AD-A141 886 RANDOM FIELD SATISFYING A LINEAR PARTIAL DIFFERENTIAL EQUATION WITH RANDOM FORCING TERM(U) NORTH CAROLINA UNIV AT CHAPEL HILL DEPT OF STATISTICS 1/1

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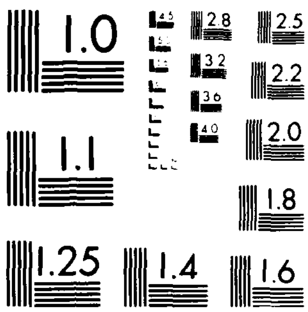

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REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER	2. GOVT ACCESSION NO. AD A141586 · 3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Random Field Satisfying a Linear Partial Differential Equation with Random Forcing Term	5. TYPE OF REPORT & PERIOD COVERED Technical
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) D. deBrucq and C. Olivier	8. CONTRACT OR GRANT NUMBER(s) N00014-75-C-0491 N00014-81-K-0373
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics University of North Carolina Chapel Hill, North Carolina 27514	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR 042-269 SRO 105
11. CONTROLLING OFFICE NAME AND ADDRESS Statistics & Probability Office of Naval Research Arlington, VA 22217	12. REPORT DATE 1984
	13. NUMBER OF PAGES 23
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report) UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Poisson process; Levy process; random fields; wave propagation in underwater acoustics.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

~~We~~ The authors first solve the equation dX + aXdt = dN, where dN represents a Poisson process, and then generalize to a Levy process. Finally, ~~we~~ they solve a linear partial differential equation DX = dL in strong distribution, meaning that the second member dL is a distribution process, generalization of Levy process on $\mathbb{R}$. The results are then applied to wave propagation in underwater acoustics, and spatial correlation is determined.

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RANDOM FIELD SATISFYING A LINEAR PARTIAL DIFFERENTIAL EQUATION WITH RANDOM FORCING TERM

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SUMMARY

We first solve the equation $dX + aXdt = dN$, where dN represents a Poisson process, and then generalize to a Levy process. Finally, we solve a linear partial differential equation $DX = dL$ in strong distribution, meaning that the second member dL is a distribution process, generalization of Levy process on $\mathbb{R}$. The results are then applied to wave propagation in underwater acoustics, and spatial correlation is determined.

INTRODUCTION

The theory of linear random fields is well developed; for example S. Badrikian and S. Chevet [2] consider the cylindrical measures and the associated linear random field. Some applications to economy and industry are presented in S. Ben Soussan [3]. We consider the linear random process X which is the solution of a linear partial differential equation formally written DX = dT, where dT is a random process; in the simplest case dT is white noise. The equation Dp = 0 in the last chapter of this paper will be the propagation equation of the pressure p in deep sea water. We consider only linearized propagation equation from general ones (cf. Poirée [11]); our equation is thus an approximation in the sea medium. In D. de Brucq [4], the second member of the equation is a Wiener measure or Gaussian measure, denoted by dW, defined on $S(\mathbb{R}^4)$ the space of indefinitely differentiable functions on $\mathbb{R}^4$ decreasingly quickly. The second member describes the random approximation. This equation is written

$$Dp \triangleq \Delta p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} + \frac{b}{c^2}\frac{\partial}{\partial t}\Delta p = dW$$

where Δ is the Laplacian in $\mathbb{R}^3$. The solution in the sense of strong distribution meaning satisfies for any f of $S(\mathbb{R}^4)$:

$$p(f) = \int \overline{F}\left[\frac{F\cdot\overline{f}}{\overline{A}}\right] dW \qquad \text{where } F \text{ is the Fourier transform on } \mathbb{R}^4$$

and $\overline{F}$ its inverse and where A is the function defined by the equality $F\cdot D = AF$. In this particular case $A(\omega,x) = 1_2(\omega)\{x^2-[k(\omega)-i\gamma(\omega)]^2\}$ where k is the wave number, γ is the absorption. By hypothesis, for any f and g of $S(\mathbb{R}^4)$, the correlation of the noise dW is $\Gamma(f,g) = E(W(f)W(g)) = \sigma^2\int\bar{f}g\,d\lambda$ with σ^2 a normalization constant and $d\lambda$ Lebesgue measure on $\mathbb{R}^4$. The spatial correlation s at a given frequency $\nu = \frac{\omega}{2\pi}$ of the pressure p is given (cf. Th III-3-7) by:

$$s(\omega,\ell) = \frac{\pi^2\sigma^2}{|1_2|^2} \times \frac{e^{-\gamma\ell}}{\gamma} \times \frac{\sin k\ell}{k\ell}$$

here ℓ is the distance between the points where the pressure is measured; by physical approximation 1_2 may be taken equal to 1.

For $\ell = 0$, the experimental spectral density function of the random fluctuations of the pressure in deep sea satisfy to a good approximation the relation

$$s(\omega,0) = \frac{\pi^2\sigma^2}{|1_2|^2} \frac{1}{\gamma} .$$

In the ocean, point processes and Poisson processes dN appear to be more accurate than Gaussian processes to describe the random sources of the noise. We generalize as much as possible to introduce a Levy linear process dL as second member of the equation. The expression $s(\omega,\ell)$ does not change with that extension.

In the first section, we explain the method with the linear differential equation

$$dX + aXdt = dN \quad a > 0 \tag{1}$$

where dN is the centered Poisson process. We compute the characteristic function of the X process so that the Gaussian or the Poisson laws of the forcing term can be separated. Not withstanding the correlation function Γ of X and in the stationary situation the spectral density power function are given.

In the second section, we introduce a Levy process L with stationary increments and recall a decomposition theorem for this indefinitely divisible process. Then we solve equation (1) using L as forcing term. We are now in the situation to consider spatio-temporal problems. In the third section, we use general theorems (A. Badrikian [1]) to construct a measure L on $S(\mathbb{R}^4)$ generalization of the Levy processes. We solve in the sense of strong distribution the equation DX = dL. The case where L is Gaussian measure is known (D. de

Brucq and C. Olivier [5]). Poisson measure is well known. (J. Neveu [10]) but $DX = dL$ with L Poisson measure is new, a fortiori with L our generalization of a Levy process.

Finally, we apply the results to the propagation equation and we obtain the spatial correlation at a given frequency.

Some classical notations and results will be useful. We will write the Fourier transform

$$F(f)(z) \triangleq \int_{\mathbf{R}^n} e^{i<z,Z>} f(x)\, dz$$ where the function f is integrable

and where $<z,Z>$ is the scalar product in $\mathbf{R}^n$. In the spatio-temporal application of propagation, for physical reasons

$$F(f)(\omega,\xi,\eta,\xi) \triangleq \int_{\mathbf{R}^4} e^{(-\omega t+\xi x+\eta y+\xi z)} f(t,x,y,z)\,dtdxdydz$$

Fourier transformations are isomorphism of distribution spaces $S(\mathbf{R}^n)$, $S(\mathbf{R}^4)$.

The distributions space $S(\mathbf{R}^n)$ is nuclear and countably semi-normed; these notions are for example defined in I.M. Gelfand and N. Yu. Vilenkin [6]. We denote by S this space and S' its topological dual, the tempered distribution space. The topology of S is defined by the semi-norms:

$$\forall k, N \in \mathbf{N}, \quad s_{k,N}(f) \triangleq \sup_{\substack{z\ \mathbf{R}^n \\ |\alpha| \leq N}} \left|(1+|z|^{\alpha})^k \mathcal{D}^{\alpha} f(z)\right|$$

we have the topological inclusion $\mathcal{D} \subset S \subset L^p$ with $\mathcal{D}$ the space of indefinitely derivable functions with compact support in $\mathbf{R}^n$.

If Γ is a real or complex function on $\mathbf{R}^n$, continuous in zero and of positive definite type then the Bochner theorem asserts the existence of a bounded positive measure ρ such as

$$\forall t \in \mathbf{R}^n \quad \Gamma(t) = \int e^{i\nu t} d\rho(\nu)$$

The random processes of the first two paragraphs are defined on a probability space (Ω,A,P). We denote by R^n the tribe of the borelians of $\mathbf{R}^n$.

I. SOLUTION OF THE EQUATION $dX + aXdt = dN$

We take a to be strictly positive real number and N to be a Poisson process.

I.1 Poisson Processes

Consider a family of random variables $\mathcal{N}(\tau_1,\tau_2)$ with $\tau_1,\tau_2 \in \mathbb{R}$ $\tau_1 < \tau_2$. Each has Poisson law and represents the number of discontinuities of a random phenomenon in the interval $[\tau_1,\tau_2]$. Let $\sigma(\tau)$ be a non-decreasing function defined by the relation

$$E(\mathcal{N}(\tau_1,\tau_2)) = \int_{\tau_1}^{\tau_2} d\sigma(\tau) .$$

We suppose σ continuous on $\mathbb{R}$ so σ is almost bounded. There exists a sequence $(T_n, n\in\mathbb{N})$ of intervals such as

a) $T_n \subset T_{n+1}$ and $\cup_n T_n = \mathbb{R}$

b) $\int_{T_n} d\sigma(z) < \infty$

The Poisson process $\mathcal{N}$ of parameter σ, is the family of random variables:

$$\mathcal{N}(\tau) = \mathcal{N}(0,\tau) \quad \text{for } \tau > 0 \text{ and}$$

$$\mathcal{N}(\tau) = -\mathcal{N}(0,\tau) \quad \text{for } \tau < 0 .$$

The centered Poisson process N is

$$\forall\tau \in \mathbb{R} \quad N(\tau) = \mathcal{N}(\tau) - \sigma(\tau) \quad \text{and we write}$$

$$dN = d\mathcal{N} - d\sigma$$

The solution X of the equation

$$dX + a\,X\,dt = d\mathcal{N} \quad \text{is}$$

$$\forall t \in \mathbb{R} \quad X(t) = \int_{-\infty}^{t} e^{-a(t-\tau)} d\mathcal{N}(\tau)$$

The integral is the almost sure limit of $\int_{t_0}^{t} e^{-a(t-\tau)} d\mathcal{N}(\tau)$ when t_o converges to $-\infty$.

We suppose that $\int_{-\infty}^{t} e^{-a(t-\tau)} d\sigma(\tau) < \infty$ and also that $\int_{-\infty}^{t} e^{-2a(t-\tau)} d\sigma(\tau) < \infty$.

Since X is a second order process, straightforward computation gives

$$E(X^2(t)) = \int_{-\infty}^{t} e^{-2a(t-\tau)} d\sigma(\tau) + \left(\int_{-\infty}^{t} e^{-a(t-\tau)} d\sigma(\tau)\right)^2$$

and the variance is

$$E(X^2(t)) - [E(X(t))]^2 = \int_{-\infty}^{t} e^{-2a(t-\tau)} d\sigma(\tau).$$

For the centered Poisson process N, the solution

$$\forall t \in \mathbb{R} \quad X(t) = \int_{-\infty}^{t} e^{-a(t-\tau)} dN(\tau) \text{ is centered and}$$

$$E(X(t)^2) = \int_{-\infty}^{t} e^{-2a(t-\tau)} d\sigma(\tau)$$

if N has stationary increments then σ is a Haar measure on $\mathbb{R}$ and $d\sigma = \lambda_0 d\tau$ with $d\tau$ Lebesgue measure and λ_0 constant. For $\lambda_0 = 1$, $E(N(t)^2) = |t|$ and $E(X(t)^2) = \int_{-\infty}^{t} e^{-2a(t-\tau)} d\tau$. If we change the Poisson process N to a Gaussian process W such as $E(W(t)) = 0$ and $E(W(t)^2) = t$ then the solution is centered and has the same variance.

The characteristic functions of $N(\tau)$ and of $N(\tau)$ are:

$$\phi_{N(\tau)}(u) \triangleq E(e^{iuN(\tau)}) = \exp[\sigma(\tau)(\exp iu-1)]$$

$$\phi_{N(\tau)}(u) \triangleq E(e^{iuN(\tau)}) = \exp[\sigma(\tau)(\exp(iu)-1-iu)]$$

I.2 Probability Law of the Solution X

Lemma I.2.1: The complex function ψ defined by

$$\forall f \in L^2(\mathbb{R}, R, d\sigma) \quad \psi_1(f) \triangleq \int_{\mathbb{R}} [e^{if(z)} - 1 - i\, f(z)] d\sigma(z) \text{ is continuous on } L^2(\mathbb{R}, R, d\sigma)$$

Proof: For every f of $L^2(\mathbb{R}, R, d\sigma)$, we have

$$|\psi_1(f)| \leq \tfrac{1}{2}\int f^2(\tau) d\sigma(\tau).$$

Then if (f_n) converges to zero in $L^2(\mathbb{R}, R, d\sigma)$ then $\psi_1(f_n)$ converges to zero in $\mathbb{C}$. □

<u>Theorem I.2.2</u>: There exists a linear centered process

$$N = (\Omega, A, P, (N(f))_{L^2(\mathbb{R}, R, d\sigma)}, \mathbb{R}, R)$$

with characteristic function $\forall f \in L^2(\mathbb{R}, R, d\sigma)$

$$\phi_f \triangleq E(e^{iN(f)}) = \psi_1(f)$$

Proof: The application.

$$1_{[\tau_1,\tau_2]} \in L^2(\mathbb{R}, R, d\tau) \rightarrow N(\tau_2) - N(\tau_1) \in L^2(\Omega, A, P)$$

is isometric and conserves the scalar product. The family $1_{[\tau_1,\tau_2]}\ \tau_1,\tau_2 \in \mathbb{R}$ is total. We note N the unique extension. So for every f in $L^2(\mathbb{R}, R, d\sigma)$, N(f) is a centered second order random variable. We have to compute its characteristic function. For any expression

$f \triangleq a_j\ 1_{[\tau_j,\tau_{j+1}]}$ in $L^2(\mathbb{R}, R, d\sigma)$, we have $N(f) = \sum a_j[N(\tau_{j+1}) - N(\tau_j)]$ and

$$\text{Log}\ \phi_{N(f)} = \sum_j \log E(e^{i\,a_j(N(\tau_{j+1}) - N(\tau_j))})$$

$$= \sum_j [e^{ia_j} - 1 - i\,a_j][\sigma(\tau_{j+1}) - \sigma(\tau_j)]$$

$$= \int_{\mathbb{R}} \left\{ e^{i\sum a_j 1_{[\tau_j,\tau_{j+1}]}(\tau)} - 1 - i\sum a_j 1_{[\tau_j,\tau_{j+1}]}(\tau) \right\} d\sigma(\tau)$$

$$= \int_{\mathbb{R}} \{e^{i\,f(\tau)} - 1 - i\,f(\tau)\} d\sigma(\tau)\ .$$

By continuity, the expression is valid for any f of $L^2(\mathbb{R}, R, d\sigma)$. □

Corollary I.2.3: The characteristic function ϕ_X of the process X solution of the equation

$$dX + aXdt = dN \quad \text{satisfies } \forall u,\ t \in \mathbb{R}\ .$$

$\phi_{X(t)}(u) = E(e^{iuX(t)}) \triangleq \exp \psi_{X(t)}(u)$ with

$$\psi_{X(t)}(u) = \int_{-\infty}^{t} [\exp iu\, e^{-a(t-\tau)} - 1 - iu\, e^{-a(t-\tau)}] d\sigma(\tau)$$

Proof: The expression of $X(t)$ is

$$X(t) = \int_{-\infty}^{t} e^{-a(t-\tau)} dN(\tau) \text{ with } 1_{[-\infty,t]}(\cdot) e^{-a(t-\cdot)}$$

in $L^2(\mathbb{R}, R, d\sigma)$ then $uX(t) = N(u\, 1_{[-\infty,t]}(\cdot)\, e^{-a(t-\cdot)})$

Corollary I.2.4: The covariance function Γ of the process X solution of the equation

$$dX + aXdt = dN \text{ satisfies } \forall t_1, t_2 \in \mathbb{R}$$

$$\Gamma(t_1,t_2) = E(X(t_1)X(t_2)) = \int_{-\infty}^{t_1 \wedge t_2} e^{-a(t_1+t_2-2\tau)} d\sigma(\tau)$$

Proof: As $X(t) = N(1_{[-\infty,t]}(\cdot)\, e^{-a(t-\cdot)})$ we have

$$\Gamma(t_1,t_2) = \int_{\mathbb{R}} 1_{[-\infty,t_1]}(\tau)\, e^{-a(t_1-\tau)}\, 1_{[-\infty,t_2]}(\tau)\, e^{-a(t_2-\tau)} d\sigma(\tau) \qquad \square$$

Corollary I.2.5: In the stationary case, the covariance function Γ is equal to

$\Gamma(t_1,t_2) = \frac{\lambda_0}{2a} e^{-a|t_1-t_2|}$, t_1,t_2 in $\mathbb{R}$, and the power spectral density function is

$$\gamma(\nu) = \frac{1}{2\pi} \frac{\lambda_0}{a^2+\nu^2}, \ \forall\, \nu \in \mathbb{R}.$$

Proof: In that case $d\sigma = \lambda_0 d\tau$ and $\Gamma(t_1,t_2) = \frac{\lambda_0}{2a} e^{-a[t_1+t_2-2(t_1 \wedge t_2)]}$, the Fourier transform of which is

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\nu\tau} \frac{\lambda_0}{2a} e^{-a|\tau|} d\tau = \frac{1}{2\pi} \frac{\lambda_0}{a^2+\nu^2} \qquad \square$$

II. SOLUTION OF THE EQUATION $dX + aXdt = dL$

We generalize the forcing term to be a Levy process. We have to give some essential properties of these processes (T. Hida [9]).

II.1 Classical Properties of the Levy Processes

Definition II.1.1: A process

$L = (\Omega, A, P, (L(t))_{\mathbf{R}}, \mathbf{R}, \mathcal{R})$ is a levy process if L satisfies the properties

(a) L has independent increments.

(b) L is continuous in probability.

(c) L has trajectories almost surely continuous to the right an· with limit to the left.

We will show how to obtain the generality of such processes.

Proposition II.1.2: Let $N \triangleq \{N_I; I=[a,b] \subset \mathbf{R}\}$ a family of processes. For each interval I, the process N_I is Poisson such that

(a) $\forall t \in \mathbf{R}$ $E[N_I(t)] = t\, n(I)$ where n is a positive measure on $\mathbf{R}$ satisfying

$$\int_{|z|>0} \frac{z^2}{1+z^2}\, dn(z) < \infty$$

(b) For I_1, I_2 disjoin ($I_1 \quad I_2 = \emptyset$) then the Poisson processes N_{I_1}, N_{I_2} are independent.

(c) For every partition $(I_k; k \in \mathbf{N})$ of $I=[a,b]$ then $\forall t \in \mathbf{R}\, N_I(t) = \sum_{k \in \mathbf{N}} N_{t_k}(t)$ almost surely.

As $I \to N_I(t)$ is a measure with value in $L^\circ(\Omega, A, P)$, we use the notation $\int_I z N_{dz}(t)$. The process

$$\forall t \in \mathbf{R} \quad L_1(t) \triangleq \lim_{p\to\infty} \int_{p>|z|>\frac{1}{p}} [z\, N_{dz}(t) - \frac{tz}{1+z^2}\, dn(z)]$$

is then defined and is a Levy process with stationary increments. The measure n is the Levy measure of the process L_1. The convergence with p is almost surely uniform in t.

Remark II.1.3: Every Poisson process is a Levy process and the vector span of such processes is composed also of Levy processes. We need only to be able to pass through the limits. Let $\{N_k, k \in \mathbb{N}\}$ a sequence of independent processes with $(\forall t \in \mathbb{R}) \sum_{k \in \mathbb{N}} \sigma_k |(t)| < \infty$; is $\sum_{k \in \mathbb{N}} z_k N_k$ a Levy process? If $(\forall t \in \mathbb{R}) \sum_{k \in \mathbb{N}} z_k^2 \sigma_k(t) < \infty$ then $\sum_{k \in \mathbb{N}} z_k N_k$ exists and is a Levy process.

Another construction is possible. Let N be a Poisson process and $(\eta_k, k\ \mathbb{Z})$ a sequence of independent real random variables, independent of N. All the discontinuities, the jumps of N, are equal to +1, we change the value +1 of the k^{th} jump in $+\eta_k$, random in $\mathbb{R}$ and we obtain a Levy process (T. Hida [9]). We recall the P. Levy's decomposition theorem:

Theorem II.1.4: Let L be a Levy-process with stationary increments. Then there exist two constants α, β of $\mathbb{R}$, a Wiener process W, and a Levy process L_1 such that $\forall t \in \mathbb{R}$ $L(t) = \alpha + \beta W(t) + L_1(t)$. The decomposition is unique.

The forcing term of the equation dX + aXdt = dL will be a Levy process so that we generalize the Gaussian and the Poisson case.

II.2 Solution of the Equation DX + aXdt = dL

We assume the Levy process L with stationary increments and the Levy measure n satisfying $\int_{\mathbb{R}} z^2 dn(z) < \infty$. We can suppose the process L centered.

Proposition II.2.1: The process

$$(\forall t \in \mathbb{R}) L_c(t) \triangleq \lim_{p \to \infty} \int_{p > |z| > \frac{1}{p}} [z N_{dz}(t) - tz dn(z)] ,$$

where N_{dz} is the Poisson process of Proposition II.1.2 and $\int_{\mathbb{R}} z^2 dn(z) < \infty$, is a centered Levy process with stationary increments.

Proof: The limit $\int_{p>|z|>\frac{1}{p}} z dN_z$ exists (T. Hida [9], p. 39) and $\int_{p>|z|>\frac{1}{p}} z dn(z) = \int_{-p}^{-1} z dn(z) + \int_{1}^{p} z dn(z) + \int_{1>|z|>\frac{1}{p}} z dn(z)$ the first two terms converge noting for example that $|\int_{-p}^{-1} z dn(z)| \leq \int_{-p}^{-1} z^2 dn(z) \leq \int_{\mathbb{R}} z^2 dn(z) < \infty$. For every $p \in \mathbb{R}$, $E[\int_{p>|z|>\frac{1}{p}} z N_{dz}(t)] = t \int_{p>|z|>\frac{1}{p}} z\, dn(z)$ so passing through the limit $E(L_c(t)) = 0$. □

<u>Lemma II.2.2</u>: The complex function ψ_2 define by $\forall f \in L^2(\mathbb{R}, R, d\tau)$

$$\psi_2(f) = \int_{\mathbb{R}} \int_{|z|>0} [\exp i\, z\, f(\tau) - 1 - i\, z\, f(\tau)] dn(z) d\tau$$

is continuous on $L^2(\mathbb{R}, R, d\tau)$.

Proof: For every f of $L^2(\mathbb{R}, R, d\tau)$, we have

$$|\psi_2(f)| \leq \int_{\mathbb{R}} \int_{|z|>0} \tfrac{1}{2} |z|^2 |f(\tau)|^2 dn(z) d\tau$$

$$\leq \int_{\mathbb{R}} |f(\tau)|^2 d\tau \int_{|z|>0} |z|^2 dn(z)$$ □

<u>Theorem II.2.3</u>: If $\int z^2 dn(z) < \infty$, there exists a linear centered process

$$L_c = (\Omega, A, P, (L_c(f))_{L^2(\mathbb{R}, R, d\tau)}, \mathbb{R}, R)$$

with characteristic function $\forall f \in L^2(\mathbb{R}, R, d\tau)$

$$\psi_f \triangleq E(e^{i\, L_c(f)}) = \exp \psi_2(f)$$

Proof: We can suppose $\int_{\mathbb{R}} z^2 dn(z) = 1$. For $f = 1_{[0,t]}$, the characteristic function of

$$L_c(t) = \lim_{p\to+\infty} \int_{p>|z|>\frac{1}{p}} [N_{dz}(t)-tzdn(z)] \text{ is } (\forall u \in \mathbb{R})$$

$$\int_{-\infty}^{t} \int_{|z|>0} [\exp iuz\, 1_{[0,t]}(\tau) - \quad - iuz\, 1_{[0,t]}(\tau)\, dn(z)d\tau\ .$$

As $\int_{-\infty}^{t}\int_{|z|>0} |i\ u\ z\ 1_{[0,t]}(\tau)\ |^2 dn(z)d\tau < \infty$, the random variable $L_c(t)$ is second order and $E(L_c(t)^2) = t \int_{|z|>0} z^2 dn(z) = t$. The same is true for $t < 0$ and $f = 1_{[t,0]}$. The process $L_c(t)$ is a Levy process; the map

$$1_{[\tau_1,\tau_2]} \in L^2(\mathbb{R},R,d\tau) \to L_c(\tau_2)-L_c(\tau_1) \in L^2(\Omega,A,P)$$

is isometric and conserves the scalar product. The family $1_{[\tau_1,\tau_2]}\tau_1,\tau_2 \in \mathbb{R}$ is total. We note L_c the unique extension. So for every f in $L^2(\mathbb{R},R,d\tau)$ $L_c(t)$ is a centered second order random variable. To obtain the characteristic function ϕ_f of $L_c(t)$, we use a plain extension of the proof of Theorem I.2.2. □

Proposition II.2.4: Let $\forall t \in \mathbb{R}$ $L_c(t) = \lim_{p\to\infty} \int_{p>|z|>\frac{1}{p}} [z\ N_{dz} - tz\ dn(z)$ and suppose $\int_{|z|>0} z^2 dn(z) < \infty$ then the process X solution of the equation $dX + a\ Xdt = dL_c$ with $a > 0$ is centered, second order, stationary with covariance

$$\forall t_1,\ t_2 \in \mathbb{R}\ \Gamma(t_1,t_2) \triangleq E(X(t_2)X(t_1)) = \frac{e^{-a|t_2-t_1|}}{2a} \int_{|z|>0} z^2 dn(z)\ .$$

Proof: The solution X satisfies

$\forall t \in \mathbb{R}(t) = \int_{-\infty}^{t} e^{-a(t-\tau)} dL_c(\tau)$. But with $a > 0$ the function $1_{[-\infty,t]}^{(\cdot)} e^{-a(t-\cdot)}$ is element of $L^2(\mathbb{R},R,d\tau)$ then $X(t) = L_c(1_{[-\infty,t]} e^{-a(t-\cdot)})$.

We need only the covariance function for L_c:

$$\Gamma(t_1,t_2) = E(X(t_1)X(t_2)) = E[L_c(1_{[-\infty,t_1]}e^{-a(t_1-\cdot)})L_c(1_{[-\infty,t_2]}e^{-a(t_2-\cdot)})]$$

$$= \int_{\mathbf{R}} \int_{|z|>0} 1_{[-\infty,t_1]}^{(\tau)} e^{-a(t_1-\tau)} 1_{[-\infty,t_2]}^{(\tau)} e^{-a(t_2-\tau)} z^2 dn(z)d\tau$$

$$= \frac{e^{-a|t_2-t_1|}}{2a} \int_{|z|>0} z^2 dn(z)$$ □

Corollary II.2.5: The power spectral density function of the solution X is $\forall \nu \in \mathbf{R}$

$$\gamma(\nu) = \frac{1}{2\pi} \frac{1}{a^2+\nu^2} \int_{|z|>0} z^2 dn(z)$$

The second order properties are the same for every Poisson, Gaussian, Levy centered processes and for the processes, solutions of $dX + aX\,dt = dL_c$. It may be useful to have the law of probability of the solution X.

Corollary II.2.6: The characteristic function ϕ_X of the solution X is $\forall u$, $t \in \mathbf{R}$

$$\phi_{X(t)}(u) \triangleq E(e^{iu\,X(t)}) = \exp \psi_{X(t)}(u)$$

$$\psi_{X(t)}(u) = \int_{-\infty}^{t} \int_{|z|>0} [\exp i\,u\,z\,e^{-a(t-\tau)}]dn(z)d\tau$$

Proof: As $uX(t) = N(u\,1_{[-\infty,t]}^{(\cdot)} e^{-a(t-\cdot)})$, we use Theorem II.2.3. □

In Proposition II.1.2, we had L_1 non-centered Levy process with stationary increments. The hypothesis $\int_{|z|>0} \frac{z^2}{1+z^2} dn(z) < \infty$ is equivalent to

$\int_{|z|\geq 1} dn(z) < \infty$ and $\int_{<|z|\leq 1} z^2 dn(z) < \infty$. The process

$L^+(t) \triangleq \lim_{p\to\infty} \int_{p\geq|z|\geq 1} [z\,N_{dz}(t) - \frac{tz}{1+z^2} dn(z)]$ is defined and the new hypothesis

$\int_{|z|\geq 1} z^2 dn(z) < \infty$ implies that $\int_{|z|\geq 1} z\,dn(z) < \infty$ for the bounded measure on

$\{z \in \mathbf{R}; |z|\geq 1\}$, then the decomposition

$$L^+(t) = \lim_{p\to\infty} \int_{p\geq|z|\geq 1} z\, N_{dz}(t) - tz\, dn(z)] + t \int_{|z|\geq 1} \frac{z}{1+z^2}\, dn(z)$$

gives $E(L^+(t)) = t \int_{|z|\geq 1} \frac{z}{1+z^2}\, dn(z)$

The process $L^-(t) \overset{\Delta}{=} \lim_{p\to\infty} \int_{1\geq|z|>\frac{1}{p}} [z\, N_{dz}(t) - \frac{tz}{1+z^2}\, dn(z)]$ is of any order and

$E(L^-(t)) = t \int_{0<|z|\leq 1} z^2 dn(z)$ without any new assumption.

Theorem II.2.6: If $\int_{|z|>0} \frac{z^2}{1+z^2}\, dn(z) < \infty$ and if $\int_{\mathbb{R}} z^2 dn(z) < \infty$, the solution X of the equation $dX + aX\, dt = dL$ with $a > 0$, is a process with characteristic function ϕ_X satisfying $\forall u,\ t \in \mathbb{R}$

$$\phi_{X(t)} \overset{\Delta}{=} E(e^{iu\, X(t)}) = \exp \psi_{X(t)}(u) \text{ with}$$

$$\psi_{X(t)}(u) = iu\frac{\gamma}{a} - \frac{1}{2}\frac{u^2\delta^2}{2a} + \int_{-\infty}^{+}\int_{|z|>0} [\exp iuze^{-a(t-\tau)} - 1 - \frac{iuz}{1+z^2}\, e^{-a(t-\tau)}]dn(z)d\gamma$$

where γ and δ are two constants.

Proof: By theorem II.1.4 for every t of $\mathbb{R}$

$$L(t) = \gamma + \delta\, W(t) + L_1(t)$$

with the hypothesis $\int_{\mathbb{R}} z^2 dn(z) < \infty$, the process L_1 is second order and $L_1(t) = L_c(t) + t \int_{|z|>0} \frac{z}{1+z^2}\, dn(z)$. Then the solution X satisfies

$$X(t) = \frac{\gamma}{a} + \delta \int_{-\infty}^{t} e^{-a(t-\tau)} dW(\tau) + \int_{-\infty}^{t} e^{-a(t-\tau)} dL_c(\tau)$$

$$+ \int_{-\infty}^{t} e^{-a(t-\tau)} d\tau \int_{|z|>0} \frac{z}{1+z^2}\, dn(z)$$

The four processes of the second member are independent. The characteristic function of each one is known then

$$\psi_{X(t)}(u) = \frac{iu\gamma}{a} - \frac{u^2\delta^2}{2}\, \frac{1}{2a}$$

$$+\int_{-\infty}^{t}\int_{|z|>0} [\exp iuze^{-a(t-\tau)}] - iuze^{-a(t-\tau)}]dn(z)d\tau$$

$$+ \int_{-\infty}^{t} \int_{|z|>0} iue^{-a(t-\tau)} \frac{z}{1+z^2} dn(z)d\tau$$ □

A random variable Y has indefinitely divisible law if its characteristic function satisfies ∀ $\mathbb{R}$

$$\log \phi(u) = i\ u\ m - \frac{u^2}{2}\sigma^2 + \int_{|z|>0} (e^{iuz}-1 - \frac{iuz}{1+z^2})d_z\sum(z)$$

where m,σ are constants and where the nondecreasing function $\sum$ is such that $\lim_{z\to-\infty} \sum(z) = 0$, $\lim_{z\to+\infty} \sum(z) = 0$, $\int_{-1}^{0-} z^2 d\sum(z) < \infty$ and $\int_{0+}^{1} z^2 d\sum(z) < \infty$ (B.V. Gnedenko and A.N. Kolmogorov [8]). For every t of $\mathbb{R}$ the random variables L(t) is indefinitely divisible.

III. SOLUTION OF THE EQUATION DX = dL IN STRONG DISTRIBUTION MEANING

The solution is known with forcing term L, a Gaussian measure (D. de Brucq and C. Olivier [5]). It is not difficult to take for L a Poisson measure defined in J. Neveu [9]. We generalize as much as possible introducing $\mathbb{L}$-measure, we should say Levy-measure but the expression is used with other acceptance (Proposition II.1.2).

We consider the space $L^2(\mathbb{R}^n, \mathcal{R}^n, d\lambda)$ where λ is a positive measure on $(\mathbb{R}^n, \mathcal{R}^n)$. The Lebesgue measure will be denoted dλ = dτ. Fourier transform is used in the theory and the space $L^2(\mathbb{R}^n, \mathcal{R}^n, d\lambda)$ is composed of functions with complex values.

<u>Definition III.1.1</u>: Let n be a positive measure on $(\mathbb{R}, \mathcal{R})$ with $\int z^2 dn(z) < \infty$. A $\mathbb{L}$-measure is a process

$$\mathbb{L} = (\Omega, A, P_a, (X(f))_{L^2(\mathbb{R}^n, \mathcal{R}^n, d\lambda)}, \mathbb{C}, C)$$

such as

(a) $f \in L^2(\mathbb{R}^n, R^n, d\lambda) \to \mathbb{L}(f) \in L^2(\Omega, A, P_a)$ is linear

(b) $\psi(f) \overset{\Delta}{=} E(e^{i\,\mathbb{L}(f)}) = \int_{\mathbb{R}^n}\int_{|z|>0} [\exp i\, z\, \overline{f}(\tau) - 1 - i\, z\overline{f}(z)]\,dn(z)\,d\lambda(\tau)$

The characteristic function $\phi_{\mathbb{L}}$ of the process $\mathbb{L}$ is given by b.

Lemma III.1.2: The complex function $f \quad L^2(\mathbb{R}^n, R^n, d\lambda) \to \psi(f)$ is continuous.

Proof: We have

$$|\psi(f)| \le |u|^2 \int_{\mathbb{R}^n} |f(\tau)|^2 d\lambda(\tau) \int_{|z|>0} |z|^2 dn(z)$$

Proposition III.1.3: The $\mathbb{L}$-measure is centered and the covariance Γ is the function $\forall f, g \in L^2(\mathbb{R}^n, R^n, d\lambda)$

$$\Gamma(f,g) = E(\mathbb{L}(f)\,\overline{\mathbb{L}(g)}) = \int_{|z|>0} z^2 dn(z) \int \overline{f(\tau)}\, g(\tau)\, d\lambda(\tau)$$

Proof: $\psi(u\ f)$ has derivatives at first and second order in $u \quad \mathbb{R}$ and

$$\psi(u\ f) = -\frac{1}{2}u^2 \int_{|z|>0} z^2 dn(z) \int_{\mathbb{R}^n} |f(\tau)|^2 d\lambda(\tau) + o(u^2)$$

Then $E(\mathbb{L}(f)) = 0$ and

$$\mathbb{E}(|\mathbb{L}(f)|^2) = \int_{|z|>0} z^2 dn(z) \int_{\mathbb{R}^n} |f(\tau)|^2 d\lambda(\tau)$$

with the linearity of $\mathbb{L}$, we obtain the covariance Γ.

III.2 Equivalent $\mathbb{L}$-Measure on $(S', \underline{\underline{S}}')$

We use a theorem for cylindrical measure (A. Badrikian [1]). We restrict the $\mathbb{L}$-process to the nuclear and countably semi-normed space S, dense sub-space of $L^2(\mathbb{R}^n, R^n, d\tau)$, here we consider the Lebesgue-measure $d\lambda(\tau) = d\tau$. The characteristic function $\phi_{\mathbb{L}} = \exp \psi$ of the linear process $\mathbb{L}$ is continuous in $t = 0$. Then exists a probability P on $(S', \underline{\underline{S}}')$ such as $\mathbb{L}$ is equivalent to the process

$$(S', \underline{\underline{S}}', P, (\langle \cdot, f \rangle)_S, ¢, S)$$

for every t of S the random variable is $T \in S' \to \langle T, f \rangle$ where $\langle , \rangle$ is the duality between S and S', its topological dual. As $\mathbb{L}(f)$ and $<\cdot,f>$ have the same probability law, $E_p(<\cdot,f>) = 0$ and $\forall f,g \in S$

$$E_p(<\cdot,f>\overline{<\cdot,g>}) \triangleq \Gamma(f,g) = \int_{|z|>0} z^2 dn(z) \int_{\mathbb{R}^n} \overline{f}(\tau)g(\tau)d\tau$$

The application $f \in S \to <\cdot,f> \in L^2(S',\underline{\underline{S}}',P)$ is linear and continuous for the topology of $L^2(\mathbb{R}^n,R^n,d\tau)$. We denote $\int \overline{f}dL$ the unique extension of this application from the Hilbert space $L^2(\mathbb{R}^n,R^n,d\tau)$ to the Hilbert space $L^2(S',\underline{\underline{S}}',P)$. For Borelian B of R^n, the application $1_B \in L^2(\mathbb{R}^n,R^n,d\tau) \to \int 1_B dL \in L^2(S',\underline{\underline{S}}',P)$ is a vectorial measure.

We suppose now that the $\mathbb{L}$-measure is the process

$$\mathbb{L} = (S',\underline{\underline{S}}',P, (\int \overline{f}dL)_S, \mathbb{C}\ C).$$

Instead of a general probability space (Ω,A,P_a) the continuity of $\phi_{\mathbb{L}}$ on S and the theorem on cylindrical measures, specify (Ω,A,P_a) into $(S',\underline{\underline{S}}',P)$.

III.3 Expression of the Solution X

The Fourier transform is defined in the introduction. We consider equation $DX = dT$ in distribution meaning $\forall\phi \in S$ $<DX,\phi> = <T,\phi>$

We limit D to be linear operators such as

(a) $F\cdot D = AF$

(b) A has derivatives of any orders

(c) multiplication by A is a linear operator on S

(d) the closed set $F = \{\tau \in \mathbb{R}^n ; A(\tau)=0\}$ is Lebesgue almost surely null.

Let $0 = \{\tau \in \mathbb{R}^n ; A(\tau) \neq 0\}$ the complement of F and let $T \triangleq F(\mathcal{D}(0))$ where $\mathcal{D}(0)$ is the distribution space of the functions with derivatives of any order and with compact supports in 0. We observe that T is dense in $L^2(\mathbb{R}^n,R^n,d\tau)$. We introduce the definition (D. De Brucq and C. Olivier [5]).

Definition III.3.1: A solution X of $DX = dT$ in strong distribution meaning, is a process

$$X = (S', \underline{\underline{S}}', P, (X(f))_T\ \mathbb{C}, C) \text{ such as}$$

$$\forall f \in T\ DX(f) = \langle T, f\rangle\ P \text{ - almost surely.}$$

We obtain the very general theorem

Theorem III.3.2: For any probability P on $(S', \underline{\underline{S}}')$ the solution X in strong distribution meaning of

$$DX = dT$$

with D satisfying properties a), b), c), d) is linear and given by

$$\forall f \in T\ X(f) = \langle T, F\left(\frac{\bar{F} f}{A}\right)\rangle$$

Proof: For f in $T = F\cdot(\mathcal{D}(0))$, the function $F\left(\frac{\bar{F}f}{A}\right)$ is also in T dense subspace of S. Then $\forall T \in S'$ $\langle T, F\left(\frac{F\cdot f}{A}\right)\rangle$ is defined. Moreover we note D* the adjoint of D, then

$$DX(f) \overset{\Delta}{=} X(D^*f) = \langle T, F\left(\frac{\bar{F}D^*f}{A}\right)\rangle$$

$$= \langle T, F\frac{A\bar{F}f}{A}\rangle = \langle T, T\bar{F}f\rangle = \langle T,f\rangle$$

When the process

$$(S', \underline{\underline{S}}', P, (\langle\cdot,f\rangle)_T, \mathbb{C}, C\rangle \text{ is additive}$$

we note $\langle T,f\rangle = \int f dT$. With $\mathbb{L}$-measure, the characteristic function of the solution X is known:

Theorem III.3.3: The characteristic function ψ_X of the solution

$$X = (S', \underline{\underline{S}}', P, (\int f dL)_T, \mathbb{C}, C)$$

of the equation DX = dL with the prior hypothesis on the linear operator D and on the $\mathbb{L}$-measure, is given by

$$\log \psi_{X(f)} = \int_{\mathbb{R}^n} \int_{|z|>0} [\exp i\, zF\left(\frac{\bar{F}f}{A}\right) - 1 - iz\, F\left(\frac{\bar{F}f}{A}\right)] dn(z) dt\ .$$

Proof: The solution X is

$$\forall f \in T\ X(f) = \int \overline{F}\left(\frac{Ff}{\overline{A}}\right) dL = L\left[F\left(\frac{Ff}{A}\right)\right]$$

with $F\left(\frac{Ff}{A}\right)$ in S and we apply property b of Definition III.1.1. □

Corollary III.3.4: The covariance Γ of the solution X is given by $\forall f,\ g \in T$

$$\Gamma(f,g) = E(X(f)\overline{X(g)}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{Ff\ \overline{Fg}}{|A|^2} d\tau \int_{|z|>0} z^2 dn(z)$$

Proof: $\Gamma(f,g) = E\left[\int F\left[\frac{Ff}{\overline{A}}\right] dL\ \overline{\int F\left[\frac{Fg}{\overline{A}}\right] dL}\right]$

$$= \int_{\mathbb{R}^n} \overline{F}\left[\frac{Ff}{\overline{A}}\right] \overline{\overline{F}\left[\frac{Fg}{\overline{A}}\right]} d\tau \int_{|z|>0} z^2 dn(z) \text{ from Proposition III.1.2}$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{Ff}{\overline{A}}\ \overline{\frac{Fg}{\overline{A}}}\ d\tau \int_{|z|>0} z^2 dn(z) \text{ from Parseval's theorem.}$$ □

The covariance Γ gives the power spectral density function

$$s(\tau) = \frac{1}{(2\pi)^n} \frac{\int_{|z|>0} z^2 dn(z)}{|A(\tau)|^2}$$

An application in the spatio-temporal space $\mathbb{R}^4$ shows how to use this result. We define the spatial correlation at a given frequency ν; in this space, the power spectral density function takes the form

$$s(\omega,\xi,\eta,\zeta) = \frac{1}{(2\pi)^n} \frac{\int_{|h|>0} h^2 dn(h)}{|A(\omega,\xi,\eta,\zeta)|^2}$$

with the Fourier transform

$$F(f)(\omega,\xi,\eta,\zeta) \triangleq \int_{\mathbb{R}^4} e^{i(-\omega t+\xi x+\eta y+\zeta z)} f(t,x,y,z)\,dt\,dx\,dy\,dz\ .$$

Definition III.3.5: For the solution X of the spatio-temporal equation $DX = dL$, the covariance function at a given frequency $\nu = \frac{\omega}{2\pi}$ is the function

$$s(\omega,x,y,z) = \frac{1}{(2\pi)^4} \int_{|h|>0} h^2 dn(h) \int_{\mathbb{R}^3} \frac{e^{i(x\xi+y\eta+z\zeta)}}{|A(\omega,\xi,\eta,\zeta)|^2} d\xi d\eta d\zeta$$

We note $c \triangleq \frac{\int_{|h|>0} h^2 dn(h)}{(2\pi)^2}$ a normalization factor. We consider the linear partial derivative operator

$$D \triangleq \sum_{k=0}^{p} \Delta^k \left(\sum_j \alpha_{k,j} \frac{\partial^j}{\partial t^j}\right) \text{ with } p \in \mathbb{N} \text{ and } \alpha_{k,j} \in \mathbb{C} .$$

We obtained $\mathcal{F}D = A\mathcal{F}$ with

$$A = \sum_{k=0}^{p} [(-1)^k (\xi^2+\eta^2+\zeta^2)^k \sum_j \alpha_{k,j} (-i)^j \omega^j] .$$

Properties a), b), c), d) for D are checked easily. We observe that A is invariant by rotations of $\mathbb{R}^3$, and is function of $\kappa^2 \triangleq \xi^2+\eta^2+\zeta^2$. Then $A(\omega,x) = \sum_{k=0}^{p} (-1)^k \kappa^{2k} [\sum_j \alpha_{k,j} (-i)^j w^j]$ is algebraic in κ with 2p complex roots $\lambda_k(\omega)$, $-\lambda_k(\omega)$, k=1,2,...,p and $A(\omega,\kappa) = I_p \prod_{k=1}^{p} (\kappa^2 - \lambda_k^2)$ with $I_p(\omega) \triangleq (-1)^p \sum_j \alpha_{p,j} (-1)^j \omega^j$

In that case, it is possible to perform the integrations that appear in the expression of the function s.

Theorem III.3.7: In the spatio-temporal space $\mathbb{R}^4$, the covariance function s at a given frequency of the solution X of the equation $DX = dL$ with

$$D = \sum_{k=0}^{p} \Delta^k \left(\sum_j \alpha_{k,j} \frac{\partial^j}{\partial t^j}\right) \; p \in \mathbb{N} \; \alpha_{k,j} \in \mathbb{C}$$

is equal to

$$s(\omega,\ell) = \frac{c}{2|1_p|^2} \sum_{k=0}^{p} \frac{e^{-b_k\ell}}{b_k\ell} \frac{1}{2a_k i} \left[\frac{e^{ia_k\ell}}{d_k} - \frac{e^{-ia_k\ell}}{\bar{d}_k} \right]$$

where a) $\ell^2 = x^2+y^2+z^2$

b) $\lambda_k = a_k - ib_k$ $a_k \geq 0$ $b_k > 0$ and $|A(\omega,\kappa)|^2 = |1_p|^2 \prod_{k=1}^{p} (\kappa^2-\lambda_k^2)(\kappa^2-\bar{\lambda}_k^2)$

$d_k = \prod_{\substack{j=1 \\ j\neq k}}^{p} (\bar{\lambda}_k^2 - \bar{\lambda}_j^2)(\bar{\lambda}_k^2 - \lambda_j^2)$ if $p \neq 1$ and $d_1 = 1$.

We have supposed the roots λ_k in the complex plane with b_k strictly positive; it is the analog of $a > 0$ in Proposition II.2.4.

Proof: The theorem is known for p=1 (D. de Brucq [4]). We have to compute

$$s(\omega,x,y,z) = c \int_{\mathbb{R}} \frac{e^{i(x\xi+y\eta+z\zeta)}}{|A(\omega,\kappa)|^2} d\xi d\eta d\zeta$$

with $|A(\omega,\kappa)|^2 = |1_p|^2 \prod_{k=1}^{p} (\kappa^2-\lambda_k)(\kappa^2-\bar{\lambda}_k^2)$. We used spherical coordinates and (I.I. Gihman and A.V. Skorohod [7])

$$s(\omega,x,y,z) = c \frac{1}{\pi i\ell} \int_{-\infty}^{+\infty} e^{i\kappa\ell} \frac{\kappa}{|A(\omega,\kappa)|^2} d\chi .$$

Let $f(\kappa) \triangleq \dfrac{\kappa\, e^{i\kappa\ell}}{\prod_{k=1}^{p} (\kappa^2-\lambda_k)^2(\kappa^2-\bar{\lambda}_k^2)}$ and $J \triangleq \int_{-\infty}^{+\infty} f(\kappa)d\kappa$

we perform the integration using residual method. The p pôles are strictly complex by hypothesis and $\bar{\lambda}_k = a_k+ib_k$, $-\lambda_k = -a_k+ib_k$ $k=1,2,\ldots,p$ are the poles in the upper half plane. Then

$$s = 2i\pi \sum_{k=1}^{p} [\text{Res}(f,\bar{\lambda}_k)+\text{Res}(f,-\lambda_k)].$$ We have

$$\mathrm{Res}(f,\bar\lambda_k) = \frac{\bar\lambda_k e^{i\bar\lambda_k \ell}}{\prod_{\substack{\ell=1\\ \ell\neq k}}^{p} (\bar\lambda_k^2 - \bar\lambda_\ell^2)(\bar\lambda_k^2 - \lambda_\ell^2)(\bar\lambda_k^2 - \lambda_k^2)} \times \frac{1}{2\bar\lambda_k}$$

$$= \frac{e^{i\lambda_k \ell}}{\prod_{\substack{\ell=1\\ \ell\neq k}}^{p} (\bar\lambda_k^2 - \bar\lambda_\ell^2)(\lambda_k^2 - \lambda_\ell^2)} \times \frac{1}{8ia_k b_k} \text{ as } \bar\lambda_k^2 - \lambda_k^2 = 4ia_k b_k$$

$$= \frac{e^{i\lambda_k \ell}}{d_k} \frac{1}{8a_k b_k} \text{ and also}$$

$$\mathrm{Res}(f, -\lambda_k) = \frac{e^{-i\lambda_k \ell}}{\bar d_k} \frac{1}{-8ia_k b_k}$$

Going back to the expression of s, we find

$$s(\omega,x,y,z) = c \frac{1}{\Pi i L} \frac{1}{|I_p|^2} J$$

$$= \frac{c}{|I_p|^2} \frac{1}{\Pi i \ell} 2i\Pi \sum_{k=1}^{p} \frac{1}{8ia_k b_k} \left[\frac{e^{i\bar\lambda_k \ell}}{d_k} - \frac{e^{-i\lambda_k \ell}}{\bar d_k} \right].$$

For Gaussian measures, for Poisson measure, results of III.1 and III.2 are valid and we have equivalent processes on $(S', \underline{\underline{S}}', P)$. These processes are linear and for every f in $L^2(\mathbb{R}^n, R^n, d\tau)$ the characteristic functions are given by

$$\log E(e^{i\omega(f)}) = -\tfrac{1}{2} \int_{\mathbb{R}^n} |f|^2 d\tau \text{ and } \log E(e^{iN(f)}) = \int_{\mathbb{R}^n} (\exp if - 1 - if) d\tau$$

These centered processes and the $\mathbb{L}$-measures have the same correlation function. Every forcing terms of these types, give solution X with the same second order properties! Moments of greater order are necessary to make differences between the forcing terms. For p = 1

$$s(\omega,\ell) = \frac{c}{2|I_1|^2} \frac{e^{-\gamma\ell}}{\gamma} \frac{\sin k\ell}{k\ell} \quad \text{where } \lambda_1 = k-i\gamma \text{ is the complex wave}$$

number. Direct verification in deep sea water of this formula is factible:

$k(\omega)$ is the wave number at pulsation ω,

$\gamma(\omega)$ is the dumping term at pulsation ω.

For two points at a distance ℓ, the correlation of the filtered observations at frequency ν of the pressure p is the function $s(\omega,\ell)$. The main assumption to obtain the result is that p satisfies any equation

$$\Delta \sum \alpha_{1,j} \frac{\partial^j}{\partial t^j} p + \sum \alpha_{0,j} \frac{\partial^j}{\partial t^j} p = dL$$

with L Gaussian, Poisson, or any centered Levy-measure!

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