

AD A147631

RECENT RESULTS ON CHARACTERIZATION OF PROBABILITY  
DISTRIBUTIONS: A UNIFIED APPROACH THROUGH EXTENSION  
OF DENY'S THEOREM

By

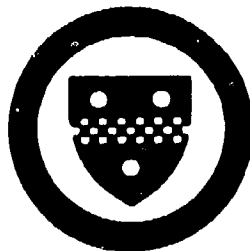
C. Radhakrishna Rao  
University of Pittsburgh  
U S A

**Center for Multivariate Analysis**  
**University of Pittsburgh**

DTIC FILE COPY

DTIC  
ELECTE  
NOV 27 1984

B



Approved for public release;  
distribution unlimited.

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH (AFSC)  
NOTICE OF TECHNICAL INFORMATION (DTIC)  
This technical report has been reviewed and is  
approved for public release under E.O. 12958-2.  
Distribution is unlimited.  
MATTHEW J. RAY  
Chief, Technical Information Division

RECENT RESULTS ON CHARACTERIZATION OF PROBABILITY  
DISTRIBUTIONS: A UNIFIED APPROACH THROUGH EXTENSIONS  
OF DELY'S THEOREM

By

C. Radhakrishna Rao  
University of Pittsburgh  
U.S.A.

and

D. N. Shanbhag  
University of Sheffield  
U.K.

September 1984

Technical Report No. 84-39

Center for Multivariate Analysis  
Fifth Floor Thackeray Hall  
University of Pittsburgh  
Pittsburgh, PA 15260

DTIC  
ELECTE  
NOV 27 1984  
B

This work was supported by the Air Force Office of Scientific Research  
under Contract F49620-82-K-0001. Reproduction in whole or in part is  
permitted for any purpose of the United States Government.

**DISTRIBUTION STATEMENT A**

Approved for public release;  
Distribution Unlimited

RECENT RESULTS ON CHARACTERIZATION OF PROBABILITY  
DISTRIBUTIONS: A UNIFIED APPROACH THROUGH EXTENSIONS  
OF DENY'S THEOREM

ABSTRACT

The problem of identifying solutions of general convolution equations relative to a group has been studied in two classical papers by Choquet and Deny (1960) and Deny (1961). Recently, Lau and Rao (1982) have considered the analogous problem relative to a certain semigroup of the real line, which extends the results of Marsaglia and Tubilla (1975) and a lemma of Shanbhag (1977). The extended versions of Deny's theorem contained in the papers by Lau and Rao, and Shanbhag (which we refer to as LRS theorems) yield as special cases improved versions of several characterizations of exponential, Weibull, stable, Pareto, geometric, Poisson and negative binomial distributions obtained by various authors during the last few years. In this paper we review some of the recent contributions to characterization of probability distributions (whose authors do not seem to be aware of LRS theorems or special cases existing earlier) and show how improved versions of these results follow as immediate corollaries to LRS theorems. We also give a short proof of Lau-Rao theorem based on Deny's theorem and thus establish a direct link between the results of Deny (1961) and those of Lau and Rao (1982). A variant of Lau-Rao theorem is proved and applied to some characterization problems.

Key Words:

Characterizations, Deny's theorem, Exchangeable random variables, Exponential, geometric, Pareto and stable distributions, Integrated Cauchy functional equations, Lau-Rao theorem, Shanbhag's lemma.

## 1. INTRODUCTION

Let  $S$  be such that it equals either  $R$  ( $=(-\infty, \infty)$ ) or  $R_+$  ( $=[0, \infty)$ ),  $\sigma$  be a measure on (the Borel  $\sigma$ -field of)  $S$  such that  $(\{0\}^c) > 0$ , and  $H : S \rightarrow R_+$  be a non-negative continuous function which satisfies the functional equation

$$(1.1) \quad H(x) = \int_S H(x+y) \sigma(dy), \quad \forall x \in S.$$

From a general theorem of Deny (1961), it follows that if  $S = R$ , then either  $H(x) \equiv 0$  or

$$(1.2) \quad H(x) = \xi_1(x) \exp(-\eta_1 x) + \xi_2(x) \exp(-\eta_2 x), \quad x \in S$$

with  $\eta_1$  such that

$$\int_S e^{-\eta_i x} \sigma(dx) = 1, \quad i = 1, 2$$

and  $\xi_i$  as non-negative periodic functions such that

$$\xi_i(x+y) = \xi_i(x), \quad \forall x \in S \text{ and } y \in \text{supp } \sigma,$$

for  $i = 1, 2$ . (Observe that if  $S = R$  and  $H \equiv 0$ , then the measure  $\sigma$  involved in (1.1), has to be a Radon measure). As a corollary of Deny's general theorem, we have Choquet and Deny (1960) theorem which has important applications in renewal theory (Feller, 1966 Vol. 2, p. 351).

Recently, Lau and Rao (1982) solved the equation (1.1) when  $S = R_+$  which subsumes partial results given by Marsaglia and Tubilla (1975), Klebanov (1977), Shanbhag (1977), Shimuzu (1978) and Ramachandran (1979). A simpler

proof of Lau-Rao theorem appears in Ramachandran (1982). More recently, Alzaid, Rao and Shanbhag (1983) used an argument based on de Finetti's theorem concerning exchangeable random variables to derive the same result. Davies and Shanbhag (1984) have given a martingale proof for an extended version of Deny's result which generalizes Lau-Rao theorem. Extensions of Deny's general theorem to the case of a semigroup have also been given via other approaches by Richards (1981) and Ressel (1984) among others. However, both Richards and Ressel were able to deal with the problem only under some stringent conditions which imply that the semigroup generated by the support of the measure in the functional equation equals the semigroup itself; additionally Richards requires the function to be bounded and Ressel requires the semigroup to be countable.

Various applications of Lau and Rao (1982) theorem and Shanbhag's (1977) lemma, which we refer to as LRS (Lau-Rao-Shanbhag) theorems have been considered by Lau and Rao (1982), M. B. Rao and Shanbhag (1982), Rao (1983), Alzaid (1983) and Davies and Shanbhag (1984) with special reference to damage models, order statistics, record values, lack of memory, reliability and renewal theories. The main purpose of this paper is to indicate further applications of LRS theorems by reviewing some of the recent contributions to characterizations of probability distributions, e.g., Dallas (1981), Deheuvels (1984), Gupta (1984) and Grosswald, Kotz and Johnson (1980), whose authors do not seem to be aware of LRS theorems or the special cases existing earlier. We show that LRS theorems not only provide a unified approach to a wide variety of characterizations of distributions such as Poisson, Pareto, Weibull, stable, geometric and negative binomial, but their application leads in many cases to improved versions of the results already available in the

literature. In this paper we also give a simple proof of Lau-Rao theorem via Deny's theorem and thus obtain a direct link between the two theorems. In addition, we investigate the problem of solving the integral equation

$$(1.3) \quad \alpha + \beta f(x) = \int_{R_+} f(x+y)\mu(dy) \text{ a.e.}[L] \quad \forall x \in R_+$$

where  $\alpha$  and  $\beta$  are constants and indicate its applications to characterization problems.



|                    |                                     |
|--------------------|-------------------------------------|
| Accession For      |                                     |
| NTIS GRA&I         | <input checked="" type="checkbox"/> |
| DTIC TAB           | <input type="checkbox"/>            |
| Unannounced        | <input type="checkbox"/>            |
| Justification      |                                     |
| By                 |                                     |
| Distribution/      |                                     |
| Availability Codes |                                     |
| Dist               | Avail and/or<br>Special             |
| A-1                |                                     |

## 2. LAU-RAO THEOREM

Lau-Rao theorem. Let  $H$  be a non-negative real locally integrable measurable function on  $R_+$ , which is not a function identically equal to zero a.e.  $[L]$ . ( $L$  indicating Lebesgue measure), satisfying the functional equation

$$(2.1) \quad H(x) = \int_{R_+} H(x+y)\sigma(dy) \text{ a.e. } [L] \text{ for } x \in R_+$$

where  $\sigma$  is a  $\sigma$ -finite measure on  $R_+$  such that  $\sigma(\{0\}) < 1$ . Then one of the two possibilities hold.

(a)  $\sigma$  in (2.1) is arithmetic with some span  $\lambda$  and

$$H(x+n\lambda) = H(x)b^n, \quad n = 0, 1, \dots, \text{ a.e. } [L] \text{ for } x \in R_+$$

with  $b$  such that

$$\sum_{n=0}^{\infty} b^n \sigma(\{n\lambda\}) = 1.$$

(b)  $\sigma$  in (2.1) is non-arithmetic and

$$H(x) \propto \exp\{-\eta x\} \text{ a.e. } [L] \text{ for } x \in R_+$$

with  $\eta$  such that

$$\int_{R_+} \exp\{-\eta x\} \sigma(dx) = 1.$$

Lau and Rao (1982) gave a self-contained real variable proof of the above theorem. We now present an alternative proof based on Deny's (1961) theorem, which provides an important link and at the same time brings out the main difference between the two theorems.

Proof There is no loss of generality in assuming that  $\sigma(R_+) > 1$ . Consider some  $d > 0$ , and define

$$(2.2) \quad \hat{H}(x) = \int_0^d H(x+y)dy, \quad x \in R_+.$$

Clearly  $\hat{H}$  is continuous and in view of Fubini's theorem satisfies (2.1) with the statement 'a.e.[L]' deleted. From elementary Lemma 1 of Davies and Shanbhag (1984), it immediately follows that for every support point  $s$  of  $\sigma$

$$(2.3) \quad \hat{H}(x+2s)\hat{H}(x) \geq [\hat{H}(x+s)]^2, \quad x \in R_+.$$

We can choose  $d$  sufficiently large so that  $\hat{H}(0) > 0$  and hence  $\hat{H}(s_0) > 0$  for some positive support point  $s_0$  of  $\sigma$ . From (2.3), it follows that  $\hat{H}(2s_0) > 0$ . Consequently, for sufficiently large  $d$ , we have  $\hat{H}(x) > 0 \quad \forall x \in [0, 2s_0]$  for some positive support point  $s_0$  of  $\sigma$ . We shall now fix the  $s_0$  in question. From (2.3), we can then claim that  $\hat{H}(x) > 0 \quad \forall x \in R_+$  and

$$\left\{ \frac{\hat{H}(x + ns_0)}{\hat{H}(x + (n-1)s_0)} : n = 1, 2, \dots \right\}$$

is an increasing sequence for each  $x \in R_+$ . Clearly then we have

$$(2.4) \quad \hat{H}(x + s_0) \geq v \hat{H}(x), \quad x \in R_+,$$

where

$$v = \inf \left\{ \frac{\hat{H}(x + s_0)}{\hat{H}(x)} : 0 \leq x \leq s_0 \right\} > 0.$$

There is no loss of generality in assuming  $\sigma(\{0\}) = 0$ . If  $\sigma$  is arithmetic, or more generally if there exists a constant  $c > 0$  such that  $\sigma((0, c)) = 0$  (i.e., if 0 is not a cluster point of the support of  $\sigma$ ), define  $\hat{\sigma} = \sigma$ . Otherwise,

considering  $c$  such that  $\sigma((0,c)) < 1$  and  $0 < c < s_0$ , define

$$\hat{\sigma}(\cdot) = \sum_{n=1}^{\infty} \sigma_n([c, \infty) \cap \cdot)$$

to be a measure on  $R_+$ , where  $\sigma_1 = \sigma$ , and for each  $n \geq 2$ ,  $\sigma_n$  is the convolution of the measures  $\sigma_{n-1}([0,c) \cap \cdot)$  and  $\sigma$ . It is then obvious that  $\hat{\sigma}([0,c)) = 0$  and

$$(2.5) \quad \hat{H}(x) = \int_{R_+} \hat{H}(x+y) \hat{\sigma}(dy), \quad x \in R_+.$$

Observe that the measure  $\hat{\sigma}$  defined here is such that it is arithmetic if  $\sigma$  is arithmetic, and in that case, both have the same support; also  $\hat{\sigma}$  is non-arithmetic if  $\sigma$  is non-arithmetic. Define now inductively  $\hat{H}(x)$  for  $x \in [-(n+1)c, -nc)$  for  $n = 0, 1, 2, \dots$  such that

$$(2.6) \quad \hat{H}(x) = \int_{[c, \infty)} \hat{H}(x+y) \hat{\sigma}(dy).$$

It is then easily seen, especially in view of (2.4), (2.5), and (2.6), that we have a continuous function  $\hat{H} : R \rightarrow R_+$  such that its restriction to  $R_+$  agrees with our original  $\hat{H}$  and

$$(2.7) \quad \hat{H}(x) = \int_{R_+} \hat{H}(x+y) \hat{\sigma}(dy), \quad x \in R.$$

From Deny's (1961) theorem, it then follows that

$$\hat{H}(x) = \xi(x) e^{-\eta x}, \quad x \in R$$

for some  $\eta > 0$  and some function  $\xi$  satisfying the condition  $\xi(x+s) = \xi(x)$ ,  $x \in R$  for each support point  $s$  of  $\hat{\sigma}$ . The required result now follows on noting that

$$\int_x^\infty H(y) dy = \frac{\eta \xi(0)}{1 - \exp(-\eta d)} \int_x^\infty e^{-\eta y} dy \quad \forall x \geq 0$$

if  $\sigma$  is non-arithmetic, and

$$= > \int_x^\infty H(y + n\lambda) dy = \int_x^\infty H(y) \exp(-n\lambda\eta) dy \quad \forall x \geq 0, n = 0, 1, \dots$$

if  $\sigma$  is arithmetic with span  $\lambda$  and

$$\int_{R_+} \exp(-\eta x) \sigma(dx) = 1.$$

Remark 1 If the conditions in Lau-Rao theorem are met with  $R_+$  replaced by  $R$ , then it follows at once from Dery's theorem that

$$(2.8) \quad n \int_0^{1/n} H(x+y) dy = \xi_1^{(n)}(x) \exp(-\eta_1 x) + \xi_2^{(n)}(x) \exp(-\eta_2 x)$$

for  $n = 1, 2, \dots$ , a.e.[L] for  $x \in R$ , where  $\eta_1$  and  $\eta_2$  are as defined in (1.2) and  $\xi_1^{(n)}$  and  $\xi_2^{(n)}$  are of the form of  $\xi_1$  and  $\xi_2$  in (1.2) with  $S = R$ . Since  $H$  is locally integrable, it follows that

$$\lim_{n \rightarrow \infty} n \int_0^{1/n} H(x+y) dy = H(x), \text{ a.e.}[L] \text{ for } x \in R$$

and hence that if  $s_0$  is any nonzero support point of  $\sigma$  (which clearly exists), then

$$(2.9) \quad n \int_0^{1/n} H(x+y) dy \rightarrow H(x) \text{ and } n \int_0^{1/n} H(x+s_0+y) dy \rightarrow H(x+s_0)$$

as  $n \rightarrow \infty$ , a.e.[L] for  $x \in \mathbb{R}$ .

Consequently, in view of (2.8), it follows that there exist functions  $\xi_1$  and  $\xi_2$  as in (1.2) such that  $\xi_1^{(n)} \rightarrow \xi_1$  and  $\xi_2^{(n)} \rightarrow \xi_2$  as  $n \rightarrow \infty$ , a.e.[L] on  $\mathbb{R}$ , and hence such that

$$(2.10) \quad H(x) = \xi_1(x) \exp(-\eta_1 x) + \xi_2(x) \exp(-\eta_2 x) \text{ a.e.}[L] \text{ for } x \in \mathbb{R}.$$

The result (2.10) was established by Lau and Rao (1984a).

Incidentally, it may be noted here that in the case of nonarithmetic  $\sigma$ , the form of  $H$  in (2.10) simplifies to

$$(2.11) \quad H(x) = \beta \exp(-\eta_1 x) + (1 - \beta) \exp(-\eta_2 x) \text{ a.e.}[L] \text{ for } x \in \mathbb{R},$$

where  $\beta$  is some constant in  $[0,1]$ .

Remark 2 From Davies and Shanbhag (1984), it is evident that at least in the case of continuous  $H$ , Lau-Rao theorem remains valid even when the requirement of  $\sigma$ -finiteness of the measure  $\sigma$  is dropped. (This is so more clearly in the case of Remark 1.) However, the following example shows that the general result of Lau-Rao theorem does not remain valid if  $\sigma$  is not  $\sigma$ -finite.

Example 1 Let  $\sigma$  be that measure on  $R_+$  for which its restriction to  $(1,2]$  agrees with the counting measure on  $(1,2]$  and  $\sigma([0,1] \cup (2,\infty)) = 0$ . Define a function  $H : R_+ \rightarrow R_+$  such that

$$H(x) = \begin{cases} 1 & \text{if } x \in [0,1) \text{ and } x = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that we have here

$$H(x) = \int_{R_+} H(x+y) \sigma(dy) \text{ a.e. } [L] \text{ for } x \in R_+$$

but  $H$  is not of the form as in Lau-Rao theorem.

Remark 3 As observed by Lau and Rao (1982) and Alzaid, Rao and Shanbhag (1983), Lau-Rao theorem yields the following modified version of Shanbhag's (1977) lemma.

Lemma Let  $\{(v_n, w_n) : n = 0, 1, \dots\}$  be a sequence of vectors with non-negative real components such that  $v_n \neq 0$  for at least one  $n$ ,  $w_0 < 1$ , and the largest common divisor of the set  $\{n : w_n > 0\}$  is unity. Then

$$v_m = \sum_{n=0}^{\infty} v_{m+n} w_n, \quad m = 0, 1, \dots$$

if and only if

$$v_n = v_0 b^n, \quad n = 0, 1, 2, \dots \text{ and } \sum_{n=0}^{\infty} w_n b^n = 1$$

for some  $b > 0$ .

The modified version of Shanbhag's lemma yields somewhat improved versions of the general characterization theorems for the univariate and

bivariate cases given in Shanbhag (1977) as discussed in Shanbhag (1983). Further variants and extensions of Shanbhag's (1977) lemma have been considered, among others, by Alzaid, Rao and Shanbhag (1983) and Lau and Rao (1984b).

## 3. COMMENTS ON RECENT RESULTS

In this section we review some recent contributions to characterization of probability distributions, comment on the gaps in the proofs and show how improved versions of the results can be obtained by using LRS theorems.

3.1 Gupta (1984)

One of the main theorems (Theorem 3.1) of Gupta (1984) is that " $E[(R_{j+1} - R_j)^r | R_j = y] = c$  (independent of  $y$ ) for fixed  $j$  and  $r \geq 1$  iff  $F$  is exponential, where  $R_1, R_2, \dots$  are record values from a continuous distribution function  $F$  such that  $F(0) = 0$ ." We have the following comments on the statement and proof of Gupta's theorem.

Gupta mentions that the condition on conditional expectation in his theorem implies that

$$(3.1.1) \quad c = \int_0^{\infty} ru^{r-1} \frac{S(u+y)}{S(y)} du$$

where  $S(x) = 1 - F(x)$ , or

$$(3.1.2) \quad c S(y) = \int_0^{\infty} ru^{r-1} S(u+y) du.$$

But for (3.1.2) to be valid for all  $y \in (0, \infty)$ , it is necessary to assume, besides continuity of  $F$ , that  $F(x) > 0$  for  $x > 0$ , which is not explicitly mentioned in the statement of the theorem. Once (3.1.2) is assumed to be valid for all  $y$ , then an application of Lau-Rao theorem immediately shows that  $S(x) = e^{-\lambda x}$  which is the required result.

However, Gupta obtains the solution in a different way by considering Mellin's transform of both sides of (3.1.2), deriving an equation of the form

$$(3.1.3) \quad h(t) - Ah(t - r) = 0 \text{ for } t > r,$$

and writing its solution as  $h(t) = ke^{bt}$  attributed to Bellman and Cooke (1963, p. 54). Unfortunately (3.1.3) has no unique solution; for instance,

$$h(t) = \exp\left(\lambda \sin \frac{2\pi t}{r}\right), \quad \lambda \neq 0$$

is also a solution, which shows that further arguments are necessary to justify Gupta's solution.

The same remark applies to the alternative proof given by Srivastava and Singh (1975, p. 273) for the Rao-Rubin (1964) theorem, quoting the Bellman-Cooke result.

As observed above, the statement of Gupta's theorem needs the additional condition,  $0 < F(x)$  for  $x > 0$ . Some extensions of Gupta's result are as follows:

- (i) The result is true even if  $0 < r < 1$  since Lau-Rao theorem is still applicable.
- (ii) If  $F$  is such that  $F(a) = 0$  and  $F(x) > 0$  for  $x > a$ , then the characterization is valid but for a modification of  $F$  as exponential with a shift in the origin.
- (iii) Lau-Rao theorem also implies that the same characterization is obtained if in Gupta's condition, the expression  $(R_{j+1} - R_j)^r$  is replaced by  $\phi(R_{j+1} - R_j)$  where  $\phi$  is an increasing or decreasing real function on  $R_+$  with  $\phi(x) \neq \phi(0+) \forall x > 0$  and such that  $E[|\phi(R_{j+1} - R_j)|] < \infty$ . As a special case of this result, it follows that

$$E[1 - \exp(R_{j+1} - R_j) | R_j] = \text{constant a.e.}$$

characterizes an exponential distribution (but for a shift).

- (iv) If  $F$  is arithmetic with its support as  $\{0, 1, 2, \dots\}$ , then the condition  $E[(R_{j+1} - R_j)^r | R_j = y] = c$  (independent of  $y$ ) implies that

$$F(s) - F(s-) \begin{cases} = \text{arbitrary for } s = 0, \dots, j-1 \\ \propto b^s \text{ for } s = j, j+1, \dots \end{cases}$$

The result is obtained by an application of Shanbhag's lemma. The last result remains valid even when the expression  $(R_{j+1} - R_j)^r$  is replaced by  $\phi(R_{j+1} - R_j)$  where  $\{\phi(n): n = 0, 1, \dots\}$  is an increasing or decreasing real sequence with  $\phi(0) \neq \phi(1) \neq \dots \neq \phi(j)$  and such that  $E\{|\phi(R_{j+1} - R_j)|\} < \infty$ .

### 3.2 Grosswald, Kotz and Johnson (1980)

Grosswald et al (1980) proved that if  $F_2$  is a distribution function on  $R_+$  with survivor function  $S_2$  satisfying  $S_2(0) = 1$  and having a power series expansion, then

$$(3.2.1) \quad \int_{[0,t]} S_2(t-x) F_1(dx) = \int_{[0,t]} [S_2(t)/S_2(x)] F_1(dx), \quad t \in R_+$$

for every distribution function  $F_1$  on  $R_+$  with  $F_1(0) = 0$  if and only if  $F_2$  is exponential. (In (3.2.1),  $S_2(t)/S_2(x)$  is interpreted as zero if  $S_2(x) = 0$ ). They conjectured that the result is still true when  $S_2$  (or equivalently  $F_2$ ) is merely assumed to be continuous. More recently, Westcott (1981) used a probabilistic argument to show that the conjecture is correct.

However, an improved version of the above result follows trivially from the result of Marsaglia and Tubilla (1975): Let  $F_2$  be a probability distribution on  $R_+$  such that  $S_2(0) > 0$  and  $x_1, x_2$  be two positive numbers such that  $S_2(x_2) > 0$ ,  $x_1 < x_2$  and  $x_1/x_2$  is irrational. If (3.2.1) is satisfied for

any two distinct probability distributions  $F_1$  concentrated on  $\{x_1, x_2\}$ , then  $F_2$  is exponential. (If  $F_2$  is exponential, then (3.2.1) is satisfied for any  $F_1$  on  $R_+$ .)

It is interesting to point out that this result does not hold when the condition 'any two distinct probability distributions' is replaced by 'a probability distribution.' This is illustrated by the following example.

Example 2 Let  $x_2 < 2x_1$  and  $F_1$  be a probability distribution on  $R_+$  such that  $F_1(x_1) - F_1(x_1-) = \alpha$  and  $F_1(x_2) - F_1(x_2-) = 1 - \alpha$  where  $0 \leq \alpha \leq 1$ . Define a probability distribution function  $F_2$  on  $R_+$  such that

$$F_2(x) = \begin{cases} 0 & \text{if } x < x_1, \\ \beta & \text{if } x_1 \leq x \leq x_2, \quad 0 < \beta < 1, \\ \beta + (1 - \beta)\{\alpha F_2(x - x_1) + (1 - \alpha)F_2(x - x_2)\} & \text{if } x > x_2. \end{cases}$$

Clearly  $F_2$  is a distribution function and satisfies (3.2.1).

It is possible to give several other variants of our modified version of the result of Grosswald et al (1980) such as the arithmetic analogue characterizing the geometric distribution.

### 3.3 Dallas (1981)

In this paper, it is shown that if  $R_0, R_1, \dots$  are record values from a continuous distribution function  $F(x)$  and  $0 \leq i < j$  are fixed integers, then the independence of  $R_j - R_i$  and  $R_i$  implies that  $F$  is exponential or shifted exponential.

There is an implicit assumption in the proof given by Dallas that every point of  $[a, \infty)$  is a support point of  $F$ . Further, the derivation of the

conditional distribution (Dallas, 1981, p. 950) needs some justification especially if one is dealing with a distribution having a singular continuous component in its Lebesgue decomposition. We provide the sketch of an alternative and more satisfactory proof based on LRS theorems.

Let for any  $c < b$ , the right extremity of the distribution function  $F$ ,  $\{R_n^{(c)}, n = 1, 2, \dots\}$  be a sequence of record values from the distribution

$$F_c(x) = \begin{cases} \frac{F(x) - F(c)}{1 - F(c)} & \text{if } x > c, \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that  $R_j - R_1$  is independent of  $R_1$ , if and only if the distribution of  $R_{j-1}^{(c)} - c$  is independent of  $c$  a.e.[ $F$ ]. The distribution of  $R_{j-1}^{(c)} - c$  is computed to be

$$(3.3.1) \quad P\{R_{j-1}^{(c)} - c \leq x\} = \begin{cases} [(j-1-1)!]^{-1} \int_c^{c+x} (-\log y)^{j-1-1} dy & \text{if } c \leq c+x < b \\ 1 & \text{if } c+x \geq b \\ 0 & \text{if } x < 0, \end{cases}$$

where  $\alpha = [1 - F(c+x)]/[1 - F(c)]$ . (A rigorous proof of (3.3.1) follows from a lemma in Kotz and Shanbhag, 1980.) Consequently, it follows that  $R_j - R_1$  is independent of  $R_1$  if and only if  $b = \infty$  and  $[1 - F(c+x)]/[1 - F(c)]$  is independent of  $c$  a.e.[ $F$ ]. Then from the result of Marsaglia and Tubilla (1975) (and not from the usual lack of memory property of an exponential distribution as mentioned by Dallas), it then follows that  $F$  is either exponential or shifted exponential. (Observe that if  $[1 - F(c+x)]/[1 - F(c)]$ ,  $x \geq 0$  is independent of  $c$  a.e.[ $F$ ], then the left extremity of  $F$  should be some  $a > -\infty$  and for every  $c \in \text{supp } [F]$  and  $x \in R_+$

$$1 - F(c+x) = [1 - F(x+a)][1 - F(c)] \text{ a.e.}[F].$$

### 3.4 Deheuvels (1984)

This paper reviews some of the characterization results on the exponential and geometric distributions based on properties of order statistics and record values. As mentioned in our present discussion and elsewhere, several of these results and improved versions in some cases can be deduced directly from LRS theorems, which provide a unified approach to a wide variety of problems. However, we make a comment on Theorem 2 of Deheuvels (1984) which is, perhaps, only of minor relevance to this discussion. The theorem mentioned is not valid unless in the statement above equation (7) on p. 329 of the paper, 'for all  $z$ ' is replaced by 'for  $z \in R$  a.e.[ $F$ ]' and  $1 - F(x)$  on the right hand side of (7) is changed to  $1 - F(z)$ . (The latter of the two errors in question appears to be a misprint).

### 3.5 Rao (1983)

We give here slight refinements and extensions of some of the results mentioned in Rao (1983), which again follow from Lau-Rao theorem.

Theorem 5.1 of Rao (1983) states: Let the distribution function  $F$  of a r.v.  $X$  be continuous and such that  $F(0) = 0 \leq F(x) < 1$  for all  $x \in [0, \infty)$ . Then  $F(x) = 1 - e^{-\lambda x}$  if and only if  $R_{j+1} - R_j$  and  $R_1$  have the same distribution, where  $R_1, R_2, \dots$ , are record values. This theorem remains true even if  $F$  is assumed to be such that  $F(0) = 0$ , the right extremity is not a discontinuity point, and at least one of its support points is a continuity point.

Theorem 5.2 of Rao (1983) states: Let  $X$  be a discrete r.v. taking values  $0, 1, \dots$  such that  $p_i = P(X = i) > 0$  for all  $i$ . Then  $X$  has a geometric distribution if  $R_{j+1} - R_j$  has the same distribution as  $R_1 + 1$ . This theorem remains true if instead of  $p_i > 0$  for all  $i$ , we have only  $\sup\{i : p_i > 0\} = \infty$  and  $p_i > 0$  for  $i = 0, \dots, j + 1$ .

Theorem 4.3 of Rao (1983) states: Let  $X_{(1)} \leq X_{(2)}$  be order statistics in a sample of size 2 from a discrete distribution on  $\{0, 1, 2, \dots\}$  with  $P(X = i) = p_i \neq 0$  for all  $i$ . Then

$$(3.5.1) \quad E(X_{(2)} - X_{(1)} | X_{(1)} = x) = \mu \text{ for } x = 0, 1, \dots$$

iff  $X$  has a geometric distribution.

The condition (3.5.1) implies that

$$(3.5.2) \quad \mu(G_r + G_{r+1}) = 2(G_{r+1} + G_{r+2} + \dots), \quad r = 0, 1, \dots$$

where  $G_r = p_r + p_{r+1} + \dots$ . Clearly (3.5.2) is equivalent to

$$(3.5.3) \quad \mu G_r = 2G_{r+1} + \mu G_{r+2}, \quad r = 0, 1, \dots$$

and the desired result follows from (3.5.3) by applying Shanbhag's lemma. (The expression (4.4) in Rao (1983) should be as in (3.5.2).)

A stronger version of the above result is obtained by replacing the condition (3.5.1) by

$$E(\phi(X_{(2)} - X_{(1)}) | X_{(1)} = x) = \mu \text{ for } x = 0, 1, \dots$$

where  $\phi$  is such that  $E[|\phi(X_{(2)} - X_{(1)})|] < \infty$ ,  $\phi(1) > \phi(0)$  and  $\phi(r+2) - 2\phi(r+1) + \phi(r) \geq 0$  for all  $r$ , i.e., the second differences of  $\phi$  are non-negative.

Another version of Theorem 4.3 is obtained by considering only samples without ties, in which case  $X_{(2)} > X_{(1)}$ . Let  $\phi$  be an increasing function such that  $\phi(2) - \phi(1) > \phi(1)$ . Then

$$E(\phi(X_{(2)} - X_{(1)}) | X_{(1)} = x) = \mu, \quad x = 0, 1, \dots$$

implies that

$$p_i = \beta^i, \quad i = 1, 2, \dots$$

for some  $\beta \in (0, 1)$  and  $p_0$  is arbitrary. (Slightly stronger results than those discussed here follow via the extended version of Shanbhag's lemma given in section 2; the results are also valid when  $-\phi$  meets the requirements of  $\phi$ ).

Finally Theorem 6.2 of Rao (1983), in which some assumptions are not explicitly mentioned, can be stated as follows: Let  $x$  be a non-negative random variable with a continuous distribution function  $F$ , and  $h$  be a real function on  $[1, \infty)$  such that it is either increasing or decreasing with  $h(x) \neq h(1+)$  for each  $x > 1$ . If

$$E[h\left(\frac{x}{a}\right) | X \geq a] = \text{constant } \forall a \in (0, \infty)$$

then  $X$  has a Pareto distribution. (The result in question remains valid even when the assumption of continuity of  $F$  is replaced by  $F(0) = 0$ ; also the extended result remains valid when the assumption that  $h(x) \neq h(1+)$  for each  $x > 1$  is replaced by that there exist points  $x_1, x_2 > 1$  such that  $h(x + x_1) \neq h(x_1-)$ ,  $i = 1, 2$  for each  $x > 0$  and  $\log x_1 / \log x_2$  is irrational).

## 4. A VARIANT OF THE LAU-RAO THEOREM

Consider the following equation which is a variant of the one discussed by Lau and Rao (1982):

$$(4.1) \quad \int_{R_+} f(x+y) \mu(dy) = f(x) + c \quad \text{a.e.}[L] \text{ for } x \in R_+$$

where  $f : R_+ \rightarrow R$  is a locally integrable Borel measurable function and  $\mu$  is a  $\sigma$ -finite measure on  $R_+$  with  $\mu(\{0\}) < 1$ . (The identity in (4.1) is understood as the one for which the left hand side exists and equals the right hand side.) This may clearly be viewed as an integrated version of the equation  $f(x+y) = f(x) + f(y)$  which is derived from the Cauchy equation by taking logarithms. In this case, by analogy with Lau-Rao theorem, one would be tempted to conjecture that a solution of (4.1) is a.e. of the same form as the logarithm of a positive solution of Lau-Rao equation. However, we have the following counter example to show that such a conjecture cannot hold.

Example 3 Consider  $\mu$  to be a probability measure on  $R_+$  such that it is determined by an infinitely divisible probability distribution with an entire characteristic function. From Picard's theorem (c.f. Titchmarsh, 1949, p. 277) and the fact that the characteristic function involved here does not vanish, we can conclude that there exist infinitely many points  $(a_r, b_r)$  of  $R^2$  such that

$$\int_{R_+} e^{a_r x + i b_r x} \mu(dx) = 1$$

or equivalently such that

$$(4.2) \quad \int_{R_+} e^{a_r x} \cos(b_r x) \mu(dx) = 1$$

and

$$(4.3) \quad \int_{R_+} e^{a_r x} \sin(b_r x) \mu(dx) = 0.$$

If we now define

$$f_r(x) = e^{a_r x} \cos(b_r x), \quad x \in R_+,$$

it follows immediately, in view of (4.2) and (4.3) that

$$\int_{R_+} f_r(x+y) \mu(dy) = f_r(x), \quad x \in R_+$$

which shows that the conjecture cannot be true.

If we replace  $R_+$  by  $R$  in the problem considered, we arrive at the variant of Lau-Rao problem mentioned in Remark 1. In this latter case, we have a simpler counter example on taking  $f(x) = x^2$  and  $\mu$  as any probability distribution with zero mean and finite variance. Clearly we have then

$$\int_R f(x+y) \mu(dy) = f(x) + c, \quad x \in R$$

with

$$c = \int_R x^2 \mu(dx).$$

(It may be worth pointing out here that the counter example in the case of  $R_+$  given above also serves as a counter example in the present case if  $R_+$  is replaced by  $R$ .)

We shall now establish the following theorem answering the question of identification of the solution of (4.1) partially.

**Theorem** If  $f$  in (4.1) is not a function which is identically equal to a constant a.e.[L] on  $R_+$  and  $f$  is either increasing a.e.[L] on  $R_+$  or decreasing a.e.[L], then the equation cannot be valid unless either  $\mu$  is a non-arithmetic measure and  $f$  is of the form

$$f(x) = \begin{cases} \gamma + \alpha(1 - e^{-\eta x}) & \text{a.e.}[L] \text{ if } \eta \neq 0 \\ \gamma + \beta x & \text{a.e.}[L], \text{ if } \eta = 0 \end{cases}$$

or  $\mu$  is arithmetic with span  $\lambda$  for some  $\lambda$ , and  $f$  is of the form for which

$$f(x + n\lambda) = \begin{cases} f(x)e^{-n\lambda\eta} + \alpha'(1 - e^{-n\lambda\eta}) & \text{a.e.}[L] \text{ if } \eta \neq 0 \\ f(x) + \beta'n & \text{a.e.}[L], \text{ if } \eta = 0 \end{cases}$$

where  $\alpha, \beta, \gamma, \alpha', \beta'$  are constants and  $\eta$  is such that

$$\int_{R_+} \exp(-\eta x) \mu(dx) = 1.$$

(From the statement of the theorem, it is implicit that if  $\mu(R_+) = \infty$ , then there is no solution to (4.1); this is also so if  $\int_{R_+} x \mu(dx) = \infty$  when  $\eta = 0$  and  $\mu$  is non-arithmetic.)

**Proof** There is no loss of generality in assuming that  $f$  is increasing. Define for each  $x \in R_+$

$$H_x(y) = f(x + y) - f(y), \quad y \in R_+.$$

In view of the discussion in Remark 1, it follows that there is no loss of generality in taking  $f$  to be continuous. In that case, we get  $H_x(\cdot)$  to be a continuous function on  $R_+$  such that

$$\int_{R_+} H_x(y+z) \mu(dz) = H_x(y), \quad y \in R_+.$$

From Lau-Rao theorem, we conclude that

$$(4.4) \quad H_x(y) = E_x(y) \exp(-ny), \quad y \in R_+,$$

where  $E_x$  is such that  $E_x(y+s) = E_x(y) \quad \forall y \in R_+$  and every support point  $s$  of  $\mu$ . In the case of non-arithmetic  $\mu$ , (4.4) implies

$$(4.5) \quad f(x+ny) = f(x + \overline{n-1} y) + E_y(0) \exp[-n(x + \overline{n-1} y)]$$

$$= f(x) + E_y(0) \exp(-nx) \sum_{k=1}^n \exp[-n(k-1)y]$$

$$= f(x) + E_{ny} \exp(-nx), \quad x, y \in R_+, \quad n \geq 1.$$

It is easy to check that if (4.5) is valid, then  $E_y(0) = [1 - \exp(-ny)]$ ,  $y \in R_+$  if  $n \neq 0$  and  $E_y(0) = y$  if  $n = 0$ . Consequently, it follows that if  $\mu$  is non-arithmetic we have for every  $x \in R_+$

$$(4.6) \quad f(x) - f(0) = H_x(0) \begin{cases} 1 - \exp(-nx) & \text{if } n \neq 0, \\ x & \text{if } n = 0. \end{cases}$$

In the case of arithmetic  $\mu$  with span  $\lambda$ , we have directly from (4.4)

$$f(x + n) - f(n\lambda) = \xi_x(\lambda) \exp(-nn\lambda)$$

$$= [f(x) - f(0)] \exp(-nn\lambda), \quad n = 0, 1, \dots, x \in R_+$$

and hence for  $x \in R_+$  and  $n = 0, 1, \dots$  we have

$$(4.7) \quad f(x + n\lambda) - f(x) = [f(n\lambda) - f(0)] + [f(x) - f(0)](e^{-n\lambda} - 1)$$

$$= \begin{cases} (f(x) - f(0) + \xi^*)(e^{-n\lambda} - 1) & \text{if } n \neq 0 \\ n[f(\lambda) - f(0)] & \text{if } n = 0 \end{cases}$$

where  $\xi^* = [f(0) - f(\lambda)]/[1 - \exp(-\lambda)]$ .

The part assertion for the arithmetic case of  $\mu$  is now obvious.

Remark 4 Isham et al (1975) considered a special case of the above theorem with the additional conditions that  $f$  is non-negative and right continuous with  $f(0) = 0$ , and  $\mu$  is a probability measure. This special case was used in obtaining a certain characterization of the Poisson process and its discrete analogue.

Remark 5 If  $R_+$  in (4.1) is replaced by  $R$ , then under the assumption that  $f$  is not a function that is equal to a constant a.e.[L] on  $R$  and  $f$  is either increasing or decreasing a.e.[L] on  $R$ , it follows that every solution  $f$  of the equation (4.1) can be expressed as a convex combination of functions  $f_1$  and  $f_2$  of the form arrived at in the theorem above with  $n$  replaced respectively by  $n_1$

and  $\eta_2$  satisfying the conditions

$$\int_{\mathbb{R}} \exp(-\eta_i x) \mu(dx) = 1, \quad i = 1, 2.$$

#### 4.1 Dugue's problem

Rossberg (1972) and more recently in an unpublished article Wolinstawez and Szynal (1984) have considered the problem of identifying characteristic functions  $\phi_1$  and  $\phi_2$  (of probability distributions on  $\mathbb{R}$ ) for which the following equation holds

$$(4.1.1) \quad (1 - c)\phi_1(t) + c\phi_2(t) = \phi_1(t)\phi_2(t), \quad t \in \mathbb{R}$$

with  $0 < c < 1$ . This is indeed an extended version of the problem posed earlier by Dugue for  $c = 1/2$ .

Rossberg (1972) solved the problem when at least one of the  $\phi_i$ 's is non-arithmetic and Wolinstawez and Szynal (1984) when both  $\phi_1$  and  $\phi_2$  are arithmetic. In both these papers, there is an assumption that the left extremity of the distribution corresponding to  $\phi_1$  is non-negative and the right extremity of the distribution corresponding to  $\phi_2$  is non-positive. We shall now show that under the assumptions made by these authors, the problem of identifying the solutions to (4.1.1) reduces to a straight-forward application of LRS theorems.

Let  $F_1$  and  $F_2$  be the distribution functions corresponding to  $\phi_1$  and  $\phi_2$  respectively. Assume that  $F_1(0-) = 1 - F_2(0) = 0$ . It is then obvious that

(4.1.1) yields

$$(4.1.2) \quad \alpha F_2(-x) = \int_{R_+} F_2(-x-y) dF_1(y), \quad x \in R_+ - \{0\}.$$

If  $F_1$  is nonarithmetic, (4.1.2) implies, in view of Lau-Rao theorem, that  $F_2(-x) \sim \exp(-ax)$  for  $x > 0$  and some  $a > 0$ , and consequently (4.1.1) yields  $F_2(-x) = \exp(-ax)$ ,  $x \in R_+$  for some  $a > 0$ . (This follows since under the given assumptions, the fact that  $F_2(-x) \sim \exp(-ax)$ ,  $x \in R_+ - \{0\}$  when used in (4.1.1) gives the following equation relative to probability measures of  $\{0\}$  on both sides

$$(1 - \alpha)F_1(0) + \alpha[F_2(0) - F_2(0-)] = F_1(0)[F_2(0) - F_2(0-)]$$

and hence  $F_2(0) = F_2(0-)$ . From this it follows that if (4.1.1) is valid, then under the assumption that at least one  $F_i$  is non-arithmetic (and hence without loss of generality that  $F_1$  is non-arithmetic), we have

$$F_1 = 1 - e^{-bx}, \quad x \in R_+, \quad F_2(-x) = e^{-ax}, \quad x \in R_+$$

with  $a > 0$  and  $b$  such that  $b = \alpha/(1 - \alpha)$ . (The converse of the assertion is obvious.) This is the result of Rossberg (1972) but for his apriori restriction that  $F_1(0) = 0$ . On the other hand if  $F_1$  is arithmetic, in view of LRS theorems, (4.1.2) implies that

$$(4.1.3) \quad \phi_2(t) = 1 - \alpha + \alpha \frac{(1 - \beta)\exp(-ibt)}{1 - \beta \exp(-ibt)}, \quad -\infty < t < \infty$$

for some  $b > 0$ ,  $\alpha \in [0, 1]$  and some  $\beta \in [0, 1)$  with an additional requirement that the corresponding characteristic function  $\phi_1$  satisfies

$$(4.1.4) \quad \phi_1(t)\{(c - \alpha)e^{ibt} - c\beta + \alpha\} = c\{(1 - \alpha)e^{ibt} - \beta + \alpha\}, \quad -\infty < t < \infty.$$

If  $X_1$  is a random variable corresponding to  $\phi_1$ , then from (4.1.4) we have for  $n \geq 1$

$$(c - \alpha)P\{X_1 = nb\} = (c\beta - \alpha)P\{X_1 = (n + 1)b\}$$

implying that either  $P\{X_1 = nb\} = 0$  for  $n \geq 1$  or  $\alpha \geq c$ . Further (4.1.4) gives

$$(\alpha - c\beta)P\{X_1 = 0\} = c(\alpha - \beta)$$

which yields that  $\alpha \geq \beta$  whenever  $\alpha \geq c$ . Thus, it follows that if (4.1.4) holds, then either  $\phi_1(t) \equiv 1$ , or for  $\alpha \geq \max\{\beta, c\}$  and for some  $b > 0$

$$(4.1.5) \quad \phi_1(t) = c \frac{\alpha - \beta + (1 - \alpha) \exp(ibt)}{\alpha - c\beta - (\alpha - c) \exp(ibt)}, \quad -\infty < t < \infty$$

which is clearly a characteristic function satisfying (4.1.4). Then it easily follows that if (4.1.1) is satisfied with at least one of the  $\phi_i$ 's as arithmetic and the extremity assumptions are satisfied, then either  $\phi_1 \equiv 1$  and  $\phi_2 \equiv 1$  or  $\phi_1$  and  $\phi_2$  are of the type given respectively by (4.1.5) and (4.1.3) for some  $\beta \in [0, 1)$  and  $\alpha \in [\max\{\beta, c\}, 1]$ . This is indeed the result arrived at by Wolinsta-Welez and Szynal using a different approach.

The following example illustrates that Rossberg-Wolinsta Welez-Szynal characterization of  $(\phi_1, \phi_2)$  satisfying (4.1.1) does not remain valid if the assumption that  $F_1(0-) = 1 - F_2(0) = 0$  is dropped.

Example For a real  $\theta \neq 0, 1$ , let

$$\phi_1(t) = [(1 + it)(1 - \theta it)(1 - \frac{\theta it}{\theta - 1})]^{-1}, -\infty < t < \infty$$

and

$$\phi_2(t) = \phi_1(-t), -\infty < t < \infty.$$

Observe that (4.1.1) is satisfied with  $\alpha = 1/2$  and  $\phi_1$  and  $\phi_2$  are non-arithmetic. However,  $\phi_1$  and  $\phi_2$  are not of the form given by Rossberg.

In the above counter example, we have  $F_1(0-) = 1 - F_2(0) > 0$ . It may be noted that there also exist examples illustrating the point with either  $F_1(0-) = 0$  or  $1 - F_2(0) = 0$ . In particular, if we take  $\alpha = \alpha/(1 - \alpha)$

$$\phi_1(t) = (1 - it)^{-2}, -\infty < t < \infty$$

$$\phi_2(t) = (1 + \beta \sqrt{\alpha} it)^{-1} (1 - \beta^{-1} \sqrt{\alpha} it)^{-1}, -\infty < t < \infty$$

with  $\alpha = (\beta^2 - 1)^2/4\beta^2$  and  $\beta > 1$ , we have an example with  $F_1(0-) = 0$ . (The existence of an example with  $1 - F_2(0) = 0$  follows by symmetry.)

## REFERENCES

- Alzaid, A. A. (1983). Some contributions to characterization theory. Ph.D. thesis, University of Sheffield.
- Alzaid, A. A., Rao, C. R. and Shanbhag, D. N. (1983). Solutions of certain functional equations and related results on probability distributions. (Submitted for publication)
- Bellman, R. and Cooke, K. L. (1963). Differential-Difference Equations. Academic Press, New York.
- Choquet, G. and Dery, J. (1960). Sur l'equation de convolution  $\mu * \sigma = \mu * \sigma$ . C. R. Acad. Sc., Paris, t, 250, 799-801.
- Dallas, A. C. (1981). Record values and exponential distribution. J. Appl. Prob. 18, 949-951.
- Davies, P. L. and Shanbhag, D. N. (1984). A generalization of a theorem of Dery with applications in characterization theory. (Submitted for publication.)
- Deheuvels, P. (1984). The characterizations of distributions by order statistics and record values - A unified approach. J. Appl. Prob. 21, 326-334.
- Dery, J. (1961). Sur l'equation de convolution  $\mu * \sigma$ . Semin. Theor. Potent. M. Brelot, Fac. Sci. Paris, 1959-60, 4 e ann.
- Feller, W. (1966). An Introduction to Probability Theory and its Applications. Vol. 2, John Wiley, New York.
- Gupta, R. C. (1984). Relationships between order statistics and record values and some characterization results. J. Appl. Prob. 21, 433-438.
- Isham, V., Shanbhag, D. N. and Westcott, M. (1975). A characterization of the Poisson process using forward recurrence times. Math. Proc. Camb. Philos. Soc. 78, 513-515.
- Kotz, S. and Shanbhag, D. N. (1980). Some new approaches to probability distribution. Adv. in Appl. Prob. 12, 903-921.
- Lau, Ka-Sing and Rao, C. Radhakrishna (1982). Integrated Cauchy functional equation and characterizations of the exponential law. Sankhyā A 44, 72-90.
- Lau, Ka-Sing and Rao, C. Radhakrishna (1984). Integrated Cauchy functional equation on the whole line. Sankhyā 46 A (in press).
- Lau, Ka-Sing and Rao, C. Radhakrishna (1984). On the integral equation  $f(x) = \int_R f(x+y)du(y)$ ,  $x \geq 0$ . Technical Report #84-09, University of Pittsburgh.

- Marsaglia, G. and Tubilla, A. (1975). A note on the lack of memory property of the exponential. Ann. Prob. 3, 352-354.
- Rao, C. Radhakrishna (1983). An extension of Deny's theorem and its application to characterization of probability distributions. In "A Festschrift for Eric Lehmann" (Eds. Bickel, Doksum and Hodges Jr.), Wadsworth, 348-365.
- Rao, C. Radhakrishna and Rubin, H. (1964). On a characterization of the Poisson distribution. Sankhyā A 26, 295-298.
- Rao, M. B. and Shanbhag, D. N. (1982). Damage Models. In Encyclopedia of Statistical Sciences, Vol. 2, John Wiley, 262-265.
- Ramachandran, B. (1979). On the strong memoryless property of the exponential and geometric laws. Sankhyā A 41, 244-251.
- Ramachandran, B. (1982). On the equation  $f(x) = \int_{[0, \infty)} f(x+y) d\mu(y)$ . Sankhyā 44 A, 364-371.
- Ressel, Paul (1984). DeFinetti-type theorems: An analytical approach. Technical Report.
- Richards, Donald, St. P. (1981). Exponential distributions on partially ordered semiabelian groups. Technical Report, University of North Carolina.
- Shanbhag, D. N. (1977). An extension of the Rao-Rubin characterization. J. Appl. Prob. 14, 640-646.
- Shanbhag, D. N. (1983). Review No: 62038 on C. R. Rao and Srivastava.
- Shimuzu, R. (1978). Solution to a functional equation and its application to some characterization problems. Sankhyā 40, 319-332.
- Srivastava, R. C. and Singh, J. (1975). On characterizations of the binomial and Poisson distributions based on a damage model. Statistical Distributions in Scientific Work, 3, Eds. G. P. Patil, S. Kotz and J. K. Ord, 271-278.
- Titchmarsh, E. C. (1949). The Theory of Functions, Oxford University Press.
- Westcott, M. (1981). Letter to the Editor. J. Appl. Prob. 18, 568.
- Wolinska-Welez, A. and Szynal, D. (1984). On a solution of Dugue's problem for a class of couples of lattice distributions. Unpublished.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

## REPORT DOCUMENTATION PAGE

|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |       |                                            |                                                                                                                                                                                        |                                              |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------|--------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------|
| 1a. REPORT SECURITY CLASSIFICATION<br><b>UNCLASSIFIED</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |       |                                            | 1b. RESTRICTIVE MARKINGS                                                                                                                                                               |                                              |
| 2a. SECURITY CLASSIFICATION AUTHORITY                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |       |                                            | 3. DISTRIBUTION/AVAILABILITY OF REPORT<br>Approved for public release; distribution unlimited.                                                                                         |                                              |
| 2b. DECLASSIFICATION/DOWNGRADING SCHEDULE                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |       |                                            |                                                                                                                                                                                        |                                              |
| 4. PERFORMING ORGANIZATION REPORT NUMBER(S)<br><b>84-39</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        |       |                                            | 5. MONITORING ORGANIZATION REPORT NUMBER(S)<br><b>AFOSR-TR- 84-1056</b>                                                                                                                |                                              |
| 6a. NAME OF PERFORMING ORGANIZATION<br>University of Pittsburgh                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |       | 6b. OFFICE SYMBOL<br>(If applicable)       | 7a. NAME OF MONITORING ORGANIZATION<br>Air Force Office of Scientific Research                                                                                                         |                                              |
| 6c. ADDRESS (City, State and ZIP Code)<br>Center for Multivariate Analysis<br>515 Thackeray Hall, Pittsburgh PA 15260                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |       |                                            | 7b. ADDRESS (City, State and ZIP Code)<br>Directorate of Mathematical & Information<br>Sciences, Bolling AFB DC 20332-6448                                                             |                                              |
| 8a. NAME OF FUNDING/SPONSORING ORGANIZATION<br>AFOSR                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               |       | 8b. OFFICE SYMBOL<br>(If applicable)<br>NM | 9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER<br>F49620-82-K-0001                                                                                                                    |                                              |
| 8c. ADDRESS (City, State and ZIP Code)<br>Bolling AFB DC 20332-6448                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                |       |                                            | 10. SOURCE OF FUNDING NOS.                                                                                                                                                             |                                              |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |       |                                            | PROGRAM<br>ELEMENT NO.<br>61102F                                                                                                                                                       | TASK<br>NO.<br>A5                            |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |       |                                            | PROJECT<br>NO.<br>2304                                                                                                                                                                 | WORK UNIT<br>NO.                             |
| 11. TITLE (Include Security Classification)<br><b>RECENT RESULTS ON CHARACTERIZATION OF PROBABILITY DISTRIBUTIONS: A UNIFIED APPROACH<br/>THROUGH EXTENSIONS OF DENY'S THEOREM</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 |       |                                            |                                                                                                                                                                                        |                                              |
| 12. PERSONAL AUTHOR(S)<br>C. Radhakrishna Rao and D.N. Shanbhag                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |       |                                            |                                                                                                                                                                                        |                                              |
| 13a. TYPE OF REPORT<br>Technical                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |       | 13b. TIME COVERED<br>FROM _____ TO _____   |                                                                                                                                                                                        | 14. DATE OF REPORT (Yr., Mo., Day)<br>SEP 84 |
| 15. PAGE COUNT<br>31                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               |       |                                            |                                                                                                                                                                                        |                                              |
| 16. SUPPLEMENTARY NOTATION                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                         |       |                                            |                                                                                                                                                                                        |                                              |
| 17. COSATI CODES                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |       |                                            | 18. SUBJECT TERMS (Continue on reverse if necessary; and identify by block number)                                                                                                     |                                              |
| FIELD                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              | GROUP | SUB GR.                                    | Characterizations; Deny's theorem; exchangeable random<br>variables; exponential; geometric; Pareto and stable distri-<br>butions; integrated Cauchy functional equations; (CONTINUED) |                                              |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |       |                                            |                                                                                                                                                                                        |                                              |
| 19. ABSTRACT (Continue on reverse if necessary and identify by block number)<br>The problem of identifying solutions of general convolution equations relative to a group has been studied in two classical papers by Choquet and Deny (1960) and Deny (1961). Recently, Lau and Rao (1982) have considered the analogous problem relative to a certain semigroup of the real line, which extends the results of Marsaglia and Tubilla (1975) and a lemma of Shanbhag (1977). The extended versions of Deny's theorem contained in the papers by Lau and Rao, and Shanbhag (which the authors refer to as LRS theorems) yield as special cases improved versions of several characterizations of exponential, Weibull, stable, Pareto, geometric, Poisson and negative binomial distributions obtained by various authors during the last few years. In this paper the authors review some of the recent contributions to characterization of probability distributions (whose authors do not seem to be aware of LRS theorems or special cases existing earlier) and show how improved versions of these results follow as immediate corollaries to LRS theorems. The authors also give a short proof of Lau-Rao theorem based on Deny's theorem and thus establish a direct link between the results of Deny (1961) and those of Lau and Rao (1982). (CONTINUED) |       |                                            |                                                                                                                                                                                        |                                              |
| 20. DISTRIBUTION/AVAILABILITY OF ABSTRACT<br>UNCLASSIFIED/UNLIMITED <input checked="" type="checkbox"/> SAME AS RPT. <input type="checkbox"/> DTIC USERS <input type="checkbox"/>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |       |                                            | 21. ABSTRACT SECURITY CLASSIFICATION<br>UNCLASSIFIED                                                                                                                                   |                                              |
| 22a. NAME OF RESPONSIBLE INDIVIDUAL<br>MAJ Brian W. Woodruff                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |       |                                            | 22b. TELEPHONE NUMBER<br>(Include Area Code)<br>(202) 767- 5027                                                                                                                        | 22c. OFFICE SYMBOL<br>NM                     |

DD FORM 1473, 83 APR

EDITION OF 1 JAN 73 IS OBSOLETE.

84 11 26 247

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

ITEM #18, SUBJECT TERMS, CONTINUED: Lau-Rao theorem; Shanbhag's lemma.

ITEM #19, ABSTRACT, CONTINUED: A variant of Lau-Rao theorem is proved and applied to some characterization problems.

UNCLASSIFIED