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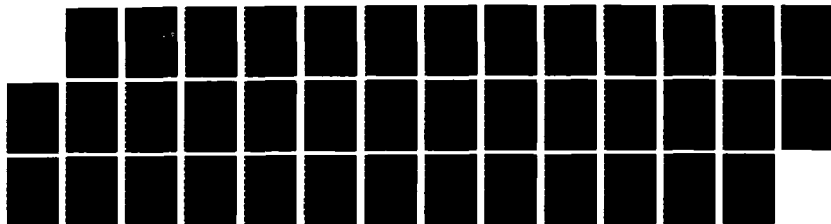
THE RENEWAL EQUATION FOR MARKOV RENEWAL PROCESSES WITH
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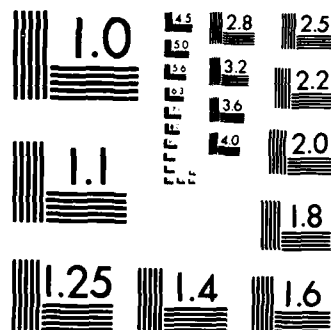
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THE RENEWAL EQUATION FOR MARKOV RENEWAL
PROCESSES WITH APPLICATIONS TO STORAGE MODELS

by

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Abstract

For Markov renewal processes in which the sojourn times are controlled by an imbedded, denumerable state Markov chain, it is shown that there exists a random time at which the Markov renewal process regenerates. The basic renewal theorem is then applied to determine the limiting behavior of the Markov renewal process. These results are applied to a particular two compartment storage model to determine the limiting behavior of the amounts in storage.

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The Renewal Equation for Markov Renewal
Processes with Applications to Storage Models

1. INTRODUCTION

Let J be a denumerable set, and let $\{X_n, n=0,1,\dots\}$ be a stationary, positive recurrent, aperiodic, irreducible Markov chain with state space J . Let π be the stationary measure of $\{X_n\}$. For an arbitrary space $(S, \mathcal{F})$, let $\{Z_n, n=0,1,2,\dots\}$ be a process defined on $(S, \mathcal{F})$ such that $\{(X_n, Z_n), n=0,1,2,\dots\}$ is also a stationary Markov chain with transition probabilities

$$P^n(i, y: (j, A)) = P(X_n = j, Z_n \in A | X_0 = i, Z_0 = y), \quad (1.1)$$

for $i, j \in J, y \in S, A \in \mathcal{F}$.

Let $0 \equiv T_0 \leq T_1 \leq T_2 \leq \dots$ be a sequence of random variables defined such that $\{((X_n, Z_n), T_n), n=0,1,\dots\}$ is a Markov renewal process, where for $t \geq 0, A \in \mathcal{F}$,

$$\begin{aligned} P(T_n \leq t, Z_n \in A | X_{n-1}, X_n) \\ = P(T_n \leq t | X_{n-1}, X_n) P(Z_n \in A | X_{n-1}, X_n). \end{aligned} \quad (1.2)$$

That is, the sequences $\{T_n\}$ and $\{Z_n\}$ are conditionally independent given $\{X_n\}$.

The first moment of the sojourn time in state (i, z) is independent of z , and given by

$$m_i = \int_0^{\infty} t \sum_{j \in J} dP(Z_1=j, T_1 \leq t | X_0=i).$$

The average sojourn time we define by

$$\beta = \sum_{i \in J} \pi_i m_i. \quad (1.3)$$

Finally, we define

$$(X(t), Z(t)) = (X_{N(t)}, Z_{N(t)}),$$

where

$$N(t) = \sup\{n: T_n \leq t\}. \quad (1.4)$$

There has been a substantial body of work on semi-Markov processes on arbitrary state spaces, in general directed at the asymptotic behavior of the process, and this paper is no exception. Typically, the authors attempt to establish conditions sufficient to guarantee that the basic renewal theorem can be applied to the process. The approaches have been varied (see Çinlar (1969), Athreya, McDonald and Ney (1978a,b), Athreya and Ney (1978), Kesten (1974), and Nummelin (1978)), but in general seem to be directed at the creation of a stopping time, independent of the future process. Athreya, McDonald and Ney used the properties of so-called C-sets of ϕ -irreducible Markov chains (see Orey (1971)) to propose a method for the creation of an artificial renewal point of the process. Unfortunately, the method does not generalize to all Markov renewal processes. However, we will establish in section 2 that for a process as

defined above, a renewal point can be created. Therefore a renewal equation is available, and results follow from application of the basic renewal theorem.

In the subsequent section, these results are applied to a storage model. In a simpler form, the model was first proposed as a single compartment model by Senturia and Puri (1973), with subsequent research by Senturia and Puri (1974), Puri and Senturia (1975), Puri (1978), Balagopal (1979), Puri and Woolford (1981), and Puri and Tollar (1985). The model was extended to an arbitrary compartment model defined on a Markov chain by Tollar (1985a,b) and was considered with two compartments when defined on a semi-Markov process by Tollar (1986). However, in the last cited paper, the case where both compartments were subcritical was left as an open question. Using the results of section 2, the asymptotic behavior of the storage model when both compartments are subcritical is determined via the basic renewal theorem.

2. RENEWAL EQUATIONS FOR THE SEMI-MARKOV PROCESS

While the structure of $\{X_n, Z_n\}$ is crucial in this paper, for ease of development, let us temporarily discuss Markov chains on arbitrary spaces. Let $\{Y_n, n=0,1,\dots\}$ be a Markov chain which takes values on some arbitrary state space (S,F) , with $P(y,\cdot)$ a regular version of the stationary transition probabilities. Then for $z \in S$, $A \in F$, we define the n -step transition probabilities recursively by

$$P^n(z,A) = \int_S P(z,dy) P^{n-1}(y,A). \quad (2.1)$$

Let ϕ be a non-trivial σ -finite measure on (S, F) .

Definition 1. $\{Y_n\}$ is ϕ -irreducible if, whenever $\phi(A) > 0$ for $A \in F$, then $\sum_{n=1}^{\infty} 2^{-n} P^n(y, A) > 0$, for all $y \in S$.

Definition 2. A σ -finite, non-trivial measure μ on F is called sub-invariant for $\{Y_n\}$ if $\mu(A) \geq \int_S \mu(dy) P(y, A)$ for all $A \in F$, and called invariant if equality holds.

Definition 3. If there is a finite invariant measure μ on F with $\mu(S) = 1$, we call $\{Y_n\}$ ergodic, and μ the stationary measure of $\{Y_n\}$.

Let $\{(Y_n, T_n)\}$ be a semi-Markov process defined on the state space (S, F) , where for all $y \in S$, $A \in F$, $t \geq 0$, $H_{yA}(t)$ is a regular version of the transition function with respect to ϕ . That is,

$$H_{yA}(t) = P(Y_n \in A, T_n - T_{n-1} \leq t | Y_{n-1} = y), \quad (2.2)$$

(for details, see Çinlar (1969)).

The concept of a splitting technique using C-sets to establish asymptotic convergence of the semi-Markov process has been proposed by Nummelin (1978), Athreya, McDonald and Ney (1978a,b), and Athreya and Ney (1978). The pertinent results we summarize below.

HYPOTHESIS. There exists an $A \in F$, an integer $k > 0$, a probability measure ψ on $S \cap A$, a family of probability measures $\nu(x, 0)$ on $\mathbb{R}^+$ for all $x \in A$, and a constant λ , $0 < \lambda < 1$, such that for all $x \in A$, $E \in F$ and $D \in \mathbb{B}^+$,

$$P(Y_k \in E, T_k \in D | Y_0 = x) \geq \lambda \psi(E) v(x, D).$$

If this hypothesis is valid, then Athreya and Ney (1978) establish the following result:

LEMMA 2.1. Subject to the hypothesis, there exists a semi-Markov process
 $\{(Y_n^*, T_n^*), n=1,2,\dots\}$ distributed as $\{(Y_n, T_n), n=0,1,2,\dots\}$, and a random
time N such that for all $B \in F, C \in \mathcal{B}^+, z \in S$,

$$\begin{aligned} P(Y_N^* \in B, T_N^* \in C, N < \infty | Y_0^* = z) \\ = \psi(B) P(T_N^* \in C, N < \infty | Y_0^* = z). \end{aligned}$$

For any probability measure $v(\cdot)$, any random variable X , and any set $A \in F$, we define

$$E_v(X) = \int_S P(X | Y_0 = z) dv(z), \quad (2.3)$$

and

$$P_v(X \in A) = E_v(I_A(X)). \quad (2.4)$$

From lemma 2.1 it can be easily seen that

$$\begin{aligned} P_\psi(Y(t) \in B) &= P_\psi(Y^*(t) \in B, T_N^* > t) \\ &+ \int_0^t P_\psi(Y(t-\tau) \in B) dP_\psi(T_N^* \leq \tau), \end{aligned} \quad (2.5)$$

which implies that renewal theory can be applied. While it is not clear in general whether or not the hypothesis is valid for all ϕ -irreducible Markov renewal processes, we will establish that it follows for $Z_n = \{(X_n, Y_n)\}$ as proposed in section 1. The following lemma is a modification of the proof of the existence of C-sets for ϕ -irreducible Markov chains (see Orey (1971)).

LEMMA 2.2. Let $\{X_n, Z_n\}$ be a ϕ -irreducible Markov chain as defined in Section 1. Then for any set $E \subset S$ and any $j_0 \in J$ where $\phi(j_0, E) > 0$, there is a $k > 0$, a $p > 0$, a sequence $j_1, j_2, \dots, j_{k-1} \in J$, and a set $A \subset E$ with $\phi(j_0, A) > 0$ such that for all $z \in A$, and all $B \subset S$,

$$P(X_n = j_0, X_{k-1} = j_{k-1}, \dots, X_1 = j_1, Z_k \in B | X_0 = j_0, Z_0 = z) \geq p\phi(j_0, B \cap A).$$

PROOF. For convenience of notation, let $\phi_j(C) = \phi(j, C)$ for all $C \subset S$. Further, for any set $U \subset S \times S$ let

$$\text{i) } U_1(x) = \{y: (x, y) \in U\}, \quad (2.6)$$

$$\text{ii) } U_2(y) = \{x: (x, y) \in U\}.$$

Let $i_{1,m}$ stand for a general sequence $i_1, i_2, \dots, i_{m-1}$ (if a particular sequence is necessary, it will be specified). Let $p_{i_{1,m}}(x, y)$ be the Radon-Nikodym derivative of

$$P(Z_m \in (\cdot) | X_0 = j_0, Z_0 = x, X_1 = i_1, \dots, X_{m-1} = i_{m-1}, X_m = j_0)$$

with respect to ϕ_{j_0} . Finally, for our set E , let

$$H_{i_{1,m}}^{(n)} = \{(x,y) \in E \times E : p_{i_{1,m}}(x,y) \geq 1/n\}, \quad (2.7)$$

and

$$H = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{\{i_{1,m} : i_j \in J, 1 \leq j \leq m-1\}} H_{i_{1,m}}^{(n)} \quad (2.8)$$

Clearly, by the ϕ -irreducibility of $\{(X_n, Z_n)\}$, we have for all $x \in E$ that $\phi_{j_0}(H_1(x)) > 0$, for $H_1(x)$ as in 2.6i. Then, by Fubini's theorem, we have

$$\int \phi_{j_0}(H_1(x)) \phi_{j_0}(dx) = \int \phi_{j_0}(H_2(y)) \phi_{j_0}(dy), \quad (2.9)$$

so $\phi_{j_0}(\{y : \phi_{j_0}(H_2(y)) > 0\}) > 0$. This implies that there must be an

n_1, i_{1,m_1} and an n_2, i_{1,m_2} where for $F = H_{i_{1,m_1}}^{(n_1)}$ and $G = H_{i_{1,m_2}}^{(n_2)}$,

$$\phi_{j_0}(\{y : \phi_{j_0}(F_2(y)) > 0, \phi_{j_0}(G_1(y)) > 0\}) > 0. \quad (2.10)$$

Consider finite partitions $\{E_\alpha^{(n)}\}$ of E , becoming finer as n increases.

Let $E_{\alpha,\beta}^{(n)} = E_\alpha^{(n)} \times E_\beta^{(n)}$. Clearly $\{E_{\alpha,\beta}^{(n)}\}$ is a finite partition of $E^2 = E \times E$.

Let $i(n,x)$ be the unique index for the element of the partition $\{E_\alpha^{(n)}\}$ where $x \in E_{i(n,x)}^{(n)}$. By a differentiation theorem (see Doob (1953)), we

have that for each measurable set $B \subset E^2$,

$$[\phi_{j_0}^2(E_{i(n,x),i(n,y)}^{(n)})]^{-1} \phi_{j_0}^2(B \cap E_{i(n,x),i(n,y)}^{(n)}) \rightarrow I_B(x,y), \quad (2.11)$$

for all $x, y \in E^2 - N$, where $\phi_{j_0}^2(N) = 0$ and $\phi_{j_0}^2 = \phi_{j_0} \times \phi_{j_0}$.

Therefore, for F and G as in (2.10), there is an x_0, y_0 and z_0 with $x_0 \in F_2(y_0) - N_2(y_0)$, and $z_0 \in G_1(y_0) - N_1(y_0)$, which satisfy (2.11). Let $\alpha = i(m, x_0)$, $\beta = i(m, y_0)$ and $\gamma = i(m, z_0)$. Then there is an N where for all $m > N$,

$$\begin{aligned} \phi_{j_0}^2(F \cap E_{\alpha, \beta}^{(m)}) &\geq (3/4) \phi_{j_0}^2(E_{\alpha}^{(m)}) \phi_{j_0}^2(E_{\beta}^{(m)}), \\ \phi_{j_0}^2(G \cap E_{\beta, \gamma}^{(m)}) &\geq (3/4) \phi_{j_0}^2(E_{\beta}^{(m)}) \phi_{j_0}^2(E_{\gamma}^{(m)}). \end{aligned} \quad (2.12)$$

For $n > N$, let

$$A = \{x \in E_{\alpha}^{(n)} : \phi_{j_0}^2(E_{\beta}^{(n)} \cap F_1(x)) \geq (3/4) \phi_{j_0}^2(E_{\beta}^{(n)})\}, \quad (2.13)$$

$$B = \{z \in E_{\gamma}^{(n)} : \phi_{j_0}^2(E_{\beta}^{(n)} \cap G_2(z)) \geq (3/4) \phi_{j_0}^2(E_{\beta}^{(n)})\}.$$

Clearly $\phi_{j_0}(A) > 0$ and $\phi_{j_0}(B) > 0$ (otherwise (2.12) would be violated), and for $x \in A$, $z \in B$, we have

$$\phi_{j_0}(F_1(x) \cap G_2(z)) \geq \phi_{j_0}(E_{\beta}^{(n)})/2. \quad (2.14)$$

Therefore, for $y \in F_1(x) \cap G_2(z)$, $(x, y) \in F$ and $(y, z) \in G$. The definition of the Radon-Nikodym derivative yields

$$\begin{aligned}
P_{i_{1,m_1} i_{1,m_2}}(x,z) &\geq \int_S P_{i_{1,m_1}}(x,y) P_{i_{1,m_2}}(y,z) \phi_{j_0}(dy) \\
&\geq \int_{F_1(x) \cap G_2(z)} P_{i_{1,m_1}}(x,y) P_{i_{1,m_2}}(y,z) \phi_{j_0}(dy) \quad (2.15) \\
&\geq \phi(E_\beta^{(n)})/2n_1 n_2.
\end{aligned}$$

Since $\{(X_n, Z_n)\}$ is ϕ -irreducible, there exists an $m > 0$ and a $c > 0$ where $\phi_{j_0}(C_{m,c}) > 0$ for

$$C_{m,c} = \{x \in B : P((X_m, Z_m) \in (j_0, A) | (X_0, Z_0) = (j_0, x)) > c\}. \quad (2.16)$$

If we consider only those paths $i_{1,m}$ where

$P(X_m = j_0, X_{m-1} = i_{m-1}, \dots, X_1 = i_1 | X_0 = j_0) > 0$, then there must be a particular $i_{1,m}$ with $\phi_{j_0}(C^*) > 0$, where

$$C^* = \{x \in B : P_{i_{1,m}}^{(m)}(x, A) > c\}, \quad (2.17)$$

for

$$P_{i_{1,m}}^{(m)}(x, A) = P(Z_m \in A | X_0 = j_0, Z_0 = x, X_1 = i_1, \dots, X_m = j_0).$$

Therefore, from (2.15) we have for $x \in C^*$, $y \in C^*$ that

$$\begin{aligned}
P_{i_{1,m} i_{1,m_1} i_{1,m_2}}(x,y) &\geq \int_A P_{i_{1,m}}^{(m)}(x, dz) P_{i_{1,m_1} i_{1,m_2}}(z,y) \\
&\geq \int_A P_{i_{1,m}}^{(m)}(x, dz) \phi_{j_0}(E_\beta^{(n)})/2n_1 n_2 \geq c \phi_{j_0}(E_\beta^{(n)})/2n_1 n_2.
\end{aligned}$$

Thus, for $k = m + m_1 + m_2$, for $D \subset S$, and for

$(j_1, j_2, \dots, j_{k-1}) = (i_{1,m}, j_0, i_{1,m_1}, j_0, i_{1,m_2})$, it follows that for all $x \in C^*$,

$$\begin{aligned} P(Z_k \in D | X_0 = j_0, Z_0 = x, X_1 = j_1, \dots, X_{k-1} = j_{k-1}, X_k = j_0) \\ \geq c \phi_{j_0}(D \cap C^*) \phi_{j_0}(E_\beta^{(n)}) / 2n_1 n_2. \end{aligned} \quad (2.18)$$

From (2.18) the lemma follows, where $A = C^*$, and

$$p = c \phi_{j_0}(E_\beta^{(n)}) P(X_k = j_0, \dots, X_1 = j_1 | X_0 = j_0) / 2n_1 n_2. \quad \square$$

COROLLARY. If $\{(X_n, Z_n)\}$ is a ϕ -irreducible Markov chain as in lemma 2.2,
and if $\{T_n\}$ and $\{Z_n\}$ are conditionally independent given $\{X_n\}$, then for
 $E, \{j_i\}, k, A$, and p as in lemma 2.2, for $x \in A, C \in \mathbb{B}^+, D \subset S$,

$$P(X_k \in j_0, Z_k \in B, T_k \in C | X_0 = j_0, Z_0 = x) \geq p \phi(j_0, B \cap A) v(c),$$

where $v([0, x]) = P(T_k \leq x | X_0 = j_0, X_1 = j_1, \dots, X_k = j_0)$.

PROOF. Certainly, for all $x \in S$,

$$\begin{aligned}
& P(X_k=j_0, Z_k \in B, T_k \in C | X_0=j_0, Z_0=x) \\
& \geq P(X_1=j_1, \dots, X_k=j_0 | X_0=j_0) P(Z_k \in B, T_k \in C | X_0=j_0, Z_0=x, X_1=j_1, \dots, X_k=j_0) \\
& = P(X_1=j_1, \dots, X_k=j_0 | X_0=j_0) P(Z_k \in B | X_0=j_0, Z_0=x, X_1=j_1, \dots, X_k=j_0) \quad (2.19) \\
& \quad \cdot P(T_k \in C | X_0=j_0, X_1=j_1, \dots, X_k=j_0),
\end{aligned}$$

where the equality follows from the conditional independence of T_k and Z_k given $\{X_i : i = 0, 1, \dots, k\}$.

Therefore, for $x \in A$,

$$\begin{aligned}
& P(X_k=j_0, Z_k \in B, T_k \in C | X_0=j_0, Z_0=x) \\
& \geq P(X_1=j_1, \dots, X_k=j_0 | X_1=j_0) P(Z_k \in B | X_0=j_0, Z_0=x, X_1=j_1, \dots, X_k=j_0) \nu(C) \quad (2.20) \\
& \geq p \nu(C) \phi(j_0, B \cap A). \quad \square
\end{aligned}$$

From the application of lemma 2.1, it can be shown that a renewal equation can be created for the Markov renewal process $\{X_n, Z_n\}$, as in expression (2.5). Once the behavior of $E_\psi(T_N^*)$ is determined, the basic renewal theorem can be applied.

Let us temporarily let $Y_n = \{X_n, Z_n\}$. To create the renewal point, we will make an inconsequential change in the definition of the process $\{Y_n^*\}$ as proposed by Athreya, McDonald and Ney (1978a,b).

For k as in lemma 2.2, we note that $\{(Y_{nk}, T_{nk}), n = 0, 1, 2, \dots\}$ is also a Markov renewal process. From this process, we define another Markov renewal process $\{(\delta_{nk}, Y_{nk}^*, T_{nk}^*)\}$ by

$$P(\delta_{nk} = \delta, Y_{nk}^* \in B, T_{nk}^* - T_{nk-k}^* \leq t | \delta_{nk-k} = \gamma, Y_{nk-k}^* = y) \\ = \begin{cases} I(\delta=0)P(Y_{nk} \in B, T_{nk} - T_{nk-k} \leq t | Y_{nk-k} = y) & \text{if } y \in (j_0, A)^c \\ p\phi(Bn(j_0, A))\nu([0, t]) + I(\delta=0)[P(Y_{nk} \in B, T_{nk} - T_{nk-k} \leq t | Y_{nk-k} = y) \\ - 2p\phi(Bn(j_0, A))\nu([0, t])] & \text{if } y \in (j_0, A), \end{cases} \quad (2.21)$$

with p, ν as in corollary 2.2, and γ and δ are either 0 or 1. We can then define $\{(\delta_n, Y_n^*, T_n^*)\}$ through the process $\{(\delta_{nk}, Y_{nk}^*, T_{nk}^*)\}$ by

$$P(\bigcap_{j=1}^{k-1} (\delta_{nk-j} = \delta_j, Y_{nk-j}^* \in A_j, T_{nk-j}^* - T_{nk-j-1}^* \leq t_j) | Y_{nk-k}^* = y_0, Y_{nk}^* = y_1, T_{nk}^* - T_{nk-k}^* = t) \\ = (\prod_{j=1}^{k-1} I(\delta_j = 0))P(\bigcap_{j=1}^{k-1} (Y_{nk-j} \in A_j, T_{nk-j} - T_{nk-j-1} \leq t_j) | Y_{nk-k} = y_0, T_{nk} = y_1, T_{nk} - T_{nk-k} = t). \quad (2.22)$$

While the process $\{\delta_n, Y_n^*, T_n^*\}$ is not a Markov renewal process, from (2.21) and (2.22) it is clear that $\{(Y_n^*, T_n^*)\} \stackrel{d}{=} \{(Y_n, T_n)\}$. Also, letting

$$N = \inf\{n > 1: \delta_n = 1\}, \quad (2.23)$$

it follows from (2.21) and (2.22) that for

$$\psi(B) = [\phi(j_0, A)]^{-1} \phi(B \cap (j_0, A)),$$

$$P_{\psi}(Y^*(t) \in C) = P_{\psi}(Y^*(t) \in C, T_N^* > t) + \int_0^t P_{\psi}(Y^*(t-\tau) \in A) dP_{\psi}(T_N^* \leq \tau), \quad (2.24)$$

and the renewal equation is satisfied.

THEOREM 2.3. If $\beta < \infty$, and $\{Y_n\}$ is ergodic then for ϕ -almost all y and any set C ,

$$\lim_{t \rightarrow \infty} P_y(Y(t) \in C) = [E_{\psi}(T_N^*)]^{-1} \int_0^{\infty} P_{\psi}(Y^*(t) \in C, T_N^* > t) dt,$$

where $E_{\psi}(T_N^*) < \infty$.

PROOF. First we note that since the distribution of $\{T_n\}$ depends only on $\{X_n\}$, if T_N^* is arithmetic, then our Markov renewal process is equivalent to an appropriate Markov chain. As such we will assume T_N^* is non-arithmetic. From (2.22) it is clear that $E_{\psi}(T_N^*)$ can be determined from $\{\delta_{nk}, Y_{nk}^*, T_{nk}^*\}$. Let μ be the stationary measure of $\{Y_n\}$. Define μ' for $B \in \mathcal{F}$ by

$$\mu'(1, B) = \mu(j_0, A) \phi(B \cap (j_0, A)),$$

$$\mu'(0, B) = \mu(B) - \mu(j_0, A) \phi(B \cap (j_0, A)).$$

Let us show that μ' is the stationary measure of $\{\delta_{nk}, Y_{nk}^*\}$. It can be trivially verified from the fact that for $y \in (j_0, A)$, $B \in \mathcal{F}$ $P^k(y, B) \geq p\phi(B \cap (j_0, A))$ that μ' is indeed a measure. Therefore, we need only check definition 3 for stationarity. For $\mu'(1, B)$

$$\begin{aligned} & \sum_{j=0}^1 \int_S \mu'(j, dx) P(\delta_k = 1, Y_k^* \in B | \delta_0 = j, Y_0^* = x) \\ &= \int_S \left[\sum_{j=0}^1 \mu'(j, dx) \right] P(\delta_k = 1, Y_k^* \in B | Y_0^* = x) \\ &= \int_S \mu(dx) P(\delta_k = 1, Y_k^* \in B | Y_0^* = x) = \mu(j_0, A) p\phi(B \cap (j_0, A)), \end{aligned}$$

where the last equality follows from (2.21). For $\mu'(0, B)$

$$\begin{aligned} & \sum_{j=0}^1 \int_S \mu'(j, dx) P(\delta_k = 0, Y_k^* \in B | \delta_0 = j, Y_0^* = x) \\ &= \int_S \mu(dx) P(\delta_k = 0, Y_k^* \in B | Y_0^* = x) \\ &= \int_{(j_0, A)} \mu(dx) [P(Y_k \in B | Y_0 = x) - p\phi(B \cap (j_0, A))] + \int_{(j_0, A)^c} \mu(dx) P(Y_k \in B | Y_0 = x) \\ &= \int_S \mu(dx) P(Y_k \in B | Y_0 = x) - \mu(j_0, A) p\phi(B \cap (j_0, A)) = \mu'(0, B), \end{aligned}$$

where the last equality follows from the stationarity of μ for $\{Y_n\}$.

Since μ' is stationary for $\{\delta_n, Y_n^*\}$, it follows from Puri and Tollar (1985) that

$$\int_{(j_0, A)} \mu'(1, dy) E_{(1, y)}(T_N^*) \leq \sum_{j=0}^1 \int_S \mu'(j, dy) E_{(j, y)}(T_k^*) \quad (2.25)$$

Since $\{T_{nk}^*\}$ is independent of δ_0 , we see that

$$\int_{(j_0, A)} \mu'(1, dy) E_{(1, y)}(T_N^*) = \mu(j_0, A) p_{\psi} E_{\psi}(T_N^*). \quad (2.26)$$

Also,

$$\begin{aligned} \sum_{j=0}^1 \int_S \mu'(j, dy) E_{(j, y)}(T_k^*) &= \int_S \mu'(dy) E_y(T_k^*) \\ &= \int_S \mu(dy) E_y(T_k) = k E_{\mu}(T_1). \end{aligned} \quad (2.27)$$

Since $E_{\mu}(T_1) = \sum_{j \in J} \pi_j E_j(T_1) = \beta$, combining (2.25), (2.26) and (2.27), we see that

$$E_{\psi}(T_N^*) \leq [\mu(j_0, A)]^{-1} k \beta.$$

Therefore, from (2.24) and the basic renewal theorem (see Karlin and Taylor (1975)) we have that

$$\lim_{t \rightarrow \infty} P_{\psi}(Y(t) \in C) = [E_{\psi}(T_N^*)]^{-1} \int_0^{\infty} P_{\psi}(Y^*(t) \in C, T_N^* > t) dt.$$

The set $\{y: P(N=\infty | Y_0^*=y) > 0\}$ must have ϕ -measure zero, or else $E_{\psi}(T_N^*) = \infty$ by simple arguments using ϕ -irreducibility. Therefore

$$\lim_{t \rightarrow \infty} P_y(Y(t) \in C) = \lim_{t \rightarrow \infty} P_{\psi}(Y(t) \in C),$$

completing the proof. $\square$

In the subsequent section, these results will be used to prove that a certain storage model converges in distribution asymptotically.

3. THE STORAGE MODEL

For all $j \in J$, let $\{(U_n(i), V_n(i), W_n(i)), n=1, 2, \dots\}$ be an i.i.d. triplet sequence, independent of $\{(X_n, T_n)\}$ as in section 1, and of all $\{(U_n(j), V_n(j), W_n(j)), n=1, 2, \dots\}$ for $j \neq i$. Define a two compartment storage model recursively by

$$(Z_{1,n}, Z_{2,n}) = (\max[U_n(X_n) + Z_{1,n-1} - V_n(X_n), 0], \quad (3.1)$$

$$\max[\min[U_n(X_n) + Z_{1,n-1}, V_n(X_n)] + Z_{2,n-1} - W_n(X_n), 0]),$$

with the amount in storage at time t being given by

$$(Z_1(t), Z_2(t)) \equiv (Z_{1,N(t)}, Z_{2,N(t)}),$$

for $N(t)$ as in (1.4).

Note that $\{X_n, Z_{1,n}, Z_{2,n}\}$ is a Markov chain on some subset of $J \times [0, \infty) \times [0, \infty)$.

Equation (3.1) is the two compartment storage model considered by Tollar (1986), which has been widely analyzed in various forms by Puri, Balagopal, Senturia and Woolford, among others.

Let us define $E_\pi U$ by

$$E_\pi U \equiv \sum_{i \in J} \pi_i E U_1(i),$$

with similar definitions for $E_{\pi}V$ and $E_{\pi}W$. We will assume $E_{\pi}U < \infty$, $E_{\pi}V < \infty$, $E_{\pi}W < \infty$. Tollar (1986) analyzed the asymptotic behavior of $(Z_1(t), Z_2(t))$ for the various orderings of $E_{\pi}U$, $E_{\pi}V$ and $E_{\pi}W$. However, the case where $E_{\pi}U < E_{\pi}V$ and $E_{\pi}U < E_{\pi}W$ was left as an open question. Using the results of section 2, we will establish the main result of this section.

THEOREM 3.1. If $\beta < \infty$, $E_{\pi}U < E_{\pi}V$ and $E_{\pi}U < E_{\pi}W$, then for arbitrary initial distribution $(X_0, Z_{1,0}, Z_{2,0})$

$$\begin{aligned} \lim_{t \rightarrow \infty} P(Z_1(t) \leq z_1, Z_2(t) \leq z_2) \\ = [E_{\psi}(T_N^*)]^{-1} \int_0^{\infty} P_{\psi}(Z_1^*(t) \leq z_1, Z_2^*(t) \leq z_2, T_N^* > t) dt, \end{aligned}$$

for ψ , $Z_1^*(t)$, $Z_2^*(t)$ and T_N^* as in section 2.

The majority of the proof is devoted to the non-trivial task of illustrating there is a measure ϕ for which $\{X_n, Z_{1,n}, Z_{2,n}\}$ is ϕ -irreducible, and then establishing ergodicity. After this is completed, theorem 2.3 can be used to establish the result.

Observe from Tollar (1986) that for initial values $(Z_{1,0}, Z_{2,0})$, $(Z_{1,n}, Z_{2,n})$ can be written in closed form as

$$Z_{1,n}(Z_{1,0}) = \max(Z_{1,0} + S_n, \max_{1 \leq j \leq n} (S_n - S_j)), \quad (3.2)$$

$$Z_{2,n}(Z_{1,0}, Z_{2,0}) = \max[Z_{1,0} + Z_{2,0} + R_n, Z_{1,0} + \max_{1 \leq k \leq n} (S_k + R_n - R_k)], \quad (3.3)$$

$$\max_{1 \leq j \leq k \leq n} (S_k - S_j + R_n - R_k)] - Z_{1,n},$$

where $S_n = \sum_{i=1}^n (U_i(X_i) - V_i(X_i))$, and $R_n = \sum_{i=1}^n (U_i(X_i) - W_i(X_i))$. Typically,

$Z_{1,n}(Z_{1,0})$ and $Z_{2,n}(Z_{1,0}, Z_{2,0})$ will be written simply as $Z_{1,n}$, $Z_{2,n}$ with the $Z_{1,0}$ and $Z_{2,0}$ being understood.

Using (3.2) and (3.3), the first step of the construction of the measure ϕ is the following lemma.

LEMMA 3.2. If $E_\pi U < E_\pi V$ and $E_\pi U < E_\pi W$, then there exists a z and a j_0 such that for every (x_0, y_0) , there is an n_0 with the property that

$$P(X_{n_0} = j_0, Z_{1,n_0} = 0, Z_{2,n_0} \leq z | X_0 = j_0, Z_{1,0} = x_0, Z_{2,0} = y_0) > 0.$$

PROOF. By a straightforward alteration of lemma 3.1 of Puri and Tollar (1985), since $E_\pi U < E_\pi V$, $E_\pi U < E_\pi W$, it follows that there exists an $\epsilon > 0$, an n , and a sequence $j_0, j_1, \dots, j_{n-1}$ such that

$$P(S_n < -\epsilon, R_n < -\epsilon, \max_{0 \leq j \leq n} (S_n - S_j) = 0, X_1 = j_1, \dots, X_{n-1} = j_{n-1}, X_n = j_0 | X_0 = j_0) > 0.$$

Let M be an integer where $M \geq \epsilon^{-1}(x_0 + y_0 + \epsilon)$, and let

$$A_j = \{\omega: S_{nj} - S_{nj-n} < -\epsilon, R_{nj} - R_{nj-n} < -\epsilon, \max_{nj-n \leq k \leq nj} (S_{nj} - S_k) = 0,$$

$$X_{nj-n+1} = j_1, \dots, X_{nj} = j_0\}.$$

Then $P(\bigcap_{j=1}^M A_j | X_0 = j_0) > 0$. Note for $\omega \in A \equiv \bigcap_{j=1}^M A_j$, that for $n_0 = nM$

$$i) \quad S_{n_0} < -M\epsilon < -x_0 - y_0 - \epsilon,$$

$$ii) \quad \max_{1 \leq j \leq n_0} (S_{n_0} - S_j) = \max_{1 \leq k \leq M} \max_{n_k - n \leq j \leq n_k} (S_{n_k} - S_j) + (S_{n_0} - S_{n_k}) = 0.$$

Thus, for all $\omega \in A$,

$$Z_{1,n_0}(\omega) = \max_{1 \leq j \leq n_0} (S_{n_0} - S_j), x_0 + S_{n_0} = 0. \quad (3.4)$$

Define n_j by

$$n_j = \{i: n_i \leq j < n(i+1)\}.$$

Then, since $\max_{0 \leq j \leq n_k} (S_{n_k} - S_j) = 0$ for all k , all $\omega \in A$, we have

$$\max_{0 \leq j \leq k} (S_k - S_j) = \max_{0 \leq j \leq n_k} (S_{n_k} - S_j) + S_k - S_{n_j}, \max_{n_k \leq j \leq k} (S_k - S_j) = \max_{n_k \leq j \leq k} (S_k - S_j). \quad (3.5)$$

If we in addition observe for $\omega \in A$ that $R_{n_0} < -x_0 - y_0 - \epsilon$, we have from (3.3) and (3.4) that

$$Z_{2,n_0}(\omega) = \max_{1 \leq j \leq k \leq n_0} (S_k - S_j + R_{n_0} - R_k), x_0 + \max_{1 \leq k \leq n_0} (S_k + R_{n_0} - R_k). \quad (3.6)$$

Clearly, from (3.5)

$$\begin{aligned}
\max_{1 \leq j \leq k \leq n_0} (S_k - S_j + R_{n_0} - R_k) &\leq \max_{0 \leq k \leq n_0} (R_{n_0} - R_k + \max_{0 \leq j \leq k} (S_k - S_j)) \\
&= \max_{0 \leq k \leq n_0} (R_{n_0} - R_k + \max_{n_k \leq j \leq k} (S_k - S_j)) \\
&= \max_{1 \leq l \leq M} [\max_{nl-n \leq k \leq nl} (R_{n_0} - R_k + \max_{nl-n \leq j \leq k} (S_k - S_j))] \\
&\leq \max_{1 \leq l \leq M} [\max_{nl-n \leq j \leq k \leq nl} (R_{nl} - R_k + S_k - S_j)],
\end{aligned} \tag{3.7}$$

where the last inequality follows from $R_{n_0} - R_{nl} < -(M-l)\epsilon$.

Also, we have that for $\omega \in A$,

$$\max_{1 \leq k \leq n_0} (S_k + R_{n_0} - R_k) = \max_{1 \leq k \leq n_0} [S_{n_k} + (S_k - S_{n_k}) + (R_{n_k+n} - R_k) + (R_{n_0} - R_{n_k+n})].$$

Since $S_{n_k} < -\epsilon n_k n^{-1}$, and $R_{n_0} - R_{n_k+n} < -\epsilon(n_0 - n_k - n)n^{-1}$,

$$\max_{1 \leq k \leq n_0} (S_k + R_{n_0} - R_k) < \max_{1 \leq k \leq n_0} (S_k - S_{n_k} + R_{n_k+n} - R_k) - \epsilon(M-1).$$

Since $\epsilon(M-1) > x_0$, we see that

$$\begin{aligned}
x_0 + \max_{1 \leq k \leq n_0} (S_k + R_{n_0} - R_k) &< \max_{1 \leq k \leq n_0} (S_k - S_{n_k} + R_{n_k+n} - R_k) \\
&\leq \max_{1 \leq l \leq M} [\max_{nl-n \leq k \leq nl} (S_k - S_{nl-n} + R_{nl} - R_k)].
\end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), from (3.6) we see that for $\omega \in A$,

$$Z_{2,n_0}(\omega) \leq \max_{1 \leq l \leq M} \left(\max_{nl-n \leq j \leq k \leq nl} (S_k - S_j + R_{nl} - R_k) \right). \quad (3.9)$$

Therefore, if z is such that

$$P(S_n < -\epsilon, R_n < -\epsilon \mid \max_{0 \leq j \leq n} (S_n - S_j) = 0, \max_{0 \leq j \leq k \leq n} (R_n - R_k + S_k - S_j) \leq z,$$

$$X_1 = j_1, \dots, X_n = j_n \mid X_0 = j_0) = \delta > 0,$$

it follows from (3.4) and (3.9) that

$$P(X_{n_0} = j_0, Z_{1,n_0} = 0, Z_{2,n_0} \leq z \mid X_0 = j_0, Z_{1,0} = x_0, Z_{2,0} = y_0) > \delta^M > 0. \quad \square$$

THEOREM 3.3. If $E_\pi U < E_\pi V$ and $E_\pi U < E_\pi W$, then there exists a z and a j_0 , such that for every (i, z_1, z_2) there is an n_1 with the property that

$$P(X_{n_1} = j_0, Z_{1,n_1} = 0, Z_{2,n_1} \leq z \mid X_0 = i, Z_{1,0} = z_1, Z_{2,0} = z_2) > 0.$$

PROOF. From (3.1) it is clear that for all n , x , and j , whenever $x_1 \leq y_1$ and $x_2 \leq y_2$, then

$$P(X_n=j_0, Z_{1,n}=0, Z_{2,n} \leq x | X_0=j, Z_{1,0}=x_1, Z_{2,0}=x_2) \quad (3.10)$$

$$\geq P(X_n=j_0, Z_{1,n}=0, Z_{2,n} \leq x | X_0=j, Z_{1,0}=y_1, Z_{2,0}=y_2).$$

Since for any i, z_1, z_2 , there must be an m, B_1 and B_2 where

$$P(X_m=j_0, Z_{1,m} \leq B_1, Z_{2,m} \leq B_2 | X_0=i, Z_{1,0}=z_1, Z_{2,0}=z_2) > 0,$$

then it follows from the Markov nature of $(X_n, Z_{1,n}, Z_{2,n})$ and from (3.9) that lemma 3.2 implies for $n_1 = n_0 + m$,

$$\begin{aligned} &P(X_{n_1}=j_0, Z_{1,n_1}=0, Z_{2,n_1} \leq z | X_0=i, Z_{1,0}=z_1, Z_{2,0}=z_2) \\ &\geq P(X_{n_0}=j_0, Z_{1,n_0}=0, Z_{2,n_0} \leq z | X_0=j_0, Z_{1,0}=B_1, Z_{2,0}=B_2) \\ &\cdot P(X_m=j_0, Z_{1,m} \leq B_1, Z_{2,m} \leq B_2 | X_0=i, Z_{1,0}=z_1, Z_{2,0}=z_2), \end{aligned}$$

which completes the proof. $\square$

To accomplish the next step in showing that $(X_n, Z_{1,n}, Z_{2,n})$ is ϕ -irreducible, we need the following lemma about cyclic permutations.

Let $X_1, \dots, X_n$ be a sequence of values. We define the cyclic permutation of $X_1, \dots, X_n$ about k_0 by

$$X_i^{(k_0)} = \begin{cases} X_{k_0+i} & 1 \leq i \leq n - k_0 \\ X_{i-n+k_0} & n - k_0 < i \leq n. \end{cases}$$

LEMMA 3.4. Let $\{(X_i, Y_i), i=1, 2, \dots, n\}$ be a sequence of numbers satisfying

$$\sum_{i=1}^n X_i \leq 0, \sum_{i=1}^n Y_i \leq 0, \max_{0 \leq j \leq n} \left(\sum_{i=j+1}^n X_i \right) = 0. \text{ Then there exists a } k_0 \text{ where}$$

$$\max_{0 \leq j \leq k \leq n} \left(\sum_{i=j+1}^k X_i^{(k_0)} + \sum_{i=k+1}^n Y_i^{(k_0)} \right) - \max_{0 \leq j \leq n} \left(\sum_{i=j+1}^n X_i^{(k_0)} \right) = 0$$

holds.

The proof of the above is omitted. It is easily verified that for

$$k_0 \text{ equal to the integer where } \sum_{i=k_0+1}^n Y_i + \max_{0 \leq j \leq k_0} \left(\sum_{i=j+1}^{k_0} X_i \right) = \max_{0 \leq j \leq k \leq n} \left(\sum_{i=j+1}^k X_i + \sum_{i=k+1}^n Y_i \right)$$

that the lemma is true. Using this lemma, we establish the next step in ϕ -irreducibility.

THEOREM 3.5. If $E_\pi U < E_\pi V$ and $E_\pi U < E_\pi W$, then for z and j_0 as in theorem 3.3, there exists an n_1 , j_1 and a measure μ_z where $\mu_z([0, \infty)) > 0$ such that for all $A \in \mathcal{B}^+$, and $0 \leq x \leq z$,

$$P(X_{n_1} = j_1, Z_{1, n_1} \in A, Z_{2, n_1} = 0 | X_0 = j_0, Z_{1, 0} = 0, Z_{2, 0} = x) \geq \mu_z(A).$$

PROOF. From lemma 3.2 it is clear there must be a y where for

$$A(i) = \{\omega: S_{ni} - S_{ni-n} < -\epsilon, R_{ni} - R_{ni-n} < -\epsilon, \max_{ni-n \leq k \leq ni} (S_{ni} - S_k) = 0, \quad (3.11)$$

$$\max_{ni-n \leq j \leq k \leq ni} (R_{ni} - R_k + S_k - S_j) \leq y, X_{ni-n+1} = j_1, \dots, X_{ni} = j_0\},$$

$$P(A(i) | X_{ni-n} = j_0) > 0.$$

Let us define $j_0^{(k)}, \dots, j_{n-1}^{(k)}$ by

$$j_i^{(k)} = \begin{cases} j_{i+k} & 0 \leq i < n-k \\ j_{i-n+k} & n-k \leq i < n \end{cases},$$

and define

$$B(i, k) = \{\omega: S_{n+i} - S_n < -\epsilon, R_{n+1} - R_n < -\epsilon, \max_{i \leq j \leq k \leq n+i} (R_{n-k} - R_k + S_k - S_j) \leq 2y, \\ X_{i+1} = j_1^{(k)}, \dots, X_{n+2} = j_0^{(k)}\}. \quad (3.12)$$

Then from (3.11) it follows that for all i, k ,

$$P(B(i, k) | X_i = j_0^{(k)}) > 0.$$

Clearly, for $A = \bigcap_{i=1}^M A(i)$, where $M > \epsilon^{-1}(z+3y)+1$, we have $P(A | X_0 = j_0) > 0$,

and for all $\omega \in A$, $S_{Mn} < -M\epsilon$, $R_{Mn} < -M\epsilon$, $\max_{0 \leq j \leq Mn} (S_{Mn} - S_j) = 0$. Thus, from

lemma 3.4 we have there is a k_0 where

$$P(A, R_{Mn} - R_{k_0} + \max_{0 \leq j \leq k_0} (S_{k_0} - S_j) = \max_{0 \leq j \leq k \leq Mn} (R_{Mn-k} - R_k + S_k - S_j) | X_0 = j_0) > 0. \quad (3.13)$$

Let $m = k_0 - n \cdot \max\{i: n_i \leq k_0\}$ and $n_1 = Mn + m$. It is clear from (3.12) and (3.13) that for

$$B = \left\{ \bigcap_{i=0}^{M-1} B(ni+m, m) \right\} \cap \{\omega: \max_{m \leq j \leq k \leq n_1} (R_{n_1-k} - R_k + S_k - S_j) - \max_{m \leq j \leq n_1} (S_{n_1} - S_j) = 0\},$$

$$P(B | X_m = j_m) > 0. \quad (3.14)$$

Also, for

$$C = \{\omega: \max_{0 \leq j \leq k \leq m} (R_m - R_k + S_k - S_j) \leq y, X_m = j_m\},$$

$$P(C \cap B | X_0 = j_0) > 0.$$

Let us show for $Z_{1,0} = 0$, $Z_{2,0} = z$ for all $\omega \in C \cap B$, $Z_{2,n_1} = 0$. It follows from (3.3) that

$$Z_{2,n_1} = \max(\max_{0 \leq j \leq k \leq n_1} (R_{n_1} - R_k + S_k - S_j), z + R_{n_1}) - \max_{0 \leq j \leq n_1} (S_{n_1} - S_j).$$

From the definition of B and C, we see

$$z + R_{n_1} = z + R_m + R_{n_1} - R_m \leq z + y - M\epsilon < 0,$$

and therefore

$$Z_{2,n_1} = \max[\max_{m \leq j \leq k \leq n_1} (R_{n_1} - R_k + S_k - S_j), \max_{\substack{0 \leq j \leq k \leq n_1 \\ j < m}} (R_{n_1} - R_k + S_k - S_j)] - \max_{0 \leq j \leq n_1} (S_{n_1} - S_j).$$

From (3.14), we have that

$$\max_{m \leq j \leq k \leq n_1} (R_{n_1} - R_k + S_k - S_j) - \max_{0 \leq j \leq n_1} (S_{n_1} - S_j) \leq 0,$$

and therefore to show $Z_{2,n_1} = 0$ for $\omega \in B \cap C$, it is sufficient that for $0 \leq j < m$, $0 \leq j \leq k \leq n_1$, $R_{n_1} - R_k + S_k - S_j \leq 0$. If $k < m$, from the definition of B and C,

$$R_{n_1} - R_k + S_k - S_j = (R_{n_1} - R_m) + (R_m - R_k + S_k - S_j) \leq -M\epsilon + y < 0.$$

If $k \geq m$, letting $i = \max \{i: ni + m \leq k\}$,

$$R_{n_1} - R_k + S_k - S_j = (R_{n_1} - R_{ni+n+m}) + (R_{ni+n+m} - R_k + S_k - S_{ni+m}) + (S_{ni+m} - S_m) + (S_m - S_j)$$

$$\leq -\epsilon[M - (i+1)] + 2y - \epsilon i + y < 0.$$

Since $Z_{2,n_1} = 0$ for $\omega \in B \cap C$, it follows that

$$P(X_{n_1} = j_m, Z_{2,n_1} = 0 | X_0 = j_0, Z_{1,0} = 0, Z_{2,0} = z) > 0.$$

For $A \in \mathcal{B}^+$, let

$$\mu_z(A) = P(X_{n_1} = j_m, Z_{1,n_1} \in A, Z_{2,n_1} = 0 | X_0 = j_0, Z_{1,0} = 0, Z_{2,0} = z).$$

Then to complete the theorem, we need only note that for $x \leq z$,

$$P(X_{n_1} = j_m, Z_{1,n_1} \in A, Z_{2,n_1} = 0 | X_0 = j_0, Z_{1,0} = 0, Z_{2,0} = x)$$

$$= P(X_{n_1} = j_m, \max_{0 \leq j \leq n_1} (S_{n_1} - S_j) \in A,$$

$$\max_{0 \leq j \leq k \leq n_1} (R_{n_1} - R_k + S_k - S_j), x + R_{n_1}] - \max_{0 \leq j \leq n_1} (S_{n_1} - S_j) = 0 | X_0 = j_0 \} \geq \mu_z(A),$$

since $\max_{0 \leq j \leq k \leq n_1} (R_{n_1} - R_k + S_k - S_j), x + R_{n_1}]$ is monotone in x . $\square$

From the previous theorems, we can now establish that $\{X_n, Z_{1,n}, Z_{2,n}\}$ is ϕ -irreducible.

THEOREM 3.6. If $E_\pi U < E_\pi V$ and $E_\pi U < E_\pi W$, then $\{X_n, Z_{1,n}, Z_{2,n}\}$ is ϕ -irreducible with respect to the measure ϕ defined by

$$\phi(B) = \int \sum_{n=1}^{\infty} 2^{-n} P[(X_n, Z_{1,n}, Z_{2,n}) \in B | X_0 = j_1, Z_{1,0} = x, Z_{2,0} = 0] \mu_z(dx),$$

for j_1 and μ_z as in theorem 3.5.

PROOF. From definition 1 of ϕ -irreducibility (see section 2), we see we need for all B where $\phi(B) > 0$, that for all i, z_1, z_2 ,

$$\sum_{n=1}^{\infty} 2^{-n} P((X_n, Z_{1,n}, Z_{2,n}) \in B | X_0 = i, Z_{1,0} = z_1, Z_{2,0} = z_2) > 0.$$

If $\phi(B) > 0$, there is an n_1 where

$$\int P((X_{n_1}, Z_{1,n_1}, Z_{2,n_1}) \in B | X_0 = j_1, Z_{1,0} = x, Z_{2,0} = 0) \mu_z(dx) > 0.$$

From theorem 3.5 there is an n_2 where for $x \leq z$,

$$P(X_{n_2} = j_1, Z_{1,n_2} \in A, Z_{2,n_2} = 0 | X_0 = j_0, Z_{1,0} = 0, Z_{2,0} = x) \geq \mu_z(A).$$

From theorem 3.3 there is an n_3 where

$$P(X_{n_3} = j_0, Z_{1,n_3} = 0, Z_{2,n_3} \leq z | X_0 = i, Z_{1,0} = z_1, Z_{2,0} = z_2) = \beta > 0.$$

Therefore, for $n = n_1 + n_2 + n_3$ we find from theorem 3.3 and 3.5 that

$$\begin{aligned}
& P((X_n, Z_{1,n}, Z_{2,n}) \in B | X_0 = i, Z_{1,0} = z_1, Z_{2,0} = z_2) \\
& \geq P((X_{n_1+n_2}, Z_{1,n_1+n_2}, Z_{2,n_1+n_2}) \in B | X_0 = j_0, Z_{1,0} = 0, Z_{2,0} = z_2) \beta \\
& \geq \int P((X_{n_1}, Z_{1,n_1}, Z_{2,n_1}) \in B | X_0 = j_1, Z_{1,0} = x, Z_{2,0} = 0) \mu_z(dx) \beta > 0. \quad \square
\end{aligned}$$

Now that ϕ -irreducibility has been established, the ergodicity of $\{X_n, Z_{1,n}, Z_{2,n}\}$ can be established by appealing to the large body of literature on ϕ -irreducible Markov chains.

THEOREM 3.7. If $E_\pi U < E_\pi V$ and $E_\pi U < E_\pi W$, then $\{(X_n, Z_{1,n}, Z_{2,n})\}$ is ergodic.

PROOF. Since $\{X_n, Z_{1,n}, Z_{2,n}\}$ is ϕ -irreducible, from Jain and Jamison (1967) it follows that there exists a subinvariant measure μ where $\mu \gg \phi$. (see definition 2 in section 2). From Tweedie (1975), it follows that if

$$\lim_{n \rightarrow \infty} n^{-1} \sum_{i=1}^n P((X_i, Z_{1,i}, Z_{2,i}) \in A | X_0 = j_0, Z_{1,0} = 0, Z_{2,0} = 0) > 0 \quad (3.15)$$

for some A where $0 < \mu(A) < \infty$, then $\{X_n, Z_{1,n}, Z_{2,n}\}$ is ergodic.

It was established in Puri and Tollar (1985) that there is a j_0, N_0, M and $\delta > 0$ such that for all $n > N_0$,

$$P(X_n \in B, Z_{1,n} = 0 | X_0 = j_0, Z_{1,0} = 0, Z_{2,0} = 0) > \delta,$$

where $B = \{j_0\} \cup \{1, 2, \dots, M\}$. Also, from Tollar (1985a), we have that $Z_{2,n} \xrightarrow{d} Z_2$. As such, there is a w where for all $n > N_0$,

$$P(Z_{2,n} > w | X_0 = j_0, Z_{1,0} = 0, Z_{2,0} = 0) < \delta/2.$$

Thus, for all $n > N_0$,

$$P(X_n \in B, Z_{1,n} = 0, Z_{2,n} \leq w | X_0 = j_0, Z_{1,0} = 0, Z_{2,0} = 0) > \delta/2.$$

As such, from (3.15) we need only establish that $0 < \mu(B, 0, [0, w]) < \infty$ to prove ergodicity. From theorem 3.3, it is clear for some w that $\phi(B, 0, [0, w]) > 0$, which implies $\mu(B, 0, [0, w]) > 0$, since $\mu \gg \phi$.

To show that $\mu(B, 0, [0, w]) < \infty$, it is sufficient that $\mu(i, 0, [0, w]) < \infty$ for all $i \in B$. Since μ is σ -finite, there are sets where $\mu(S_n) < \infty$, and $\bigcup_{n=1}^{\infty} S_n = J \times [0, \infty) \times [0, \infty)$. In the construction of these sets in the proofs that μ is σ -finite (see Jain and Jamison (1967), and Orey (1971)), we see that the sets are of the form

$$S_n = \{(j, x_1, x_2) : \sum_{i=1}^n P((X_i, Z_{1,i}, Z_{2,i}) \in A | X_0 = j, Z_{1,0} = x_1, Z_{2,0} = x_2) \geq \frac{1}{n}\} \quad (3.16)$$

for A a specified set where $\phi(A) > 0$.

Since $\phi(A) > 0$, there is an n_1 where

$$\int_s P((X_{n_1}, Z_{1,n_1}, Z_{2,n_1}) \in A | X_0 = j_1, Z_{1,0} = x, Z_{2,0} = 0) \mu_z(dx) = \alpha > 0. \quad (3.17)$$

From theorem 3.5, there is an n_2 where for $x \leq z$, all $D \in \mathcal{B}^+$,

$$P(X_{n_2} = j_1, Z_{1,n_2} \in D, Z_{2,n_2} = 0 | X_0 = j_0, Z_{1,0} = 0, Z_{2,0} = x) \geq \mu_z(D). \quad (3.18)$$

Also, from theorem 3.3 and (3.1), it is apparent that there is an n_j where for all $u \leq w$,

$$\begin{aligned} & P(X_{n_j} = j_0, Z_{1,n_j} = 0, Z_{2,n_j} \leq z | X_0 = j, Z_{1,0} = 0, Z_{2,0} = u) \\ & \geq P(X_{n_j} = j_0, Z_{1,n_j} = 0, Z_{2,n_j} \leq z | X_0 = j, Z_{1,0} = 0, Z_{2,0} = w) = \delta_j > 0. \end{aligned} \quad (3.19)$$

Therefore from (3.17), (3.18) and (3.19) for $n = n_j + n_1 + n_2$, it is clear that for all $u \leq w$

$$P((X_n, Z_{1,n}, Z_{2,n}) \in A | X_0 = j, Z_{1,0} = 0, Z_{2,0} = u) \geq \alpha \delta_j > 0.$$

It is therefore clear from (3.16) that $(j, 0, [0, w]) \subset S_{n_j^*}$ where $n_j^* > (\alpha \delta_j)^{-1}$.

As such $\mu(B, 0, [0, w]) < \sum_{j \in B} \mu(S_{n_j^*}) < \infty$, which completes the proof. $\square$

Once the ϕ -irreducibility and ergodicity of $\{X_n, Z_{1,n}, Z_{2,n}\}$ is established, the proof of theorem 3.1 follows quickly.

PROOF OF THEOREM 3.1. The majority of the proof is accomplished by simply noting that $\{X_n, Z_{1,n}, Z_{2,n}, T_n\}$ has the desired form of section 2. As such, theorem 2.3 can be applied to yield for ϕ -almost all (i_0, x_0, y_0) ,

$$\begin{aligned} \lim_{t \rightarrow \infty} P_{(i_0, x_0, y_0)}((Z_1(t) \leq z_1, Z_2(t) \leq z_2) \\ = [E_{\psi}(T_N^*)]^{-1} \int_0^{\infty} P_{\psi}(Z_1^*(t) \leq z_1, Z_2^*(t) \leq z_2, T_N^* > t) dt. \end{aligned} \quad (3.20)$$

To complete the theorem, it is sufficient to show that for all (i_0, x_0, y_0) , (3.20) holds.

Since (3.20) holds for ϕ -almost all initial values, for any $i_0 \in J$, there is an (x_1, y_1) where for initial value (i_0, x_1, y_1) , (3.20) holds. As such, to complete the proof it is sufficient to show that

$$P[(Z_{1,n}(x_0), Z_{2,n}(x_0, y_0)) \neq (Z_{1,n}(x_1), Z_{2,n}(x_1, y_1)) \text{ i.o. } | X_0 = i_0] = 0,$$

for $Z_{1,n}(x)$, $Z_{2,n}(x, y)$ as in (3.2) and (3.3).

From (3.2) and (3.3) we can see that

$$\begin{aligned} P[(Z_{1,n}(x_0), Z_{2,n}(x_0, y_0)) \neq (Z_{1,n}(x_1), Z_{2,n}(x_1, y_1)) \text{ i.o. } | X_0 = i_0] \\ \leq P([S_n > \min(-x_0, -x_1)] \cup [\max_{1 \leq j \leq n} (S_j + R_n - R_j) > -\min(-x_0, -x_1)]) \quad (3.21) \\ \cup [R_n > \min(-x_0 - y_0, -x_1 - y_1)] \text{ i.o. } | X_0 = i_0). \end{aligned}$$

From Chung (1967),

$$n^{-1}S_n \rightarrow E_{\pi}U - E_{\pi}V \text{ a.s.}, \quad n^{-1}R_n \rightarrow E_{\pi}U - E_{\pi}W \text{ a.s.},$$

and

$$n^{-1} \max_{1 \leq j \leq n} (S_j + R_n - R_j) = n^{-1}R_n + n^{-1} \max_{1 \leq j \leq n} (S_j - R_j)$$

$$\rightarrow E_{\pi}U - E_{\pi}W + \max(0, E_{\pi}W - E_{\pi}V), \text{ a.s..}$$

Since $E_{\pi}U < E_{\pi}V$ and $E_{\pi}U < E_{\pi}W$, we see that

$$S_n \rightarrow -\infty, \quad R_n \rightarrow -\infty, \quad \max_{1 \leq j \leq n} (S_j + R_n - R_j) \rightarrow -\infty, \text{ a.s.},$$

and therefore from (3.21) we have that

$$P[(Z_{1,n}(x_0), Z_{2,n}(x_0, y_0)) \neq (Z_{1,n}(x_1), Z_{2,n}(x_1, y_1)) \text{ i.o.} | X_0 = i_0] = 0,$$

which completes the theorem. $\square$

As is usually the case, the construction of the measure ϕ was the major difficulty in dealing with the storage model as a Markov chain. The technical details unfortunately obscure the simplicity of the concept. When there is a renewal point, the measure ϕ is readily constructed for general Markov chains. While there is no renewal point in the present model we make the process act like one exists by first visiting $(i_0, 0, \cdot)$ and then visiting $(i_1, \cdot, 0)$. In this manner it "forgets" the initial values $Z_{1,0}, Z_{2,0}$.

4. CONCLUSION

While the results in section 2 are useful for the typical storage models defined on denumerable state Markov renewal processes, they are not particularly satisfying for the more general Markov renewal process on an arbitrary state space. Perhaps more structure on $\{T_n\}$ (for example, absolute continuity on all sojourn times with respect to a single measure) would allow a suitable modification of the C-set proof to establish that ergodicity and an appropriate finite sojourn moment are sufficient to satisfy the hypothesis of section 2. If this were the case, the proof that the arbitrary semi-Markov process converges would be complete. This area remains an open question.

The multitude of steps in section 3 point out a recurring problem in Markov chains, the construction of ϕ . The techniques of section 3 shed little light on how to construct such measures. As of now, the technique is very model specific. For applications, it would be very useful to have conditions which guarantee ϕ -irreducibility for Markov chains.

As far as the actual model under consideration in section 3, there are a variety of areas for further research. The most obvious is to take the arbitrary compartment model in Tollar (1985a,b), and extend the continuous time results as in the present paper and Tollar (1986). Perhaps of more interest would be to alter the model to more realistically accomodate two-way flow between compartments. Finally, it would be nice to have a more useful characterization of the limiting distribution of $(Z_1(t), Z_2(t))$ than the renewal equation results found in section 3. Unfortunately, the present techniques seem to be of little help in these directions.

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