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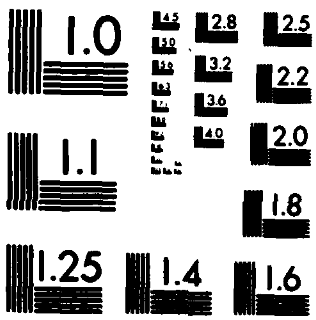

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MRC Technical Summary Report #2588

ON ARMA REPRESENTATIONS FOR WHITE NOISE IN A MARKOVIAN ENVIRONMENT

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October 1983

(Received April 1, 1983)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
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North Carolina 27709

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MATHEMATICS RESEARCH CENTER

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ABSTRACT

Consider an environment having d possible states, where the state of the environment evolves through time according to a stationary Markov chain. A natural model for noise in such an environment is to assume that the disturbance is driven by a white noise process that depends on the current state of the environment. In this note, it is shown that such a noise process may be represented by a (d+1)th order ARMA model.

AMS (MOS) Subject Classifications: 62M15, 93E03

Key Words: autoregressive moving average processes, random environment

Work Unit Number 5 - Optimization and Large Scale Systems

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER #2588	2. GOVT ACCESSION NO. AD-A136 354	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) On ARMA Representations for White Noise in a Markovian Environment		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Peter W. Glynn		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 5 - Optimization and Large Scale Systems
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE October 1983
		13. NUMBER OF PAGES 7
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

autoregressive moving average processes, random environment

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SIGNIFICANCE AND EXPLANATION

Autoregressive moving average (ARMA) models are frequently used to describe disturbances associated with signals. Usually, such processes are discussed in the context of finite dimensional linear systems. In this note, ~~we~~ the author shows that ARMA models also occur in a rather different setting, namely as descriptions of white noise in a randomly varying environment. Such a result is useful in better understanding the proper role of ARMA processes in systems theory.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON ARMA REPRESENTATIONS FOR WHITE NOISE IN A MARKOVIAN ENVIRONMENT

Peter W. Glynn

1. Introduction

Autoregressive moving average (ARMA) models are frequently used to describe disturbances associated with signals. Usually, such processes are discussed in the context of finite dimensional linear systems. In this note, we shall show that ARMA models also occur in a rather different setting, namely as descriptions of white noise in a randomly varying environment. Such a result is useful in better understanding the proper role of ARMA processes in systems theory.

To be precise, we consider an environment possessing d possible states, labelled 1 to d. The state of the environment at time n is given by a stationary Markov chain $\{X_n : -\infty < n < \infty\}$ having an aperiodic irreducible transition matrix P. Associated with each state i is a sequence of independent identically distributed finite variance random variables $\{Z_n(i) : -\infty < n < \infty\}$; the sequences are independent of one another and $\{X_n\}$. The noise process $\{Y_n : -\infty < n < \infty\}$ is then defined by the rule $Y_n = Z_n(X_n)$. In other words, Y_n is driven by the white noise $Z_n(i)$ whenever X_n equals i.

As an application, consider the following simple model for signal disturbances caused by atmospheric distortion. Assume that the atmosphere has a finite number of states (e.g. high and low humidity) and that state transitions occur according to a Markov chain. If the disturbance is assumed to be a white noise process that depends on current atmospheric conditions, then the noise can be realized as a special case of the above model.

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2. Calculation of the Spectral Density

We first review some basic properties of the transition matrix P (see Chapter 4 of [1]). The matrix P has a unique $d \times 1$ stationary vector π solving $\pi'P = \pi'$ (y' denotes the transpose of y). If Π is the $d \times d$ matrix with each row identical to π', then $\Pi P = P\Pi = \Pi^2 = \Pi$ and $P - \Pi$ has spectral radius less than 1.

Turning to the calculation of the spectral density, let μ and σ^2 be $d \times 1$ vectors with i'th component given by $\mu(i) = EZ_n(i)$ and $\sigma^2(i) = \text{var } Z_n(i)$, respectively. Then,

$$EY_n = \sum_i E\{Y_n \mid X_n = i\}P\{X_n = i\} = \pi'\mu$$

and

$$\text{var } Y_n = \sum_i \pi_i EZ_n^2(i) - (\sum_i \pi_i EZ_n(i))^2$$

$$= \pi'\sigma^2 + \mu'T(I-\Pi)\mu$$

where T is a diagonal matrix with $T_{ii} = \pi_i$. Furthermore, for $m > 1$

$$\text{cov}(Y_n, Y_{n+m}) = \mu'T(P^m - \Pi)\mu \quad .$$

The spectral density of $\{Y_n\}$ is given by

$$\begin{aligned} f(\lambda) &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{i\lambda k}\text{cov}(Y_0, Y_k) \\ &= \frac{1}{2\pi} \{\pi'\sigma^2 + \mu'T(I-\Pi + D(\lambda) + D(-\lambda))\mu\} \end{aligned} \tag{1}$$

where $D(\lambda) = \sum_{k=1}^{\infty} e^{i\lambda k}(P^k - \Pi)$.

<u>Lemma 1</u>. The inverse matrix $F(\lambda) = (\Pi - P + e^{i\lambda}I)^{-1}$ exists for all λ, and $D(\lambda) = PF(-\lambda) - e^{i\lambda}\Pi$.

Proof. Observe that $D(\lambda) = \sum_{k=1}^{\infty} e^{i\lambda k}(P - \Pi)^k$. Since $P - \Pi$ has spectral radius less than 1, it follows that the matrix sum for $D(\lambda)$ converges absolutely, and hence $D(\lambda)$ exists. Letting $n \to \infty$ in the identity

$$\sum_{k=1}^{n} e^{i\lambda k}(P^k - \Pi)(\Pi - P + e^{-i\lambda}I)$$

$$= P - \Pi - e^{i\lambda n}(P-\Pi)^{n+1}$$

yields

$$D(\lambda)(\Pi - P + e^{-i\lambda}I) = P - \Pi \tag{2}$$

so that

$$e^{i\lambda}(D(\lambda) + I)(\Pi - P + e^{-i\lambda}I) = I$$

and thus $F(-\lambda)$ exists. It is easily seen that $\Pi F(-\lambda) = e^{i\lambda}\Pi$, from which $D(\lambda) = PF(-\lambda) - e^{i\lambda}I$ follows from (2).

As an immediate consequence of (1) and Lemma 1, we obtain a closed form for $f(\lambda)$.

Theorem 1. The spectral density $f(\lambda)$ of $\{Y_n\}$ is given by

$$f(\lambda) = \frac{1}{2\pi} \{\pi'\sigma^2 + \mu'T(I-\Pi + PF(\lambda) + PF(-\lambda) - \Pi e^{i\lambda} - \Pi e^{-i\lambda})\mu\} \quad . \tag{3}$$

By Cramer's rule for calculation of matrix inverses, $g(e^{i\lambda})F(\lambda)$ is a matrix of polynomials in $e^{i\lambda}$ of degree at most $r < d$, where $g(e^{i\lambda})$ is the d'th order polynomial $\det(e^{i\lambda} - (P-\Pi))$. Since $f(\lambda)$ is real, even, and non-negative, it follows from (3) that $f(\lambda)$ may be written as

$$f(\lambda) = \frac{\alpha^2}{2\pi} \left|\sum_{k=0}^{q} a_k e^{i\lambda(q-k)}\right|^2 \Big/ \left|\sum_{\ell=0}^{p} b_\ell e^{i\lambda(p-\ell)}\right|^2 \quad , \tag{4}$$

where $a_0 = b_0 = 1$, the a_k's and b_ℓ's are real, and $q < p+1 < d+1$; here the numerator polynomial has no roots outside the unit disc, the denominator has all roots inside the unit disc (recall that $g(e^{i\lambda}) \neq 0$ for all λ), and the two polynomials have no roots in common.

3. ARMA Representations for Y_n

Our main result is the following.

Theorem 2. There exists an orthogonal mean zero stationary sequence $\{\varepsilon_n : -\infty < n < \infty\}$ and a constant β such that

$$Y_n + b_1 Y_{n-1} + \ldots + b_p Y_{n-p} = \varepsilon_n + a_1 \varepsilon_{n-1} + \ldots + a_q \varepsilon_{n-q} + \beta \ .$$

Proof. Let $W_n = \hat{Y}_n + b_1 \hat{Y}_{n-1} + \ldots + b_p \hat{Y}_{n-p}$ $(\hat{Y}_n \triangleq Y_n - EY_n)$ and observe that the spectral density of W_n is given by $\alpha^2 |\sum_{k=0}^{q} a_k e^{i\lambda(q-k)}|^2/2\pi$ (see (4)). The spectral representation theorem for the weakly stationary sequence W_n can be utilized to write W_n as

$$W_n = \varepsilon_n + a_1 \varepsilon_{n-1} + \ldots + a_q \varepsilon_{n-q}$$

where $\{\varepsilon_n : -\infty < n < \infty\}$ is an orthogonal weakly stationary sequence (see [2], p. 504, for details).

It is natural to ask whether all ARMA processes may be regarded as white noise in a Markovian environment. However, it is clear, from (4), that if the order of the moving average component is two or more than the order of the autoregressive part, that such a representation is impossible.

4. Martingale Difference Representations for Y_n

If $\sigma^2 = 0$, then $Y_n = \mu(X_n)$ is a d state Markov chain. In the case $d = 2$, the ARMA representation may be calculated explicitly. First observe that since P is irreducible, the eigenvalue 1 has algebraic multiplicity 1; since P is aperiodic, 1 is the only eigenvalue of unit modulus (see [3], p. 536-551). Thus the second eigenvalue λ of P for $d = 2$ has modulus less than unity and is clearly real. We claim that

$$\varepsilon_n = Y_n - \lambda Y_{n-1} - (1-\lambda)\pi'\mu \qquad (5)$$

is a sequence of mean zero orthogonal random variables. Let $v = \mu - \pi'\mu e$ where $e = (1,1)'$ is the right eigenvector associated with 1, and note that

$$E\{\varepsilon_n \mid X_k : k \le n-1\} = (Pv)(X_{n-1}) - \lambda v(X_{n-1}) \quad .$$

Then, if x is the right eigenvector for λ, $v = \alpha_1 e + \alpha_2 x$ for some α_1, α_2, so that $v = \alpha_2 x$ (use $\pi' v = 0 = \pi' P v$). Thus $Pv = \lambda v$ and $E\{\varepsilon_n \mid X_k : k \le n-1\} = 0$; a sequence ε_n having this property is called a martingale difference sequence, and it follows from basic properties of conditional expectation that ε_n is a mean zero orthogonal sequence, proving (5).

In general, one might hope that for a stationary d state Markov chain $Y_n = \mu(X_n)$, there exist constants $b_1,\ldots,b_p$, $a_1,\ldots,a_q$, β such that

$$Y_n + b_1 Y_{n-1} + \ldots + b_p Y_{n-p} = \varepsilon_n + a_1 \varepsilon_{n-1} + \ldots + a_q \varepsilon_{n-q} + \beta \qquad (6)$$

where the ε_n's are martingale differences. Condition both sides of (6) on X_k, $k \le n-1$, to obtain

$$(P\mu)(X_{n-1}) + b_1 Y_{n-1} + \ldots + b_p Y_{n-p} = a_1 \varepsilon_{n-1} + \ldots + a_q \varepsilon_{n-q} + \beta \quad . \qquad (7)$$

From (6) and (7), it follows that $\varepsilon_n = \mu(X_n) - (P\mu)(X_{n-1})$. By the stationarity of Y_n, one may back-solve in (6) to obtain

$$Y_n = \sum_{k=0}^{\infty} \gamma_k \varepsilon_{n-k} + \beta \qquad (8)$$

where $\gamma_0 = 1$. Substituting $\varepsilon_n = \mu(X_n) - (P\mu)(X_{n-1})$, it follows from (8) that $(P\mu)(X_{n-1}) - \gamma_1 \mu(X_{n-1})$ is a function of X_k, $k \le n-2$. This, in general, holds only if μ is a constant vector, and therefore martingale difference representations of the form (6) are usually nonexistent for $d \ge 3$.

5. Conclusions

We have investigated ARMA representations for white noise in a Markovian environment. Such representations are always possible, although the converse is incorrect (not all ARMA processes can be realized as white noise in a

Markovian environment). We have also shown that in a special setting, the moving average innovations are martingale differences.

Acknowledgement

The author would like to thank Thomas G. Kurtz for pointing out (5).

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