clear that $Z_1(t)$ is of the desired form of (3.3). From Puri and Tollar (1985), $Z_1(t) \xrightarrow{d} Z_1$, and there exists a $Z_\tau(t)$ of the form (3.4).

From (1.5), we have that

$$Z_2(t) = R_{M(t)} + \max[Z_{1,0} + Z_{2,0},\ Z_{1,0} + \max_{1\le j\le M(t)} (S_k - R_k),$$

$$\max_{1\le j\le k\le M(t)} (S_k - S_j - R_k)] - Z_1(t).$$

Since $E_\pi W - E_\pi U < 0$, $E_\pi W - E_\pi V < 0$, we have from Tollar (1985a) that as $t \to \infty$,

$$\max[Z_{1,0} + Z_{2,0},\ Z_{1,0} + \max_{1\le k\le M(t)} (S_k - R_k),\ \max_{1\le j\le k\le M(t)} (-S_j - S_k - R_k)] \uparrow Z^* < \infty \text{ a.s.}.$$

Therefore, as $t \to \infty$

$$t^{-\frac{1}{2}}[Z_2(t) - P_{M(t)}] \xrightarrow{P} 0.$$

Let

$$Y(t) = R_{M(t)} - t(E_\pi U - E_\pi W)\beta^{-1}.$$

As established in Puri and Tollar (1985), for any fixed τ,

$$t^{-\frac{1}{2}}[R_{M(t)} - R_{M(t-\tau)}] \xrightarrow{P} 0.$$

As such, it is clear that Y(t) fulfills (3.1b). It is equally clear from the strong Markov property that (3.1c) is met by $Z_\tau(t)$ and $Y(t-\tau)$. As such, since

$$\lim_{t\to\infty} P(R_{M(t)} - t(E_\pi U - E_\pi W)\beta^{-1} \le y[\sigma_1^2 t\pi_{i_0}\beta^{-1}]^{\frac{1}{2}}) = \Phi(y)$$

(see Puri and Woolford (1981)), (3.1) completes the theorem □

The proofs of the subsequent theorems will be essentially the same as the previous theorem. It will be shown that the subcritical compartment is asymptotically equivalent to a form like (3.3). Therefore a $Z_\tau(t)$ will exist. $Y(t-\tau)$ will be essentially the supercritical or critical compartment, therefore (3.1) will be satisfied. The arguments of Puri and Tollar (1985) give us the marginal behavior of the subcritical compartment, and arguments entirely analagous to the discrete time arguments of Tollar (1985b) can be extended as was done in theorem 2.2 to deal with the continuous time behavior of the critical or supercritical compartment. As such, in the following theorems, we will only show that the subcritical cell can be reduced to the form of (3.3), then simply state the desired $Y(t)$ and its convergence properties.

THEOREM 3.2. If $E_\pi U < E_\pi V$, $E_\pi U = E_\pi W$, $\beta < \infty$, $E[(\tilde{Y}^{(1)}_{u-v})^2 | X_0 = i_0) < \infty$, then for all continuity points x of $P(Z_1 \le x)$,

$$\lim_{t\to\infty} P(Z_1(t) \le x,\ Z_2(t) \le y(\sigma_1^2 t\pi_{i_0}\beta^{-1})^{\frac{1}{2}}) = P(Z_1 \le x)|\Phi|(y),$$

where $\sigma_1^2 = E(Y^{(1)}_{u-w})^2 | X_0 = i_0)$.

PROOF. Since $Z_1(t)$ is clearly of the form (3.3), letting

$$Y(t) = \max_{0\le j\le k\le M(t)} (S_k - S_j + R_{M(t)} - R_k),$$

by arguments identical to the discrete time arguments of Tollar (1985b), it can be shown that

$$t^{-\frac{1}{2}}[Y(t) - \max_{0\le k\le M(t)} (R_{M(t)} - R_k)] \xrightarrow{p} 0,$$

and from Puri and Woolford (1981), it follows that

$$\lim_{t\to\infty} P(Z_2(t) \le y(\sigma_1^2 t\pi_{i_0}\beta^{-1})^{\frac{1}{2}}) = |\Phi|(y). \quad \square$$

THEOREM 3.3. If $E_\pi U = E_\pi V$, $E_\pi V < E_\pi W$, $\beta < \infty$, $E[(\tilde{Y}^{(1)}_{u-v})^2 | X_0 = i_0) < \infty$, then for all continuity points y of $P(Z_2 \le y)$,

$$\lim_{t\to\infty} P(Z_1(t) \le x(\pi_{i_0}\sigma_1^2\beta^{-1}t)^{\frac{1}{2}},\ Z_2(t) \le y) = |\Phi|(x)P(Z_2 \le y),$$

where $\sigma_1^2 = E[(Y^{(1)}_{u-v})^2 | X_0 = i_0)$.

PROOF. Let $Y(t) = \max_{0\le j\le M(t)} (S_{M(t)} - S_j)$. It can be shown by renewal arguments as in Puri and Tollar (1985) that $t^{-\frac{1}{2}}(Y(t) - Y(t-\tau)) \xrightarrow{p} 0$, and we have from Puri and Woolford (1981) that

$$\lim_{t\to\infty} P(Z_1(t) \le x(\pi_{i_0}\sigma_1^2\beta^{-1}t)^{\frac{1}{2}}) = |\Phi|(x).$$

Therefore, to complete the proof via (3.1) we need to find an appropriate $Z_\tau(t)$.

Let $Q_n = R_n - S_n$,

and

$$Z_\tau(t) = \max_{M(t-\tau)\le j\le M(t)} (Q_{M(t)} - Q_j). \tag{3.9}$$

To show that for each $\varepsilon > 0$, there is a $\tau > 0$ and a $T > 0$ where for $t > T$,

$$P(|Z_\tau(t) - Z_2(t)| > \varepsilon) < \varepsilon,$$

we define three sets for specified τ and t;

$$A(t) = \{\omega:\ R_{M(t)} + Z_{1,0} + Z_{2,0} < 0\},$$

$$B_\tau(t) = \{\omega:\ \max_{M(t-\tau)\le j\le M(t)} (Q_{M(t)} - Q_j) = \max_{1\le j\le M(t)} (Q_{M(t)} - Q_j)\},$$

$$C_\tau(t) = \{\omega:\ \max_{1\le j\le M(t-\tau)} (S_{M(t)} - S_j) = \max_{1\le j\le M(t)} (S_{M(t)} - S_j)\}. \tag{3.6}$$

Since $E_\pi U - E_\pi W < 0$, it is clear that

$$\lim_{t\to\infty} P(A(t)) = 1 \tag{3.7}$$

Also, from lemma 5.1 of Puri and Tollar (1985), it follows for all $\varepsilon > 0$, there is a T_1 and a T_2 where for all $t > T_1$, $\tau > T_2$,

$$P(B_\tau(t)) > 1 - \varepsilon \tag{3.8}$$

To examine $C_\tau(t)$, we observe from Puri and Woolford (1981), that for $\delta > 0$ such that $|\Phi|(\delta) < \varepsilon$,

$$\lim_{t\to\infty} P\Big(\max_{1\le j\le M(t)} (S_{M(t)} - S_j) \ge \delta(\pi_{i_0} \beta^{-1}\sigma_1^2 t)^{\frac{1}{2}}\Big) = 1 - |\Phi|(\delta) > 1 - \varepsilon. \tag{3.9}$$

From Puri and Tollar (1985) we observe that for fixed τ, as $t \to \infty$,

$$t^{-\frac{1}{2}} \max_{M(t-\tau)\le j\le M(t)} (S_{M(t)} - S_j) \le t^{-\frac{1}{2}} \sum_{i=M(t-\tau)+1}^{M(t)} |U_i(X_i) - V_i(X_i)| \xrightarrow{P} 0. \tag{3.10}$$

Therefore,

$$P(C_\tau(t)) \ge P\Big[\max_{1\le j\le M(t)} (S_{M(t)} - S_j) \ge \delta(\pi_{i_0} \beta^{-1}\sigma_1^2 t)^{\frac{1}{2}} > \max_{M(t-\tau)\le j\le M(t)} (S_{M(t)} - S_j)\Big],$$

which, when coupled with (3.9) and (3.10) yields that for fixed τ,

$$\lim_{t\to\infty} P(C_\tau(t)) > 1 - \varepsilon. \tag{3.11}$$

From (3.7), (3.8) and (3.11), let us select a τ and a T such that for $t > T$,

$$P(A(t) \cap B_\tau(t) \cap C_\tau(t)) > 1 - 3\varepsilon.$$

Then, for $\omega \in A(t) \cap B_\tau(t) \cap C_\tau(t)$, we have from (1.5) and (3.5) that

$$Z_2(t)(\omega) = \max[Z_{1,0} + Z_{2,0} + R_{M(t)},\ Z_{1,0} + \max_{1\le k\le M(t)} (R_{M(t)} + S_k - R_k),$$

$$\max_{1\le j\le k\le M(t)} (S_k - S_j + R_{M(t)} - R_k)] - \max\ (Z_{1,0} + S_{M(t)},\ \max_{1\le j\le M(t)} (S_{M(t)} - S_j))$$

$$= \max[0, Z_{1,0} + S_{M(t)} + \max_{1\le k\le M(t)} (Q_{M(t)} - Q_k),$$

$$\max_{1\le j\le k\le M(t)} (S_{M(t)} - S_j + Q_{M(t)} - Q_k)] - \max[Z_{1,0} + S_{M(t)},\ \max_{1\le j\le M(t)} (S_{M(t)} - S_j)]$$

$$= \max[Z_{1,0} + S_{M(t)} + \max_{M(t-\tau)\le k\le M(t)} (Q_{M(t)} - Q_k),$$

$$\max_{1\le j\le M(t-\tau)} (S_{M(t)} - S_j) + \max_{M(t-\tau)\le k\le M(t)} (Q_{M(t)} - Q_k)]$$

$$- \max[Z_{1,0} + S_{M(t)},\ \max_{1\le j\le M(t-\tau)} (S_{M(t)} - S_j)] = \max_{M(t-\tau)\le k\le M(t)} (Q_{M(t)} - Q_k) = Z_\tau(t).$$

Therefore, we have that for any $\varepsilon > 0$,

$$\lim_{t\to\infty} P(Z_2(t) = Z_\tau(t)) > 1 - 3\varepsilon,$$

which from (3.2) implies the theorem. □

THEOREM 3.4. If $E_\pi U > E_\pi V$, $E_\pi V < E_\pi W$, $\beta < \infty$, and $\sigma_1^2 = E((Y_{u-v}^{(1)})^2 | X_0 = i_0) < \infty$, then for all continuity points y of $P(Z_2 \le y)$

$$\lim_{t\to\infty} P(Z_1(t) - t\beta^{-1}(E_\pi U - E_\pi V) \le x(t\pi_{i_0}\beta^{-1}\sigma_1^2)^{1/2},$$

$$Z_2(t) \le y) = \Phi(x)P(Z_2 \le y).$$

PROOF. Letting

$$Y(t) = S_{M(t)} - t\beta^{-1}(E_\pi U - E_\pi V),$$

it can be easily established that $t^{-1/2}|Y(t) - Z_1(t)| \xrightarrow{p} 0$, and that $t^{-1/2}|Y(t-\tau) - Y(t)| \xrightarrow{p} 0$.

Define $Z_\tau(t)$ as in (3.5).

As in the proof of theorem 3.3, for $\tau > 0$, $t > 0$, define

$$A(t) = \{\omega: Z_{1,0} + Z_{2,0} + R_{M(t)} - S_{M(t)} < 0\},$$

$$B_\tau(t) = \{\omega: \max_{M(t-\tau)\le j\le M(t)} (Q_{M(t)} - Q_j) = \max_{1\le j\le M(t)} (Q_{M(t)} - Q_j)\},$$

$$C_\tau(t) = \{\omega: \max_{1\le j\le M(t)} (-S_j) = \max_{1\le j\le M(t-\tau)} (-S_j)\}.$$

Since $E_\pi V < E_\pi W$, it follows that

$$\lim_{t\to\infty} P(A(t)) = 1.$$

Again, from Puri and Tollar (1985) we know for all $\varepsilon > 0$, there is a T_1 and a T_2 where for all $t > T_1$, $\tau > T_2$,

$$P(B_\tau(t)) > 1 - \varepsilon.$$

Since $E_\pi U > E_\pi V$, $-S_n \to -\infty$ a.s., and thus for any fixed τ,

$$\lim_{t\to\infty} P(C_\tau(t)) = 1.$$

Again choosing a τ and a T where for all $t > T$,

$$P(A(t) \cap B_\tau(t) \cap C_\tau(t)) > 1 - 3\varepsilon,$$

we have for all $\omega \in A(t) \cap B_\tau(t) \cap C_\tau(t)$,

$$Z_2(t)(\omega) = \max[Z_{1,0} + Z_{2,0} + R_{M(t)},\ Z_{1,0} + \max_{1\le k\le M(t)} (R_{M(t)} + S_k - R_k),$$

$$\max_{1\le j\le k\le M(t)} (S_k - S_j + R_{M(t)} - R_k)]$$

$$- \max(Z_{1,0} + S_{M(t)},\ \max_{1\le j\le M(t)} (S_{M(t)} - S_j))$$

$$= \max[Z_{1,0} + Z_{2,0} + R_{M(t)} - S_{M(t)},\ Z_{1,0} + \max_{1\le j\le M(t)} (Q_{M(t)} - Q_k),$$

$$\max_{1\le j\le k\le M(t)} (-S_j + Q_{M(t)} - Q_k)] - \max(Z_{1,0},\ \max_{1\le j\le M(t)} (-S_j)).$$

By arguments identical to those of theorem 3.3, we find for $\omega \in A(t) \cap B_\tau(t) \cap C_\tau(t)$,

$$Z_2(t)(\omega) = \max_{M(t-\tau)\le j\le M(t)} (Q_{M(t)}-Q_j) = Z_\tau(t)(\omega),$$

which completed the proof. □

This concludes the examination of the cases where one compartment is subcritical, and the other is either critical or supercritical. It is of interest to note that when one compartment is subcritical, its limit behavior does not depend on whether the other compartment was critical or supercritical.

4. CONCLUSION

In this two compartment model, eight cases have been examined. The remaining case, where both compartments are subcritical, is left to a subsequent paper since the present techniques seem to be insufficient to establish the results. There are of course a variety of generalizations possible. The most obvious is to examine the arbitrary k compartment model in continuous time, analagous to the discrete time behavior of Tollar (1985a,b). Clearly, the critical and supercritical compartments pose no problem, and results analagous to the previous results will still hold. However, the subcritical compartments pose substantial problems. In the case of exactly

one subcritical compartment, results similar to those of section 3 can be established. However, when two or more compartments are subcritical, the techniques in this paper do not seem directly applicable.

There are of course a myriad of other directions for generalization. We will only mention two. One that appears useful would be to define $Z_1(t)$ and $Z_2(t)$ in such a way that two-way flow would be allowed, so assumption (1.3) could be eliminated. Unfortunately, two-way flow does not appear to allow a closed form expression for $Z_1(t)$ and $Z_2(t)$, so the techniques here offer little hope of success. Another area of interest would be to eliminate the assumption of moments, and determine limit behavior in this case. This is left for subsequent work.

LIST OF REFERENCES

Balagopal, K. (1979). Some limit theorems for the general semi-Markov storage model, J. Appl. Prob., 16, 607-617.

Billingsley, P. (1968). Convergence of Probability Measures, J. Wiley and Sons, New York.

Çinlar, E. (1969). Markov renewal theory, Adv. Appl. Prob., 1, 123-187.

Donsker, M. (1951). An invariance principle for certain probability limit theorems, Mem. Amer. Math. Soc., 6.

Karlin, S. and Taylor, H. M. (1975). A First Course in Stochastic Processes, Academic Press, New York.

Puri, P. S. and Senturia, J. (1975). An infinite depth dam with Poisson input and Poisson release. Scand. Actuarial J. 193-202.

Puri, P. S. and Tollar, E. S. (1985). On the limit behavior of certain quantities in a subcritical storage model, Adv. Appl. Prob., 17, 443-459.

Puri, P. S. and Woolford, S. W. (1981). On a generalized storage model with moment assumptions, J. Appl. Prob., 18, 473-481.

Senturia, J. and Puri, P. S. (1973). A semi-Markov storage model, Adv. Appl. Prob., 5, 362-378.

Senturia, J. and Puri, P. S. (1974). Further aspects of a semi-Markov storage model, Sankhyā, A, 36, 369-378.

Tollar, E. S. (1985a). On the limit behavior of a multi-compartment model with an underlying Markov Chain I: without normalization, FSU Statistics Report M 693, Florida State University, Tallahassee, FL.

Tollar, E. S. (1985b). On the limit behavior of a multi-compartment storage model with an underlying Markov chain II: with normalization, FSU Statistics Report M 694, Florida State University, Tallahassee, FL.

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