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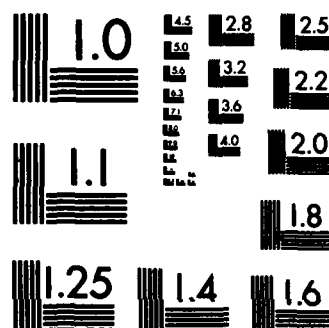
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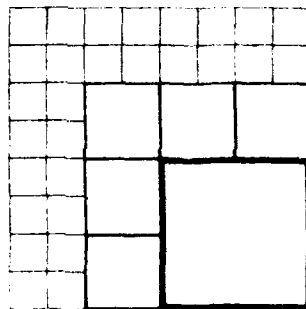


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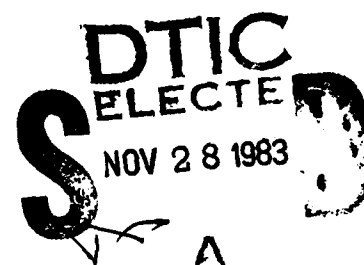
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Nyquist Frequency

Abstract

Nyquist frequency is related to aliasing and the sampling theorem. This report explains the relationship among these terms.

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Key words: Aliasing, Nyquist frequency, spectrum analysis, the sampling theorem

pi/delta

The Nyquist frequency is half the sampling frequency when a continuous-time function is sampled at equally spaced time points. That is, the Nyquist frequency is π/Δ (in radians per unit time) where Δ is the time interval between two successive sampled data. *delta*
this report discusses the
In this article, we discuss the basic ideas of the Nyquist frequency and a relevant and very *explains its relationship between*
aliasing and the sampling theorem
~~important theorem - the sampling theorem.~~

In many applications involving processing a continuous-time signal, it is often preferable to convert the continuous-time signal to a discrete-time signal since discrete-time signal processing can be implemented with a digital computer. It is important to examine whether the discrete-time signal preserves all the information in the original continuous-time signal. We first consider the case that the signal $x(t)$ is a real-valued function. Assume that its Fourier transform $X(\omega)$
 $= \int_{-\infty}^{\infty} x(t) \exp(-i\omega t) dt$ exists and that $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \exp(i\omega t) d\omega$
for all t . The signal $x(t)$ is sampled at $t = n\Delta$, $n = \dots, -1, 0, 1, \dots$. We are interested in interpolating $x(t)$ from its samples $x(n\Delta)$. A natural question arises: Under what conditions can $x(t)$ be perfectly reconstructed from $x(n\Delta)$? The samples $x(n\Delta)$ can be related to $X(\omega)$ as follows.

$$x(n\Delta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \exp(i\omega n\Delta) d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi/\Delta}^{\pi/\Delta} \left[\sum_{k=-\infty}^{\infty} X(\omega + 2\pi k/\Delta) \right] \exp(i\omega n\Delta) d\omega$$

Therefore, $X_d(\omega) = \sum X(\omega + 2\pi k/\Delta)$ ($|\omega| < \pi/\Delta$) is the discrete Fourier transform of the sequence $x(n\Delta)$. Obviously, $X_d(\cdot)$ is obtained by folding $X(\cdot)$ every π/Δ radians per unit time. (Here we identify ω with $-\omega$.) This frequency π/Δ is the Nyquist frequency, and hence also called the folding frequency. It is easy to see that $X(\cdot)$ is not uniquely determined by $X_d(\cdot)$. In other words, some sinusoidal components of different frequencies (e.g. $2\pi k/\Delta \pm \omega_0$, $k = \dots, -1, 0, 1, \dots$) in $x(t)$ cannot be distinguished from one another by the observations $x(n\Delta)$. This is called aliasing. Aliasing is the effect of under-sampling. This effect is the principle on which the stroboscopic effect is based [6, Section 8.3].

When $x(t)$ is a band-limited signal with $X(\omega) = 0$ for $|\omega| > \omega_M$, $X_d(\omega)$ is identical to $X(\omega)$ (i.e. no aliasing) if $\omega_M < \pi/\Delta$. In other words, from the uniqueness property of Fourier transform, $x(t)$ is uniquely determined by its samples $x(n\Delta)$ under the condition that $X(\omega) = 0$ for $|\omega| > \pi/\Delta$. This is usually called the (Shannon) sampling theorem on information theory [5]. From the sampling theorem, if we sample the signal $x(t)$ at a rate at least twice the highest frequency in $x(t)$, then $x(t)$ can be completely recovered from the samples. This sampling rate (twice the highest frequency in $x(t)$) is commonly referred to as the Nyquist rate. Actually $x(t)$ can be explicitly written, in terms of $x(n\Delta)$, as

$$x(t) = \sum_{n=-\infty}^{\infty} x(n\Delta) \frac{\sin \pi \{(t/\Delta) - n\}}{\pi \{(t/\Delta) - n\}}$$

It should be noted that band-limited signals are generally not realizable physically, for $x(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} X(\omega) \exp(i\omega t) d\omega$ is analytic in t , as a complex variable, and therefore cannot vanish for all $t < -T$ for arbitrarily large T . Therefore, aliasing is inevitable in practice. A discussion on error bounds for aliasing can be found in [5].

In some applications, the signal $x(t)$ is assumed to be bandpass, i.e. there exist $0 \leq \omega_0 < \omega_1$ such that $X(\omega) = 0$ outside the intervals $[\omega_0, \omega_1]$ and $[-\omega_1, -\omega_0]$. The sampling theorem says that $x(t)$ can be recovered from equally spaced sampling at a rate of $2\omega_1$. Actually, this rate $2\omega_1$ is too conservative. It has been shown [4, Section 8.5] that a sampling rate of $2\omega_1/v$ is enough to recover $x(t)$ where v is the largest integer not beyond $\omega_1/(\omega_1 - \omega_0)$.

The sampling theorem has been generalized to many situations such as random signals. When $x(t)$ ($-\infty < t < \infty$) is a wide-sense stationary stochastic process, possessing a spectral density which vanishes outside the interval $[-\pi/\Delta, \pi/\Delta]$, Balakrishnan showed [1] that $x(t)$ has the representation

$$x(t) = \lim_{N \rightarrow \infty} \sum_{n=-N}^N x(n\Delta) \frac{\sin \pi \{(t/\Delta) - n\}}{\pi \{(t/\Delta) - n\}}$$

for every t , where $\lim$ stands for limit in the mean square. Gardner [3] derived a similar result for non-stationary stochastic processes. Obviously, the Nyquist frequency and

Nyquist rate can be similarly defined in the random signal case.

Blackman and Tukey [2, Section 12] provided good interpretations on aliasing. Jerri [5] gave an excellent review of the sampling theorem and its various extensions and applications. He discussed topics such as unequally spaced sampling, higher-dimensional functions, nonband-limited functions and error bounds for the truncation, aliasing, and jitter. A very exhaustive bibliography can be found therein.

References

- [1] Balakrishnan, A.V. (1957). *IRE Trans. Inform. Theory*, IT-3, pp. 143-146.
- [2] Blackman, R.B. and Tukey, J.W. (1958). *The Measurement of Power Spectra*, Dover, New York.
- [3] Gardner, W.A. (1972). *IEEE Trans. Inform. Theory*, IT-18, pp. 808-809.
- [4] Gregg, W.D. (1977). *Analog and Digital Communication*, John Wiley, New York.
- [5] Jerri, A.J. (1977). *Proc. IEEE*, 65, pp. 1565-1596.
- [6] Oppenheim, A.V., Willsky, A.S. and Young, I.T. (1983). *Signals and Systems*, Prentice-Hall, Englewood Cliffs, N.J.

(INFORMATION THEORY
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