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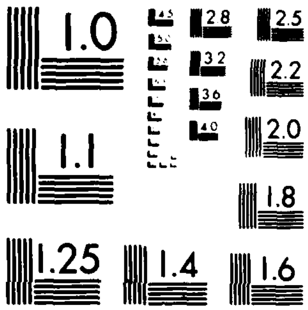

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DIFFERENTIAL GAMES, OPTIMAL CONTROL AND DIRECTIONAL DERIVATIVES OF VISCOSITY SOLUTIONS OF BELLMAN'S and ISAACS' EQUATIONS

P. L. Lions and P. E. Souganidis

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

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ABSTRACT

Recent work by the authors and others has demonstrated the connections between the dynamic programming approach to optimal control theory and to two-person, zero-sum differential games problems and the new notion of "viscosity" solutions of Hamilton-Jacobi PDE's introduced by M. G. Crandall and P. L. Lions. In particular, it has been proved that the dynamic programming principle implies that the value function is the viscosity solution of the associated Hamilton-Jacobi-Bellman and Isaacs equations. In the present work, it is shown that viscosity super- and subsolutions of these equations must satisfy some inequalities called super- and subdynamic programming principle respectively. This is then used to prove the equivalence between the notion of viscosity solutions and the conditions, introduced by A. Subbotin, concerning the sign of certain generalized directional derivatives.

AMS (MOS) Subject Classifications: 35F30, 35L60, 90D25, 49C20

Key Words: Differential games, optimal control, Hamilton-Jacobi equations, directional derivatives, viscosity solutions

Work Unit Number 1 (Applied Analysis)

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[1]Université Paris IX-Dauphine, Place de Lattre de Tassigny, 75775, Paris, Cedex 16, France.

[2]Lefschetz Center for Dynamical Systems and Division of Applied Mathematics, Brown University, Providence, Rhode Island 02912. This work was supported in part by National Science Foundation Grant No. MCS-8002946.

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DIFFERENTIAL GAMES, OPTIMAL CONTROL AND DIRECTIONAL DERIVATIVES OF VISCOSITY SOLUTIONS OF BELLMAN'S AND ISAACS' EQUATIONS

P. L. Lions[1] and P. E. Souganidis[2]

INTRODUCTION

Recent work by the authors and others has demonstrated the connections between the dynamic programming approach to optimal control theory problems and to two-person, zero-sum differential games and the new notion of "viscosity" solutions of Hamilton-Jacobi partial differential equations introduced by M. G. Crandall and P. L. Lions [6].

The formal relationships here are (cf. W. H. Fleming and R. Rishel [15], R. Isaacs [18]): if the values of various optimal control problems and differential games are regular, then they solve certain first order partial differential equations with "min", "max", "max-min" or "min-max" type nonlinearity. The problem is that usually the value functions are not smooth enough to make sense of the above in any obvious way. Many papers in the subject have worked around this difficulty: see Fleming [12], [13], [14], Friedman [15], [16], Elliott-Kalton [8], [9], Krassovski-Subbotin [20], Subbotin [28], etc.

Recently, however, the new notion of "viscosity" solution for first order partial differential equations was introduced by M. G. Crandall and P. L. Lions [6]. (Also see M. G. Crandall, L. C. Evans and P. L. Lions [5]). This solution was proved to be unique under some very general assumptions. Moreover, it was observed by P. L. Lions [21] that the dynamic programming condition for the value in control theory problems implies that this value is the viscosity solution of the associated Hamilton-Jacobi-Bellman partial

[1]Université Paris IX-Dauphine, Place de Lattre de Tassigny, 75775, Paris, Cedex 16, France.

[2]Lefschetz Center for Dynamical Systems and Division of Applied Mathematics, Brown University, Providence, Rhode Island 02912. This work was supported in part by National Science Foundation Grant No. MCS-8002946.

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differential equation. These considerations extend to the theory of differential games. It follows, in particular, that the dynamic programming conditions imply that the values are viscosity solutions of the associated Hamilton-Jacobi-Isaacs partial differential equations. See P. Souganidis [26], [27] for a proof of this based on both the Fleming and the Friedman definitions of upper and lower values for a differential game, N. Barron, L. C. Evans and R. Jensen [1] for a different proof for the Friedman definition, L. C. Evans and P. Souganidis [11] for the Elliott-Kalton values in $\mathbb{R}^N$, L. C. Evans and H. Ishii [10] for the Elliott-Kalton values in bounded domains. Some related papers are: P. L. Lions [23], P. L. Lions and M. Nisio [25], I. Capuzzo Dolcetta and L. C. Evans [3], I. Capuzzo Dolcetta [2], I. Capuzzo Dolcetta and H. Ishii [4], H. Ishii [19], etc.

The present paper is concerned with the relation between the notion of viscosity sub- and supersolutions of the Hamilton-Jacobi-Bellman and Isaacs equations and the sign which must be assumed at every point by certain generalized directional derivatives. In particular, we show that super- and subsolutions of Hamilton-Jacobi-Bellman and Isaacs equations satisfy certain inequalities which are related to the optimality principle of dynamic programming. Under some assumptions this implies a particular sign for certain generalized directional derivatives. Finally, this sign suffices to characterize functions as viscosity super- and subsolutions of the appropriate equations. This is motivated by a work of A. Subbotin [28]. In particular, Subbotin gives a necessary and sufficient condition for a function to be the value of a differential game. This condition, which is not within the context of the viscosity solution, roughly says that at every point certain generalized derivatives must have a particular sign. L. C. Evans and H. Ishii [10], using a "blow-up" argument, showed that the value of an infinite horizon control problem satisfies Subbotin's condition, as it applies to control problems. The techniques used here are different than the ones in [10]. One direction of the equivalence claimed above is straightforward. The other is closely related to the principle of dynamic programming and requires some arguments of P. L. Lions [22], [24], which treat optimal control problems of diffusion processes.

The paper is organized as follows: The rest of the introduction recalls the definition of the viscosity solution. Section 1 is devoted to optimal control problems. Section 2 deals with differential games. In the Appendix we make some observations concerning the existence of directional derivatives of the value function. All the definitions and results from other papers are recalled when necessary.

We conclude the introduction with the definition of viscosity solutions.

Definition 0.1 [6]. Let $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $z : \partial\Omega \to \mathbb{R}$ be continuous functions, where Ω is an open subset of $\mathbb{R}^N$. A continuous function $u : \bar{\Omega} \to \mathbb{R}$ is a viscosity subsolution of

$$(0.1)\quad \begin{cases} H(x,u,Du) = 0 & \text{in } \Omega \\ u(x) = z(x) & \text{on } \partial\Omega \end{cases}$$

if $u(x) \leq z(x)$ on $\partial\Omega$ and, moreover, for every $\phi \in C^\infty(\Omega)^{(*)}$, if $x_0 \in \Omega$ is a local max of $u - \varphi$, then

$$(0.2)\quad H(x_0,u(x_0),D\varphi(x_0)) \leq 0$$

A continuous function $u : \bar{\Omega} \to \mathbb{R}$ is a viscosity supersolution of (1.1), if $u(x) \geq z(x)$ on $\partial\Omega$ and, moreover, for every $\phi \in C^\infty(\Omega)$, if $x_0 \in \Omega$ is a local min of $u - \varphi$, then

(*) $C^\infty_{(0)}(\mathcal{O})$ denotes the set of real valued infinitely many times continuously differentiable functions (of compact support) defined on $\mathcal{O}$.

$$(0.3) \quad H(x_0,u(x_0),D\phi(x_0)) \geq 0$$

A continuous function $u : \bar{\Omega} \to \mathbb{R}$ is a viscosity solution of (1.1), if it is both sub- and supersolution of (1.1).

<u>Definition 0.2 ([6])</u>. Let $H : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $z : \partial\Omega \times [0,T] \to \mathbb{R}$ and $u_0 : \bar{\Omega} \to \mathbb{R}$ be continuous functions, where Ω is an open subset of $\mathbb{R}^N$. A viscosity subsolution (respectively, supersolution, solution) of

$$(0.4) \begin{cases} \frac{\partial u}{\partial t} + H(t,x,u,Du) = 0 & \text{in } \Omega \times (0,T] \\ u(x,t) = z(x,t) & \text{on } \partial\Omega \times (0,T] \\ u(x,0) = u_0(x) & \text{on } \Omega \end{cases}$$

is a function $u \in C(\bar{Q}_T)^{(*),(**)}$ such that:

$$(0.5) \quad u \leq z \text{ on } \partial\Omega \times [0,T], \; u(x,0) \leq u_0(x) \text{ in } \Omega$$

(respectively,

$$(0.6) \quad u \geq z \text{ on } \partial\Omega \times [0,T], \; u(x,0) \geq u_0(x) \text{ in } \Omega$$

respectively (0.5) and (0.6)).

and, for every $\phi \in C^\infty(Q_T)$,

$$(0.7) \begin{cases} \text{if } (x_0,t_0) \in Q_T \text{ is a local max of } u - \varphi, \text{ then} \\ \frac{\partial \phi}{\partial t}(x_0,t_0) + H(t_0,x_0,u(x_0,t_0),D\phi(x_0,t_0)) \leq 0 \end{cases}$$

(*) $C(\mathcal{O})$ is the set of continuous real valued functions defined on $\mathcal{O}$.

(**) $Q_T = \Omega \times (0,T]$, $\bar{Q}_T = \bar{\Omega} \times [0,T]$.

(respectively,

$$(0.8)\quad \begin{cases} \text{if } (x_0,t_0) \in Q_T \text{ is a local min of } u-\varphi, \text{ then} \\ \frac{\partial\varphi}{\partial t}(x_0,t_0) + H(t_0,x_0,u(x_0,t_0),D\phi(x_0,t_0)) \geq 0 \end{cases}$$

respectively (0.7) and (0.8)).

For a detailed account of the recent developments in the theory of viscosity solutions as well as references, we refer to the book by P. L. Lions [21] and the article by M. G. Crandall and P. Souganidis [7].

SECTION 1

In this section we consider Hamilton-Jacobi-Bellman equations associated with optimal control problems. In particular, we look at problems of the form

$$(1.1)\quad \begin{cases} u + \sup_{y\in Y} \{-f(x,y)\cdot Du - \ell(x,y)\} = 0 \quad \text{in } \Omega \\ u = g \quad \text{on } \partial\Omega \end{cases}$$

where Ω is an open subset of $\mathbb{R}^N$, $g \in C(\partial\Omega)$, Y is some separable metric space and $f : \overline{\Omega} \times Y \to \mathbb{R}^N$, $\ell(x,y) : \overline{\Omega} \times Y \to \mathbb{R}$ are continuous functions such that

$$(1.2)\quad \begin{cases} \text{There exists a constant } C > 0 \text{ such that} \\ |f(x,y)|, |\ell(x,y)| \leq C \text{ for every } (x,y) \in \overline{\Omega} \times Y \\ \text{and} \\ |f(x,y)-f(\hat{x},y)|, |\ell(x,y)-\ell(\hat{x},y)| \leq C|x-\hat{x}| \\ \text{for every } (x,\hat{x},y) \in \overline{\Omega}\times\overline{\Omega} \times Y \end{cases}$$

(1.1) corresponds to an infinite horizon control problem (for the details we refer to [21] and W. Fleming and R. Rishel [15]) with dynamics given by

$$(1.3)\qquad \begin{cases} \dfrac{dx}{d\tau} = f(x(\tau), y(\tau)) & \text{for } 0 < \tau \\ x(0) = x \in \Omega \end{cases}$$

where $y : [0,\infty) \to Y$ is measurable. For notational simplicity in what follows let

$$(1.4)\qquad M = \{y : [0,\infty) \to Y,\ y(\cdot) \text{ measurable}\}.$$

Let u be the unique viscosity solution of (1.1) if it exists. It is known (P. L. Lions [21], I. Capuzzo Dolcetta and L. C. Evans [3], L. C. Evans and H. Ishii [10]) that u satisfies the optimality principle of dynamic programming, that is

$$(1.5)\qquad \begin{cases} u(x) = \inf_M \{e^{-(t\wedge t_x)} u(x(t\wedge t_x)) + \int_0^{t\wedge t_x} e^{-s} \ell(x(s), y(s))\, ds\} \text{ for} \\ \text{every } t > 0 \text{ and } x \in \Omega^{(*)} \end{cases}$$

where, for $x \in \Omega$ and $y \in M$, $t_x = t_x(y)$ is the exit time from Ω of the solution of (1.3) for the particular x,y, i.e.,

$$(1.6)\qquad t_x = \inf\{t > 0 : x(t) \in \mathbb{R}^N - \Omega\}$$

The first result of this section concerns viscosity supersolutions of (1.1). In particular, we show that every viscosity supersolution of (1.1) satisfies some inequality, which, in view of (1.5), may be called superoptimality principle of dynamic programming. This was first proved by P.L. Lions in [22], [24], in the general context of optimal stochastic control. Here we give two proofs related to those given in [22], [24], but slightly more adapted

(*) $r \wedge s = \min\{r,s\}$

to the special situation at hand. The first proof uses the fact that a viscosity supersolution of (1.1) is a viscosity supersolution of an appropriately defined time dependent problem. The second proof is based on the fact that a viscosity supersolution of (1.1) is a viscosity solution of an obstacle problem, which can be solved easily using differential games. The first step in both proofs is a localization argument introduced in [24]. It consists of multiplying by suitable cut-off function. This allows us to reduce to the case $\Omega = \mathbb{R}^N$.

We have

Proposition 1.1. Let $v \in C(\bar{\Omega})$ be a viscosity supersolution of (1.1). Then, for every $t > 0$ and $x \in \Omega$, we have

$$\text{(1.6)} \qquad v(x) \geq \inf_M \{e^{-(t \wedge t_x)} v(x(t \wedge t_x)) + \int_0^{t \wedge t_x} e^{-s} \ell(x(x), y(s)) ds\}$$

Proof 1. The first step in the proof is to modify the problem so that it is defined in $\mathbb{R}^N$. To this end, for $\delta > 0$ let Ω_δ be defined by

$$\Omega_\delta = \{x \in \Omega : |x| < \frac{1}{\delta} \text{ and } \operatorname{dist}(x, \partial\Omega) > \delta\}$$

Moreover, choose $\xi, \varphi \in C^\infty(\mathbb{R}^N)$ such that: $0 \leq \xi \leq 1$ on $\bar{\Omega}_\delta$, $0 \leq \xi \leq 1$, $\xi \equiv 0$ on $\mathbb{R}^N \setminus \bar{\Omega}_{\delta/2}$, $\varphi \equiv 1$ on a neighborhood of $\operatorname{supp}\xi$, $0 \leq \varphi \leq 1$ and $\varphi \equiv 0$ on $\mathbb{R}^N \setminus \bar{\Omega}_{\delta/4}$. Then the function $\tilde{v} : \mathbb{R}^N \to \mathbb{R}$ which is defined by

$$\tilde{v}(x) = \begin{cases} (\varphi v)(x) & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

is a viscosity supersolution of the problem

$$\text{(1.7)} \qquad \xi(x)\tilde{u} + \sup_{y \in Y}\{-\xi(x) f(x,y) \cdot D\tilde{u} - \xi(x) \ell(x,y)\} = 0 \quad \text{in } \mathbb{R}^N.$$

Next for $t > 0$ fixed consider the initial value problem

$$(1.8)\quad \begin{cases} \frac{\partial w}{\partial s} + \sup_{y\in Y} \{-\xi(x) f(x,y)\cdot Dw - \xi(x)\,\ell(x,y)\} + \xi(x)w = 0 & \text{in } Q_t \\ w(x,0) = \tilde{v}(x) & \text{in } \mathbb{R}^N \end{cases}$$

In view of the results of [6] and [21], (1.8) has a unique viscosity solution $w \in C(\mathbb{R}^N \times [0,t])$ given by

$$(1.9)\quad w(x,s) = \inf_M \{v(\tilde{x}(s))e^{-\int_0^s \xi(\tilde{x}(\tau))d\tau} + \int_0^s e^{\int_0^\tau \xi(\tilde{x}(\lambda))d\lambda}\, \ell(\tilde{x}(\tau),y(\tau))d\tau\}$$

where $\tilde{x}(\cdot)$ is the solution of

$$\begin{cases} \frac{d\tilde{x}}{d\tau} = -\xi(\tilde{x}(\tau))f(\tilde{x}(\tau),y(\tau)) & \text{for } 0 < \tau < t \\ \tilde{x}(0) = x \end{cases}$$

$\tilde{v}$, however, is a viscosity supersolution of (1.8). The uniqueness estimates of [6] imply

$$\tilde{v}(x) \geq w(x,s) \quad \text{for every } (x,s) \in \mathbb{R}^N \times [0,t]$$

Next observe that for $x \in \Omega$ and $y \in M$, if $t < t_x$, then $\tilde{x}(s) = x(s)$ for $0 \leq s \leq t$, where $x(\cdot)$ is the solution of (1.3), provided that δ is sufficiently small. Moreover, $x(\cdot) \in \{x \in \Omega : \xi(x) = 1\}$. These observations together with (1.9) imply (1.6) for $t < t_x$. If $t \geq t_x$, choose $t_n \uparrow t_x$. Then

$$v(x) \geq \inf_{y\in M} \{v(x(t_n))e^{-t_n} + \int_0^{t_n} e^{-s}\, \ell(x(s),y(s))ds\}$$

As $n \to \infty$ we obtain (1.6), since $v \in C(\bar{\Omega})$.

Proof 2. (Obstacle problem method). Here, in order to exhibit the main ideas, for simplicity, we are going to assume $\Omega = \mathbb{R}^N$. The general case follows by appropriate use of the localization technique explained in Proof 1.

It is easy to see that v is the unique viscosity solution of the problem

$$v + \min\{\sup_{y\in Y}\{-f(x,y)\cdot Dv - \ell(x,y)\}, -v\} = 0 \quad \text{in } \mathbb{R}^N$$

which can be rewritten as

$$v + \min_{z\in Z}\sup_{y\in Y}\{-\tilde{f}(x,y,z)\cdot Dv - \tilde{h}(x,y,z)\} = 0 \quad \text{in } \mathbb{R}^N \tag{1.10}$$

with $Z = \{1,2\}$ and

$$\begin{cases} \tilde{f}(x,y,z) = \begin{cases} 0, & \text{if } z = 1 \\ f(x,y), & \text{if } z = 2 \end{cases} \\ \tilde{h}(x,y,z) = \begin{cases} v(x), & \text{if } z = 1 \\ -\ell(x,y), & \text{if } z = 2 \end{cases} \end{cases}$$

(1.10) corresponds to an infinite horizon differential game, thus v must satisfy the dynamic programming principle, as it is shown in the first part of L. C. Evans and H. Ishii [10].

We need some more notation. In particular, let

$$N = \{z : [0,\infty) \to Z,\ z(\cdot) \text{ measurable}\}$$

Moreover, denote by Γ the set of mappings $\alpha : N \to M$, which, for every $t > 0$, satisfy the following condition:

$$\begin{cases} \text{If } z(x) = \hat{z}(s) \text{ for a.e. } 0 \le s \le t, \text{ then} \\ \alpha[z](s) = \alpha[\hat{z}](s) \text{ for a.e. } 0 \le s \le t \end{cases}$$

In view of Theorem 3.1 of [10], for every $t > 0$, we obtain

$$(1.11) \qquad v(x) = \inf_{\alpha\in\Gamma} \sup_{z\in N} \{\int_0^t e^{-s}\tilde{h}(\tilde{x}(s),\alpha[z](s),z(s))ds + e^{-t}v(\tilde{x}(t))\}$$

where for $x \in \mathbb{R}^N$, $z \in N$ and $\alpha \in \Gamma$, $\tilde{x}(\cdot)$ is the unique solution of

$$\begin{cases} \frac{d\tilde{x}}{ds} = -\tilde{f}(\tilde{x}(s),\alpha[z](s),z(s)) \quad \text{for } 0 < s \\ \tilde{x}(0) = x \end{cases}$$

Choose $\bar{z} \in N$ such that $\bar{z} \equiv 2$. Then (1.11) implies

$$v(x) \ge \inf_{\alpha\in\Gamma} \{v(x(t))e^{-t} + \int_0^t e^{-\tau}\ell(x(\tau),\alpha[\bar{z}](\tau))d\tau\}.$$

since, in this case, $\tilde{x}(\cdot)$ is the solution of (1.3). But

$$\{\alpha[\bar{z}] : \alpha \in \Gamma\} \subset M$$

thus the result.

The next proposition deals with viscosity subsolutions of (1.1). In particular, we show that a viscosity subsolution of (1.1) satisfies an inequality, which we call the <u>suboptimality principle</u> of dynamic programming. The proof relies on the fact that viscosity subsolutions of (1.1) are viscosity subsolutions of an appropriately defined time dependent problem.

We have

<u>Proposition 1.2</u>. Let $w \in C(\bar{\Omega})$ be a viscosity subsolution of (1.1). Then, for every $x \in \mathbb{R}^N$ and $t \ge 0$, we have

(1.12) $$w(x) \leq \inf_M \{ e^{-(t\wedge t_x)} w(x(t\wedge t_x)) + \int_0^{(t\wedge t_x)} e^{-s} \ell(x(s),y(s))ds \}$$

Proof. Here we give the proof in the case $\Omega = \mathbb{R}^N$. For the general case, one has to use first a localization argument as in Proof 1 of Proposition 1.1.

For $t > 0$ consider the problem

$$\begin{cases} \frac{\partial z}{\partial s} + \max_{y\in Y} \{-f(x,y)\cdot Dz - \ell(x,y)\} + z = 0 & \text{in } Q_t \\ z(x,0) = w(x) & \text{in } \mathbb{R}^N. \end{cases}$$

w is a viscosity subsolution of this problem, therefore, for every $x \in \mathbb{R}^N$,

$$w(x) \leq z(x,t) = \inf_M \{w(x(t))e^{-t} + \int_0^t e^{-s} \ell(x(s),y(s))ds\}$$

The above are justified as in Proof 1 of Proposition 1.1.

Next we want to use Propositions 1.1 and 1.2 to obtain a kind of infinitesimal version of the super- and sub optimality principle of the dynamic programming, satisfied by viscosity super- and subsolutions of (1.1). To do this, we have to assume that we work with sub- and supersolutions which are locally Lipschitz. Moreover, we need to introduce some notation.

(1.13) $$(FL)(x) = \overline{co}\{(f(x,y),\ell(x,y)) : y \in Y\}$$

We have:

Theorem 1.3. Let $v \in C^{0,1}_{loc}(\Omega)$ (*) be a viscosity supersolution of (1.1). Then, for every $x \in \Omega$, it is:

(1.14) $$v(x) + \varliminf_{\delta\downarrow 0} \sup_{(f,\ell)\in(FL)(x)} \left\{ \frac{v(x)-v(x+\delta f)}{\delta} - \ell \right\} \geq 0$$

(*) $C^{0,1}_{loc}(\mathcal{O})$ denotes the set of real valued (locally) Lipschitz continuous functions defined on $\mathcal{O}$.

and the inequality is achieved as $\delta \downarrow 0$ uniformly on compact sets.

Proof: For $x \in \Omega$ fixed let K be the Lipschitz constant of v in a ball of radius C centered at x, where C is given by (1.2). Moreover, assume that $0 < \delta < 1$ is small enough so that $\delta < t_x$. In view of Proposition 1.1, we have:

$$\sup_M \left\{ \frac{v(x) - e^{-\delta} v(x + \int_0^\delta f(x(s),y(s))ds)}{\delta} - \frac{1}{\delta}\int_0^\delta e^{-s} \ell(x(s),y(s))ds \right\} \geq 0$$

Therefore

$$\frac{1-e^{-\delta}}{\delta} v(x) + e^{-\delta} \sup_M \left\{ \frac{v(x) - v(x + \delta \frac{1}{\delta}\int_0^\delta f(x,y(s))ds)}{\delta} - \frac{1}{\delta}\int_0^\delta \ell(x,y(s))ds \right\}$$

$$\geq -(K+1)C^2\delta - C\left(1 + \frac{e^{-\delta}-1}{\delta}\right)$$

But

$$\left(\frac{1}{\delta}\int_0^\delta f(x,y(x))ds,\ \frac{1}{\delta}\int_0^\delta \ell(x,y(s))ds\right) \in (FL)(x)$$

The above inequality implies

$$\frac{1-e^{-\delta}}{\delta} v(x) + e^{-\delta} \sup_{(f,\ell)\in(FL)(x)} \left\{ \frac{v(x) - v(x+\delta f)}{\delta} - \ell \right\} \geq -(K+1)C^2\delta - C\left(1 + \frac{e^{-\delta}-1}{\delta}\right)$$

Letting $\delta \downarrow 0$ we obtain the result. The uniformity claimed in the statement is an immediate consequence of the fact that the above also holds for every $y \in \mathbb{R}^N$ in a neighborhood of x of radius $C/2$.

As a consequence of Theorem 1.3 we have

Corollary 1.4. Let $v \in C^{0,1}_{loc}(\Omega)$ be a viscosity supersolution of (1.1). Then, for every $x \in \Omega$, it is:

$$(1.15) \qquad v(x) + \sup_{(f,\ell)\in(FL)(x)} \left\{ \overline{\lim_{\delta\downarrow 0}} \frac{v(x) - v(x+\delta f)}{\delta} - \ell \right\} \geq 0$$

and the inequality is achieved as $\delta \downarrow 0$ uniformly on compact sets.

Remark 1.5. In the second part of [10] L. C. Evans and H. Ishii proved that if $\{f(x,y) : y \in Y\}$ is convex and $\ell(x,y) \equiv 0$, then a locally Lipschitz viscosity supersolution of (1.1) satisfies

$$\inf_{y \in Y} \varliminf_{\delta \downarrow 0} \frac{e^{-\delta} v(x+\delta f) - v(x)}{\delta} \leq 0$$

which, under their assumptions, is equivalent to (1.15). As mentioned in the Introduction, they used a "blow-up" argument. The proof we give here is based completely on Proposition 1.1 and Theorem 1.3.

Proof of Corollary 1.4: (1.14) implies that there is a subsequence $\delta_k \downarrow 0$ as $k \to \infty$ such that

$$v(x) + \lim_{k \to \infty} \sup_{(f,\ell) \in (FL)(x)} \left\{ \frac{v(x) - v(x+\delta_k f)}{\delta_k} - \ell \right\} \geq 0$$

Then for $\varepsilon > 0$ fixed but arbitrary there is a $k_0 = k_0(\varepsilon) > 0$ such that

$$v(x) + \sup_{(f,\ell) \in (FL)(x)} \left\{ \frac{v(x) - v(x+\delta_k f)}{\delta_k} - \ell \right\} \geq -\varepsilon$$

for $k \geq k_0$.

Next for each $k \geq k_0$ choose $(f_k, \ell_k) \in (FL)(x)$ such that

$$\frac{v(x) - v(x+\delta_k f_k)}{\delta_k} - \ell_k = \sup_{(f,\ell) \in (FL)(x)} \left\{ \frac{v(x) - v(x+\delta_k f)}{\delta_k} - \ell \right\}$$

The compactness of $(FL)(x)$ implies that along some subsequence of $\delta_k \downarrow 0$ (which again for simplicity is denoted by δ_k) we have

$$(f_k, \ell_k) \to (f, \ell) \in (FL)(x)$$

This, together with the Lipschitz property of v, implies

$$v(x) + \overline{\lim_{k\to\infty}} \left\{ \frac{v(x)-v(x+\delta_k f)}{\delta_k} - \ell \right\} \geq 0$$

and, therefore, the result.

Remark 1.6. It is of some interest to know whether, in the case at hand, (1.15) holds with $\overline{\lim}$ replaced by $\underline{\lim}$.

For the case of a viscosity subsolution of (1.1) we have:

Theorem 1.7. Let $w \in C^{0,1}_{loc}(\Omega)$ be a viscosity subsolution of (1.1). Then, for every $x \in \Omega$, we have

$$(1.16) \qquad w(x) + \overline{\lim_{\delta\downarrow 0}} \sup_{(f,\ell)\in(FL)(x)} \left\{ \frac{w(x)-w(x+\delta f)}{\delta} - \ell \right\} \leq 0$$

and

$$(1.17) \qquad w(x) + \sup_{(f,\ell)\in(FL)(x)} \overline{\lim_{\delta\downarrow 0}} \left\{ \frac{w(x)-w(x+\delta f)}{\delta} - \ell \right\} \leq 0$$

and the inequality is achieved as $\delta\downarrow 0$ uniformly on compact sets.

Proof: (1.17) follows immediately from (1.16). To prove (1.16) observe that, in view of Proposition 1.2, we have:

$$\frac{w(x)-e^{-\delta}w(x+\int_0^\delta f(x(s),y(s))ds}{\delta} - \int_0^\delta e^{-s}\ell(x(s),y(s))ds \leq 0$$

and, therefore

$$(1.18) \qquad \frac{1-e^{-\delta}}{\delta} w(x) + e^{-\delta}\, \frac{w(x)-w(x+\delta\frac{1}{\delta}\int_0^\delta f(x,y(s))ds)}{\delta} - \frac{1}{\delta}\int_0^\delta \ell(x,y(s))ds$$

$$\leq (K+1)C^2\delta + C\,\left(1+\frac{e^{-\delta}-1}{\delta}\right)$$

for every $y \in M$, where K is the Lipschitz constant of w in the ball of radius C centered at x.

In view of the general geometrical fact

$$\{(\frac{1}{\delta}\int_0^\delta f(x,y(s))ds, \frac{1}{\delta}\int_0 \ell(x,y(s))ds) \ : \ y \in M\} = \overline{co}\{(f(x,y), \ell(x,y)) \ : \ y \in Y\}$$

(1.18) implies

$$\frac{1-e^{-\delta}}{\delta} w(x) + e^{-\delta} \sup_{(f,\ell)\in(FL)(X)} \left\{ \frac{w(x)-w(x+\delta f)}{\delta} - \ell \right\} \le 0(1)$$

where $0(1) \to 0$ as $\delta \downarrow 0$ and thus (1.16). The uniformity follows from the fact that all the above holdswith the same constants for all points in an appropriate neighborhood of x.

Combining Corollary 1.4 and Theorem 1.6 we obtain

Corollary 1.8. Let $u \in C^{0,1}_{loc}(\Omega)$ be a viscosity solution of (1.1). Then

$$(1.19) \qquad u(x) + \sup_{(f,\ell)\in(FL)(x)} \left\{ \overline{\lim_{\delta\downarrow 0}} \left\{ \frac{u(x)-u(x+\delta f)}{\delta} \right\} - \ell \right\} = 0, \ \forall x \in \Omega$$

We conclude this section with a result which is the inverse of Corollary 1.5 and Theorem 1.7. In particular, it says that (1.15) and (1.17) together with appropriate boundary conditions characterize continuous functions as viscosity super-respectively subsolution of (1.1). We have

Proposition 1.9. (a) Let $v \in C(\overline{\Omega})$ satisfy (1.15) for every $x \in \Omega$. Then v satisfies (0.3) with H as in (1.1).

(b) Let $w \in C(\overline{\Omega})$ satisfy (1.17) for every $x \in \Omega$. Then w satisfies (0.2) with H as in (1.1).

Proof: (a) For $\phi \in C^\infty(\Omega)$ let $x_0 \in \Omega$ be a local minimum of $v-\phi$. We want to show that

$$v(x_0) + \sup_{y\in Y} \{-f(x_0,y)\cdot D\phi(x_0,y) + \ell(x_0,y)\} \ge 0$$

But for δ sufficiently small we have

$$\frac{\phi(x_0)-\phi(x_0+\delta f)}{\delta} - \ell \geq \frac{v(x_0)-v(x_0+\delta f)}{\delta} - \ell \quad \text{for all} \quad (f,\ell) \in (FL)(x_0)$$

This inequality and (1.15) imply

$$v(x_0) + \sup_{(f,\ell) \in (FL)(x_0)} \{-f \cdot D\phi(x_0) - \ell\} \geq 0$$

Finally, since

$$\sup_{\lambda \in \Lambda} \lambda = \sup_{\lambda \in \overline{co}\Lambda} \lambda$$

we have the result.

(b) The proof is similar to the one of part (a), therefore we omit it.

Remark 1.10. All the results of this section extend to several other cases including time-dependent problems. The type of statements that one obtains are similar to the ones of Section 2.

SECTION 2

In this section we consider Hamilton-Jacobi equations which are related to theory of two-player, zero-sum differential games. Since in Section 1 we looked at stationary problems, here to show the generality of the arguments involved, we work with time dependent ones, in particular, we consider the following problems

$$(2.1) \quad \begin{cases} \dfrac{\partial U}{\partial t} + \inf_{y \in Y} \sup_{z \in Z} \{-f(t,x,y,z) \cdot DU - \ell(t,x,y,z)\} = 0 & \text{in} \quad \Omega \times (0,T] \\ U(x,t) = g(x,t) & \text{on} \quad \partial\Omega \times [0,T] \\ U(x,0) = u_0(x) & \text{in} \quad \Omega \end{cases}$$

and

$$(2.2)\quad \begin{cases} \dfrac{\partial V}{\partial t} + \sup\limits_{z\in Z}\inf\limits_{y\in Y}\{-f(t,x,y,z)\cdot DV - \ell(t,x,y,z)\} = 0 & \text{in } \Omega\times(0,T] \\ V(x,t) = g(x,t) & \text{on } \partial\Omega\times[0,T] \\ V(x,0) = u_0(x) & \text{in } \Omega \end{cases}$$

where Y,Z are compact sets and $f : [0,T]\times\overline{\Omega}\times Y\times Z \to \mathbb{R}^N$, $\ell : [0,T]\times\overline{\Omega}\times Y\times Z \to \mathbb{R}$, $g : \partial\Omega\times[0,T] \to \mathbb{R}$, $u_0 : \overline{\Omega} \to \mathbb{R}$ are bounded continuous functions. Moreover, they satisfy

$$(2.3)\quad \begin{cases} \text{There exists a constant } C > 0 \text{ such that} \\ |f(t,x,y,z)|, |\ell(t,x,y,z)| \le C \quad \text{for every } (t,x,y,z)\in[0,T]\times\overline{\Omega}\, Y\times Z \\ \text{and} \\ |f(t,x,y,z)-f(\hat{t},\hat{x},y,z)|, |\ell(t,x,y,z)-\ell(\hat{t},\hat{x},y,z)| \le C(|t-\hat{t}| + |x-\hat{x}|) \\ \text{for every } (t,x,y,z), (\hat{t},\hat{x},y,z) \in [0,T]\times\overline{\Omega}\times Y\times Z \end{cases}$$

(2.1) and (2.2) correspond to a finite horizon two-player, zero-sum differential game (for details we refer to W. Fleming [12], [13], [14], Elliott and Kalton [8], A. Friedman [16], [17]) with dynamics given by

$$(2.4)_{T-t}\quad \begin{cases} \dfrac{dx}{d\tau} = f(\tau,x(\tau),y(\tau),z(\tau)) & \text{for } T-t < \tau < T. \\ x(T-t) = x \in \Omega \end{cases}$$

where $y : [t,T] \to Y$, $z : [t,T] \to Z$ are measurable functions. Before we continue we need to introduce some notation. In particular, for $0 \le t \le T$ define

$$M(t) = \{y : [t,T] \to Y \text{ measurable}\}$$

$$N(t) = \{z : [t,T] \to Z \text{ measurable}\}$$

Moreover, denote by $\Gamma(t), \Delta(t)$ the sets of mappings $\alpha : N(t) \to M(t)$, $\beta : M(t) \to N(t)$ respectively with the following property

For each s such that $t \leq s \leq T$

$$\begin{cases} \text{If } z(\tau) = \hat{z}(\tau) \text{ for a.e. } t \leq \tau \leq s, \text{ then} \\ \alpha[z](\tau) = \alpha[\hat{z}](\tau) \text{ for a.e. } t \leq \tau \leq s \end{cases}$$

and

$$\begin{cases} \text{If } \hat{y}(\tau) = \hat{y}(\tau) \text{ for a.e. } t \leq \tau \leq s, \text{ then} \\ \beta[y](\tau) = \beta[y](\tau) \text{ for a.e. } t \leq \tau \leq s \end{cases}$$

Let U,V be the unique viscosity solutions of (2.1),(2.2) respectively if they exist. It is known (L. C. Evans and P. Souganidis [11] for $\Omega = \mathbb{R}^N$, L. C. Evans and H. Ishii [10] for stationary problems) and it follows from the results of this section for other cases that U,V satisfy the <u>optimality principle</u> of dynamic programming, that is

$$(2.5)\quad \begin{cases} \text{For } (x,t) \in \Omega \times (0,T) \text{ and } \delta > 0 \text{ such } \delta \leq t \\ V(x,t) = \inf_{\beta \in \Delta(T-t)} \sup_{y \in M(T-t)} \{ \int_{T-t}^{(T-t+\delta)\wedge t_x} \ell(s,x(s),y(s),\beta[y](s))ds + \\ \quad + U(x((T-t+\delta)\wedge t_x), T-((T-t+\delta)\wedge t_x)) \} \\ \text{and} \\ V(x,t) = \sup_{\alpha \in \Gamma(T-t)} \inf_{z \in N(T-t)} \{ \int_{T-t}^{(T-t+\delta)\wedge t_x} \ell(s,x(s),\alpha[z](s),z(s))ds + \\ \quad + V(x((T-t+\delta)\wedge t_x), T-((T-t+\delta)\wedge t_x)) \} \end{cases}$$

where, for $x \in \Omega$, $x(\cdot)$ is the solution of $(2.3)_{T-t}$ with the appropriate $y(\cdot), z(\cdot)$ funcitons and t_x is the exit time from $\Omega \times (0,T)$ of $x(\cdot)$.

The first result of this section concerns viscosity supersolutions and subsolutions of (2.1) and (2.2). In particular, we show that they satisfy some inequalities, which, in view of (2.5), may be called super- and sub-optimality principle of dynamic programming. All the results are going to be stated as they apply to the general problems (2.1) and (2.2), the proofs, however, for simplicity are going to be given only for the special case $\Omega = \mathbb{R}^N$. To obtain the most general results one has to use the localization argument, which was described in the course of the proof of Proposition 1.1.

We have

Proposition 2.1. Let $v,w \in C(\overline{Q}_T)$ be viscosity super-respectively subsolution of (2.1) respectively (2.2). For every $(x,t) \in \Omega \times (0,T)$ and $\delta > 0$ such that $\delta \leq t$, we have

$$(2.6)\qquad v(x,t) \geq \inf_{\beta\in\Delta(T-t)} \sup_{y\in M(T-t)} \{\int_{T-t}^{(T-t+\delta)\wedge t_x} \ell(s,x(s),y(s),\beta[y](s))ds + v(x((T-t+\delta)\wedge t_x), T-(T-t+\delta)\wedge t_x))\}$$

and

$$(2.7)\qquad w(x,t) \leq \sup_{\alpha\in\Gamma(T-t)} \inf_{z\in N(T-t)} \{\int_{T-t}^{(T-t+\delta)\wedge t_x} \ell(s,x(s), \alpha[z](s),z(s))ds + w(x(T-t+\delta)\wedge t_x), T-((T-t+\delta)\wedge t_x))\}$$

Proof: Here we prove only (2.6), since (2.7) is proved in exactly the same way. As mentioned above we are going to assume $\Omega = \mathbb{R}^N$.

For $\varepsilon > 0$ choose $\xi,\phi \in C^\infty(\mathbb{R})$ such that $0 \leq \xi \leq 1$, $0 \leq \phi \leq 1$, $\xi \equiv 1$ on $[\varepsilon,T-\varepsilon]$, $\xi \equiv 0$ on $(-\infty,\varepsilon/2] \cup (T-\frac{\varepsilon}{2},\infty)$, $\phi \equiv 1$ on $[\frac{\varepsilon}{4},T-\frac{\varepsilon}{4}]$, $\phi \equiv 0$ on $(-\infty,\frac{\varepsilon}{\delta}] \cup [T-\frac{\varepsilon}{\delta},\infty)$. Moreover, let $\tilde{v} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\tilde{v}(x,s) = \begin{cases} \phi(s)v(x,s) & \text{if } T \geq s \geq 0 \\ 0 & \text{if } s < 0 \text{ or } s > T \end{cases}$$

It is easy to check that $\tilde{v}$ is a viscosity supersolution of the problem

$$\xi(s)\frac{\partial\tilde{U}}{\partial t} + \inf_{y\in Y}\sup_{z\in Z}\{-\xi(s)f(s,x,y,z)\cdot D\tilde{U} - \xi(s)\ell(s,x,y,z)\} = 0 \quad \text{in} \quad \mathbb{R}^{N+1}$$

Next let $T > t > \delta > 0$ be fixed. Then $\tilde{v}$ is also a viscosity supersolution of

$$(2.8)\quad \begin{cases} \frac{\partial W}{\partial \tau} + \inf_{y\in Y}\sup_{z\in Z}\{-\xi(x)f(s,x,y,z)\cdot DW + \xi(s)\frac{\partial W}{\partial s} - \xi(s)\ell(s,x,y,z)\} = 0 \\ \qquad\qquad \text{in} \quad \mathbb{R}^{N+1}\times(0,T-t+\delta) \\ W(x,s,0) = \tilde{v}(x,s) \quad \text{in} \quad \mathbb{R}^{N+1} \end{cases}$$

If $W \in C(\mathbb{R}^{N+1}\times[0,T-t+\delta])$ is the unique viscosity solution of (2.8), the uniqueness estimates of [6] imply

$$\tilde{v}(x,t) \geq W(x,t,\delta)$$

Moreover, the results of L. C. Evans and P. E. Souganidis [11] give

$$W(x,t,\delta) = \inf_{\beta\in\Delta(T-t)}\sup_{y\in M(T-t)}\{\int_{T-t}^{T-t+\delta}\xi(T-t+\delta-t(\rho))\,\ell(T-t+\delta-t(\rho),\tilde{x}(\rho),y(\rho),\beta[y](\rho))\,d\rho$$
$$+ \tilde{v}(\tilde{x}(T-t+\delta),\ t(T-t+\delta))\}$$

where for $y \in M(T-t)$ and $\beta \in \Delta(T-t)$, $\tilde{x}(\cdot)$, $t(\cdot)$ are the solution of

$$\begin{cases} \frac{d\tilde{x}}{d\rho} = \xi(T-t+\delta-t(\rho))f(T-t+\delta-t(\rho),\ \tilde{x}(\rho),y(\rho),\beta[y](\rho)) & \text{for} \quad T-t<\rho<T-t+\delta \\ \frac{dt}{d\rho} = -\xi(T-t+\delta-t(\rho)) & \text{for} \quad T-t<\rho<T-t+\delta \\ \tilde{x}(T-t) = x,\ t(T-t) = t \end{cases}$$

As $\varepsilon\downarrow 0$ the above observations imply the result, since as $\varepsilon\downarrow 0$

$$\tilde{x}(0) \to x(0) \quad \text{uniformly on} \quad [T-t,T-t+\delta]$$

where $x(\cdot)$ is the solution of $(2.4)_{T-t}$

The next proposition considers subsolutions of (2.1) and supersolutions of (2.2). Since the proof of the results is exactly the same as the proof of Proposition 2.1, we omit it.

Proposition 2.2. Let $v,w \in C(\bar{Q}_T)$ be viscosity sub-respectively supersolution of (2.1) respectively (2.2). For every $(x,t) \in \Omega \times (0,T)$ and $\delta > 0$ such that $\delta \le t$, we have

$$(2.9) \quad v(x,t) \le \inf_{\beta\in\Delta(T-t)} \sup_{y\in M(T-t)} \int_{T-t}^{(T-t+\delta)\wedge t_x} \ell(s,x(s),y(s),\beta[y](s))ds + \\ + v(x((T-t+\delta)\wedge t_x),T-(T-t+\delta)\wedge t_x)))\}$$

and

$$(2.10) \quad w(x,t) \ge \sup_{\alpha\in\Gamma(T-t)} \inf_{z\in N(T-t)} \{\int_{T-t}^{(T-t+\delta)\wedge t_x} \ell(s,x(s),\alpha[z](s),z(s))ds + \\ + w(x((T-t+\delta)\wedge t_x),\ T-((T-t+\delta)\wedge t_x)))\}$$

Next we want to use Proposition 1.1 to obtain a kind of infinitesimal version of the super- and suboptimality principle of the dynamic programming. To do this we have to assume, as in Section 1, that we deal with locally Lipschitz viscosity super- and subsolutions. Before we state the results we need some notation. We have:

$$(2.11) \quad \begin{cases} \text{For } (t,x,y) \in (0,T) \times \Omega\times Y \\ (FL)(t,x,y) = \overline{co}\{(f(t,x,y,z),\ell(t,x,y,z)) : z \in Z\} \end{cases}$$

and

$$(2.12)\quad \begin{cases} \text{For } (t,x,z) \in (0,T)\times\Omega\times Z \\ (FL)(t,x,z) = \overline{co}\,\{(f(t,x,y,z)\ \ell(t,x,y,z)) : y \in Y\} \end{cases}$$

The result is

Proposition 2.3. Let $v,w \in C^{0,1}_{loc}(\Omega\times(0,T)) \cap C(\overline{Q}_T)$ be super-respectively subsolution of (2.1)-respectively (2.2). Then for every $(x,t) \in \Omega\times(0,T)$ we have

$$(2.13)\quad \varliminf_{\delta\downarrow 0}\ \sup_{y\in Y}\ \inf_{(f,\ell)\in(FL)(T-t,x,z)} \left\{ \frac{w(x,t)-w(x+\delta f,t-\delta)}{\delta} - \ell \right\} \geq 0$$

and

$$(2.14)\quad \varlimsup_{\delta\downarrow 0}\ \sup_{z\in Z}\ \inf_{(f,\ell)\in(FL)(T-t,x,z)} \left\{ \frac{w(x,t)-w(x+\delta f,t-\delta)}{\delta} - \ell \right\} \leq 0$$

with the inequalities being achieved as $\delta\downarrow 0$ uniformly on compact sets.

Proof: Here we show only (2.13), since (2.14) follows in a similar way. For a fixed $(x,t) \in \Omega\times(0,T)$ let K be the Lipschitz constant of v in a neighborhood of (x,t). For $\delta > 0$ sufficiently small it is

$$T - t + \delta < t_{\overline{x}}$$

for every $y \in M(T-t)$, $\beta \in \Delta(T-t)$, and this uniformly for every $\overline{x}$ in a neighborhood of x. (2.6) then implies

$$\sup_{\beta\in\Delta(T-t)}\ \inf_{y\in M(T-t)} \left\{ \frac{v(x,t)-v(x(T-t+\delta),t-\delta)}{\delta} - \frac{1}{\delta}\int_{T-t}^{T-t+\delta} \ell(s,x(s),y(s),\beta[y](s))ds \right\} \geq 0$$

But

$$\sup_{\beta\in\Delta(T-t)}\ \inf_{y\in M(T-t)} \leq \inf_{y\in M(T-t)}\ \sup_{\beta\in\Delta(T-t)} \leq \inf_{y\in Y}\ \sup_{\beta\in\Delta(T-t)}$$

Therefore, in view of (2.3),(2.4), we have

$$\inf_{y\in Y} \sup_{(f,\ell)\in(FL)(T-t,x,y)} \left\{ \frac{v(x,t)-v(x+\delta f,t-\delta)}{\delta} - \ell \right\} \geq 0(1)$$

where $0(1) \to 0$ as $\delta \downarrow 0$ uniformly for (x,t), in a compact set. Here we used the fact that for $y \in Y$

$$\left(\frac{1}{\delta}\int_{T-t}^{T-t+\delta} f(T-t,x,y,\beta[y](s))ds, \frac{1}{\delta}\int_{T-t}^{T-t+\delta} \ell(T-t,x,y,\beta[y](s))ds\right) \in (FL)(T-t,x,y)$$

for every $\beta \in \Delta(T-t)$.

Letting $\delta \downarrow 0$ we obtain the result.

As a consequence of Proposition 2.2 we have

Corollary 2.4. Let $v,w \in C^{0,1}_{loc}(\Omega \times (0,T)) \cap C(\overline{Q}_T)$ be super-respectively subsolutions of (2.1)-respectively (2.2). For every $(x,t) \in \Omega \times (0,T)$ we have

$$(2.15) \qquad \inf_{y\in Y} \sup_{(f,\ell)\in(FL)(T-t,x,y)} \overline{\lim_{\delta\downarrow 0}} \left\{ \frac{v(x,t)-v(x+\delta f,t-\delta)}{\delta} - \ell \right\} \geq 0$$

and

$$(2.16) \qquad \sup_{z\in Z} \inf_{(f,\ell)\in(FL)(T-t,x,z)} \underline{\lim_{\delta\downarrow 0}} \left\{ \frac{w(x,t)-w(x+\delta f,t-\delta)}{\delta} - \ell \right\} \leq 0$$

with the inequalities being achieved as $\delta \downarrow 0$ uniformly on compact sets.

Since Corollary 2.3 follows from Proposition 2.2 in the same way that Corollary 1.4 follows from Theorem 1.3 we omit its proof.

We continue with a proposition and a corollary concerning viscosity sub- and supersolutions of (2.1) and (2.2) respectively. Since these results follow from Proposition 2.2 the same way as Proposition 2.3 and Corollary 2.4 follow from Proposition 2.1 we omit their proof. We should also remark, however, that one can obtain these results directly from Proposition

2.3 and Corollary 2.4, by observing that a viscosity subsolution (supersolution) of (2.1)((2.2)) is a viscosity subsolution (supersolution) of (2.2) ((2.1)). We have:

Proposition 2.5. Let $v,w \in C^{0,1}_{loc}(\Omega\times(0,T)) \cap C(\overline{Q}_T)$ be sub- respectively supersolution of (2.1) respectively (2.2). Then for every $(x,t) \in \Omega\times(0,T)$ we have

$$(2.17) \quad \overline{\lim_{\delta\downarrow 0}} \sup_{z\in Z} \inf_{(f,\ell)\in(FL)(T-t,x,z)} \left\{ \frac{v(x,t)-v(x+\delta f,t-\delta)}{\delta} - \ell \right\} \leq 0$$

and

$$(2.18) \quad \underline{\lim_{\delta\downarrow 0}} \inf_{y\in Y} \sup_{(f,\ell)\in(FL)(T-t,x,y)} \left\{ \frac{w(x,t)-w(x+\delta f,t-\delta)}{\delta} - \ell \right\} \geq 0$$

with the inequalities being achieved as $\delta\downarrow 0$ uniformly on compact sets.

Corollary 2.6. Let $v,w \in C^{0,1}_{loc}(\Omega\times(0,T)) \cap C(\overline{Q}_T)$ be sub-respectively supersolutions of (2.1) respectively (2.2). For every $(x,t) \in \Omega\times(0,T)$ we have

$$(2.19) \quad \sup_{z\in Z} \inf_{(f,\ell)\in(FL)(T-t,x,z)} \underline{\lim_{\delta\downarrow 0}} \left\{ \frac{v(x,t)-v(x+\delta f,t-\delta)}{\delta} - \ell \right\} \leq 0$$

and

$$(2.20) \quad \inf_{y\in Y} \sup_{(f,\ell)\in(FL)(T-t,x,y)} \overline{\lim_{\delta\downarrow 0}} \left\{ \frac{w(x,t)-w(x+\delta f,t-\delta)}{\delta} - \ell \right\} \geq 0$$

with the inequalities being achieved as $\delta\downarrow 0$ uniformly on compact sets.

The next result is the inverse of Corollary 2.4 and Corollary 2.5. In particular, it says that (2.15),(2.16),(2.19) and (2.20) together with appropriate boundary conditions characterize continuous functions as viscosity super- and subsolutions of (2.1) and (2.2).

We have:

Proposition 2.7. (a) Let $v \in C(\Omega \times (0,T))$ satisfy (2.15). Then v also satisfies (0.9) with H as in (2.1).

(b) Let $w \in C(\Omega \times (0,T))$ satisfy (2.16). Then w also satisfies (0.8) with H as in (2.2).

(c) Let $v \in C(\Omega \times (0,T))$ satisfy (2.19). Then v also satisfies (0.8) with H as in (2.1).

(d) Let $w \in C(\Omega \times (0,T))$ satisfy (2.20). Then w also satisfies (0.9) with H as in (2.2).

Since the proof is similar to the proof of Proposition 1.9(a), we omit it.

We conclude this section which is an immediate consequence of Corollary 2.4 and Proposition 2.7. We have

Corollary 2.8. Suppose that for every $(t,x,p) \in [0,T] \times \overline{\Omega} \times \mathbb{R}^N$ it is

(2.21) $$\sup_{z \in Z} \inf_{y \in Y} \{-f(t,x,y,z) \cdot p - \ell(t,x,y,z)\} = \inf_{y \in Y} \sup_{z \in Z} \{-f(t,x,y,z) \cdot p - \ell(t,x,y,z)\}$$

Then a function $u \in C(\overline{Q}_T) \cap C^{0,1}_{loc}(\Omega \times (0,T))$ is a viscosity solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} + \sup\limits_{z \in Z} \inf\limits_{y \in Y} \{-f(t,x,y,z) \cdot Du - \ell(t,x,y,z)\} = 0 & \text{in } \Omega \times (0,T) \\ u(x,t) = g(x,t) & \text{on } \partial\Omega \times [0,T] \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases}$$

if and only if u satisfies (2.15),(2.16) and the correct boundary conditions.

Remark 2.9. A result analogous to Corollary 2.8 is proved by Subbotin [28] but not in the context of viscosity solutions. In particular, in [28] (2.15) and

(2.16) are necessary and sufficient conditions for a locally Lipschitz continuous function to be the value of a positional differential game, under the assumption that $\ell \equiv 0$ and $\Omega = \mathbb{R}^N$. Corollary 2.8 also implies in view of the results of [26],[27], [10],[11],[1], that notion of the value of a positional differential game is the same as the value of differential game introduced by W. Fleming, A. Friedman and N. Elliott and J. Kalton.

Remark 2.10. A remark analogous to Remark 1.6 holds here too.

APPENDIX

In view of Remark 1.6 and Remark 2.10, we want to make some (classical) observations concerning the existence of directional derivatives of the value function of optimal control and differential games problems. For simplicity here we investigate the case of an infinite horizon optimal control problem in $\mathbb{R}^N$. In particular, we deal with the existence of

$$\lim_{h \downarrow 0} \frac{v(x+h\chi) - v(x)}{h}$$

for all $x, \chi \in \mathbb{R}^N$, where v is the value function. Using the notation of Section 1 let us also assume:

(1)
$$\begin{cases} \text{For every } x \in \mathbb{R}^N,\ y \in Y \text{ and } h \in \mathbb{R} \\ |f(x+h,y) - f(x,y) - D_x f(x,y) \cdot h| \le |h| \varepsilon(|h|) \\ \text{and} \\ |\ell(x+h,y) - \ell(x,y) - D_x \ell(x,y) \cdot h| \le |h| \varepsilon(|h|) \\ \text{where } \varepsilon(|h|) \to 0 \text{ as } |h| \to 0 \end{cases}$$

For every $y(\cdot) \in M$, let

(2)
$$J(x,y) = \int_0^\infty e^{-s} \ell(x(s), y(s)) ds$$

where $x(\cdot)$ is the solution of (1.3) with $x(0) = x$. Moreover, let

(3)
$$v(x) = \inf_{y \in M} J(x,y)$$

In view of the discussion in Section 1 and the references given there, v is the value function of the associated optimal control problem.

We have

Proposition A.1. Assume that (1.2) and (1) hold with

$$(4) \qquad 1 > \sup_{(x,y)\in \mathbb{R}^N \times Y} |D_x f(x,y)|$$

Let v be given by (3). Then

$$\lim_{h\downarrow 0} \frac{v(x+h\chi) - v(x)}{h}$$

exists for every $x, \chi \in \mathbb{R}^N$ and

$$(5) \quad \lim_{h\downarrow 0} \frac{v(x+h\chi)-v(x)}{h} = \inf\left\{ \lim_{n\to\infty} \frac{\partial J(x_n,y_n)}{\partial \chi} : y_n \in M,\ J(x_n,y_n) \underset{n\to\infty}{\to} v(x) \right\}$$

Proof: The proof is a consequence of the following lemma.

Lemma A.2 Let $w(x) = \inf_i w^i(x)$ with w, w^i equibounded, equicontinuous and satisfying:

$$(6) \quad \begin{cases} \forall \chi \in \mathbb{R}^N, |\chi| = 1 \text{ there exist } \dfrac{\partial w^i}{\partial \chi}(x) \text{ such that} \\ \left| \dfrac{w^i(x+h\chi) - w^i(x)}{h} - \dfrac{\partial w^i}{\partial \chi}(x) \right| \le \delta(h) \underset{h\downarrow 0}{\to} 0 \end{cases}$$

Then $\lim_{h\downarrow 0^+} \frac{w(x+h\chi)\ w(x)}{h}$ exists for all χ and is equal to

$$(7) \quad \lim_{h\downarrow 0^+} \frac{w(x+h\chi)-w(x)}{h} = \inf\left\{ \lim_{n\to\infty} \frac{\partial w^{i_n}}{\partial \chi}(x) : w^{i}(x) \underset{n\to\infty}{\to} w(x) \right\}$$

In view of our hypotheses, v and $J(\cdot,y)$ satisfy the assumptions of Lemma 2. Therefore here we only prove the lemma. We have:

Proof of Lemma 2: Let i_n be a sequence such that

$$w^{i_n}(x) \to w(x) \quad \text{as} \quad n \to \infty$$

Then

APPENDIX

In view of Remark 1.6 and Remark 2.10, we want to make some (classical) observations concerning the existence of directional derivatives of the value function of optimal control and differential games problems. For simplicity here we investigate the case of an infinite horizon optimal control problem in $\mathbb{R}^N$. In particular, we deal with the existence of

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For every $y(\cdot) \in M$, let

$$(2) \qquad J(x,y) = \int_0^\infty e^{-s} \ell(x(s), y(s))\, ds$$

where $x(\cdot)$ is the solution of (1.3) with $x(0) = x$. Moreover, let

$$(3) \qquad v(x) = \inf_{y \in M} J(x,y)$$

In view of the discussion in Section 1 and the references given there, v is the value function of the associated optimal control problem.

We have

Proposition A.1. Assume that (1.2) and (1) hold with

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Then $\lim\limits_{h\downarrow 0^+} \dfrac{w(x+h\chi)\ w(x)}{h}$ exists for all χ and is equal to

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In view of our hypotheses, v and $J(\cdot,y)$ satisfy the assumptions of Lemma 2. Therefore here we only prove the lemma. We have:

Proof of Lemma 2: Let i_n be a sequence such that

$$w^{i_n}(x) \to w(x) \quad \text{as } n \to \infty$$

Then

$$\frac{w(x+h\chi)-w(x)}{h} \le \frac{w^{i_n}(x+h\chi)-w^{i_n}(x)}{h} + \frac{w^{i_n}(x)-w(x)}{h}$$

$$\le \frac{\partial w^{i_n}}{\partial \chi}(x)+\delta(h) + \frac{|w^{i_n}(x)-w(x)|}{h}$$

$$\le \underline{\lim}_{n\to\infty} \frac{\partial w^{i_n}}{\partial \chi}(x)+\delta(h)$$

Therefore

$$\overline{\lim_{h\downarrow 0}} \frac{w(x+h\chi)-w(x)}{h} \le \alpha$$

where α is the right hand side of (7). For the other direction, let $h_n>0$ be such that $h_n\downarrow 0$ as $n\to\infty$. Choose i_n such that

$$v(x+h\chi) \le v^{i_n}(x+h_n\chi) \le v(x+h_n\chi)+\frac{h_n}{n} \quad \text{as} \quad n\to\infty$$

Then, in view of the assumptions,

$$v^{i_n}(x) \to v(x) \quad \text{as} \quad n\to\infty$$

We have

$$\frac{v^{i_n}(x+h_n\chi)-v^{i_n}(x)}{h_n} \le \frac{v(x+h_n\chi)-v(x)}{h_n} + \frac{1}{n}$$

which implies

$$\alpha \le \underline{\lim}_{h\downarrow 0} \frac{v(x+h\chi)-v(x)}{h}$$

and thus the result.

Remark 3. Results analogous to the above also hold for finite horizon control problems and differential games. In the finite horizon case, one does not have to assume (4).

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Recent work by the authors and others has demonstrated the connections between the dynamic programming approach to optimal control theory and to two-person, zero-sum differential games problems and the new notion of "viscosity" solutions of Hamilton-Jacobi PDE's introduced by M. G. Crandall and P. L. Lions. In particular, it has been proved that the dynamic programming principle implies that the value function is the viscosity solution of the associated Hamilton-Jacobi-Bellman and Isaacs equations.

(cont.)

ABSTRACT (cont.)

In the present work, it is shown that viscosity super- and subsolutions of these equations must satisfy some inequalities called super- and subdynamic programming principle respectively. This is then used to prove the equivalence between the notion of viscosity solutions and the conditions, introduced by A. Subbotin, concerning the sign of certain generalized directional derivatives.