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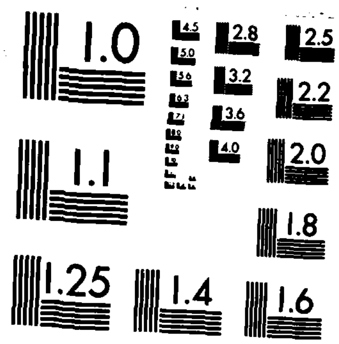

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ON THE TURNING LOZANGE OF CONSTANT SIDE WHOSE VERTICES ALTERNATE BETWEEN TWO FIXED CIRCLES OF A RING

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ABSTRACT

This note was written to call attention to the remarkable paper [1] by W. L. Black, H. C. Howland and B. Howland. Their result recalls the classical theorems of Poucelet and Steiner from the golden age of Geometry. [1] starts with two circles Γ and $\tilde{\Gamma}$ in R^3 satisfying the qualitative condition (A) and assumes that there is a closed equilateral polygon

$$\Pi_{2n} = P_1 P_2 \cdots P_{2n} \quad (n > 2) \quad ,$$

having the sides $P_k P_{k+1} = d$ $(k = 1,2,\ldots,n;\ P_{2n+1} + P_1)$ such that all vertices P_{2k-1} are on Γ, and P_{2k} are on $\tilde{\Gamma}$ $(k = 1,\ldots,n)$. It is shown that Π_{2n} can be turned around with constant d, so as to return to its initial position. This note, independent of [1], settles the case when the circles Γ and $\tilde{\Gamma}$ are coplanar and $n = 2$, when Π_4 becomes a lozange. It is shown that $d^2 = a^2 + b^2 - c^2$, where a and b are the radii of Γ and $\tilde{\Gamma}$, respectively, and c is the distance between the centers of Γ and $\tilde{\Gamma}$.

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cont
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SIGNIFICANCE AND EXPLANATION

on a theorem about zig-zags between two circles,

This note is to call attention to the ~~remarkable~~ paper [1] by W. L. Black, H. C. Howland and B. Howland, and may be ~~read~~ independently of [1]. It explains the construction ~~(Fig. 1)~~ of a linkage composed of four equal bars $P_1P_2 = P_2P_3 = P_3P_4 = P_4P_1 = d$, joined at the points P_k, with the following property: With P_1 and P_3 moving in the circular groove $\cancel{\Lambda}$ T, and P_3, P_4 moving in the circular groove $\cancel{\Lambda}$ T′, it is shown that the lozange $P_1P_2P_3P_4$ can be turned around. → cont'd keywords include: see page -A-

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON THE TURNING LOZANGE OF CONSTANT SIDE WHOSE VERTICES ALTERNATE BETWEEN TWO FIXED CIRCLES OF A RING

I. J. Schoenberg

I am writing this note to call attention to the remarkable paper [1] by W. L. Black, H. C. Howland and B. Howland. Their result which recalls the two classical theorems of V. Poucelet and J. Steiner (see e.g. [2, Chapter 14]), may be described as follows.

Let Γ and $\tilde{\Gamma}$ be two circles in the 3-dimensional space R^3 with the property

(A) There exists a number d such that each point of either circle is at the distance d from exactly two points on the other circle

Should the circles $\Gamma, \tilde{\Gamma}$ be coplanar, then (A) will be satisfied if we make the assumption

(B) The smaller circle encloses the center of the larger circle.

Furthermore, the authors assume that there is a d with the following property: There is an equilateral closed polygon

$$\Pi_{2n} = P_1P_2 \dots P_{2n} \quad (n \geq 2) \quad ,$$

having all its sides = d, such that

$$P_{2k-1} \in \Gamma \text{ , } P_{2k} \in \tilde{\Gamma} \quad (k = 1,2,\dots,n) \quad . \tag{1}$$

Then the polygon Π_{2n} can be turned around, with all its sides = d, so that the zig-zag property (1) holds, and returned to its initial position.

The authors do not describe a way of determining the (constant) size d of the side of Π_{2n}.

The present note should be regarded as an appendix to [1] that solves the case when $n = 2$ and the circles are coplanar. It can be read independently. We prove

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Theorem 1. *Let* O *and* $\tilde{O}$ *be the centers of the coplanar circles* Γ *and* $\tilde{\Gamma}$, *respectively, and let*

(2) $$a = \text{radius of } \Gamma, \quad b = \text{radius of } \tilde{\Gamma}, \quad c = O\tilde{O} \ .$$

We assume that the assumption (B) *is satisfied. Then there is a* 1-*parameter family of lozanges* Π_4, *of constant sides* $= d$, *having the zig property* (1) *for* $n = 2$, *with* d *given by the equation*

(3) $$d^2 = a^2 + b^2 - c^2 \ .$$

Proof.

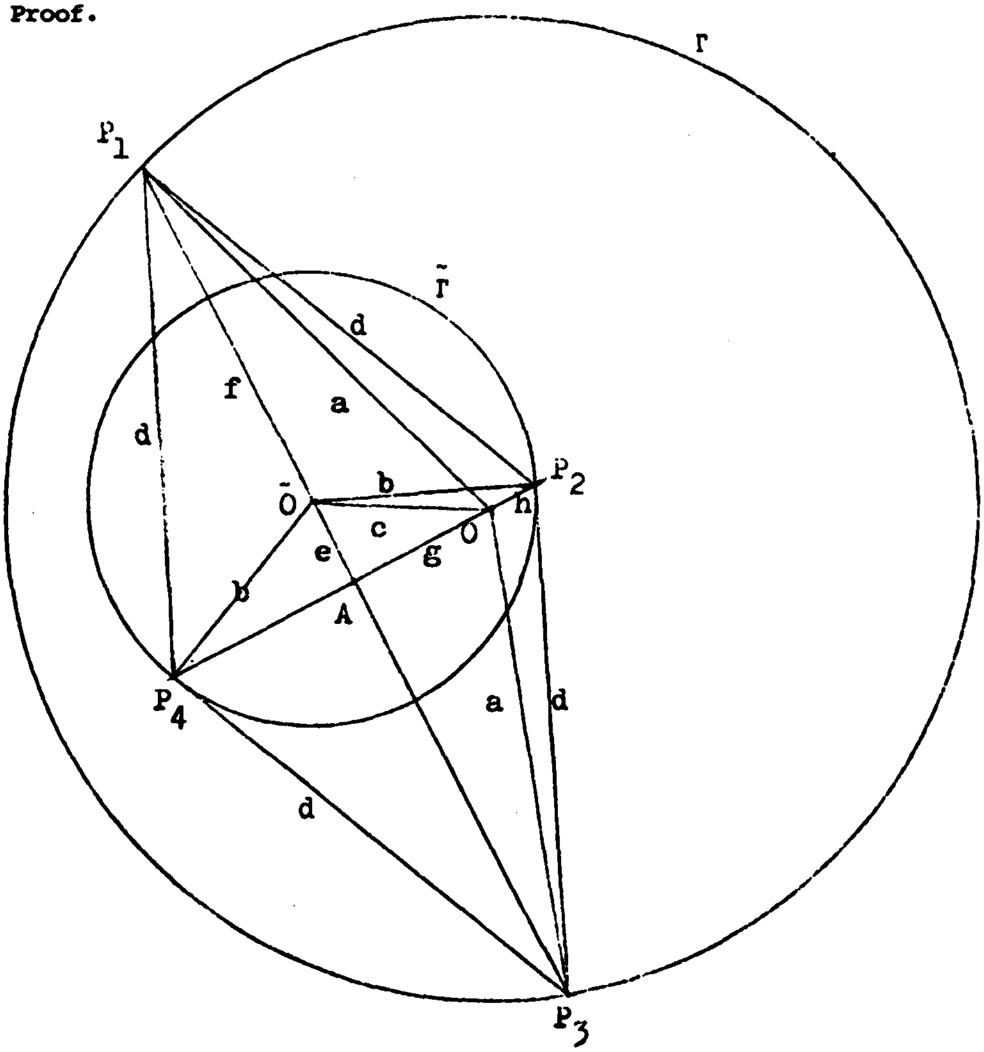

Figure 1

I. Let us first show that there is a 1-parameter family of lozanges $\Pi_4 = P_1P_2P_3P_4$ each of which enjoys the zig-zag property (1) for $n = 2$. Afterwards, we will show that the side d of Π_4 satisfies the equation (3) so that d has the same value for all members of the family. Referring to Fig. 1 let $\Pi_4 = P_1P_2P_3P_4$ be a lozange such that P_1 and P_3 are on Γ, and P_2, P_4 on $\tilde{\Gamma}$. Evidently $P_1P_2P_3$ is isosceles and so is $P_1\, 0\, P_3$. It follows that 0 is on the diagonal P_2P_4. Likewise $P_4P_1P_2$ and $P_4\, \tilde{0}\, P_2$ are isosceles triangle, and therefore $\tilde{0}$ is on the diagonal P_1P_3. Moreover P_1P_3 and P_2P_4 are evidently perpendicular to each other.

This makes the construction of $\Pi_4 = P_1P_2P_3P_4$ evident: Having picked P_1 arbitrarily on Γ, we draw the line joining P_1 to $\tilde{0}$ and intersecting Γ again in P_3. Next we drop the perpendicular from 0 onto $P_1\, \tilde{0}\, P_3$. With $A = P_1P_3 \cap P_2P_4$ we evidently have $P_1A = AP_3$ and $P_2A = AP_4$, so that $\Pi_4 = P_1P_2P_3P_4$ is a lozange having the side $d = P_1P_2$.

We recall that P_1 was an arbitrary point of Γ; the constancy of d will follow as soon as we establish the equation (3), for it shows that d does not depend on the choice of P_1 on Γ.

II. Let us prove the equation (3). To simplify our notations, and referring to Fig. 1, we write

$$e = A\tilde{0},\ f = \tilde{0}P_1,\ g = A0,\ h = 0P_2 \quad .$$

The Pythagorean theorem gives the following equations

$$d^2 = (P_1P_2)^2 = (e+f)^2 + (g+h)^2 = e^2 + f^2 + g^2 + h^2 + 2(ef+gh) \ ,$$

$$a^2 = (e+f)^2 + g^2 = e^2 + g^2 + f^2 + 2ef \ ,$$

$$b^2 = (g+h)^2 + e^2 = g^2 + h^2 + e^2 + 2gh \ ,$$

$$c^2 = e^2 + g^2 \quad .$$

These show that $d^2 + c^2 = a^2 + b^2$ and the equation (3) is established. The constancy of d immediately implies the turning property of the lozange Π_4.

Remarks. 1. On Fig. 1 we see that if $0 = \tilde{0}$, hence $c = 0$ then $a = AP_1$ and $b = AP_2$, and therefore the equation (3) may be regarded as a generalization of the Pythagorean theorem

$$(P_1P_2)^2 = (AP_1)^2 + (AP_2)^2 \quad .$$

2. Our result would make an attractive linkage that would be fun to handle: The four equal side of Π_4 should be linked at the vertices, while the two pairs of opposite vertices P_1, P_3 and P_2, P_4 should fit into the circular grooves Γ and $\tilde{\Gamma}$, respectively. From the general result of [1] we may similarly make a linkage from the closed equilateral polygon Π_{2n}, with P_{2k-1} fitting into the circular groove Γ, and P_{2k} into $\tilde{\Gamma}$. Clearly $n = 3$, or $n = 4$, are the most likely choices. For $n > 2$ the side d must be well approximated graphically, by trial and error, after the grooves Γ and $\tilde{\Gamma}$ have been chosen.

References

1. W. L. Black, H. C. Howland and B. Howland, A theorem about zig-zags between two circles, American Mathematical Monthly, 81 (1974), 754-757.

2. I. J. Schoenberg, Mathematical Time Exposures, Mathematical Association of America, 1982.

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ABSTRACT (continued)

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