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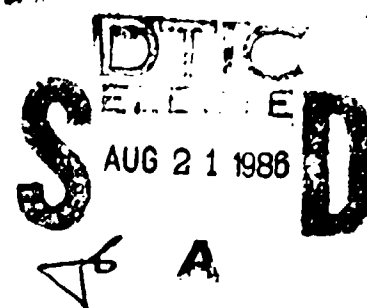
An Incomplete Lipschitz-Hankel Integral of K_0

Part I

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AN INCOMPLETE LIPSCHITZ-HANKEL INTEGRAL OF K_0

PART I

INTRODUCTION

An incomplete Lipschitz-Hankel integral of cylindrical functions of order zero, C_0 , may be defined by

$$C_{e_0}(a, z) \equiv \int_0^z e^{at} C_0(t) dt$$

Of interest in applications are the functions $J_{e_0}(a, z)$, $I_{e_0}(a, z)$, and $N_{e_0}(a, z)$ where J denotes the Bessel function of the first kind, I denotes the modified Bessel function, and N denotes the Bessel function of the second kind or Neumann function. $J_{e_0}(a, z)$ and $N_{e_0}(a, z)$ occur in problems in the theory of diffraction in optical apparatus [1, p. 227]. The function $I_{e_0}(a, z)$ plays an important role in the study of oscillating wings in supersonic flow and arises in the study of resonant absorption in media with finite dimensions [1, p. 195].

In this report we are interested in

$$K_{e_0}(a, z) \equiv \int_0^z e^{at} K_0(t) dt \quad (1)$$

where K denotes the MacDonald function or Bessel function of imaginary argument. We shall show that $K_{e_0}(a, z)$ can be written in closed form in terms of elementary functions, K_0 , K_1 , and Kampé de Fériet double hypergeometric functions. As an application it shall be shown that $K_{e_0}(\gamma, z)$ occurs when the statistical distribution of the maxima of a random function is applied to the amplitude of a sine wave in order to calculate the distribution of its ordinate. This latter distribution is of interest in the study of the scattered coherent reflected field from the sea surface [2].

Moreover we derive formulas for several integrals that are not readily available, and we exhibit some of the properties of the Kampé de Fériet functions associated with $K_{e_0}(a, z)$.

PRELIMINARY DEFINITIONS

The Pochhammer symbol $(a)_n$ is defined for nonnegative integers n as a ratio of gamma functions:

$$\begin{aligned} (a)_n &\equiv \Gamma(a+n)/\Gamma(a) = a(a+1) \dots (a+n-1) \\ (a)_0 &\equiv 1 \end{aligned} \quad (2)$$

Following Srivastava and Panda [3, p. 63] we define the Kampé de Fériet double hypergeometric functions:

$$F_{l,m;n}^{p,q;k} \left[\begin{matrix} (a_p); (b_q); (c_k); \\ (\alpha_l); (\beta_m); (\gamma_n); \end{matrix} x, y \right] \equiv \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{r+s} \prod_{j=1}^q (b_j)_r \prod_{j=1}^k (c_j)_s}{\prod_{j=1}^l (\alpha_j)_{r+s} \prod_{j=1}^m (\beta_j)_r \prod_{j=1}^n (\gamma_j)_s} \frac{x^r}{r!} \frac{y^s}{s!}$$

where the Pochhammer symbols $(a)_n$ are defined by Eq. (2). For convergence

$$p + q < l + m + 1, \quad p + k < l + n + 1, \quad |x| < \infty, \quad |y| < \infty, \text{ or}$$

$$p + q = l + m + 1, \quad p + k = l + n + 1, \text{ and}$$

$$\begin{cases} |x|^{1/(p-l)} + |y|^{1/(p-l)} < 1 & p > l \\ \max\{|x|, |y|\} < 1 & p \leq l \end{cases}$$

As special cases we define

$$L[\alpha, \beta; \gamma, \delta; x, y] \equiv \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\beta)_n}{(\gamma)_{m+n} (\delta)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \quad |x| < \infty, |y| < \infty \quad (3)$$

$$M[\alpha, \beta; \gamma, \delta; x, y] \equiv \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_n}{(\gamma)_{m+n} (\delta)_m} \frac{x^m}{m!} \frac{y^n}{n!} \quad |x| < \infty, |y| < 1 \quad (4)$$

We may then write

$$L[\alpha, \beta; \gamma, \delta; x, y] = F_{2,0;0}^{0,1,1} \left[\begin{matrix} -; \alpha; \beta; \\ \gamma, \delta; -; -; \end{matrix} x, y \right]$$

$$M[\alpha, \beta; \gamma, \delta; x, y] = F_{1,0;0}^{1,0,1} \left[\begin{matrix} \alpha; -; \beta; \\ \gamma; \delta; -; \end{matrix} x, y \right]$$

SOME ELEMENTARY PROPERTIES OF $M[\alpha, \beta; \gamma, \delta; x, y]$

Substituting [4, p. 266]

$$\frac{(\alpha)_p}{(\gamma)_p} = \frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\gamma - \alpha)} \int_0^1 t^{p+\alpha-1} (1-t)^{\gamma-\alpha-1} dt$$

where $\operatorname{Re} \gamma > \operatorname{Re} \alpha > 0$, and $p = m + n$ into Eq. (4), we deduce an integral representation for M :

$$M[\alpha, \beta; \gamma, \delta; x, y] = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 {}_0F_1[-; \delta; xt] t^{\alpha-1} (1-t)^{\gamma-\alpha-1} (1-yt)^{-\beta} dt$$

$$= \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} x^{-\frac{\delta-1}{2}} \int_0^1 I_{\delta-1}(2\sqrt{xt}) t^{\alpha-\frac{\delta+1}{2}} (1-t)^{\gamma-\alpha-1} (1-yt)^{-\beta} dt$$

Here we have used the equation

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} {}_0F_1[-; \nu+1; z^2/4] \quad (5)$$

We obtain directly from Eq. (4) the generating relation

$$M[\alpha, \beta; \gamma, \delta; x, y] = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n (\delta)_n} \frac{x^n}{n!} {}_2F_1[n+\alpha, \beta; n+\gamma; y] \quad (6)$$

We now prove the following

THEOREM: Suppose $-1 < \operatorname{Re}(\gamma - \alpha - \beta) < 0$, $|\arg y| < \pi$, $|\arg(1-y)| < \pi$. Then for $y \rightarrow 1$,

$$M[\alpha, \beta; \gamma, \delta; x, y] = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} {}_1F_2[\alpha; \gamma-\beta, \delta; x]$$

$$+ \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} (1-y)^{\gamma-\alpha-\beta} {}_0F_1[-; \delta; x] + O(1-y) \quad (7)$$

or

$$M[\alpha, \beta; \gamma, \delta; x, y] = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} {}_1F_2[\alpha; \gamma-\beta, \delta; x] + O(1-y)$$

$$+ \Gamma(\alpha+\beta-\gamma) \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\alpha)\Gamma(\beta)} x^{-\frac{\delta-1}{2}} (1-y)^{\gamma-\alpha-\beta} I_{\delta-1}(2\sqrt{x}) \quad (8)$$

Proof: The following result is found in [4, Eq. (9.5.7), p. 249]: for $\alpha + \beta - \gamma \neq 0, \pm 1, \pm 2, \dots$, $|\arg z| < \pi$, $|\arg(1-z)| < \pi$

$${}_2F_1[\alpha, \beta; \gamma; z] = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} {}_2F_1[\alpha, \beta; 1+\alpha+\beta-\gamma; 1-z]$$

$$+ (1-z)^{\gamma-\alpha-\beta} \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} {}_2F_1[\gamma-\alpha, \gamma-\beta; 1-\alpha-\beta+\gamma; 1-z]$$

Hence

$${}_2F_1[n + \alpha, \beta; n + \gamma; y] = \frac{\Gamma(n + \gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(n + \gamma - \beta)} {}_2F_1[n + \alpha, \beta; 1 + \alpha + \beta - \gamma; 1 - y] \\ + (1 - y)^{\gamma - \alpha - \beta} \frac{\Gamma(n + \gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(n + \alpha)\Gamma(\beta)} {}_2F_1[\gamma - \alpha, n + \gamma - \beta; 1 - \alpha - \beta + \gamma; 1 - y]$$

Now suppose that $-1 < \operatorname{Re}(\gamma - \alpha - \beta) < 0$. Then for $y \rightarrow 1$ we have

$${}_2F_1[n + \alpha, \beta; n + \gamma; y] = \frac{\Gamma(n + \gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(n + \gamma - \beta)} + \frac{\Gamma(n + \gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(n + \alpha)\Gamma(\beta)} (1 - y)^{\gamma - \alpha - \beta} + O(1 - y) \\ = \frac{(\gamma)_n}{(\gamma - \beta)_n} \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \beta)\Gamma(\gamma - \alpha)} + \frac{(\gamma)_n}{(\alpha)_n} \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)} (1 - y)^{\gamma - \alpha - \beta} + O(1 - y)$$

Substituting this result into Eq. (6) gives

$$M[\alpha, \beta; \gamma, \delta; x, y] = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma - \beta)_n (\delta)_n} \frac{x^n}{n!} \\ + \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)} (1 - y)^{\gamma - \alpha - \beta} \sum_{n=0}^{\infty} \frac{x^n}{(\delta)_n n!} + O(1 - y)$$

from which we obtain Eq. (7). Then using Eq. (5) we obtain Eq. (8).

Employing series rearrangement we deduce

$$M[\alpha, \beta; \gamma, \delta; x, tx] = \sum_{p=0}^{\infty} \frac{(\alpha)_p (\beta)_p}{(\gamma)_p} \frac{x^p}{p!} {}_1F_2[-p; \delta, 1 - \beta - p; 1/t] \quad (9)$$

Using a general result of Srivastava [3, Eq. (30), p. 145] we find Eq. (9) in a different form, viz,

$$M[\alpha, \beta; \gamma, \delta; tx, t] = \sum_{p=0}^{\infty} \frac{(\alpha)_p (\beta)_p}{(\gamma)_p} \frac{t^p}{p!} {}_1F_2[-p; \delta, 1 - \beta - p; x]$$

From Eq. (9) it follows that

$$M[\alpha, \beta; \gamma, \delta; x, tx] = \sum_{p=0}^{\infty} \frac{(\alpha)_p}{(\gamma)_p} \frac{x^p}{p!} {}_3F_0[\beta, -p, 1 - \delta - p; -; t] \quad (10)$$

Equation (10) may be obtained directly from [3, Eq. (60.ii), p. 194].

We remark that it may be shown that $M[\alpha, 1; \gamma, \delta; x, y]$ converges on the unit circle $|y| = 1$ if and only if ${}_2F_1[\alpha, 1; \gamma; y]$ converges on $|y| = 1$.

SOME ELEMENTARY PROPERTIES OF $L[\alpha, \beta; \gamma, \delta; x, y]$

Using series rearrangement we find

$$L[\alpha, \beta; \gamma, \delta; x, tx] = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n(\delta)_n} \frac{x^n}{n!} {}_2F_1[\beta, -n; 1 - \alpha - n; t]$$

This can also be obtained from [3, Eq. (30), p. 145] in a different form. Using Vandermonde's theorem [5, Eq. (1.7.7), p. 28]

$${}_2F_1[a, -n; c; 1] = (c - a)_n / (c)_n$$

$${}_2F_1[\beta, -n; 1 - \alpha - n; 1] = \frac{(1 - \alpha - \beta - n)_n}{(1 - \alpha - n)_n} = \frac{(\alpha + \beta)_n}{(\alpha)_n}$$

so that we have a reduction formula for L , viz.,

$$L[\alpha, \beta; \gamma, \delta; x, x] = \sum_{n=0}^{\infty} \frac{(\alpha + \beta)_n}{(\gamma)_n(\delta)_n} \frac{x^n}{n!} = {}_1F_2[\alpha + \beta; \gamma, \delta; x] \quad (11)$$

This result can be obtained also by using the following general result of Srivastava [3, Eq. (20), p. 55] applied to Eq. (3):

$$\sum_{m, n=0}^{\infty} c_{m+n}(\rho)_m(\sigma)_n \frac{x^{m+n}}{m!n!} = \sum_{n=0}^{\infty} c_n(\rho + \sigma)_n \frac{x^n}{n!}$$

provided each series is absolutely convergent.

We obtain directly from Eq. (3) the generating relation

$$L[\alpha, \beta; \gamma, \delta; x, y] = \sum_{m=0}^{\infty} \frac{(\alpha)_m}{(\gamma)_m(\delta)_m} \frac{x^m}{m!} {}_1F_2[\beta; m + \gamma, m + \delta; y] \quad (12)$$

Finally, using [3, Eq. (43), p. 150] we obtain

$$L[\alpha, \beta; \gamma, \delta; -x, x \tan^2 \theta] = (\cos^2 \theta)^\beta \sum_{n=0}^{\infty} \frac{(\beta)_n (\sin^2 \theta)^n}{n!} {}_1F_2[\alpha - n; \gamma, \delta; -x]$$

A CLOSED FORM FOR $K_{e_0}(a, z)$

From Eq. (1) we write

$$K_{e_0}(\alpha/\beta, \beta) = \beta \int_0^1 e^{\alpha t} K_0(\beta t) dt \quad (13)$$

Using [6, p. 89] we find the following formulas:

$$\begin{aligned} \int_0^1 s^m K_0(zs) ds &= \frac{K_0(z)}{m+1} {}_1F_2 \left[1; \frac{m+1}{2}, \frac{m+3}{2}; \frac{z^2}{4} \right] \\ &+ \frac{z K_1(z)}{(m+1)^2} {}_1F_2 \left[1; \frac{m+3}{2}, \frac{m+3}{2}; \frac{z^2}{4} \right] \quad m = 0, 2, 4, \dots \end{aligned} \quad (14)$$

$$\begin{aligned} \int_0^1 s^m K_0(zs) ds &= \frac{2^{m+1} \Gamma \left(\frac{m+1}{2} \right) \Gamma \left(\frac{m+1}{2} \right)}{z^{m+1}} \\ &- \frac{(m-1) K_0(z)}{z^2} {}_3F_0 \left[1, \frac{1-m}{2}, \frac{3-m}{2}; -; \frac{4}{z^2} \right] \\ &- \frac{K_1(z)}{z} {}_3F_0 \left[1, \frac{1-m}{2}, \frac{1-m}{2}; -; \frac{4}{z^2} \right] \quad m = 1, 3, 5, \dots \end{aligned} \quad (15)$$

Integrating term by term we find

$$\begin{aligned} \int_0^1 \exp(\alpha t) K_0(\beta t) dt &= \int_0^1 \sum_{n=0}^{\infty} \frac{\alpha^n t^n}{n!} K_0(\beta t) dt = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \int_0^1 t^n K_0(\beta t) dt \\ &= \sum_{n=0,2,4,\dots} \frac{\alpha^n}{n!} \int_0^1 t^n K_0(\beta t) dt + \sum_{n=1,3,5,\dots} \frac{\alpha^n}{n!} \int_0^1 t^n K_0(\beta t) dt \\ &= \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n)!} \int_0^1 t^{2n} K_0(\beta t) dt + \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n+1)!} \int_0^1 t^{2n+1} K_0(\beta t) dt \end{aligned}$$

so that using Eqs. (14) and (15)

$$\begin{aligned}
 \int_0^1 \exp(\alpha t) K_0(\beta t) dt &= \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n)!} \frac{K_0(\beta)}{2n+1} {}_1F_2\left[1; \frac{2n+1}{2}, \frac{2n+3}{2}; \frac{\beta^2}{4}\right] \\
 &+ \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n)!} \frac{\beta K_1(\beta)}{(2n+1)^2} {}_1F_2\left[1; \frac{2n+3}{2}, \frac{2n+3}{2}; \frac{\beta^2}{4}\right] \\
 &+ \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n+1)!} 2^{2n} \frac{\Gamma(n+1)\Gamma(n+1)}{\beta^{2n+2}} \\
 &- \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n+1)!} (2n) \frac{K_0(\beta)}{\beta^2} {}_3F_0[1, -n, 1-n; -; 4/\beta^2] \\
 &- \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n+1)!} \frac{K_1(\beta)}{\beta} {}_3F_0[1, -n, -n; -; 4/\beta^2]
 \end{aligned} \tag{16}$$

We shall consider each of the above five sums in the order in which they appear. We find

$$\begin{aligned}
 &\sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n+1)!} {}_1F_2[1; n+1/2, n+3/2; \beta^2/4] \\
 &= \sum_{n=0}^{\infty} \frac{1}{(3/2)_n} \frac{(\alpha^2/4)^n}{n!} {}_1F_2[1; n+1/2, n+3/2; \beta^2/4] = L[1/2, 1; 1/2, 3/2; \alpha^2/4, \beta^2/4]; \\
 &\sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n+1)(2n+1)!} {}_1F_2[1; n+3/2, n+3/2; \beta^2/4] \\
 &= \sum_{n=0}^{\infty} \frac{(1/2)_n}{(3/2)_n(3/2)_n} \frac{(\alpha^2/4)^n}{n!} {}_1F_2[1; n+3/2, n+3/2; \beta^2/4] = L[1/2, 1; 3/2, 3/2; \alpha^2/4, \beta^2/4]
 \end{aligned}$$

where in the latter two cases we have used Eq. (12);

$$\sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n+1)!} 2^{2n} \frac{\Gamma(n+1)\Gamma(n+1)}{\beta^{2n+2}} = \frac{\sin^{-1}(\alpha/\beta)}{\sqrt{\beta^2 - \alpha^2}} \quad |\alpha/\beta| \leq 1, \quad \alpha \neq \pm\beta$$

we have used [9, Eq. (9.121-14), p. 1041] the result: ${}_2F_1[1, 1; 3/2; \sin^2 z] = z/\sin z \cos z$:

$$\sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n+1)!} (n) {}_3F_0[1, -n, 1-n; -; 4/\beta^2] = \sum_{n=0}^{\infty} \frac{\alpha^{2n+3}}{(2n+3)!} (2n+2) {}_3F_0[1, -1-n, -n; -; 4/\beta^2]$$

$$\sum_{n=0}^{\infty} \frac{1}{(5/2)_n} \frac{(\alpha^2/4)^n}{n!} {}_3F_0[1, -1-n, -n; -; 4/\beta^2] = \frac{\alpha^3}{3} M \left[2, 1; \frac{5}{2}, 2; \frac{\alpha^2}{4}, \frac{\alpha^2}{\beta^2} \right];$$

and finally

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n+1)!} {}_3F_0[1, -n, -n; -; 4/\beta^2] \\ &= \alpha \sum_{n=0}^{\infty} \frac{1}{(3/2)_n} \frac{(\alpha^2/4)^n}{n!} {}_3F_0[1, -n, -n; -; 4/\beta^2] = \alpha M \left[1, 1; \frac{3}{2}, 2; \frac{\alpha^2}{4}, \frac{\alpha^2}{\beta^2} \right] \end{aligned}$$

where in the latter two cases we have used Eq. (10).

Defining

$$L_0(x, y) \equiv \sum_{m, n=0}^{\infty} \frac{(1/2)_m (1)_n}{(3/2)_{m+n} (3/2)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} = L[1/2, 1; 3/2, 3/2; x, y]$$

$$L_1(x, y) \equiv \sum_{m, n=0}^{\infty} \frac{(1/2)_m (1)_n}{(1/2)_{m+n} (3/2)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} = L[1/2, 1; 1/2, 3/2; x, y]$$

$$M_0(x, y) \equiv \sum_{m, n=0}^{\infty} \frac{(1)_{m+n}}{(3/2)_{m+n}} \frac{(1)_n}{(1)_m} \frac{x^m}{m!} \frac{y^n}{n!} = M[1, 1; 3/2, 1; x, y]$$

$$M_1(x, y) \equiv \sum_{m, n=0}^{\infty} \frac{(2)_{m+n}}{(5/2)_{m+n}} \frac{(1)_n}{(2)_m} \frac{x^m}{m!} \frac{y^n}{n!} = M[2, 1; 5/2, 2; x, y]$$

we have from Eq. (16) and the above results

$$\begin{aligned} \int_0^1 \exp(\alpha t) K_0(\beta t) dt &= K_1(\beta) \left[\beta L_0(\alpha^2/4, \beta^2/4) - \frac{\alpha}{\beta} M_0(\alpha^2/4, \alpha^2/\beta^2) \right] \\ &+ K_0(\beta) \left[L_1(\alpha^2/4, \beta^2/4) - \frac{\alpha^3}{3\beta^2} M_1(\alpha^2/4, \alpha^2/\beta^2) \right] + \frac{\sin^{-1}(\alpha/\beta)}{\sqrt{\beta^2 - \alpha^2}} \quad (17) \end{aligned}$$

which we may write using Eq. (13)

$$\begin{aligned}
 K_{e_0}(a, z) = & z K_1(z) \left[z L_0 \left[\frac{a^2 z^2}{4}, \frac{z^2}{4} \right] - a M_0 \left[\frac{a^2 z^2}{4}, a^2 \right] \right] \\
 & + z K_0(z) \left[L_1 \left[\frac{a^2 z^2}{4}, \frac{z^2}{4} \right] - \frac{a^3 z}{3} M_1 \left[\frac{a^2 z^2}{4}, a^2 \right] \right] + \frac{\sin^{-1} a}{\sqrt{1-a^2}}
 \end{aligned} \quad (18)$$

We have then given $K_{e_0}(a, z)$ in terms of elementary, MacDonald, and Kampé de Fériet functions.

We remark that in view of Eq. (6) and the definitions of M_0 and M_1

$$M_0(x, y) = \sum_{n=0}^{\infty} \frac{1}{(3/2)_n} \frac{x^n}{n!} {}_2F_1[1, n+1; n+3/2; y]$$

$$M_1(x, y) = \sum_{n=0}^{\infty} \frac{1}{(5/2)_n} \frac{x^n}{n!} {}_2F_1[1, n+2; n+5/2; y]$$

Since each of the Gauss hypergeometric functions above is conditionally convergent on the unit circle $|y| = 1$ except at $y = 1$ we see that $M_0(x, y)$ and $M_1(x, y)$ are conditionally convergent on $|y| = 1$ except at $y = 1$.^{*} Hence Eq. (17) is valid for $|\alpha/\beta| \leq 1$, $\alpha \neq \pm\beta$ and Eq. (18) is valid for $|a| \leq 1$, $a \neq \pm 1$. We shall show shortly that Eq. (17) is valid in the limit even when $\alpha = \pm\beta$. See Ref. 1 for other representations of $K_{e_0}(a, z)$.

In a future report (Part II) it shall be shown that Eq. (18) is easily extended to the entire complex a -plane in terms of elementary, MacDonald and Kampé de Fériet functions.

KING'S INTEGRAL

Using properties of L and M we have derived earlier we shall derive (a formula for) King's integral [6, Eq. (12), p. 123]:

$$\int_0^{\infty} \exp t K_0(t) dt = \alpha \exp \alpha [K_0(\alpha) + K_1(\alpha)] - 1 \quad (19)$$

that is we shall show Eq. (17) is valid in the limit for $\alpha = \beta$. Using Eq. (8) we find for $\alpha \rightarrow \beta$

$$M_0(\alpha^2/4, \alpha^2/\beta^2) = \frac{\pi}{2} \frac{I_0(\alpha)}{\sqrt{1-\alpha^2/\beta^2}} - \cosh \alpha + O(1-\alpha^2/\beta^2)$$

$$M_1(\alpha^2/4, \alpha^2/\beta^2) = \frac{3}{\alpha} \left[\frac{\pi}{2} \frac{I_1(\alpha)}{\sqrt{1-\alpha^2/\beta^2}} - \sinh \alpha + O(1-\alpha^2/\beta^2) \right]$$

^{*}Also see remark on top of p. 5

Substituting these equations into Eq. (17) gives

$$\int_0^1 \exp(\alpha t) K_0(\beta t) dt = \frac{\sin^{-1}(\alpha/\beta) - \frac{\pi}{2} [\alpha K_1(\beta) I_0(\alpha) + (\alpha^2/\beta) K_0(\beta) I_1(\alpha)]}{\sqrt{\beta^2 - \alpha^2}} + \frac{\alpha}{\beta} K_1(\beta) \cosh \alpha$$

$$+ \frac{\alpha^2}{\beta^2} K_0(\beta) \sinh \alpha + K_0(\beta) L_1(\alpha^2/4, \beta^2/4) + \beta K_1(\beta) L_0(\alpha^2/4, \beta^2/4) + O(1 - \alpha^2/\beta^2) \quad (20)$$

Using the reduction formula Eq. (11) for L we deduce

$$L_0(x^2/4, x^2/4) = \frac{\sinh x}{x}$$

$$L_1(x^2/4, x^2/4) = \cosh x$$

Now holding β fixed and letting $\alpha \rightarrow \beta$ we obtain after simplification

$$\int_0^1 \exp(\beta t) K_0(\beta t) dt = [K_0(\beta) + K_1(\beta)] \exp \beta + \lim_{\alpha \rightarrow \beta} J(\alpha, \beta)$$

where $J(\alpha, \beta)$ is the first term on the right-hand side of Eq. (20). We find however that

$$\lim_{\alpha \rightarrow \beta} \{\text{numerator } J(\alpha, \beta)\} = \frac{\pi}{2} [1 - \alpha K_1(\beta) I_0(\beta) - \beta K_0(\beta) I_1(\beta)] = 0$$

$$\lim_{\alpha \rightarrow \beta} \{\text{denominator } J(\alpha, \beta)\} = 0$$

so that on applying L'Hospital's rule we have

$$\lim_{\alpha \rightarrow \beta} J(\alpha, \beta) = -1/\beta$$

Hence

$$\int_0^1 \exp(\beta t) K_0(\beta t) dt = [K_0(\beta) + K_1(\beta)] \exp \beta - 1/\beta$$

and a simple transformation now gives Eq. (19). We may perform a similar analysis for $\alpha \rightarrow -\beta$ to obtain

$$\int_0^1 \exp(-\beta t) K_0(\beta t) dt = [K_0(\beta) - K_1(\beta)] \exp(-\beta) + 1/\beta$$

A DISTRIBUTION FOR THE ELEVATION OF A SINE WAVE

Consider the random variable $y = H \sin \theta$, where H is a random variable with density $K(H, \epsilon)$, $|H| < \infty$, and θ is a random variable, independent of H , with density

$$U(\theta) = \begin{cases} \pi^{-1} & |\theta| \leq \pi/2 \\ 0 & |\theta| > \pi/2 \end{cases}$$

Let $D(y, \epsilon)$ be the density function for y . It is shown in Ref. 2 that

$$D(y, \epsilon) = \frac{1}{\pi} \int_{-\infty}^{-|y|} \frac{K(H, \epsilon) dH}{\sqrt{H^2 - y^2}} + \frac{1}{\pi} \int_{|y|}^{\infty} \frac{K(H, \epsilon) dH}{\sqrt{H^2 - y^2}} \quad (21)$$

Rice [7] and Cartwright and Longuet-Higgins [8] have derived an expression for the statistical distribution of the maxima of a random function that may be expressed in the form

$$K(H, \epsilon) = \frac{\epsilon}{\sigma_H \sqrt{2\pi}} \exp\left(\frac{-H^2}{2\epsilon^2 \sigma_H^2}\right) + \frac{\sqrt{1-\epsilon^2}}{2\sigma_H^2} H \exp\left(\frac{-H^2}{2\sigma_H^2}\right) \left[1 + \operatorname{erf}\left(\frac{\sqrt{2}}{2} \frac{H}{\sigma_H} \frac{\sqrt{1-\epsilon^2}}{\epsilon}\right)\right] \quad (22)$$

Here σ_H is the standard deviation of H , and $0 < \epsilon < 1$ is known as the spectral width parameter. It is shown in Ref. 2 that the standard deviation σ of y is given by

$$\sigma = \sigma_H / (\sqrt{2}\eta)$$

where η is defined by

$$\eta \equiv \left[1 + \frac{\pi}{2}(1 - \epsilon^2)\right]^{-1/2}$$

Substituting Eq. (22) into Eq. (21) and using the latter result gives

$$D(y, \epsilon) = \frac{\epsilon}{2\pi^{3/2}\eta\sigma} \exp\left(\frac{-y^2}{8\epsilon^2\eta^2\sigma^2}\right) K_0\left(\frac{y^2}{8\epsilon^2\eta^2\sigma^2}\right) + \frac{\sqrt{1-\epsilon^2}}{\pi\eta\sigma} \exp\left(\frac{-y^2}{4\eta^2\sigma^2}\right) \Psi\left(\frac{\sqrt{1-\epsilon^2}}{\epsilon}, \frac{y}{2\eta\sigma}\right) \quad (23)$$

where the function $\Psi(k, u)$ is defined by

$$\Psi(k, u) \equiv \int_0^\infty \exp(-s^2) \operatorname{erf}(k\sqrt{u^2 + s^2}) ds \quad (24)$$

For real u and k it is shown in Ref. 2 that

$$\pi^{1/2} \int_0^\infty \exp(-s^2) \operatorname{erf}(k\sqrt{u^2 + s^2}) ds = \tan^{-1} k + \frac{k}{1+k^2} \int_0^{\frac{1}{2}u^2(1+k^2)} \exp\left(\frac{1-k^2}{1+k^2} s\right) K_0(s) ds$$

Using Eqs. (1) and (24) this may be written

$$\Psi(k, u) = \frac{\tan^{-1}(k)}{\pi^{1/2}} + \frac{1}{\pi^{1/2}} \frac{k}{1+k^2} K_{e_0} \left(\frac{1-k^2}{1+k^2}, \frac{1}{2} u^2 (1+k^2) \right)$$

We may then write Eq. (23)

$$D(y, \epsilon) = \frac{\epsilon}{2\pi^{3/2}\eta\sigma} \exp \left(\frac{-y^2}{8\epsilon^2\eta^2\sigma^2} \right) K_0 \left(\frac{y^2}{8\epsilon^2\eta^2\sigma^2} \right) \\ + \frac{\sqrt{1-\epsilon^2}}{\pi^{3/2}\eta\sigma} \exp \left(\frac{-y^2}{4\eta^2\sigma^2} \right) [\cos^{-1}\epsilon + \epsilon\sqrt{1-\epsilon^2} K_{e_0}(2\epsilon^2-1, y^2/8\epsilon^2\eta^2\sigma^2)]$$

where $K_{e_0}(a, z)$ is given by Eq. (18).

SOME INTEGRALS RELATED TO $K_{e_0}(a, z)$

The following integrals can easily be obtained from Eq. (17):

$$\int_0^1 \sin(\alpha t) K_0(\beta t) dt = \frac{\sinh^{-1}(\alpha/\beta)}{\sqrt{\alpha^2 + \beta^2}} - \frac{\alpha}{\beta} K_1(\beta) M_0(-\alpha^2/4, -\alpha^2/\beta^2) \\ + \frac{\alpha^3}{3\beta^2} K_0(\beta) M_1(-\alpha^2/4, -\alpha^2/\beta^2) \quad |\alpha/\beta| \leq 1, \alpha \neq \pm i\beta$$

$$\int_0^1 \cos(\alpha t) K_0(\beta t) dt = \beta K_1(\beta) L_0(-\alpha^2/4, \beta^2/4) + K_0(\beta) L_1(-\alpha^2/4, \beta^2/4)$$

Further, using the result [2] for $0 < |\alpha| < 1, 0 < x$

$$K_{e_0}(\alpha, x) = \operatorname{sgn} \alpha \left\{ \exp(\alpha x) \left[\int_0^\infty \frac{\cos(\alpha x t) dt}{(1+t^2)\sqrt{1+\alpha^2 t^2}} + \int_0^\infty \frac{t \sin(\alpha x t) dt}{(1+t^2)\sqrt{1+\alpha^2 t^2}} \right] - \frac{\cos^{-1}(|\alpha|)}{\sqrt{1-\alpha^2}} \right\}$$

we find for $0 < \alpha < \beta$

$$\int_0^\infty \frac{\cos(\alpha x) dx}{(1+x^2)\sqrt{\beta^2 + \alpha^2 x^2}} = \cosh \alpha \left\{ \frac{\pi/2}{\sqrt{\beta^2 - \alpha^2}} - \frac{\alpha}{\beta} K_1(\beta) M_0(\alpha^2/4, \alpha^2/\beta^2) - \frac{\alpha^3}{3\beta^2} K_0(\beta) M_1(\alpha^2/4, \alpha^2/\beta^2) \right\} \\ - \sinh \alpha \left\{ \beta K_1(\beta) L_0(\alpha^2/4, \beta^2/4) + K_0(\beta) L_1(\alpha^2/4, \beta^2/4) \right\}$$

$$\int_0^\infty \frac{x \sin(\alpha x) dx}{(1+x^2)\sqrt{\beta^2+\alpha^2 x^2}} = \cosh \alpha \left\{ \beta K_1(\beta) L_0(\alpha^2/4, \beta^2/4) + K_0(\beta) L_1(\alpha^2/4, \beta^2/4) \right\} \\ - \sinh \alpha \left\{ \frac{\pi/2}{\sqrt{\beta^2-\alpha^2}} - \frac{\alpha}{\beta} K_1(\beta) M_0(\alpha^2/4, \alpha^2/\beta^2) - \frac{\alpha^3}{3\beta^2} K_0(\beta) M_1(\alpha^2/4, \alpha^2/\beta^2) \right\}$$

In addition [9, Eq. (3.367), p. 316] we have

$$\int_0^\infty \frac{e^{-pt} \sin \theta dt}{(1+t+\cos \theta)\sqrt{t^2+2t}} = \exp \left[2p \cos^2 \frac{\theta}{2} \right] [\theta - \sin \theta K_{e_0}(-\cos \theta, p)] \quad \text{Re } p > 0$$

CONCLUSIONS

The Kampé de Fériet functions have been used to put in closed form the incomplete Lipschitz-Hankel integral $K_{e_0}(a, z)$ and several related integrals that are not readily available and are of interest in mathematical physics and applications. Some of the properties of the Kampé de Fériet functions associated with $K_{e_0}(a, z)$ are derived. These properties are useful in deriving additional results quickly. As an example we have given an elementary derivation of a closed form for King's integral based on generating function techniques.

In addition, the utility of a closed form for $K_{e_0}(a, z)$ is indicated by deriving a certain density function that is associated with the scattered coherent return from the sea surface.

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