

AD-R163 702

B-SPLINES WITHOUT DIVIDED DIFFERENCES(U) WISCONSIN  
UNIV-MADISON MATHEMATICS RESEARCH CENTER  
C DE BOOR ET AL. DEC 85 MRC-TSR-2896 DAA629-88-C-0041

1/1

UNCLASSIFIED

F/G 12/1

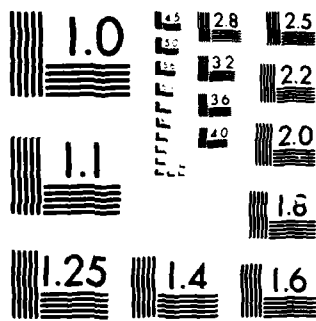
NL



END

FBI/DOJ

1/1



MICROCOPY RESOLUTION TEST CHART  
NATIONAL BUREAU OF STANDARDS-1963-A

2

MRC Technical Summary Report #2896

B-SPLINES WITHOUT DIVIDED DIFFERENCES

Carl de Boor and Klaus Höllig

AD-A163 702

Mathematics Research Center  
University of Wisconsin—Madison  
610 Walnut Street  
Madison, Wisconsin 53705

December 1985

DTIC  
ELECTE  
FEB 5 1986  
S D  
B

DTIC FILE COPY (Received December 11, 1985)

Approved for public release  
Distribution unlimited

Sponsored by

U. S. Army Research Office  
P. O. Box 12211  
Research Triangle Park  
North Carolina 27709

National Science Foundation  
Washington, DC 20550

86 2 5 000

UNIVERSITY OF WISCONSIN-MADISON  
MATHEMATICS RESEARCH CENTER

B-SPLINES WITHOUT DIVIDED DIFFERENCES

Carl de Boor<sup>(1)</sup> and Klaus Höllig<sup>(1,2)</sup>

Technical Summary Report #2896

December 1985

ABSTRACT

This note develops the basic B-spline theory without using divided differences. Instead, the starting point is the definition of B-splines via recurrence relations. This approach yields very simple derivations of basic properties of spline functions and algorithms.

AMS (MOS) Subject Classification: 41A15

Key Words: B-splines, recurrence relations, algorithms

Work Unit Number 3 (Numerical Analysis and Scientific Computing)

---

(1) Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

(2) Supported by International Business Machines Corporation and National Science Foundation Grant No. DMS-8351187.

## SIGNIFICANCE AND EXPLANATION

The standard definition of the univariate B-spline involves the divided difference and the calculus of divided differences has been heavily used in developing the univariate spline theory.

In this report we discuss an alternative approach. We choose the recurrence relations for B-splines as the basic concept. This leads to very simple derivations of the main properties of spline functions as well as of the basic algorithms.

Accession For	
NTIS GRA&I	<input checked="" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By _____	
Distribution/	
Availability Codes	
Dist	Avail and/or Special
A-1	

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

# B-SPLINES WITHOUT DIVIDED DIFFERENCES

Carl de Boor<sup>(1)</sup> and Klaus Höllig<sup>(1,2)</sup>

## 1. Basic Properties

Let  $t := \dots, t_{-1}, t_0, t_1, \dots$  be a nondecreasing, biinfinite sequence of real numbers with  $\lim_{i \rightarrow \pm \infty} t_i = \pm \infty$ . The **B-splines** corresponding to the "knot sequence"  $t$  are defined by the recurrence relation

$$B_{i,k} := \omega_{i,k} B_{i,k-1} + (1 - \omega_{i+1,k}) B_{i+1,k-1} \quad (1a)$$

with

$$B_{i,1}(t) := \begin{cases} 1, & \text{if } t_i \leq t < t_{i+1} \\ 0, & \text{otherwise} \end{cases} \quad (1b)$$

$$\omega_{i,k}(t) := \begin{cases} \frac{t - t_{i+k-1}}{t_i - t_{i+k-1}}, & \text{if } t_i < t_{i+k-1} \\ 0, & \text{otherwise} \end{cases}$$

This gives  $B_{i,k}$  in the form

$$B_{i,k} = \sum_{j=1}^{i+k-1} b_{j,k} B_{j,1} \quad (2)$$

with each  $b_{j,k}$  a polynomial of degree  $< k$  since it is the sum of products of  $k-1$  linear polynomials. From this we read off that  $B_{i,k}$  is a piecewise polynomial of degree  $< k$  which vanishes outside the interval  $[t_i, t_{i+k})$  and has possible breakpoints  $t_i, \dots, t_{i+k}$ . In particular,  $B_{i,k}$  is just the zero function in case  $t_i = t_{i+k}$ . Also, by induction,  $B_{i,k}$  is positive on the open interval  $(t_i, t_{i+k})$ , since both  $\omega_{i,k}$  and  $1 - \omega_{i+1,k}$  are positive there. But it is a bit of a miracle that the recurrence relations produce smooth functions. We discuss this question in Theorem 2 below.

A **spline of order  $k$  with knot sequence  $t$**  is, by definition, a linear combination of the B-splines  $B_{i,k}$  associated with that knot sequence. Let

$$S_{k,t} := \left\{ \sum_{i=-\infty}^{\infty} a_i B_{i,k} : a_i \in \mathbf{R} \right\} \quad (3)$$

denote the collection of all such splines. We now explore this space.

- (1) Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.
- (2) Supported by International Business Machines Corporation and National Science Foundation Grant No. DMS-8351187.

We deduce from the recurrence relation that

$$\sum a_i B_{i,k} = \sum (a_i \omega_{i,k} + a_{i-1} (1 - \omega_{i,k})) B_{i,k-1}. \quad (4)$$

On the other hand, arguing as in [2], for the special sequence

$$a_i := \psi_{i,k}(\tau) := (t_{i-1} - \tau) \cdots (t_{i-k-1} - \tau)$$

(with  $\tau \in \mathbf{R}$ ), we find for  $B_{i,k-1} \neq 0$ , i.e., for  $t_i > t_{i-k-1}$  that

$$\begin{aligned} a_i \omega_{i,k} + a_{i-1} (1 - \omega_{i,k}) - \psi_{i,k-1}(\tau) \left( (t_{i-k-1} - \tau) \omega_{i,k} - (t_i - \tau) (1 - \omega_{i,k}) \right) \\ = \psi_{i,k-1}(\tau) (\cdot - \tau) \end{aligned}$$

since  $f(t_{i-k-1}) \omega_{i,k} + f(t_i) (1 - \omega_{i,k})$  is the straight line which agrees with  $f$  at  $t_{i+k-1}$  and  $t_i$ . This shows that

$$\sum \psi_{i,k}(\tau) B_{i,k} = (\cdot - \tau) \sum \psi_{i,k-1}(\tau) B_{i,k-1},$$

hence, by induction, that

$$\sum \psi_{i,k}(\tau) B_{i,k} = (\cdot - \tau)^{k-1} \sum \psi_{i,1}(\tau) B_{i,1}.$$

This proves the following identity due to Marsden.

**Theorem 1.** For any  $\tau \in \mathbf{R}$ ,

$$(\cdot - \tau)^{k-1} = \sum_1 \psi_{i,k}(\tau) B_{i,k}, \quad (5)$$

with  $\psi_{i,k}(\tau) := (t_{i-1} - \tau) \cdots (t_{i-k-1} - \tau)$ .

Since  $\tau$  in (5) is arbitrary, it follows that  $S_{k,t}$  contains all polynomials of degree  $< k$ . More than that, we can even give an explicit expression for the required coefficients, as follows.

By differentiating (5) with respect to  $\tau$ , we obtain the identities

$$\frac{(\cdot - \tau)^\nu}{\nu!} = \sum_1 \frac{(D)^{k-1-\nu} \psi_{i,k}(\tau)}{(k-1)!} B_{i,k}, \quad \nu < k, \quad (6)$$

with  $Df$  the derivative of the function  $f$ . On using this identity in the Taylor formula

$$p = \sum_1 \frac{D^\nu p(\tau)}{\nu!} (\cdot - \tau)^\nu,$$

for a polynomial  $p$  of degree  $< k$ , we conclude that any such polynomial can be written in the form

$$p = \sum_i \lambda_{i,k} p B_{i,k}. \quad (7)$$

with  $\lambda_{i,k}$  given by the rule

$$\lambda_{i,k} f := \sum_{\nu=0}^{k-1} \frac{(-D)^{k-1-\nu} \psi_{i,k}(\tau)}{(k-1)!} D^\nu f(\tau). \quad (8)$$

Here are two special cases of particular interest. For  $p = 1$ , we get

$$1 = \sum_i B_{i,k} \quad (9)$$

since  $D^{k-1} \psi_{i,k} = (-1)^{k-1} (k-1)!$ , and this shows that the  $B_{i,k}$  form a **partition of unity**. Further, since  $D^{k-2} \psi_{i,k}$  is a linear polynomial which vanishes at  $t_i' := (t_{i-1} + \dots + t_{i-k-1}) / (k-1)$ .

$$\ell = \sum_i \ell(t_i') B_{i,k} \text{ for every linear polynomial } \ell. \quad (10)$$

The identity (5) also gives us various **piecewise** polynomials contained in  $S_{k,t}$ : If  $\tau = t_j$ , then  $\psi_{i,k}(\tau)$  vanishes for  $i = j - k - 1, \dots, j - 1$ . Since  $B_{i,k}(t) = 0$  for  $i < j - k + 1$  and  $t < t_j$ , it follows that

$$(\cdot - t_j)_+^{k-1} = \sum_{i=j}^{\infty} \psi_{i,k}(t_j) B_{i,k} \quad (11)$$

with  $\alpha_+ := \max\{\alpha, 0\}$  the positive part of the number  $\alpha$ . The same observation applied to (6) shows that

$$(\cdot - t_j)_+^{k-\mu} \in S_{k,t} \text{ for } 1 \leq \mu \leq \#t_j := \#\{t_i : t_i = t_j\}. \quad (12)$$

**Theorem 2.** If  $t_i < t_{i+k}$ , then the B-splines  $B_{i,k}$  are linearly independent and the space  $S_{k,t}$  coincides with the space  $\tilde{S}$  of all piecewise polynomials of degree  $< k$  with breakpoints  $t_i$  which are  $k-1-\#t_i$  times continuously differentiable at  $t_i$ .

**Proof.** It is sufficient to prove that, for any finite interval  $I := [a, b]$ , the restriction  $\tilde{S}_I$  of the space  $\tilde{S}$  to the interval  $I$  coincides with the restriction of  $S_{k,t}$  to that interval. The latter space is spanned by all the B-splines having some support in  $I$ , i.e., all  $B_{i,k}$  with  $(t_i, t_{i+k}) \cap I \neq \emptyset$ . The space  $\tilde{S}_I$  has a basis consisting of the functions

$$(\cdot - a)_+^{k-\nu}, \nu = 1, \dots, k; (\cdot - t_i)_+^{k-\nu}, \nu = 1, \dots, \#t_i, \text{ for } a < t_i < b. \quad (13)$$

This follows from the observation that a piecewise polynomial function  $f$  with a breakpoint at  $t_i$  which is  $k-1-\#t_i$  times continuously differentiable there can be written uniquely as

$$f = p + \sum_{\nu=1}^{\#t_i} a_\nu (\cdot - t_i)_+^{k-\nu}.$$



with  $p$  a suitable polynomial of degree  $\leq k$  and suitable coefficients  $a_j$ . Since each of the functions in (13) lies in  $S_{k,t}$ , by (6) and (12), we conclude that

$$\hat{S}_I = (S_{k,t})_I. \quad (14)$$

On the other hand, the dimension of  $\hat{S}_I$ , i.e., the number of functions in (13), equals the number of B-splines with some support in  $I$  (since it equals  $k \cdot \sum_{a < t_i < b, b \neq t_i}$ ), hence is an upper bound on the dimension of  $(S_{k,t})_I$ . This implies that equality must hold in (14), and that the set of B-splines having some support in  $I$  must be linearly independent over  $I$ .

**Corollary 1.** All B-splines having some support on a given interval are linearly independent over that interval.

**Corollary 2.** If  $\hat{t}$  is a refinement of the knot sequence  $t$ , then  $S_{k,\hat{t}} \subset S_{k,t}$ .

**Corollary 3.** If  $t_i < t_{i+k-j}$ , then the derivative of a spline in  $S_{k,t}$  is a spline of degree  $\leq k-1$  with respect to the same knot sequence, i.e.,  $DS_{k,t} = S_{k-1,t}$ .

The identity (7) can be extended to all spline functions. For this, we agree, consistent with (1b), that all derivatives in (8) are to be taken as limits from the right in case  $\tau$  coincides with a knot.

**Theorem 3.** If  $\tau$  in definition (8) of  $\lambda_{i,k}$  is chosen in the interval  $(t_i, t_{i+k})$ , then

$$\lambda_{i,k} \left( \sum_j a_j B_{j,k} \right) = a_i. \quad (15)$$

It is remarkable that  $\tau$  can be chosen arbitrarily in the interval  $(t_i, t_{i+k})$ . The reason behind this is that  $\lambda_{i,k} f$  does not depend on  $\tau$  at all if  $f$  is a polynomial of degree less than  $k$ .

**Proof.** Assume that  $\tau \in (t_l, t_{l+1}) \subset (t_i, t_{i+k})$  and let  $p_j$  be the polynomial which agrees with  $B_{j,k}$  on  $(t_l, t_{l+1})$ . Then

$$\lambda_{i,k} B_{j,k} = \lambda_{i,k} p_j.$$

On the other hand,

$$p_j = \sum_{i=l-1-k}^l \lambda_{i,k} p_j p_i.$$

since this holds by (7) on  $(t_l, t_{l+1})$ , while, by Corollary 1 or directly from (7),  $p_{l-1-k}, \dots, p_l$  are linearly independent. Therefore necessarily  $\lambda_{i,k} p_j$  equals 1 if  $i = j$  and 0 otherwise.

## 2. Algorithms

In this section, the basic algorithms for computing with the B-spline representation are derived. Unless otherwise stated, all algorithms refer to the spline function

$$s = \sum_i a_i B_{i,k}. \quad (16)$$

In the following, the subscript  $k$  is omitted whenever possible, e.g.  $B_i := B_{i,k}$ ,  $v_i := v_{i,k}$ , etc.

The spline  $s$  can be evaluated using the recurrence relation. By (4),

$$\sum_i a_i B_{i,k} = \sum_i \left( \omega_{i,k} a_i + (1 - \omega_{i,k}) a_{i-1} \right) B_{i,k-1} =: \sum_i a_i^1 B_{i,k-1}.$$

Iterating this identity one finally arrives at

$$s = \sum_i a_i^{k-1} B_{i,1}$$

and, by the definition of  $B_{i,1}$ , the right hand side equals  $a_j^{k-1}$  on  $[t_j, t_{j+1})$ . This yields

**Algorithm 1.** On the interval  $[t_j, t_{j+1})$ ,  $s = a_j^{k-1}$ , with the polynomials  $a_i^r$  computed as follows:

$$\begin{aligned} a_i^1 &:= a_i \\ a_i^{r-1} &:= \omega_{i,k-r} a_i^r + (1 - \omega_{i,k-r}) a_{i-1}^r, \quad j - k - r + 1 < i \leq j. \end{aligned}$$

For the relevant range of indices,  $\omega_{i,k-r}(t) \in [0, 1]$  so that  $s(t)$  is computed by repeatedly forming convex combinations of the B-spline coefficients.

It follows from Theorem 2 that the derivative of  $s$  is a spline of degree  $< k - 1$  with respect to the same knot sequence, i.e.

$$Ds =: \sum_i a_i' B_{i,k-1}. \quad (17)$$

By Theorem 3,

$$a_i' = \lambda_{i,k-1}(Ds) \quad (18)$$

if  $\tau$  is chosen in the interval  $(t_i, t_{i+k-1})$ . To relate  $a'$  to  $a$ , we express  $\lambda_{i,k-1}D$  as a linear combination of the functionals  $\lambda_{i,k}$ , making use of the fact that  $\lambda_{i,k}$  depends linearly on  $v_{i,k}$  and that

$$(t_{i+k-1} - t_i) v_{i,k-1} = v_{i,k} - v_{i-1,k}. \quad (19)$$

From the definition (8),

$$(\lambda_{i,k} - \lambda_{i-1,k}) f(\tau) = \sum_{\nu=0}^{k-1} \frac{(D)^\nu (v_{i,k} - v_{i-1,k})(\tau)}{(k-1)!} D^\nu f(\tau)$$

and

$$\lambda_{i,k-1} Df(\tau) = \sum_{\nu=0}^{k-2} \frac{(-D)^{k-2-\nu} \psi_{i,k-1}(\tau)}{(k-2)!} D^{\nu+1} f(\tau) = \sum_{\nu=0}^{k-1} \frac{(-D)^{k-1-\nu} \psi_{i,k-1}(\tau)}{(k-2)!} D^{\nu} f(\tau),$$

the last equality by setting  $\mu := \nu + 1$  and using the fact that  $D^{k-1} \psi_{i,k-1} = 0$ . Comparison of these two lines shows with the aid of (19) that

$$\lambda_{i,k-1} D = \frac{k-1}{t_{i+k-1} - t_i} (\lambda_{i,k} - \lambda_{i-1,k}).$$

Assuming that  $B_{i,k-1} \neq 0$ , i.e., that  $t_i < t_{i+k-1}$ , we can choose  $\tau \in (t_i, t_{i+k-1}) = (t_{i-1}, t_{i+k-1}) \cap (t_i, t_{i+k})$ . By Theorem 3, this yields

**Algorithm 2.** Compute the coefficients for  $\sum a'_i B_{i,k-1} := D \sum a_i B_{i,k}$  by

$$a'_i = \frac{a_i - a_{i-1}}{(t_{i+k-1} - t_i)(k-1)}, \text{ if } t_i < t_{i+k-1}.$$

By Corollary 3,  $S_{\hat{\mathbf{t}}} \subset S_{\mathbf{t}}$  for any refinement  $\hat{\mathbf{t}}$  of the knot sequence  $\mathbf{t}$ , and therefore any spline  $s \in S_{\hat{\mathbf{t}}}$  can be written as a linear combination  $\sum \hat{a}_i \hat{B}_i$  of the B-splines  $\hat{B}_i$  which correspond to the refined knot sequence. The computation of the new coefficients  $\hat{a}_i$  from the  $a_i$  constitutes the **knot insertion** or **subdivision algorithm** used in CAGD [1,4]. For this, we need to express  $\hat{a}_i$  in terms of the  $a_i$ . By Theorem 3, this is equivalent to comparing the corresponding  $\hat{\lambda}_i$  with  $\lambda_i$ . Since  $\lambda_i$  depends linearly on  $\psi_i$ , this requires nothing more than to express

$$\hat{\psi}_i = (\hat{t}_{i+1} - \cdot) \cdots (\hat{t}_{i+k-1} - \cdot)$$

as a linear combination of the  $\psi_i$ .

This is particularly easy when  $\hat{\mathbf{t}}$  is obtained from  $\mathbf{t}$  by adding just one knot, say the point  $\hat{t}_j = (t_j, t_{j+1})$  so that

$$\hat{t}_i = \begin{cases} t_i, & \text{if } i \leq j; \\ \hat{t}_j, & \text{if } i = j+1; \\ t_{i-1}, & \text{if } i > j+1. \end{cases}$$

Then

$$\hat{\psi}_i = \begin{cases} \psi_i, & i < j - k + 2; \\ \psi_{i-1}, & i = j. \end{cases}$$

hence there is some actual computing necessary only for  $j - k + 2 \leq i \leq j$ . For this case, observe that  $\hat{t}_{i-1} = \hat{t}_i = \hat{t}_{j-1} = \hat{t}_{i+k-1}$ , hence

$$\frac{\alpha \psi_i + \beta \psi_{i-1}}{\hat{\psi}_i} = \frac{(t_{i-1} - \cdot) \cdots (t_{i+k-2} - \cdot)}{(t_{i+k-1} - \cdot)} \alpha + \beta (t_i - \cdot)$$

provided  $\alpha(t_{i+k-1} - \cdot) + \beta(t_i - \cdot) = (\hat{t} - \cdot)$ , i.e.,

$$\alpha = \omega_i(\hat{t}) \text{ and } \beta = (1 - \omega_i(\hat{t})).$$

Since  $\hat{t}_{i-1} \leq \hat{t} \leq \hat{t}_{j-2} \leq \hat{t}_{i-k}$ , we can choose  $\tau$  in the definition (8) in the interval  $(\hat{t}_i, \hat{t}_{i+k}) = (t_{i-1}, t_{i+k-1}) \cap (t_i, t_{i+k})$ . This proves

**Algorithm 3.** If the knot sequence  $\hat{\mathbf{t}}$  is obtained from the knot sequence  $\mathbf{t}$  by addition of the point  $\hat{t} \in (t_j, t_{j+1})$ , then the coefficients  $\hat{a}_i$  for the spline  $s$  with respect to the refined knot sequence are given by

$$\hat{a}_i = \begin{cases} a_i, & \text{if } i = j - k - 2; \\ \omega_i(\hat{t}) a_i + (1 - \omega_i(\hat{t})) a_{i-1}, & \text{if } j - k - 2 \leq i < j; \\ a_{i-1}, & \text{if } i > j. \end{cases} \quad (20)$$

Observe that  $\omega(\hat{t}) \in [0, 1]$  for the relevant range of indices and thus the coefficients  $\hat{a}$  are convex combinations of the coefficients  $a$ .

If  $r := \# \hat{t} \leq k - 1$ , then, after just  $(k - 1 - r)$ -fold insertion of  $\hat{t}$ , we obtain a knot sequence  $\tilde{\mathbf{t}}$  in which the number  $\hat{t}$  occurs exactly  $k - 1$  times. This means that there is exactly one B-spline for that knot sequence which is not zero at  $\hat{t}$ . Hence it must equal 1 at  $\hat{t}$  and its coefficient must provide the value of  $s$  at  $\hat{t}$ . This makes it less surprising that the calculations in Algorithms 1 and 3 are identical.

## References

- 1 W. Böhm. Inserting new knots into a B-spline curve. *Computer-Aided Design* **12** (1980). 199-201.
- 2 C. de Boor. On calculating with B-splines. *J. Approximation Theory* **6** (1972). 50-62.
- 3 C. de Boor. *A practical guide to splines*. Springer Verlag, 1978.
- 4 E. Cohen, T. Lyche and R. Riesenfeld. Discrete B-splines and subdivision techniques in Computer-Aided Design and Computer Graphics. *Computer Graphics and Image Processing* **14** (1980). 87-111.

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2896	2. GOVT ACCESSION NO. AD A163702	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) B-SPLINES WITHOUT DIVIDED DIFFERENCES		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Carl de Boor and Klaus Höllig		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041 DMS-8351187
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Madison, Wisconsin 53705		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 3 - Numerical Analysis and Scientific Computing
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 below.		12. REPORT DATE December 1985
		13. NUMBER OF PAGES 8
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES U. S. Army Research Office P. O. Box 12211 Research Triangle Park North Carolina 27709 National Science Foundation Washington, DC 20550		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) B-splines recurrence relations algorithms		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) This note develops the basic B-spline theory without using divided differences. Instead, the starting point is the definition of B-splines via recurrence relations. This approach yields very simple derivations of basic properties of spline functions and algorithms.		

**END**

**FILMED**

3-86

**DTIC**