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Courant Institute of Mathematical Sciences

Magneto-Fluid Dynamics Division

Relativistic Broadening Near Cyclotron Resonance

Kaya Imre and Harold Weitzner

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Relativistic broadening of absorption (or emission) lines near cyclotron resonance in a warm plasma is investigated using the linearized relativistic Vlasov-Maxwell system. The unperturbed state is assumed to be isotropic, but not necessarily Maxwellian. The expansion parameter is $\eta = v_\perp/c$, v_e being the electron thermal speed. It is assumed that the wave frequency, plasma frequency, and cyclotron frequency are all comparable in magnitude, and the refractive index n = 0(1). The parameter $\alpha = \eta/n$ is of arbitrary order, thus the results are uniformly valid for all values of oblique propagation angles, although the relativistic effects are negligible for $\alpha \ll 1$. The dielectric tensor is reordered, and the dispersion relation appropriate for this problem is derived to the lowest significant order in η. The results are expressed in terms of the readily calculable (generalized) plasma dispersion function Z. In the Maxwellian case the results are algebraic in Z, and unlike the previously published results, they do not involve infinite integrals or series imposed on Z, thus leading to simple and efficient evaluations. The case of perpendicular propagation is obtained by taking the large q limit. Some inconsistencies in the literature dealing with the extraordinary mode are resolved.

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Relativistic Broadening Near Cyclotron Resonance

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ABSTRACT

Relativistic broadening of absorption (or emission) lines near cyclotron resonance in a warm plasma is investigated using the linearized relativistic Vlasov-Maxwell system. The unperturbed state is assumed to be isotropic, but not necessarily Maxwellian. The expansion parameter is $\eta = v_e/c$, v_e being the electron thermal speed. It is assumed that the wave frequency, plasma frequency, and cyclotron frequency are all comparable in magnitude, and the refractive index $n = 0(1)$. The parameter $\alpha = \eta/n$ is of arbitrary order, thus the results are uniformly valid for all values of oblique propagation angles, although the relativistic effects are negligible for $\alpha \ll 1$. The dielectric tensor is reordered, and the dispersion relation appropriate for this problem is derived to the lowest significant order in η. The results are expressed in terms of the readily calculable (generalized) plasma dispersion function Z. In the Maxwellian case the results are algebraic in Z, and unlike the previously published results, they do not involve infinite integrals or series imposed on Z, thus leading to simple and efficient evaluations. The case of perpendicular propagation is obtained by taking the large α limit. Some inconsistencies in the literature dealing with the extraordinary mode are resolved.

I. INTRODUCTION

The importance of the relativistic effects on the broadening of the cyclotron emission lines has long been recognized. Trubnikov[1] was first to point out as early as 1956 that the Doppler broadening of spectral lines emitted by a magnetized plasma in the direction perpendicular to the field is a purely relativistic effect, which must be taken into account even for relatively low temperature plasmas. Although he did not investigate this phenomenon in his well-known paper[1], he derived two distinct expressions for the plasma dielectric tensor based on the relativistic linearized Vlasov-Maxwell system of equations. The first, involving an infinite series in terms of the Bessel functions J_n, appears to be particularly useful when the spectral lines are well resolved (low temperatures and low harmonics), whereas the second form, which is expressed as an infinite integral containing the modified Bessel functions K_n, is preferable when the spectrum is continuous (high temperatures and high harmonics), as was stated in Ref. 1. The second form has the shortcoming that the unperturbed state has already been assumed to be the relativistic Maxwellian distribution so that the velocity space integrations could be carried out. No such assumption need be imposed in the derivation of the former representation.

Various heating schemes subsequently proposed for plasmas, which utilize the cyclotron resonance process, have initiated further research in which the energy absorption mechanism associated with relativistic broadening has been investigated extensively in certain ranges of the plasma parameters. Dnestrovskii et al.[2] attacked the

relatively simpler problem of perpendicular propagation in a weakly relativistic plasma, and Shkarofsky[3] extended their results to the case of nearly perpendicular propagation where the relativistic effects are still significant. In both of these studies, as well as many others which followed them[4], the integral representation of the dielectric tensor was utilized, thus restricting the system to be near thermal equilibrium. The main purpose of the present work is to remove this restriction by reformulating the problem starting from the infinite series representation for the dielectric tensor for weakly relativistic plasmas. For simplicity, the unperturbed state is assumed to be isotropic, but not necessarily Maxwellian, and only the electron cyclotron resonance is considered. The extension of the present method to the case of ion resonance, and to the higher harmonics, is straightforward. As was done in previous studies, the plasma is assumed to be warm, i.e. only the leading contributions of the parameter $\eta = v_e/c$ are considered, where v_e is the electron thermal speed $(2T_e/m_e)^{\frac{1}{2}}$, and c is the speed of light. The electron Larmor radius is also assumed to be small compared with the wave length, and their ratio is assumed to be of first order in η. The refractive index $n = kc/\omega = 0(1)$.

The formulation presented is uniform in terms of the parameter

$$\alpha = v_e/(n_{\parallel}c),$$ (1)

$n_{\parallel}$ being the longitudinal refractive index. The case of perpendicular propagation $(\alpha \to \infty)$, and the nonrelativistic limit $(\alpha \ll 1)$ are obtained by taking appropriate limits. Section II contains these results, as well as the evaluation and ordering of the dielectric

tensor near the fundamental electron resonance, viz., $1 - |\Omega_e|/\omega = O(\eta^2)$, where Ω_e denotes the electron cyclotron frequency.

In Sec.III, the zero order dispersion relation is derived. This relation differs from the cold plasma dispersion relation by the presence of a function A which generates the relativistic Doppler effects. Section IV is devoted to the special case of the Maxwellian unperturbed state. It is shown that in this case the results can be expressed algebraically in terms of the standard dispersion function Z, thus providing a considerable improvement over the form given in Ref. 3, which contains an infinite complex integral or infinite series involving the function Z. Equivalency of the present results to those previously given is demonstrated to order η. The study of the perpendicular extraordinary mode, which can be found in Sec. V, requires a second order analysis for the investigation of the lowest order Doppler effects. Certain discrepancies, stemming from an improper evaluation of the second order terms, are found in the previously published results. Section V also contains a study of the nearly perpendicular ordinary mode based on the results presented in this work. In the part C of the same section, the Doppler broadening of the Nth electron cyclotron resonance is studied, and particular attention is given to the second harmonic extraordinary mode. A summary of results, and a final discussion are found in Sec. VI.

II. EVALUATION OF THE DIELECTRIC TENSOR

The dielectric tensor for a plasma, which can be expressed for a plane wave of the form $\exp(i\underset{\sim}{k}\cdot\underset{\sim}{x} - i\omega t)$ as

$$\underset{\approx}{\epsilon} = \underset{\approx}{I} + (4\pi i/\omega)\,\underset{\approx}{\sigma}, \tag{2}$$

$\underset{\approx}{\sigma}$ being the conductivity tensor, can be derived from the linearized relativistic Vlasov equation, as was done in many texts on plasmas. It is convenient to adopt the $(+,-,\|)$ coordinate reference system, which is defined by means of the transformation matrix P, and its inverse P^{-1}, where

$$P = \begin{bmatrix} 1 & i & 0 \\ 1 & -i & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad P^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2}i & \frac{1}{2}i & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3}$$

representing contravariant and covariant transformation rules applied to the cartesian system in which the z axis is selected along the external magnetic field. One has, in this representation,

$$\frac{4\pi i}{\omega}\,\underset{\approx}{\sigma} = \sum_{j}\frac{\omega_{pj}^2}{\omega\,\Omega_j}\int d^3 u \sum_{-\infty < p < \infty} \frac{\underset{\approx}{T}}{\omega_* - p}, \tag{4}$$

where
$$\underset{\sim}{u} = \underset{\sim}{v}\gamma, \quad \gamma = (1 + u^2/c^2)^{\frac{1}{2}}, \quad \omega_* = (\omega/\Omega_j)\gamma - k_\| u_\|/\Omega_j,$$
$$\omega_{pj}^2 = 4\pi n_j e_j^2/m_j, \quad \Omega_j = e_j B_0/m_j c, \quad \underset{\sim}{B}_0 = B_0\underset{\sim}{\hat{z}}.$$

m_j, e_j, n_j denoting the rest mass, electric charge, number density (respectively) of the jth species, and [with $\int d^3 u\,F(u) \equiv 1$]

$$\underset{\approx}{T} = \begin{bmatrix} \frac{1}{2}u_\perp\frac{\partial F}{\partial u_\perp}J_{p-1}^2 & \frac{1}{2}u_\perp\frac{\partial F}{\partial u_\perp}e^{2i\psi}J_{p-1}J_{p+1} & u_\perp\frac{\partial F}{\partial u_\|}e^{i\psi}J_{p-1}J_p \\[2ex] \frac{1}{2}u_\perp\frac{\partial F}{\partial u_\perp}e^{-2i\psi}J_{p+1}J_{p-1} & \frac{1}{2}u_\perp\frac{\partial F}{\partial u_\perp}J_{p+1}^2 & u_\perp\frac{\partial F}{\partial u_\|}e^{-i\psi}J_{p+1}J_p \\[2ex] \frac{1}{2}u_\|\frac{\partial F}{\partial u_\perp}e^{-i\psi}J_p J_{p-1} & \frac{1}{2}u_\|\frac{\partial F}{\partial u_\perp}e^{i\psi}J_p J_{p+1} & u_\|\frac{\partial F}{\partial u_\|}J_p^2 \end{bmatrix}.$$

In Eq.(5), the unperturbed distribution function is assumed to be isotropic, so that $u_\perp \partial F/\partial u_\| = u_\| \partial F/\partial u_\perp$, and the argument of the Bessel functions J_p is $k_\perp u_\perp/\Omega_j$. The azimuth of the propagation vector $\underset{\sim}{k}$ is denoted by ψ. It is convenient to normalize F so that $F(u)d^3u \to F(w)d^3w$, where $\underset{\sim}{w} = \underset{\sim}{u}/v_j$, v_j denoting the thermal speed of the jth species. Near the electron fundamental resonance, one may neglec the ion dynamics, and introduce the parameter

$$\xi = (\omega - |\Omega_e|)/k_\| v_e = (1 - |\Omega_e|/\omega)\alpha\eta^{-2},$$

so that

$$\omega_* + 1 = -\frac{\eta^2}{\alpha}\frac{\omega}{\Omega_e}(w_\| - \xi - \frac{\alpha w^2}{1+\gamma}), \tag{7}$$

indicating that the resonance condition occurs at $p = -1$. One also assumes that the electron Larmor radius is small compared with the wavelength, so that the Bessel functions can be replaced by the first few terms of their power series representations. With $k_\perp v_e/\Omega_e = 0(\eta)$, one obtains

$$\epsilon_{++} = 1 + \tfrac{1}{2}X\frac{\Omega}{\omega}\int d^3w\, w_\perp\frac{\partial F}{\partial w_\perp}(-\tfrac{1}{2} + \eta^2 n_\perp^4\frac{\alpha}{64D}w_\perp^4 - \frac{\eta^2}{12}n_\perp^2 w_\perp^2 - \frac{\eta^2}{4\alpha}D),$$

$$\epsilon_{--} = 1 + \tfrac{1}{2}X\int d^3w\, w_\perp\frac{\partial F}{\partial w_\perp}\frac{\alpha}{D}(-\eta^{-2}\frac{\Omega^2}{\omega^2} + \tfrac{1}{2}n_\perp^2 w_\perp^2),$$

$$\epsilon_{\|\|} = 1 + X\int d^3w\, w_\|\frac{\partial F}{\partial w_\|}(\frac{\Omega}{\omega} - \frac{\alpha}{4D}n_\perp^2 w_\perp^2),$$

$$\epsilon_{+-}e^{-2i\psi} = \epsilon_{-+}e^{2i\psi} = -\frac{X}{16}n_\perp^2\int d^3w\, w_\perp^3\frac{\partial F}{\partial w_\perp}\frac{\alpha}{D}, \tag{8}$$

$$\epsilon_{+\|}e^{-i\psi} = 2\epsilon_{\|+}e^{i\psi} = \frac{X}{16}\eta\frac{\omega}{\Omega}n_\perp^3\int d^3w\, w_\perp^4\frac{\partial F}{\partial w_\|}\frac{\alpha}{D},$$

$$\epsilon_{-\|}e^{i\psi} = 2\epsilon_{\|-}e^{-i\psi} = \tfrac{1}{2}X\frac{\omega}{\Omega}n_\perp\int d^3w\, w_\perp^2\frac{\partial F}{\partial w_\|}\frac{\alpha}{D}(\eta^{-1} - \tfrac{1}{2}\eta n_\perp^2 w_\perp^2).$$

where

$$D = w_\| - \xi - \tfrac{1}{2}\alpha w^2, \quad \text{and} \quad X = \omega_{pe}^2/\Omega_e^2.$$

The η^2 terms are kept in ϵ_{++}, since they are needed in studying the broadening of the perpendicular extraordinary mode in Sec. V. It is clear from Eq. (8) that $\epsilon_{--} = 0(\eta^{-2})$, $\epsilon_{-\parallel} = 0(\eta^{-1})$, and ϵ_{++}, ϵ_{+-}, $\epsilon_{\parallel\parallel}$ are all of zero order in η. Hence, by writing

$$\underset{\approx}{\epsilon} = \eta^{-2} \underset{\approx}{\epsilon}^{(-2)} + \eta^{-1} \underset{\approx}{\epsilon}^{(-1)} + \underset{\approx}{\epsilon}^{(0)} + \eta \underset{\approx}{\epsilon}^{(1)} + \cdots, \tag{9}$$

one obtains the nonzero elements up to first order in η as

$$\epsilon_{--}^{(-2)} = X a_{0,0},$$

$$\epsilon_{-\parallel}^{(-1)} e^{i\psi} = 2 \epsilon_{\parallel-}^{(-1)} e^{-i\psi} = X n_\perp a_{0,1},$$

$$\epsilon_{++}^{(0)} = 1 - \tfrac{1}{2}X,$$

$$\epsilon_{--}^{(0)} = 1 - X n_\perp^2 a_{1,0} - (2\xi/\alpha) X a_{0,0}, \tag{10}$$

$$\epsilon_{\parallel\parallel}^{(0)} = 1 - X + \tfrac{1}{2} X n_\perp^2 a_{0,2},$$

$$\epsilon_{+-}^{(0)} e^{-2i\psi} = \epsilon_{-+}^{(0)} e^{2i\psi} = \tfrac{1}{4} X n_\perp^2 a_{1,0},$$

where

$$a_{n,m} = \alpha Z_{2n,m} + \frac{\alpha^2}{2n+2} \frac{d}{d\xi} Z_{2n+2,m} = -\frac{\alpha}{2n+2} \int \frac{d^3w}{D} w_\perp^{2n+1} w_\parallel^m \frac{\partial F}{\partial w_\perp},$$

$$Z_{n,m} = \int d^3w \, w_\perp^n w_\parallel^m \, F(w)/D. \tag{11}$$

It is shown in Appendix I that $Z_{2n,m}$ and its derivative (thus $a_{n,m}$) can be expressed in terms of $Z_{0,m}$, giving

$$a_{0,0} = \alpha Z_{0,0} - \alpha^2 Z_{0,1},$$

$$a_{0,1} = -\alpha - \alpha\xi Z_{0,0} + 2\alpha Z_{0,1} - \tfrac{1}{2} 3\alpha^2 Z_{0,2}, \tag{12}$$

$$a_{0,2} = -2\alpha\xi Z_{0,1} + 3\alpha Z_{0,2} - 2\alpha^2 Z_{0,3}.$$

Furthermore, the functions $Z_{0,m}$ (thus $Z_{n,m}$) can be expressed in terms of the integrals (see Appendix I for details)

$$I_n = \int_C d\zeta\, \zeta^{2n+1} z(\zeta),\tag{13}$$

where

$$z(\zeta) = \pi \int_{-\infty}^{\infty} dw\, F(w)/(w - \zeta)\tag{14}$$

with $F(w)$ extended as an even function for $w < 0$, i.e. $F(w) = F(-w)$. The normalization of $Z(\zeta)$ is chosen so that when F is Maxwellian, namely $F = \pi^{-3/2} e^{-w^2}$, it reduces to the standard plasma dispersion function. This case will be further studied in Sec.IV. One obtains, with $\langle g \rangle \equiv \int d^3w\, g(w)\, F(w)$,

$$Z_{0,0} = -\frac{2}{\alpha}\langle w^{-2}\rangle - 2I_0,$$

$$Z_{0,1} = -\frac{4}{3\alpha^2}\langle w^{-2}\rangle - 2\xi I_0 - \alpha I_1,$$

$$Z_{0,2} = -\frac{2}{3\alpha} - \frac{4}{\alpha^2}\langle w^{-2}\rangle(\frac{2}{5\alpha} - \frac{\xi}{3}) - 2\xi^2 I_0 - 2\alpha\xi I_1 - \tfrac{1}{2}\alpha^2 I_2,$$

$$Z_{0,3} = -\frac{4}{5\alpha^2} - \frac{16}{\alpha^3}\langle w^{-2}\rangle(\frac{1}{7\alpha} - \frac{\xi}{5}) - 2\xi^3 I_0 - 3\alpha\xi^2 I_1 - \tfrac{3}{2}\alpha^2\xi I_2 - \tfrac{1}{4}\alpha^3 I_3.$$

(15)

The following comments regarding Eqs.(13) and (14) are in order. Although the integral in (14) defines a sectionally holomorphic function in the complex ζ plane (cut through the real axis), it is customary to consider only the upper-half branch and analytically continue it to the lower-half plane, since the lower branch can be obtained, if needed, by $Z^*(\zeta^*)$, which follows from the Schwartz reflection principle. Moreover, since $F(w)$ is an even

function of w, the function defined by this integral is odd in ζ.
Therefore, the values of $Z(\zeta)$ in the first quadrant are related to
those in the second quadrant by $Z(\zeta) = -Z^*(-\zeta^*)$.

The contour C in Eq.(13) is defined as the image of the interval
$|\mu| \leqslant 1$ under the mapping $\alpha\zeta = \mu + (\mu^2 - \mu_0^2)^{\frac{1}{2}}$, where $\mu_0^2 = 2\xi\alpha$, (see
Appendix I). The contour lies on the real ζ axis, except when $|\mu| < \mu_0$, i
which case C lies on the upper half of the circle of radius $(2\xi/\alpha)^{\frac{1}{2}} \equiv R_0$.
If ξ is allowed to approach the real axis from below, then the entire
path C remains in the upper half plane, which enables one to use the
conventional definition of $Z(\zeta)$. The case in which ξ approaches the
real axis from above, which is of more physical interest, can be
studied by simply taking the complex conjugate of the results.

The case when $\mu_0 \geqslant 1$ is of particular interest, in which case C
lies entirely on the circular arc $\zeta = R_0 \exp(i\vartheta)$, for $|\vartheta| \leqslant \tan^{-1}(\mu_0^2 - 1)^{\frac{1}{2}}$.
It follows from the symmetry of C, and the reflection property $Z(\zeta) =$
$-Z^*(-\zeta^*)$ that $\mathrm{Im}\{I_n\} \equiv 0$, thus $Z_{n,m}$ and $a_{n,m}$ are all real in this case.
This property will be used in the following section in deriving the
condition for evanescence for plasma waves.

When $\mu_0^2 < 1$, C can be chosen as the interval $\left(\frac{-1+(1-\mu_0^2)^{\frac{1}{2}}}{\alpha}, \frac{1+(1-\mu_0^2)^{\frac{1}{2}}}{\alpha}\right)$.

In closing this section, one may point out an alternative
expression for the integral in I_n, namely

$$\int d\zeta \, \zeta^{2n+1} Z(\zeta) = -\frac{1}{2}\sum_{j=0}^{n} \frac{\zeta^{2j+1}}{2j+1} \left\langle w^{2n-2j-2}\right\rangle + \frac{1}{4}\left\langle w^{2n-1}\ln\frac{\zeta+w}{\zeta-w}\right\rangle .$$

III. DISPERSION RELATION NEAR CYCLOTRON RESONANCE

Maxwell's equation for the electric field can be written as

$$\underset{\sim}{n} \times (\underset{\sim}{n} \times \underset{\sim}{E}) + \underset{\approx}{\epsilon} \cdot \underset{\sim}{E} = 0, \tag{16}$$

where $\underset{\sim}{n} = \underset{\sim}{k}c/\omega$ is the refractive index, so that the dispersion relation for plasma can be written as

$$\det(n^2 \underset{\approx}{I} - \underset{\sim\sim}{nn} - \underset{\approx}{\epsilon}) = 0. \tag{17}$$

Generally, it is more convenient to introduce the ordering (9), together with the expansion

$$\underset{\sim}{E} = \underset{\sim}{E}^0 + \eta \underset{\sim}{E}^1 + \dots \tag{18}$$

into Eq.(16), rather than expanding the determinant (17). One has to order η^{-2}, $\underset{\approx}{\epsilon}^{(-2)} \cdot \underset{\sim}{E}^0 = 0$, or $\epsilon_{--}^{(-2)} E_-^0 = 0$, where it is assumed that $n = 0(1)$. Therefore, E_- is of first order in η. To order η^{-1}, one has $\underset{\approx}{\epsilon}^{(-1)} \cdot \underset{\sim}{E}^0 + \underset{\approx}{\epsilon}^{(-2)} \cdot \underset{\sim}{E}^1 = 0$, giving

$$\epsilon_{--}^{(-2)} E_-^1 + \epsilon_{-\|}^{(-1)} E_\|^0 = 0, \tag{19}$$

and to the next order, one obtains

$$(\epsilon_{++}^{(0)} - \tfrac{1}{2}n_\perp^2 - n_\|^2) E_+^0 + n_\| n_\perp e^{i\psi} E_\|^0 = 0,$$

$$(\tfrac{1}{2}n_\perp^2 e^{-2i\psi} + \epsilon_{-+}^{(0)}) E_+^0 + n_\| n_\perp e^{-i\psi} E_\|^0 + \epsilon_{--}^{(-2)} E_-^2 + \epsilon_{-\|}^{(-1)} E_\|^1 = 0,$$

$$\tfrac{1}{2}n_\perp n_\| e^{-i\psi} E_+^0 + (\epsilon_{\|\|}^{(0)} - n_\perp^2) E_\|^0 + \epsilon_{\|-}^{(-1)} E_-^1 = 0. \tag{20}$$

The first and the third of Eqs.(20), together with (19), form a system of equations for E_+^0, E_-^1, and $E_\|^0$, the determinant of the coefficients yielding the zero order dispersion relation. [The second of Eqs.(20) should be moved to the first order system, since it contains E_-^2 and $E_\|^1$.]

One has, by making use of Eqs.(10),

$$(1 - X - n_\perp^2 - n_\perp^2 XA)(2 - X - n_\perp^2 - 2n_\parallel^2) - n_\perp^2 n_\parallel^2 = 0, \tag{21}$$

where
$$A = -\tfrac{1}{2}(a_{0,2} - a_{0,1}^2/a_{0,0}). \tag{22}$$

The following comments can be made regarding the dispersion relation (21). First, this relation differs from the cold plasma dispersion relation (evaluated at the resonance) by the presence of an additional term in the first factor, which contains the function A. Therefore, the function A is solely responsible for the generation of the zero order relativistic Doppler effects, which will be further studied in what follows. Second, it is clear from Eqs. (12) and (22) that when $\alpha \ll 1$ (i.e. $n_\parallel \gg \eta$) A approaches zero, which indicates that the relativistic broadening becomes negligible. Third, in the case of perpendicular propagation ($n_\parallel = 0$, or $\alpha \to \infty$) the two modes represented by the two factors in the first term in (21) are decoupled. The dispersion relation for the ordinary mode (the first factor) is

$$n^2 = (1 - X)/(1 + XA_0), \tag{23}$$

where $A_0 = \lim_{\alpha \to \infty} A$. As will be shown in Sec. IV, in the Maxwellian case this limiting value coincides with the result of Dnestrovskii et al.[2]. The second factor in Eq.(21) represents the extraordinary mode, and contains no broadening effect to the zero order theory as developed here; thus the cold plasma dispersion relation applies:

$$n^2 = 2 - X. \tag{24}$$

In order to study the broadening of this branch, the second order contributions for ϵ_{++} are required. This will be done in Sec. V.

Fourth, as is demonstrated in Sec. II, the imaginary parts of the integrals I_n defined in (13) are zero whenever $2\xi\alpha \geqslant 1$, indicating that A is real in this case. Therefore, the region of evanescence for waves near cyclotron resonance is obtained as

$$|\Omega|/\omega > 1 - \tfrac{1}{2}n_{\parallel}^2 .$$ (25)

In the special case of perpendicular propagation for Maxwellian plasmas, this result (which then reduces to $\omega \leqslant |\Omega|$) has already been pointed out[2-4]. It may be of some interest to note that the condition (25) closely resembles a similar condition, namely $(\Omega/\omega)^2 - 1 + n^2 > 0$, in an earlier work[8], in which waves propagating along the external field ($n_{\perp} = 0$) were studied.

IV. MAXWELLIAN CASE

The purpose for studying this important case is two fold:
it allows one to perform quantitative evaluations of the results
presented in previous sections, and it also enables one to compare
the present results with those previously published, since, as was
pointed out earlier, previous studies invariably assumed this case
at the onset of their formulations by adopting the continuum
description for the dielectric tensor.

The relativistic equilibrium distribution function is

$$F(u) = (4\pi c^3)^{-1}[\beta/K_2(\beta)] \exp(-\beta\gamma), \tag{26}$$

where $\beta = m_e c^2/T_e = 2\eta^2$. By introducing the normalization $w = u/v_e$,
and expanding into powers of η^2, one obtains

$$F(w) = \pi^{-3/2} e^{-w^2}(1 - \frac{15}{16}\eta^2 + \frac{w^4}{4}\eta^2) + O(\eta^4). \tag{27}$$

Thus η dependence in $F(w)$ can be neglected in the first order theory.
One also observes that $\epsilon_{++}^{(2)}$, which will be used in studying the
broadening of the perpendicular extraordinary mode in Sec. V., has
no second order correction due to the relativistic effects associated
with $F(w)$, since $\epsilon_{++}^{(0)}$ involves only the zero order moment, and $\langle 1 \rangle \equiv 1$.
Therefore, one can use nonrelativistic Maxwellian averages: $\langle w^{-2} \rangle = \frac{1}{2}$,
$\langle w_\perp^{2n} w_\parallel^{2m} \rangle = n!(2m)!4^{-m}/m!$, and $\langle w^{2n} \rangle = (2n+1)!4^{-n}/n!$. Moreover, $Z(\zeta)$
is the standard plasma dispersion function[9], and satisfies $Z' = -2 - 2\zeta Z$,
with $Z(0) = i\pi^{\frac{1}{2}}$. The integral in (13) now can be performed exactly,
by making use of the recurrence relation:

$$I_n = nI_{n-1} - \frac{1}{2}\zeta^{2n}Z(\zeta) - \zeta^{2n+1}/(2n+1)$$

giving
$$I_n = -n!\sum_{k=0}^{n} (\zeta^{2k}/k!)\left[\frac{1}{2}Z(\zeta) + \zeta/(2k+1)\right] \tag{28}$$

Equations (15) now yield

$$z_{0,0} = [z],$$

$$z_{0,1} = 2 + (\xi + \tfrac{1}{2}\alpha)[z] + \tfrac{1}{2}\alpha[\zeta^2 z],$$

(29)

$$z_{0,2} = 2\alpha + \tfrac{2}{\alpha} + (\xi^2 + \alpha\xi + \tfrac{1}{2}\alpha^2)[z] + (\alpha\xi + \tfrac{1}{2}\alpha^2)[\zeta^2 z] + \tfrac{1}{4}\alpha^2[\zeta^4 z],$$

$$z_{0,3} = 4 + 3\alpha^2 - \tfrac{4\xi}{\alpha} + \tfrac{4}{\alpha^2} + (\xi^3 + \tfrac{3}{2}\alpha\xi^2 + \tfrac{3}{2}\alpha^2\xi + \tfrac{3}{4}\alpha^3)[z]$$
$$+ (\tfrac{3}{2}\alpha\xi^2 + \tfrac{3}{2}\alpha^2\xi + \tfrac{3}{4}\alpha^3)[\zeta^2 z] + (\tfrac{3}{4}\alpha^2\xi + \tfrac{3}{8}\alpha^3)[\zeta^4 z] + \tfrac{\alpha^3}{8}[\zeta^6 z].$$

where the notation $[F(\zeta)]$ denotes

$$[F(\zeta)] = F*\left(\frac{1 + (1 - 2\xi\alpha)^{\frac{1}{2}}}{\alpha}\right) - F*\left(\frac{-1 + (1 - 2\xi\alpha)^{\frac{1}{2}}}{\alpha}\right),$$

(30)

where the complex conjugate is taken to assure that ξ approaches the
real axis from above.

It is straightforward, but somewhat tedious, to show that Eqs.(29)
reduce to the nonrelativistic limit as $\alpha \to 0$, namely

$$z_{0,0} \to Z(\xi),$$

$$z_{0,1} \to 1 + \xi Z(\xi),$$

(31)

$$z_{0,2} \to \xi + \xi^2 Z(\xi),$$

$$z_{0,3} \to \tfrac{1}{2} + \xi^2 + \xi^3 Z(\xi),$$

which are more readily obtained from Eq.(A1) directly, which states
that in this limit $z_{n,m+1} \to \langle w_\perp^n w_\parallel^m \rangle + \xi z_{n,m}$.

The case $\alpha \to \infty$ is also of considerable importance, since it
corresponds to the case of perpendicular propagation, viz., $n_\parallel = 0$.
However, in taking this limit one should keep the quantity ξ/α finite,
and consider the limits of $\alpha z_{0,2m}$ and $\alpha^2 z_{0,2m+1}$ (cf. Appendix II).

Although this limit can be carried out directly using Eqs.(29), it is more convenient to use the definition (11) directly as is done in Appendix II. The results can be expressed in terms of the functions F_q introduced by Dnestrovskii et al.[2], as well as the functions $\mathcal{F}_q$ introduced by Shkarofsky[3]. One finds that

$$\lim_{\alpha \to \infty} (\alpha z_{2n,2m}) = - \frac{2n!(2m)!}{4^m m!} F_{n+m+3/2},$$

$$\lim_{\alpha \to \infty} (\alpha^2 z_{2n,2m+1}) = - \frac{2n!(2m+1)!}{4^m m!} (F_{n+m+3/2} - F_{n+m+5/2}),$$

(32)

leading to

$$\lim_{\alpha \to \infty} a_{n,2m} = - \frac{2n!(2m)!}{4^m m!} F_{n+m+5/2},$$

$$\lim_{\alpha \to \infty} (\alpha a_{n,2m+1}) = - \frac{2n!(2m+1)!}{4^m m!} (F_{n+m+5/2} - F_{n+m+7/2}).$$

(33)

The latter can be used to calculate the limiting value of the dielectric tensor, cf., Eq.(10), as well as the function A introduced in Eq.(22). One obtains

$$A_0 = \lim_{\alpha \to \infty} A = \tfrac{1}{2} F_{7/2}.$$

(34)

Equation (34), together with the dispersion relation (23), coincides with the results for the perpendicular ordinary mode first derived by Gershman[5] and Dnestrovskii et al.[2]

For arbitrary values of α, the function $A(\alpha,\xi)$ can easily be calculated by using Eqs.(29) together with Eqs.(12) and (22). Figure 1 illustrates the real and imaginary parts of A as a function of $-2\xi/\alpha$ for various values of the parameter α. The case $\alpha = 5$ agrees with the limiting value A_0 within two digits of accuracy, leading to the

conclusion that the asymptotic expansions in terms of the parameter $1/\alpha^2$ presented by Shkarofsky[3], which are further studied in Appendix II, are useful whenever $\alpha \geqslant 5$. The cutoff condition (25) is also observed in Fig.1 as Im(A) vanishes for $2\xi\alpha \geqslant 1$. Both real and imaginary parts of A monotonically decrease in general, thus making the cold plasma approximation more appropriate for $\alpha \ll 1$.

V. SOME APPLICATIONS

A. Perpendicular Extraordinary mode

In the case of the perpendicular propagation, the dispersion relation (17) can be factorized. It is more convenient, in this case, to expand directly the extraordinary branch, which reads

$$n^2 = \frac{\epsilon_{++}\epsilon_{--} - \epsilon_{+-}\epsilon_{-+}}{\frac{1}{2}\epsilon_{++} + \frac{1}{2}\epsilon_{--} + \epsilon_{+-}\exp(-2i\psi)} , \qquad (35)$$

rather than developing the second order theory in the manner of Sec. III. Introducing the ordering given in Eq.(9) into (35), and keeping terms up to order η^2, one obtains

$$n^2 = 2\epsilon_{++}^{(0)} + \eta^2 \, 2\left[\epsilon_{++}^{(2)} - \frac{(\epsilon_{++}^{(0)} + \epsilon_{+-}^{(0)}e^{-2i\psi})^2}{\epsilon_{--}^{(-2)}}\right] . \qquad (36)$$

It is important to note that the only second order component needed for the extraordinary mode is $\epsilon_{++}^{(2)}$, which manifests the convenience of the present coordinate system over the cartesian system, in which every component appearing in the dispersion relation must be evaluated to the second order. This component is obtained from the first of Eqs.(8), giving

$$\epsilon_{++}^{(2)} = \chi\left[\frac{3}{64}n_\perp^4 a_{2,0} - \frac{1}{6}n_\perp^2\langle w_\perp^2\rangle + \frac{1}{8}\langle 3w_\perp^2 + w_\parallel^2\rangle - \frac{3}{8}x^2\right],$$

which in the Maxwellian case and $n_\parallel = 0$ yields

$$\epsilon_{++}^{(2)} = -\frac{3}{16}\chi n^4 F_{9/2} + \chi\left(\frac{5}{16} - \frac{1}{6}n^2 - \frac{3x^2}{8}\right), \qquad (37)$$

where $x^2 = -2\xi/w = 2\eta^{-2}(|\Omega|/\omega - 1)$. The second term in the right-hand side of (37) is missing in Shkarofsky's "warm" elements[3]. However, its contribution to the dispersion (as will be shown later) is small, since it affects only the Doppler shift.

By making use of Eqs.(10) and (33), and also approximating successively, one obtains the dispersion relation from (36) as

$$n^2 = (2 - X) + \eta^2\left[\frac{(2 - X)^2}{4XF_{5/2}}(1 - XF_{7/2})^2 - \frac{3}{8}X(2 - X)^2F_{9/2}\right.$$
$$\left. + X(\frac{5}{8} - \frac{3}{4}x^2) - \frac{1}{3}X(2 - X)\right]. \tag{38}$$

The latter equation has notable differences compared to the one derived by Dnestrovskii et al.[2] and used extensively by Fidone et al.[4], viz.,

$$n^2_{D-F} = (2 - X) + \eta^2(2 - X)^2/(4XF_{5/2}). \tag{39}$$

The difference stems from the neglect of the second order contribution from ϵ_{++}, and also from the incorrect assumption that $\epsilon_{xx} - \epsilon_{yy} \cong 2\epsilon_{+-}e^{-2i\psi} = 0$. Unlike the Shkarofsky approximation, Eq.(39) fails to describe the relativistic Doppler broadening appropriately. To see this, consider the imaginary part of n^2, which can be written, with Eq.(A9), as

$$Im(n^2) = \eta^2(2 - X)^2\left[\frac{1}{4X}(1 - \frac{2X}{5})^2 Im(\frac{1}{F_{5/2}}) - \frac{Xx^4}{350}Im(F_{5/2})\right], \tag{40}$$

and

$$Im(n^2_{D-F}) = \eta^2\frac{(2 - X)^2}{4X}Im(\frac{1}{F_{5/2}}). \tag{41}$$

The second term in the bracket in Eq.(40) affects the damping coefficient only for larger values of x, when $Im(n^2)$ is already small. Thus the major difference between Eqs.(40) and (41) stems from the factor $(1 - 2X/5)^2$, leading to the conclusion that the discrepancy becomes increasingly important as X aproaches its cutoff value 2. For a moderate value of $X = 0.3$ the relative error is nearly 25%, and at the cutoff it could be as high as 64%. The D-F approximation is valid, however, for tenuous plasmas. In Fig. 2, a comparison

of the present result with the approximation of Ref.2 is presented for η = 0.034, ω = (2π)18Ghz., and for X = 0.3 and 1.0 is shown. An independent calculation provided by D.B. Batchelor,[6] which is based on Shkarofsky formulation[3], is in good agreement with the present result, indicating that additional Doppler shift terms found in Eq.(38) do not affect the absorption rate.

B. Nearly Perpendicular Ordinary Mode

The appropriate dispersion relation is given by Eq.(21), with the definition (22). For the Maxwellian case, Eqs.(29), together with (12) are used to calculate A. [In the non-Maxwellian case, Eqs.(15) must be used instead of (29).] Although the resulting biquadratic equation for $n_\perp$ can be solved by factorization, a simpler approximate form can be obtained by approximating successively with respect to $n_{||}$. The zero order solution is

$$(n_\perp^2)_0 = (1 - X)/(1 + AX),\tag{42}$$

leading to

$$(n_\perp^2)_1 = (n_\perp^2)_0 \left[1 - \frac{n_{||}^2}{1 + X(2 - X)A}\right].\tag{43}$$

The second term in the bracket represents the broadening due to mixing of modes, and it is expected to generate a small correction, since $n_{||}$ is small.

Numerical evaluations performed for a Maxwellian plasma based on Eqs.(42) and (43) are presented in Fig.3 for various values of the parameter α, and for $\eta = 0.034$, $\omega = (2\pi)18\text{Ghz.}$, $X = 0.3$. It is found that for $\alpha = 5$, both the real and imaginary parts of $k_\perp$ differ only negligibly from their corresponding values for $\alpha = \infty$; [6] they appear to be shifted slightly to the left. The coupling effects are also negligible in this case, since in this case $n_{||} = \eta/\alpha = 0.0068$, corresponding to a propagation angle of $\theta \cong 89.5°$. For $\alpha = 1$ ($\theta \cong 87.7°$) the shift is more pronounced, and the resonance curves are flatter. For $\alpha = 0.5$ ($\theta \cong 85.5°$) the broadening is

further extended into the region $\omega > |\Omega|$, starting from the cutoff value $\omega = 1.0023 |\Omega|$. The coupling effects are also more significant in this case, and the line width is more than doubled compared with the case of perpendicular propagation.

The study of the nearly perpendicular extraordinary mode based on the dispersion relation (22) predicts only broadening due to mode coupling effects. Calculations performed indicate that these effects are of the same (or lower) order of magnitude as those obtained in the second order analysis. Therefore, the broadening of the nearly perpendicular extraordinary mode requires the second order treatment, which will not be given here.

C. Higher Order Harmonics

In order to study the relativistic broadening of higher order harmonics, one must re-examine the ordering of the dielectric tensor. When the ratio of the Larmor radius to the wavelength is of order η (as is assumed in this paper), one observes from Eq.(5) that the contributions of the Nth cyclotron resonance to the dielectric tensor have the following ordering:

$$\epsilon_{--}^N = O(\eta^{-1}\epsilon_{-\parallel}^N) = O(\eta^{-2}\epsilon_{+-}^N) = O(\eta^{-2}\epsilon_{\parallel\parallel}^N) = O(\eta^3 \epsilon_{+\parallel}^N) = O(\eta^{-4}\epsilon_{++}^N) = O(\eta^{2N-4}).$$

Therefore, when $N \geqslant 2$, E_-^0 does not have to vanish. For $N \geqslant 3$, the zero order components do not contain the resonance effects, thus the cold plasma dispersion relation evaluated at $\omega = N|\Omega|$ prevails, viz.,

$$D_N = (1 - \frac{X}{N^2} - n_\perp^2)\left[(1 - \frac{X}{N(N+1)})(1 - \frac{X}{N(N-1)}) - (n^2 + n_\parallel^2)(1 - \frac{X}{N^2-1}) + n^2 n_\parallel^2\right]$$
$$- n_\parallel^2 n_\perp^2 (1 - \frac{X}{N^2-1} - n^2) = 0. \tag{44}$$

For $N = 2$, however, the dielectric tensor contains a resonance contribution through the component

$$\epsilon_{--} = 1 - \tfrac{1}{2}X + \tfrac{1}{2}X n_\perp^2 a_{1,0} + O(\eta), \tag{45}$$

which modifies the cold plasma dispersion relation, yielding

$$D_2 + \tfrac{1}{2}X n_\perp^2 a_{1,0} M_2 = 0, \tag{46}$$

where M_N is the cofactor determinant of the $(--)$ element, viz.,

$$M_N = (1 - \frac{X}{N(N+1)})(1 - \frac{X}{N^2}) - n^2(1 - \frac{X}{N^2}) - \tfrac{1}{2}n_\perp^2(1 - \frac{(N-1)X}{N^2(N+1)}) + \tfrac{1}{2}n_\perp^2 n^2. \tag{47}$$

The function $a_{1,0}$, which is defined in Eq.(11), can be written, by using Eqs.(A1) and (A2) as

$$a_{1,0} = -2 - 2\xi_2 Z_{0,0} + 2(1 + \alpha\xi_2)Z_{0,1} - 3\alpha Z_{0,2} + \alpha^2 Z_{0,3}, \tag{48}$$

where

$$\xi_N = \alpha\eta^{-2}(1 - N|\Omega|/\omega). \tag{49}$$

The functions $Z_{n,m}$ can be calculated from Eqs.(15), or, in the Maxwellian case, from Eqs.(29).

The dispersion relation (46) is a cubic polynomial in $n_\perp^2$, due to the presence of the relativistic term $a_{1,0}$. However, in the case of perpendicular propagation it can be factorized. The first factor leads to the cold plasma dispersion relation for the ordinary mode evaluated at the resonance, i.e.,

$$n^2 = 1 - X/4, \tag{50}$$

indicating that, unlike the case of $N = 1$, the second harmonic ordinary mode contains no zero order broadening. The extraordinary mode, which is associated with the second factor, is obtained from the following dispersion relation:

$$(1-X/6)(1-X/2) - n^2(1-X/3) + n^2 X (1-X/6 - \tfrac{1}{2}n^2) \tfrac{1}{2}a^0_{1,0} = 0, \tag{51}$$

where $a^0_{1,0}$ is the large α limit of $a_{1,0}$, which in the Maxwellian case gives $a^0_{1,0} = -2F_{7/2}$, cf., Eq.(A14). Therefore, for $N = 2$, the absorption of the extraordinary mode is more significant than that of the ordinary mode when $n_{||} = 0$ (or nearly zero). Apart from notation, Eq.(51) in the Maxwellian case is equivalent to those given in Refs. 2 and 3, and to that recently studied by Bornatici et al.[10] in connection with certain mode conversion schemes.

The two factors in the first term of the dispersion relation (51) correspond to the cold plasma right and left hand cutoff frequencies at the resonance, and the coefficient of n^2 in the second term leads to the upper hybrid resonance frequency. The last term generates the relativistic broadening effects.

When $n_{\parallel}$ is not necessarily zero, Eq.(46) can be written as

$$A_3 y^3 + A_2 y^2 + A_1 y + A_0 = 0, \tag{52}$$

where $y = n_{\perp}^2$, and with the shorthand $X_m = 1 - X/m$, the coefficients are

$$A_3 = \tfrac{1}{4} X a_{1,0},$$

$$A_2 = X_3 - A_3(X_4 + 2X_6 - n_{\parallel}^2),$$

$$A_1 = -X_6 X_2 - X_3(X_4 - n_{\parallel}^2) + X_4 n_{\parallel}^2 + A_3 2X_4(X_6 - n_{\parallel}^2),$$

$$A_0 = X_4(X_6 - n_{\parallel}^2)(X_2 - n_{\parallel}^2).$$

In Fig. 4 the function $a_{1,0}$ is illustrated for the Maxwellian case for the values of $\alpha = 5$, 1, and 0.5. The imaginary part of $a_{1,0}$ is zero when $2\xi_2\alpha \geqslant 1$, thus the region of evanescence for the second harmonic is

$$2|\Omega|/\omega > 1 - \tfrac{1}{2}n_{\parallel}^2. \tag{53}$$

For $\alpha = 5$, $a_{1,0}$ is very close to its large α limit $-2F_{7/2}$, and for $\alpha \ll 1$, both real and imaginary parts of $a_{1,0}$ are much less than 1 in magnitude, hence the broadening is negligible in the latter case.

Figure 5 illustrates the variation of the real and imaginary parts of $k_{\perp}$ for the extraordinary mode as a function of $2|\Omega|/\omega$, for the Maxwellian case with $X = 0.3$, $v_c/c = 0.034$, and $\omega = 2\pi$ 18 Ghz.

The general behavior of these curves is similar to those given in Fig. 3 for the fundamental ordinary mode. The $\alpha = 5$ case coincides with the perpendicular propagation case within at least three decimal digits. For smaller values of α, the maximum value of the imaginary part of $k_\perp$ decreases, and the line width increases.

The second harmonic ordinary branch has a negligible imaginary part and does not vary appreciably from the cold plasma value given in Eq. (50). It is important to note that the Re $k_\perp$ curves for the ordinary and the extraordinary modes intersect at the two values of the wave frequency. The one corresponding to the smaller frequency, (or the larger value of $2|\Omega/\omega|$), appears to be promising for a possible mode conversion between these branches.

The third root of Eq. (52) leads to a branch which is nearly electrostatic and heavily absorbed in the plasma.

VI. SUMMARY AND DISCUSSIONS

In the present paper, the relativistic broadening of nearly perpendicular waves near cyclotron resonance is investigated for a warm ($v_e/c \ll 1$), isotropic, but not necessarily Maxwellian, plasma. The underlying assumptions are that $\omega = O(\omega_p) = O(\Omega) = O(kc)$, so that $|\epsilon_{--}| = O(c^2/v_e^2)$ and $n_\parallel = O(v_e/c)$. When these conditions are satisfied, one also has $|\epsilon_{\parallel-}| = O(c/v_e)$, and the zero order dispersion relation is no longer the cold plasma dispersion relation near the resonance as was found in the nonrelativistic case.[7] The formulation is uniform in the parameter $v_e/(cn_\parallel)$, so that all values of $n_\parallel$ [provided that $n = O(1)$] can be studied. However, the relativistic effects are unimportant when this parameter is much less than one. The formalism used is also suitable in geometrical optics methods to study the effects of weak inhomogeneities, even though the magnitude of these effects may be altered due to the ordering assumed.[7]

The basic advantage of the formulation given here is to express the dielectric tensor in terms of a set of functions defined similar to the plasma dispersion function of the nonrelativistic theory. Various recurrence relations and reduction formulae obtained help one to evaluate these functions with relative ease; particularly so for the Maxwellian case, in which case the results are expressed without any integrations or infinite series involving the plasma dispersion function. The latter property enables one to calculate efficiently the broadening of the plasma spectrum in some range of physical space.

The formalism is applied to the fundamental extraordinary mode, (which requires the second order theory,) for $n_{\|}^2 = 0$, and the correction terms for $\text{Im}(n^2)$ are given, which generalize the previously published results for the tenuous plasmas. The study of the fundamental ordinary mode shows that the relativistic broadening is very sensitive to the angle of propagation, and it is restricted to the region within a few degrees from the perpendicular propagation.

The broadening of the higher order harmonics is of order η^{2N-4} for the Nth harmonic. The zero order theory for the second harmonic shows that the ordinary mode is nearly undamped and the extraordinary mode has a damping rate larger than the one for the fundamental ordinary mode. The possibility of the mode conversion between these branches are also pointed out. An analysis for the latter problem requires the inclusion of the spatial inhomogeneities, which will be studied elsewhere.

The region of evanescence for the Nth harmonic is

$$N \, |\Omega|/\omega < 1 - \tfrac{1}{2}n_{\|}^2. \tag{54}$$

It is of interest to note that this condition is independent of the plasma frequency, thus it may be used as a diagnostic tool in weakly inhomogeneous plasmas.

ACKNOWLEDGEMENTS

APPENDIX I: PROPERTIES OF $Z_{n,m}$ FUNCTIONS

The functions $Z_{n,m}$ are defined in Eq.(11) with $\mathbf{D}$ given in (8) are needed only for even values of n. The following recurrence relation is obtained directly from the definition

$$Z_{n+2,m} = (2/\alpha)(Z_{n,m+1} - \xi Z_{n,m} - \langle w_\perp^n w_\parallel^m \rangle) - Z_{n,m+2}, \qquad (A1)$$

which enables one to express these functions in terms of $Z_{0,m}$. Furthermore, when F is isotropic, as is assumed in this paper, the derivatives of these functions can be calculated using

$$dZ_{n,m}/d\xi \equiv Z'_{n,m} = mZ_{n,m-1} - nZ_{n-2,m+1}, \quad (n > 0), \qquad (A2)$$

which is obtained by integrating by parts after replacing F in (11) by its derivative.

Since $D = -\frac{1}{2}\alpha(w - \zeta_+)(w - \zeta_-)$, where

$$\zeta_{\mp} = (1/\alpha)\left[\mu \mp (\mu^2 - 2\xi\alpha)^{\frac{1}{2}}\right], \qquad (A3)$$

μ being the cosine of the polar angle in the velocity space, one obtains, with the use of the identity

$$\frac{w^{k+1}}{w - a} = \frac{a^{k+1}}{w - a} + \sum_{j=0}^{k} a^j w^{k-j},$$

$$Z_{2n,m} = -\frac{2\pi}{\alpha}\int_{-1}^{1}\frac{d\mu}{q}(1 - \mu^2)^n \mu^m \int_0^{\infty} dw\, F(w)\left[\frac{\zeta_+^{2n+m+2}}{w - \zeta_+} - \frac{\zeta_-^{2n+m+2}}{w - \zeta_-}\right.$$

$$\left. + \sum_{j=0}^{2n+m} (\zeta_+^{j+1} - \zeta_-^{j+1})w^{2n+m-j}\right], \qquad (A4)$$

where $q = (1/\alpha)(\mu^2 - 2\xi\alpha)^{\frac{1}{2}} = \frac{1}{2}(\zeta_+ - \zeta_-)$. Since $\zeta_{\pm}(-\mu) = -\zeta_{\mp}(\mu)$, after extending F as $F(-w) = F(w)$ and replacing $\mu \to -\mu$ and $w \to -w$, one obtains

$$z_{2n,m} = -\frac{1}{\alpha} \int_{-1}^{1} d\mu \, (1-\mu^2)^n \mu^m \sum_{j=0}^{2n+m} \langle w^{2n+m-2-j} \rangle \sum_{p=0}^{j} \zeta_+^p \, \zeta_-^{j-p}$$

$$- \frac{1}{\alpha} \int_{-1}^{1} \frac{d\mu}{q} (1-\mu^2)^n \mu^m \left[\zeta_+^{2n+m+2} Z(\zeta_+) - \zeta_-^{2n+m+2} Z(\zeta_-) \right] \, ,$$

(A5)

where the function $Z(\zeta)$ is as defined in Eq.(14). Equation (A5) can be further reduced by observing that the lower half of the μ integration in the first term of (A5) is $(-1)^{j+m}$ times that of the upper half, hence $j+m$ must be even. Moreover, since $d\mu/q = \pm\alpha d\zeta_{\pm}/\zeta_{\pm}$, with $\alpha\zeta_{\pm}^2 - 2\mu\zeta_{\pm} + 2\xi = 0$, the two terms in the second integral can be combined into one, by introducing the variable ζ as

$$\alpha\zeta = \mu + (\mu^2 - 2\xi\alpha)^{\frac{1}{2}},$$

(A6)

and the contour C defined as the image of $-1 \leq \mu \leq 1$ under the mapping (A6). One obtains, with $r = [m/2]$ being the closest integer less than or equal to m/2,

$$z_{2n,m} = -\frac{2}{\alpha} \int_{0}^{1} d\mu \, (1-\mu^2)^n \mu^m \sum_{p=0}^{n+r} \langle w^{2n+2r-2p-2} \rangle \sum_{j=0}^{2p+m-2r} \zeta_+^j \zeta_-^{2p+m-2r-j}$$

$$- 2 \int_{C} d\zeta \left[\zeta^2 - (\tfrac{1}{2}\alpha\zeta^2 + \xi)^2 \right]^n (\tfrac{1}{2}\alpha\zeta^2 + \xi)^m \zeta \, Z(\zeta).$$

(A7)

Equations (15) are obtained directly from (A7). Since the factors which multiply $\zeta Z(\zeta)$ in the integrand in (A7) are polynomials in ζ^2, the dependence of the second term on the functions I_n introduced in (13) is evident.

APPENDIX II. CONNECTION WITH F_q and $\mathcal{F}_q$ FUNCTIONS

In this Appendix the relations between $Z_{n,m}$ functions and the F_q functions introduced by Dnestrovskii et al. is given. It is also shown how they are connected to Shkarofsky's $\mathcal{F}_q$ functions, so that the results presented in this paper can be compared efficiently with those previously known.

Consider the functions, with $F(w) = \pi^{-3/2}\exp(-w^2)$,

$$F^{(m)}_{n+3/2} = \frac{2^n(m-1)!}{(2n+1)!!}\int d^3w\, \frac{w^{2n}F(w)}{(w^2-x^2)^m} \,. \tag{A8}$$

These functions have the following properties: For $m = 1$,

$$F^{(1)}_{3/2} = 2 + 2xZ(x) = -Z'(x), \text{ with } x = -(x^2)^{\frac{1}{2}},$$

$$F^{(1)}_{q+1} = (1 + x^2 F^{(1)}_q)/q. \tag{A9}$$

Since these are the recursive properties which generate the Dnestrovskii functions, one concludes that

$$F^{(1)}_q \equiv F_q. \tag{A10}$$

For $m > 1$,

$$F^{(m)}_{q+1} = \left[(m-1) F^{(m-1)}_q + x^2 F^{(m)}_q\right]/q,$$

$$\frac{d}{d(x^2)}F^{(m)}_q = F^{(m+1)}_q, \tag{A11}$$

$$F^{(m+1)}_{q+1} = F^{(m)}_q - F^{(m)}_{q+1}.$$

In order to relate these functions to $Z_{n,m}$, consider the asymptotic expansion of the latter in $1/\alpha$ which is obtained by expanding $1/D$ into power series, giving

$$Z_{n,m} = -\sum_{k=r}^{\infty} \left(\frac{2}{\alpha}\right)^{2k-m+1} \int d^3w \; \frac{w_{\perp}^n w_{\parallel}^{2k} F(w)}{(w^2 - x^2)^{2k-m+1}} \; , \tag{A12}$$

where $r=[(m+1)/2]$, and $x^2 = -2\xi/\alpha = 2\eta^{-2}(|\Omega|/\omega - 1)$.

One obtains, after some straightforward manipulations, and angular integrations,

$$Z_{2n,m} = -\frac{n!}{2^{m-1}} \sum_{k=r}^{\infty} \frac{(2k)!}{\alpha^{2k-m+1} k! \, (2k-m)!} \; F_{n+k+3/2}^{(2k-m+1)} \; . \tag{A13}$$

The large α limits given in Eqs.(32) follow directly from (A13), with the third of Eqs.(A11). Substitution of (A13) into the definition of $a_{n,m}$ in (11) yields the result

$$a_{n,m} = -\frac{n!}{2^{m-1}} \sum_{k=r}^{\infty} \frac{(2k)!}{\alpha^{2k-m} \; k! \, (2k-m)!} \; F_{n+k+5/2}^{(2k-m+1)} \; , \tag{A14}$$

from which the limits in Eqs.(33) follow. When this result is substituted into Eqs.(10), one finds that Shkarofsky's "warm" elements are obtained to the order one in η and order -2 in α.

In order to establish the relationship with Shkarofsky's $\mathcal{F}_q$ function (or W_n function used in Ref.4) consider the simplification (15) given in Ref.3

$$\mathcal{F}_q = -\frac{i}{2} \int_0^{\infty} \frac{dt}{(1-it)^q} \exp\left[-x^2 t - \frac{t^2}{\alpha^2(1-it)}\right]$$

$$= \exp(-1/\alpha^2) \sum_{j=0}^{\infty} \frac{1}{\alpha^{2j} \, j!} \; F_{q+j}^{(1)}(x^2 + 1/\alpha^2) \; . \tag{A15}$$

In the latter equation, the function $F_q^{(1)}$ is regarded as a function of x^2. Expanding this function, as well as the multiplicative exponential function, into powers of $1/\alpha$, and rearranging the terms

one obtains the asymptotic series

$$\mathcal{F}_q^{(m)} = \sum_{k=0}^{\infty} \frac{1}{\alpha^{2k} k!} F_{q+k}^{(2k+m)}, \tag{A16}$$

where

$$\mathcal{F}_q^{(1)} = \mathcal{F}_q , \text{ and } \mathcal{F}_q^{(m+1)} = \frac{d}{d(x^2)} \mathcal{F}_q^{(m)}. \tag{A17}$$

In deriving (A16), the following relation is used, which can be established by induction,

$$F_{q+k}^{(2k+m)} = \sum_{p=0}^{k} (-1)^{k-p} \binom{k}{p} \sum_{j=0}^{p} \binom{p}{j} F_{q+j}^{(p+m-j)}. \tag{A18}$$

The functions $a_{n,m}$ for $m = 0, 1,$ and 2, which appear in Eq.(10), can be expressed in terms of Shkarofsky functions as

$$a_{n,0} = -2n! \, \mathcal{F}_{n+5/2}^{(1)},$$

$$a_{n,1} = -(2n!/\alpha) \, \mathcal{F}_{n+7/2}^{(2)}, \tag{A19}$$

$$a_{n,2} = -n! \left[\mathcal{F}_{n+7/2}^{(1)} + (2/\alpha^2) \, \mathcal{F}_{n+9/2}^{(3)} \right].$$

Equations (A19) can be further reduced by making use of the recurrence relation

$$\mathcal{F}_{q+1}^{(m+1)} = \mathcal{F}_q^{(m)} - \mathcal{F}_{q+1}^{(m)}. \tag{A20}$$

The function A may also be written as

$$A = \tfrac{1}{2}\mathcal{F}_{7/2}^{(1)} + (1/\alpha^2)\left[\mathcal{F}_{9/2}^{(1)} - (\mathcal{F}_{7/2}^{(1)})^2 / \mathcal{F}_{5/2}^{(1)} \right] \tag{A21}$$

An evaluation of A based on the latter, together with (A15), is certainly much less efficient than the method presented in this work. Keeping terms up to order $1/\alpha^2$ in (A21), [using (A16),] and substituting into (42), one gets the dispersion relation given in Ref. 3.

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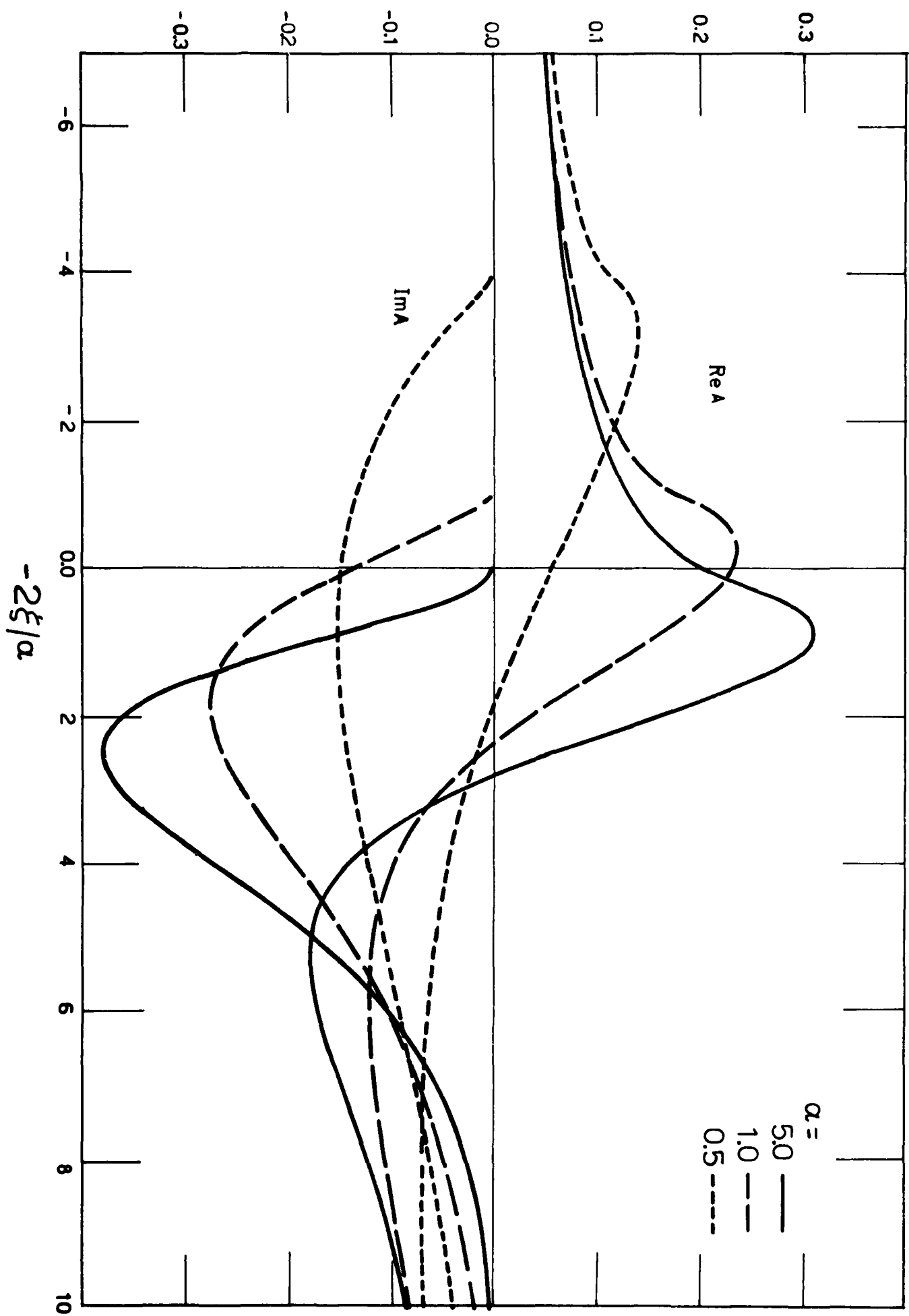
FIGURE CAPTIONS

Fig. 1   Real and imaginary parts of $A(\xi,\alpha)$ for the Maxwellian
          case as a function of $x^2 = -2\xi/\alpha = 2\eta^2(|\Omega/\omega| - 1)$, for
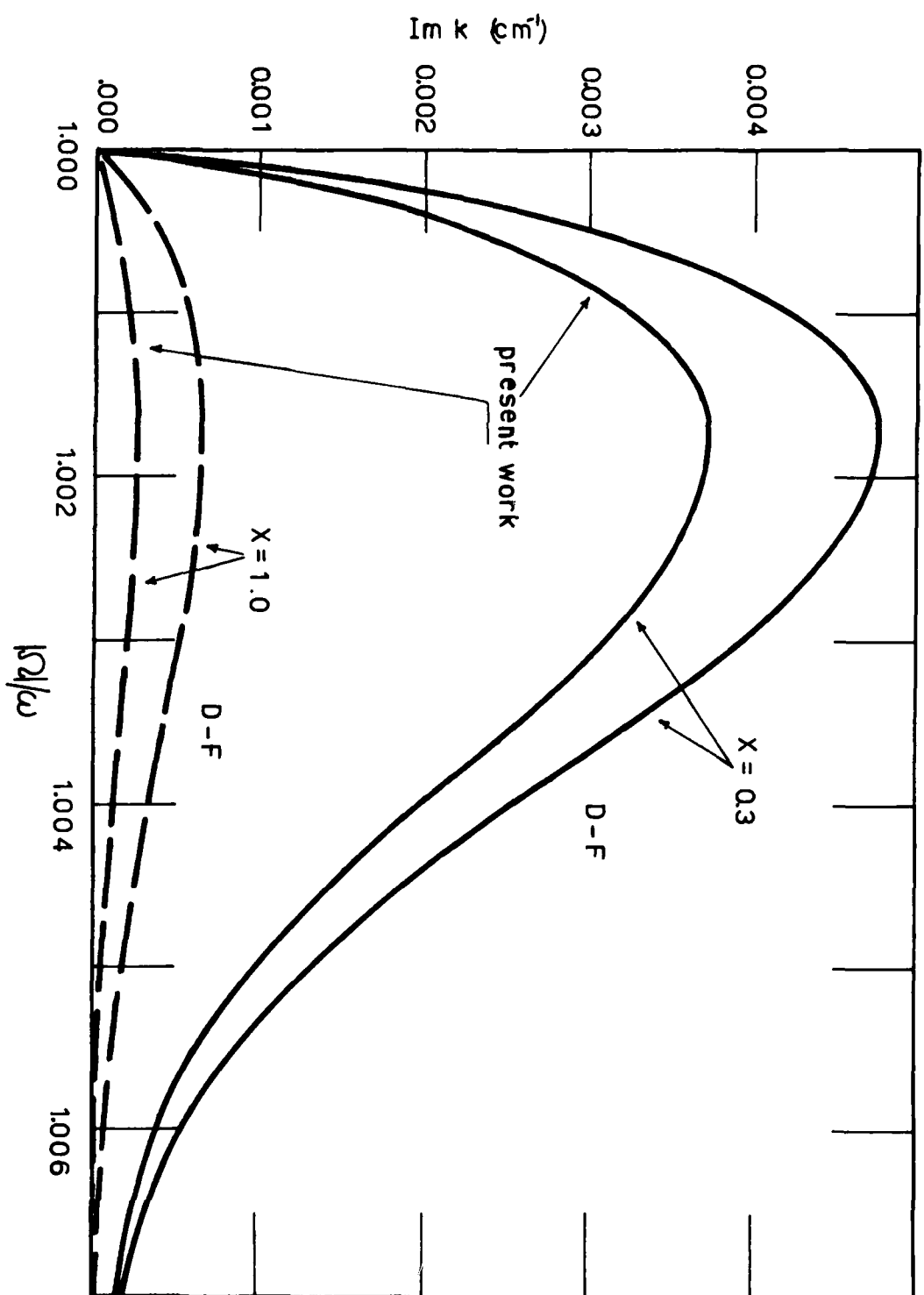 $\alpha = 5.0$, 1.0, and 0.5.

Fig. 2   A comparison of Dnestrovskii approximation with the
          present analysis: $Im(k_\perp)$ (cm^{-1}) vs. $|\Omega/\omega|$ for the
          perpendicular extraordinary mode, and for $X = 0.3$, 1.0,
          $\eta = 0.034$, $\omega = (2\pi)18Ghz.$.
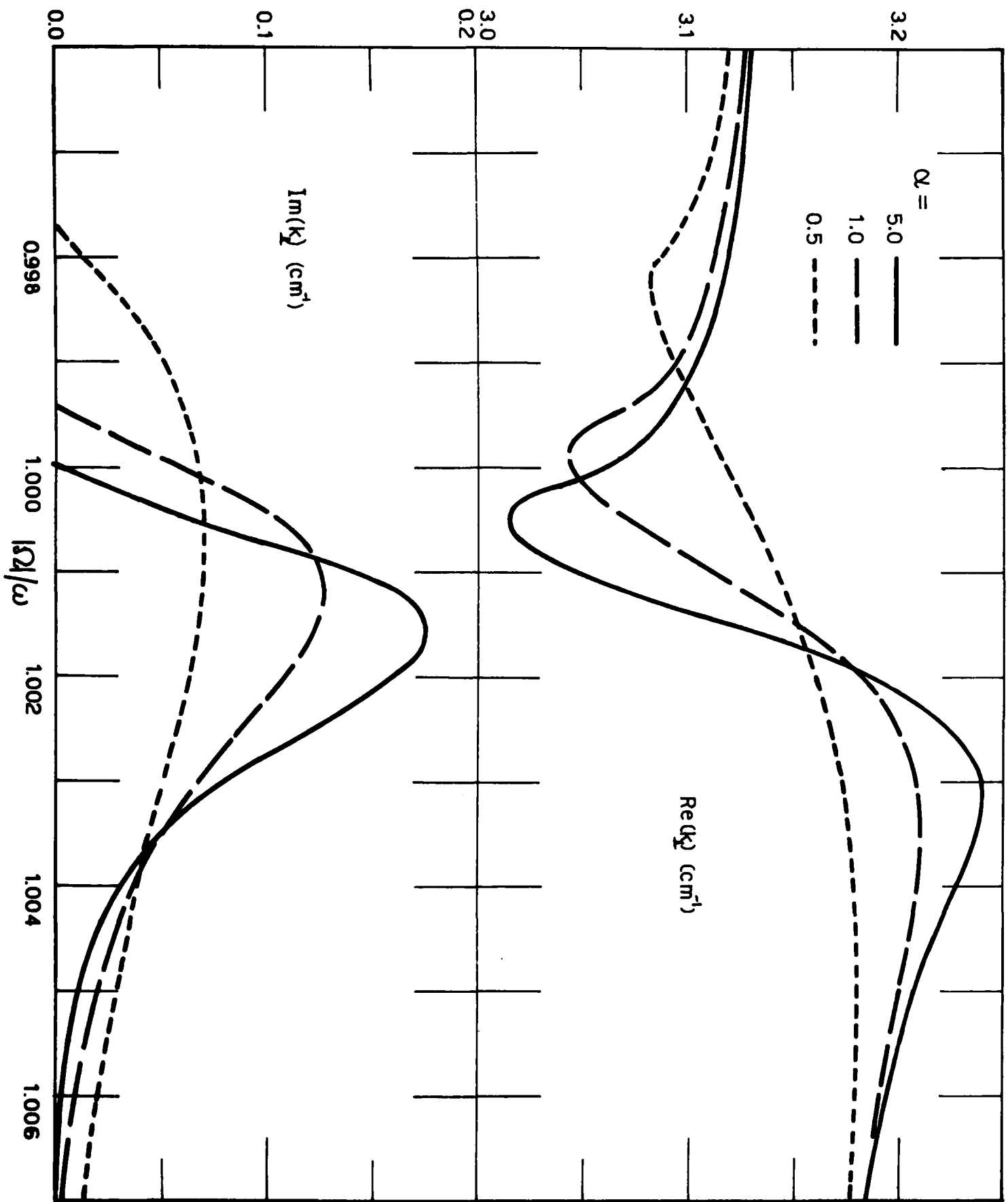
Fig. 3   Real and imaginary parts of $k_\perp$ (cm^{-1}) as a function of
          $|\Omega/\omega|$ for the nearly perpendicular ordinary mode, for
          $X = 0.3$, $\eta = 0.034$, $\omega = (2\pi)18GHz.$, and for the values
          of $\alpha = 0.5$, 1.0, 5.0. (Maxwellian case)
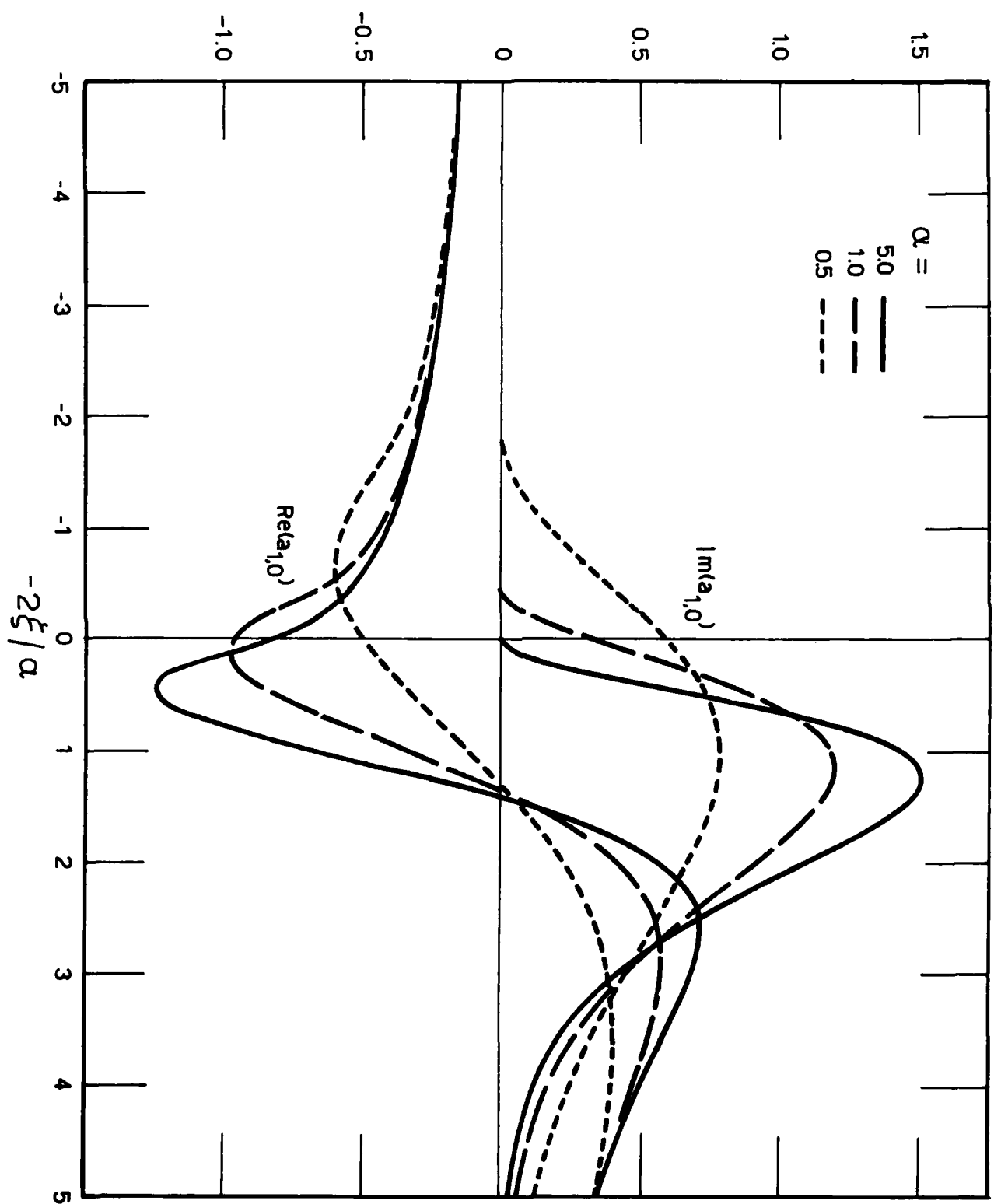
Fig. 4   Real and imaginary parts of $a_{1,0}$ for the Maxwellian
          case as a function of $x^2 = -2\xi_2/\alpha = 2\eta^2(2|\Omega|/\omega - 1)$, for
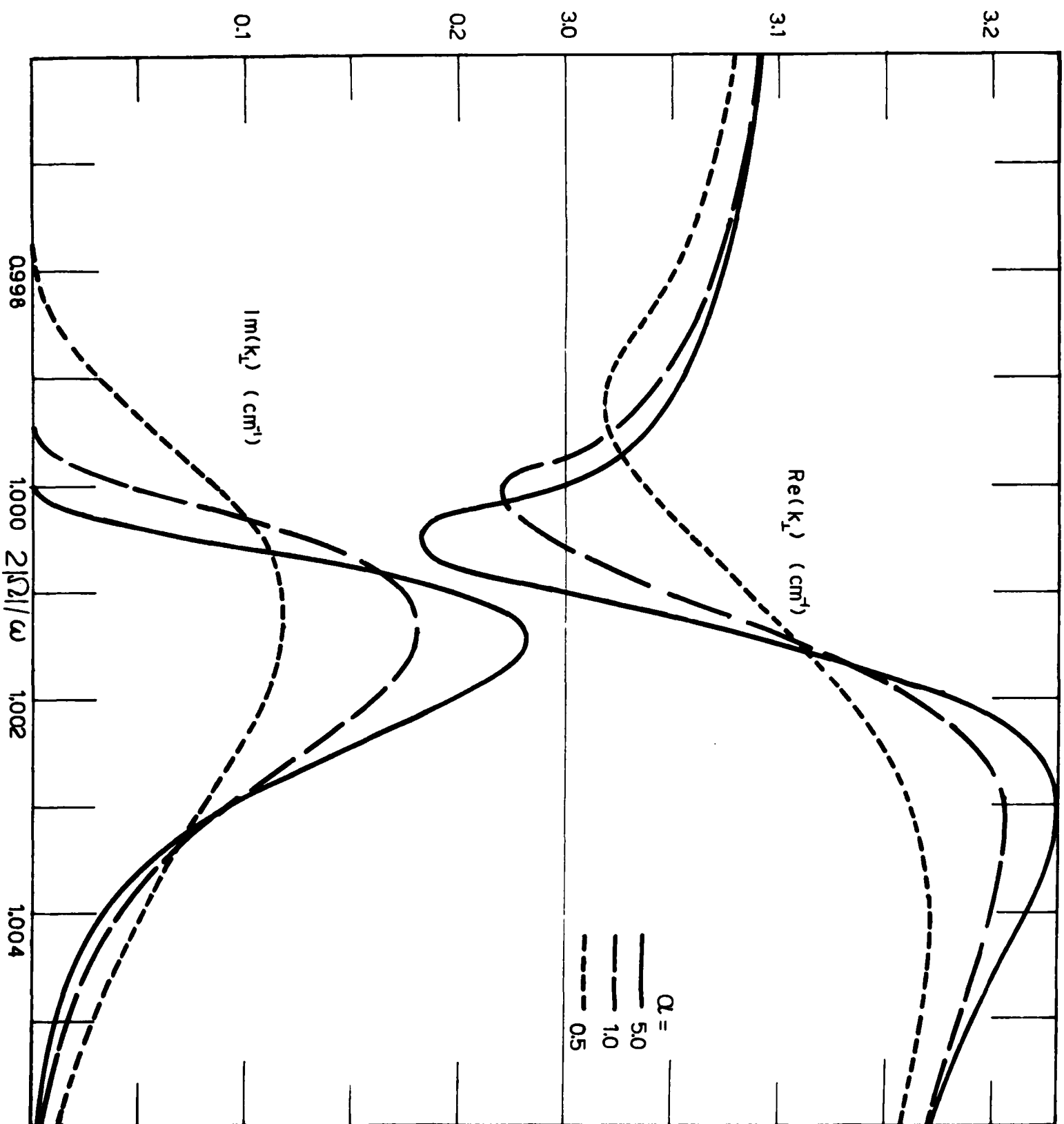 $\alpha = 5.0$, 1.0, and 0.5.

Fig. 5   Real and imaginary parts of $k_\perp$ (cm^{-1}) as a function of
          $2|\Omega|/\omega$ for the second harmonic extraordinary mode, for
          $X = 0.3$, $\eta = 0.034$, $\omega = (2\pi)18Ghz.$, and for the values
          of $\alpha = 0.5$, 1.0, 0.5. (Maxwellian case)

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