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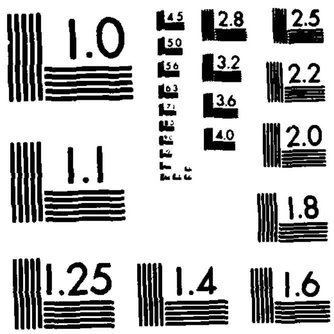
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ANALYSIS OF A DELAYED DELTA MODULATOR

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ABSTRACT

Delayed Delta Modulation (DDM) uses a second feedback loop in addition to the standard DM loop. While the standard loop compares the current predictive estimate of the input to the current sample, the new loop compares it to the upcoming sample so as to detect and anticipate slope overloading. Since this future sample must be available before the present output is determined and the estimate updated, delay is introduced at the encoding.

The performance of DDM with perfect integration and step-function reconstruction is analyzed for each of three inputs. In every case, the stochastic stability of the system is established. For a discrete time i.i.d. input, the (limiting) joint distribution of input and output is derived, and the (asymptotic) mean square sample point error MSE(SP) is computed when the input is Gaussian. For a Wiener input, the joint distribution of the sample point and predictive errors is derived, and MSE(SP) and the time-averaged MSE (MSE(TA)) are computed. For a stationary, first-order Gauss-Markov input, the joint distribution of input and output is derived, and MSE(SP) and MSE(TA) computed. Graphs of the MSE's illustrate the improvement attainable by using DDM instead of DM. With optimal setting of parameters, MSE(SP) (MSE(TA)) is reduced about 15% (35%).

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I. DELTA MODULATION AND DELAYED DELTA MODULATION

Feedback/quantization schemes, such as Pulse-Code Modulation (PCM), Differential PCM (DPCM), and Adaptive PCM and DPCM, have been studied by a number of authors (cf. [1],[3],[5-10], [14] and [15]). These schemes all have the same goal: Instead of transmitting the entire analog signal X_t, $t\geq 0$, they use only the samples X_{nT}, $n=0,1,2,\ldots$, to generate and transmit digitally an appropriately coded (e.g. quantized and possibly block coded) sequence Y_{nT}, $n=0,1,2,\ldots$, from which an analog output Y_t, $t\geq 0$, is reconstructed, which should approximate the input as closely as possible. The analog input, which we denote by X_t throughout, is usually taken to be some random process. The sampling interval $T>0$ is fixed, and we employ (generically) the notation $X_n \triangleq X_{nT}$ throughout as well.

Delta Modulation (DM) is the simplest of the differential feedback/quantization schemes. Using either perfect ($\rho=1$) or leaky ($0\leq\rho<1$) integration, the DM Y_t^{DM} of X_t is generated as follows:

$$Y_0^{DM} = 0,$$

$$Y_{n+1}^{DM} = \rho Y_n^{DM} + \Delta b_{n+1}^{DM}, \quad n=0,1,2,\ldots$$

where the step size $\Delta>0$ and $b_{n+1}^{DM} = \text{sign}(X_{n+1} - \rho Y_n)$. For $t\in[nT,(n+1)T)$, the value of Y_t^{DM} is appropriately defined, depending on the system and the input, either in terms of the present and earlier values of Y_n (predictive reconstruction) or in terms of neighboring past and future values of Y_n (interpolative reconstruction). When an interpolative reconstruction in used, the reconstruction is delayed until the required future value(s) of Y_n are available. Frequently, the "step-function" (predictive) reconstruction $Y_t = Y_{nT}$ for $t\in[nT,(n+1)T)$, $n=0,1,2,\ldots$ is considered. More sophisticated reconstructions will, of course, yield better performance.

It is well known that the size of Δ determines the trade-off between the granular, or round-off, error of the system and the slope overload error. In the

DM (with perfect integration and step-function reconstruction) realization of Figure 1, [8T,12T] is an interval of slope overload. Note how the DM response at time 7T, though locally optimal, exacerbates the effect of the slope overload suffered immediately afterwards. Clearly, an upward step would have been preferable to the downward step taken, but, a priori, there was no way of knowing this. If we knew when a slope overload was to occur we could reduce its effect by anticipating it and getting a "head start" on it. This suggests the following scheme, for $c \geq 0$,

$$Y_{-1} = 0$$

$$Y_0 = \begin{cases} 0 & \text{if } |X'_0| \leq c\Delta , \\ \Delta \, \text{sign}(X'_o) & \text{if } |X'_0| > c\Delta , \end{cases}$$

and for $n=0,1,2,\ldots,$

$$Y_{n+1} = \rho Y_n + \Delta b_{n+1},$$

where

$$b_{n+1} = \begin{cases} \text{sign}(X_{n+1} - \rho Y_n) & \text{if } |X'_{n+1}| \leq c\Delta , \\ \text{sign}(X'_{n+1}) & \text{if } |X'_{n+1}| > c\Delta . \end{cases}$$

Unfortunately, this scheme requires sampling both the signal X_t and its derivative X'_t. The following scheme is similarly motivated but requires sampling X_t only, for $c \geq 0$,

$$Y_{-1} = 0,$$

$$Y_0 = \begin{cases} 0 & \text{if } |X_1| \leq c\Delta , \\ \Delta \, \text{sign}(X_1) & \text{if } |X_1| > c\Delta , \end{cases}$$

and for $n=0,1,2,\ldots,$

$$Y_{n+1} = \rho Y_n + \Delta b_{n+1},$$

where

$$b_{n+1} = \begin{cases} \text{sign}(X_{n+1} - \rho Y_n) & \text{if } |X_{n+2} - Y_n| \leq c\Delta , \\ \text{sign}(X_{n+2} - \rho Y_n) & \text{if } |X_{n+2} - Y_n| > c\Delta . \end{cases}$$

We call this Delayed Delta Modulation (DDM) because Y_n is not determined until X_{n+1} has been sampled. Thus, delay is introduced at the encoding step (and, if we wish, at the decoding step, i.e. the reconstruction, as well). To facilitate the analysis and allow comparison with the results of Fine [5], Slepian [15] and Masry and Cambanis [14] we henceforth consider only perfectly integrated DDM, and so take $\rho=1$, with the step function reconstruction, $Y_t = Y_{nT}$ for $t \epsilon [nT,(n+1)T)$, $n=0,1,2,\ldots$. Figure 1 includes both the DM and (c=1)-DDM outputs of the given input.

Note that setting $c = \infty$ reduces DDM to DM. This enables us to check our computations against the literature. When we set $c = 0$, the DDM output is just the DM output shifted to the left one sampling interval T, i.e., $Y_n^{DDM} = Y_{n+1}^{DM}$.

A system using greater delay and additional comparisons would undoubtedly outperform this basic DDM scheme. Unfortunately, adding non-linear feedback loops to non-linear differential quantization schemes greatly increases the difficulty of their analysis.

II. STOCHASTIC STABILITY OF DDM

Several types of "stochastic stability" have been considered for feedback/quantization of random inputs. Henceforth, we let $e_n \triangleq X_n - Y_n$ denote the sample point error (at instant n).

Gersho [8] (Hayashi [9]) has shown, for DM (with perfect integration) of a stationary and continuously distributed input that has finite variance and a rational spectral density (of an input that has i.i.d., zero-mean, finite variance increments), that

(i) the distribution of (X_n, Y_n) (of e_n) converges weakly to a unique stationary steady-state distribution irrespective of the initial (output) state, and

(ii) $n^{-1}\sum_{k=0}^{n-1} E|e_k|$ is uniformly bounded, and $E|e| < \infty$, where e is the limit in distribution of e_n.

Kieffer [12] considers several other types of stochastic stability. Letting $X_k^\infty \triangleq (X_k, X_{k+1}, \ldots)$, he shows that for DM of a stationary input, $n^{-1}\sum_{k=0}^{n-1} f(X_k^\infty, Y_k^\infty)$ converges almost surely as $n \to \infty$ for every bounded measurable f.

A stronger, and more useful, notion of stability is what may be called "rth moment stochastic stability." We say an analog-digital-analog system is rth moment stochastically stable when for an input that is either stationary with finite rth moment or has i.i.d. zero-mean finite-rth-moment increments

(a) $e_n \overset{D}{\Rightarrow} e$, whose distribution does not depend on the initial state and is stationary for the sequence, and

(b) $$\lim_{n\to\infty} n^{-1}\sum_{k=0}^{n-1} |e_k|^r = E|e|^r < \infty \quad \text{a.s.}$$

It is well known that DM, and DPCM, with leaky integration satisfy (a) and generate a uniformly bounded output. Thus, for a stationary input they satisfy (b) trivially and so are r^{th} moment stochastically stable. For systems with perfect integrators (b) is far more difficult to obtain and may be replaced by (b') $\lim_{n\to\infty} E|e_n|^r = E|e|^r < \infty$, or

(b") $\lim_{n\to\infty} n^{-1}\sum_{k=0}^{n-1} E|e_k|^r = E|e|^r < \infty$,

or even for some v<r,

(b"') $\lim_{n\to\infty} n^{-1}\sum_{k=0}^{n-1} E|e_k|^v = E|e|^v < \infty$.

The averagings used by Gersho [8] and Hayashi [9] can be easily extended to show that DM, and DPCM, with perfect integration satisfy (b"') for v=r-1. In Theorem 1 below we present a uniform bound on $|Y_n^{DDM} - Y_n^{DM}|$. Hence, DDM with leaky (perfect) integration satisfies (b) ((b"') with v=r-1), and so is rth moment (v=(r-1)th moment) stochastically stable whenever (a) holds.

Henceforth, when we say simply that DDM is stochastically stable we mean only that (a) is satisfied. In the following subsections we will show that DDM with prefect integration is stochastically stable. The modifications needed to show that the same is true for the leaky integrator are straightforward.

In order to establish the results quoted earlier, Gersho [8] embeds both the input X_n and output Y_n in a "state vector" S_n, while Hayashi [9] considers $e_n^* = X_{n+1}-Y_n$, the predictive error. They show that S_n (e_n^*) is Markovian, and that $n^{-1}\sum_{k=0}^{n-1} E|Y_k|$ ($n^{-1}\sum_{k=0}^{n-1} E|e_k^*|$) is uniformly bounded. Thus, the sequence of averaged distributions of S_n (e_n^*) is tight, and so S_n (e_n^*) has a stationary distribution. By verifying certain conditions on the transition function of S_n (e_n^*), it follows that the distribution of S_n (e_n^*) converges weakly to a unique (stationary) steady-state distribution irrepsective of the initial state S_0 (e_0^*). Our approach parallels theirs.

The following uniform bound is very useful. From it we obtain tightness and finiteness of moments for DDM as a consequence of the same for DM. This bound can be shown to be tight for $c\neq 0$, but the argument will not be produced here.

<u>Theorem 1</u> For any given input, perfectly integrating systems (ρ=1), and $c\geq 0$, we have, for all n,

$$|Y_n^{DDM} - Y_n^{DM}| \leq (c+3)\Delta \ .$$

<u>Proof</u>: Let $D_n \triangleq Y_n^{(DDM)} - Y_n^{DM}$, $n\geq 0$. Clearly (i) $D_0 = 0$, (ii) $D_1 = -\Delta, 0,$ or Δ, and (iii) $D_{n+1} = D_n - 2\Delta$, D_n, or $D_n + 2\Delta$, $n\geq 1$.

The bound will be established by contradiction. Let $n_0 < \infty$ be the smallest value of n s.t. $|D_n| > (3+c)\Delta$. Then $|D_{n_0}| = (d_0+2)\Delta$, where the integer d_0 satisfies $d_0 \varepsilon (1+c, 3+c]$. Without loss of generality, assume

$$D_{n_0} = (d_0+2)\Delta \ . \tag{2.1}$$

Let n_1 denote the largest value of n less than n_0 such that $D_n = (d_0-2)\Delta$. Then

$$D_{n_1} = (d_0-2)\Delta \quad \text{and} \quad D_{n_1+1} = d_0\Delta \ , \tag{2.2}$$

and so $b_{n_1+1} = 1$ and $b_{n_1}^{DM} = -1$. This yields $\{(X_{n_1+2} > Y_{n_1} + c\Delta) \cup (|X_{n_1+2} - Y_{n_1}| \leq c\Delta$ and $X_{n_1+1} > Y_{n_1})\} \cap \{X_{n_1+1} \leq Y_{n_1}^{DM}\}$, and since $Y_{n_1} = Y_{n_1}^{DM} + (d_0-2)\Delta \geq Y_{n_1}^{DM}$, this reduces to $X_{n_1+2} > Y_{n_1} + c\Delta$ and $X_{n_1+1} \leq Y_{n_1}^{DM}$. From the first of these inequalities and (2.2) we derive

$$\begin{aligned} X_{n_1+2} &> Y_{n_1}^{DM} + (d_0-2) + c\Delta \\ &> Y_{n_1}^{DM} + (1+c-2)\Delta + c\Delta \\ &\geq Y_{n_1}^{DM} - \Delta + 2c\Delta \\ &\geq Y_{n_1}^{DM} + \Delta b_{n_1+1}^{DM} + 2c\Delta \\ &\geq Y_{n_1+1}^{DM} \ . \end{aligned}$$

Hence $b^{DM}_{n_1+2} = 1$. But by (2.1), $D_{n_1+2} = d_0\Delta$. Thus $b_{n_1+2} = 1$ as well.

This implies that

$$X_{n_1+3} > Y^{DM}_{n_1+1} + d_0\Delta - c\Delta$$

$$> Y^{DM}_{n_1+1} + (1+c)\Delta - c\Delta$$

$$\geq Y^{DM}_{n_1+1} + \Delta$$

$$\geq Y^{DM}_{n_1+2} .$$

Hence $b^{DM}_{n_1+3} = 1$. But by (2.1), $D_{n_1+3} = d_0\Delta$. Thus $b_{n_1+3} = 1$ as well.

Repeating this series of steps (n_0-n_1) times yields $D_{n_0} = D_0\Delta$, in contradiction of (2.1). □

A. Stationary input

We show first that DDM is stochastically stable for an input X_t, $t\geq 0$, that is either

(i) stationary Gaussian with rational spectral density, or

(ii) stationary Markovian with finite variance, uniformly bounded and everywhere positive multivariate densities of all orders, and uniformly bounded conditional densities of X_{n+1} given $X_n, \ldots, X_0$.

Processes of the former type satisfy all but the Markov requirement of the latter type. However, as noted by Slepian [15], they can be embedded as the first component in a stationary Gauss-Markov row-vector process $\tilde{X}_t$, $t\geq 0$, that is of the latter type. Thus, with a slight abuse of notation, we take $\tilde{X}_n$ to be a (possibly one-dimensional) row-vector that satisfies the conditions in (ii) and we put

$$S_n \triangleq (Y_n, \tilde{X}_{n+1}, \tilde{X}_{n+2}), \; n=0,1,2,\ldots \quad (2.3)$$

1. $\{S_n\}$ is Markovian

From the definition of DDM we see that

$$Y_{n+1} = g(Y_n, X_{n+1}, X_{n+2}) = g(S_n) \ , \tag{2.4}$$

and so

$$Y_n = h_n(X_{n+1}, \ldots, X_0) \ .$$

Hence, with probability one,

$$\begin{aligned} S_{n+1} | (S_n, \ldots, S_0) &= (Y_{n+1}, \tilde{X}_{n+2}, \tilde{X}_{n+3}) | (Y_n, \ldots, Y_0, \tilde{X}_{n+2}, \ldots, \tilde{X}_0) \\ &\overset{D}{=} (Y_{n+1}, \tilde{X}_{n+2}, \tilde{X}_{n+3}) | (Y_n, \tilde{X}_{n+2}, \ldots, \tilde{X}_0) \\ &\overset{D}{=} (Y_{n+1}, \tilde{X}_{n+2}, \tilde{X}_{n+3}) | (Y_n, \tilde{X}_{n+2}, \tilde{X}_{n+1}) \\ &= S_{n+1} | S_n \ , \end{aligned}$$

and so S_n is a Markov process.

2. $\{S_n\}$ has a stationary distribution G

Gersho [8] has shown that the averaged distributions of Y_n^{DM} are tight, and, as a consequence of Theorem 1, the same holds for Y_n^{DDM}. By stationarity, it is also true of $\tilde{X}_n$. Thus the averaged distributions of the components of S_n are tight, and hence so are the averaged distributions of S_n.

Let F_n denote the distribution of S_n, and $G_n \overset{\Delta}{=} n^{-1}\sum_{k=0}^{n-1} F_k$. Conditioning backwards gives us

$$F_{n+1}(A) = \int P(S_{n+1} \epsilon A | S_n = s) dF_n(s) .$$

From (2.3), (2.4), and the fact that $\tilde{X}_n$ is Markovian it follows that the transition function $P(S_{n+1} \epsilon A | S_n = s)$ does not depend on n. Hence we may write $F_{n+1} = TF_n$.

For a set A, let D_A denote the set of points s such that $P(S_{n+1} \epsilon A | S_n = s)$ is discontinuous, and let D_A^δ denote the set of points within δ of D_A. The following result is Gersho's [8], but the proof has been simplified.

<u>Theorem 2</u>. Let S_n be a Markov process with averaged distributions G_n and transition function $P(S_{n+1}\epsilon A|S_n=s)$ that does not depend on n. If

(1) the sequence G_n is tight, and

(2) for any open set A, there exists a function $c_A(\delta) \to 0$ as $\delta \to 0$ such that $P(S_n \epsilon D_A^\delta) \le c_A(\delta)$ for all n,

then S_n has a stationary distribution.

<u>Proof</u>: Because $\{G_n\}$ is tight, there exist a subsequence G_{n_i} and a distribution G such that $G_{n_i} \overset{D}{\Rightarrow} G$. But $TG_n = G_n + n^{-1}(F_n - F_0)$ and so $TG_{n_i} \underset{i}{\overset{D}{\Rightarrow}} G$.

From (2) we have $F_n(D_A^\delta) \le c_A(\delta)$ for all n, and so $G_n(D_A^\delta) \le c_A(\delta)$. Since D_A^δ is open, $G(D_A^\delta) \le c_A(\delta)$ so that $G(D_A) = 0$ for all A open.

Theorem 5.2.iii on page 31 in Billingsley [2] now gives us $TG_{n_i}(A) \underset{i}{\to} TG(A)$ for all A open, and therefore $TG_{n_i} \underset{i}{\overset{D}{\Rightarrow}} TG$. Hence TG = G. □

In order to apply this theorem, it remains only to show that the second condition is satisfied. The following lemmas prove that we may take $c_A(\delta) = c(\delta) = 8M\delta$.

<u>Lemma 1</u>. For DDM, for any Borel set A, $\delta<\Delta$, and all n, we have $P(S_n \epsilon D_A^\delta) \le 8M\delta$, where M is a uniform bound on the conditional density of X_{n+1} given $X_n, \ldots, X_0$.

<u>Proof</u>: Observe that for DDM, for any region A, $D_A \subset D = \cup_{i=-\infty}^{\infty} D_i^\delta$, where $D_i = \{(s_1,s_2,s_3)\colon\ s_1 = i\Delta$, and $(s_3^{(1)} = i\Delta \pm c\Delta$ or $s_2^{(1)} = i\Delta)\}$, and thus $D_A^\delta \subset D^\delta = \cup_{i=-\infty}^{\infty} D_i^\delta$. Since for $\delta<\Delta$,

$$P(S_n \epsilon D_i^\delta) = P(Y_n = i\Delta,\ |X_{n+2} - i\Delta - c\Delta| < \delta) + P(Y_n = i\Delta,\ |X_{n+2} - i\Delta + c\Delta| < \delta)$$
$$+ P(Y_n = i\Delta,\ |X_{n+1} - i\Delta| < \delta)\ ,$$

while

$$P(Y_n = i\Delta,\ |X_{n+1} - i\Delta| < \delta) = P(Y_{n-1} = (i-1)\Delta,\ b_n = 1,\ |X_{n+1} - i\Delta| < \delta)$$
$$+ P(Y_{n-1} = (i+1)\Delta,\ b_n = -1,\ |X_{n+1} - i\Delta| < \delta)$$
$$\le P(Y_{n-1} = (i-1)\Delta,\ |X_{n+1} - i\Delta| < \delta) + P(Y_{n-1} = (i+1)\Delta,\ |X_{n+1} - i\Delta| < \delta),$$

we see that for $\delta<\Delta$,

$$P(S_n \in D_A^\delta) \le \sum_{i=-\infty}^{\infty} \{P(Y_n=i\Delta,\ |X_{n+2}-i\Delta-c\Delta|<\delta) + P(Y_n=i\Delta,\ |X_{n+2}-i\Delta+c\Delta|<\delta) \tag{2.5}$$

$$+ P(Y_{n-1}=(i-1)\Delta,\ |X_{n+1}-i\Delta|<\delta) + P(Y_{n-1}=(i+1)\Delta,\ |X_{n+1}-i\Delta|<\delta)\}.$$

But for any k,K, and m,

$$P(Y_m=k\Delta,\ |X_{m+2}-K\ |<\delta) = P(|X_{m+2}-K| < \delta|Y_m=k\Delta)P(Y_m=k\Delta)$$

$$\le 2\delta M \cdot P(Y_m=k\Delta) \tag{2.6}$$

by (2.4), Lemma 2 below, and assumption. Hence, combining (2.5) and (2.6) we obtain our result. □

In the following lemma, U and V are i- and j-dimensional random vectors. respectively, and λ is i-dimensional Lebesgue measure.

Lemma 2. Suppose $f_{U|V}(\tilde{u}|\tilde{v}) \le M < \infty$ a.e. Then for all Borel $A\in\mathbb{R}^i$ and $B\in\mathbb{R}^j$, $P(U\in A|V\in B) \le M\lambda(A)$.

Proof: $P(U\in A,V\in B) = \iint_{AB} f_{U|V}(\tilde{u}|\tilde{v})f_V(\tilde{v})d\tilde{v}d\tilde{u}$

$$\le \iint_{AB} Mf_V(\tilde{v})d\tilde{v}d\tilde{u} = M\lambda(A)P(V\in B).$$ □

3. $\{S_n\}$ "converges" in distribution

Doob [; Theorem 5] gives, for a Markov process, conditions sufficient for a stationary distribution to be the (periodic or aperiodic) steady-state distribution. It is a simple matter to verify that these conditions are satisfied. (cf. Gersho [8] and Gerr [7]). Since the sequence $\{\Delta^{-1}Y_n\}$ alternates between odd and even values, the period two result obtains (cf. Gerr [7] for more details).

B. I.i.d. increments input

When the input to a DDM system has i.i.d. increments, such as a Wiener process, the joint distribution of input and output clearly cannot converge.

However, when the perfect integrator is used the predictive error $e_n^* \overset{\Delta}{=} X_{n+1}-Y_n$ does converge in law.

We let $I_{n+1} \overset{\Delta}{=} X_{n+1}-X_n$, n=0,1,... and assume that the I_n's form an i.i.d. sequence with mean zero, variance T, and bounded, everywhere positive density f_I. These assumptions are satisfied by the standard Wiener process.

1. $\{e_n^*\}$ is Markovian.

It is easy to see that

$$e_{n+1}^* = I_{n+2} + e_n^* - \Delta b_{n+1} , \tag{2.7}$$

where

$$b_{n+1} = \operatorname{sign}(e_n^* + I_{n+2}) 1_{\pm(c\Delta,\infty)}(e_n^* + I_{n+2}) + \operatorname{sign}(e_n^*) 1_{[-c\Delta,c\Delta]}(e_n^* + I_{n+2})$$

$$\overset{\Delta}{=} b(e_n^* , I_{n+2}) .$$

Thus

$$e_{n+1}^* = g(e_n^*, I_{n+2}) \tag{2.8}$$

and so

$$e_n^* = h_n(I_{n+1},\dots, I_1) . \tag{2.9}$$

Since the I_n's are i.i.d., it follows that e_n^* is Markovian.

2. $\{e_n^*\}$ has a stationary distribution.

Hayashi [9] has shown that for DM of Wiener process input, the sequence of averaged distributions of e_n^{*DM} is tight for all $\Delta > 0$. It is easily seen that this result can be extended to include any process that has zero-mean, finite variance, i.i.d. increments. Since $|e_n^*-e_n^{*DM}| = |Y_n^{DDM}-Y_n^{DM}| \le (c+3)\Delta$ by Theorem 1, the sequence of averaged distributions of e_n^* is tight for all $\Delta > 0$.

Letting F_n denote the distribution of e_n^*, we again have that

$$F_{n+1}(A) = \int P(e_{n+1}^* \in A | e_n^* = s) dF_n(s) .$$

From (2.7-9) we see that the transition function $P(e^*_{n+1} \in A | e^*_n = s)$ does not depend on n, and so we may write $F_{n+1} = TF_n$. It follows from (2.9) that e^*_n and I_{n+2} are independent. Since $b_{n+1} = \pm 1$, from (2.7) we see that we may take $c_A(\delta) = c(\delta) = 2M\delta$, where $M = \sup f_I$. Thus both conditions of Theorem 2 are satisfied and e^*_n has a stationary distribution G.

3. $\{e^*_n\}$ converges in distribution.

Once again the conditions of Doob's Theorem are easily verified (cf. Gerr [7]), but in this case the aperiodic result obtains.

C. I.i.d. input

We show here that if the input X_n is a discrete time i.i.d. process that has bounded everywhere positive density, the joint distribution of input and output converges weakly. Let $S_n \overset{\Delta}{=} (Y_n, X_{n+1}, X_{n+2})$.

1. $\{S_n\}$ is Markovian.

This follows from Subsection A.1, as X_n is both stationary and Markov.

2. $\{S_n\}$ has a stationary distribution G.

Because X_n is stationary, Subsection A.2 implies that the averaged distributions of S_n are tight. As the X_n are independent with bounded density the second condition of Theorem 2 is also satisfied. Hence S_n has a stationary distribution G.

3. $\{S_n\}$ "converges" in distribution.

Once again, the conditions of Doob's Theorem are easily verified, and in this case the period two result obtains.

III. DDM OF DISCRETE TIME I.I.D. INPUT

To analyze the response of DDM to a discrete time i.i.d. imput X_n, $n=0,1,2,\ldots$, we adapt the approach taken by Fine [5] in his study of DM of this input. First, we find the stationary joint distribution of input and output. This is done by solving the Chapman-Kolmogorov Equation (CKE). Then, assuming that the input has a standard normal distribution, we compute the asymptotic sample point mean squared error (MSE(SP)) as a function of c for several values of Δ and as a function of Δ for several values of c.

Recall from Section II.C that when we write

$$F_i(x) \triangleq \lim_n P(Y_n = i\Delta,\ X_n \le x)$$

we in fact mean

$$F_i(x) = \lim_n \tfrac{1}{2}\{P(Y_n = i\Delta,\ X_n \le x) + P(Y_{n+1} = i\Delta,\ X_{n+1} \le x)\}\ .$$

In the following, all expressions will be written using the simplified, rather than the precise averaged, form.

The input X_n, $n=0,1,2,\ldots$, is assumed to be i.i.d. symmetric, with finite second moment and common distribution H that has density h.

A. Analysis.

1. The asymptotic distribution of (Y_n, X_{n+1}).

Let $F_i^*(x) \triangleq \lim_n P(Y_n = i\Delta,\ X_{n+1} \le x)$, $i=0,\pm 1,\pm 2,\ldots$. The existence of these weak limits was shown in II.C. In (3.1) we show that these distributions have densities and may be written in terms of the input distribution and the constants $p_i \triangleq \lim_n P(Y_n = i\Delta)$, and $q_i \triangleq F_i^*(i\Delta)$, for which we derive an infinite system of homogeneous linear equations (3.2-5) that is solved by the Method of Reduction.

By conditioning backwards we obtain a CKE for F_i^*:

$$F_i^*(x) = \lim_n [P(Y_{n-1} = (i-1)\Delta,\ b_n = 1,\ X_{n+1} \le x) + P(Y_{n-1} = (i+1)\Delta,\ b_n = -1,\ X_{n+1} \le x)]$$

$$= p_{i-1}[H(x)-H((i-1+c)\Delta)]^+ + (p_{i-1}-q_{i-1})[H(\min(x,(i-1+c)\Delta)) - H((i-1-c)\Delta)]^+$$

$$+ p_{i+1}H(\min(x,(i+1-c)\Delta)) + q_{i+1}[H(\min(x,(i+1+c)\Delta)) - H((i+1-c)\Delta)]^+ \qquad (3.1)$$

where $[a]^+ = a$ if $a \ge 0$, $[a]^+ = 0$ if $a < 0$.

It follows from (3.1) that F_i^* has density f_i^*, and so the endpoints neglected in the derivation are of no concern. Specifically, we can write $f_i^*(x) = a_i^*(x)h(x)$, where h is the density of H and the step function a_i^* is easily identified by (3.1) (see Gerr [ ]). It is also clear that the constants $\{p_i,q_i\}_{i=-\infty}^{\infty}$ determine the distributions $\{F_i^*(\cdot)\}_{i=-\infty}^{\infty}$.

By letting $x \to \infty$ in (3.1) we obtain

$$p_i = p_{i-1}[1-H((i-1-c)\Delta)] - q_{i-1}[H((i-1+c)\Delta)-H((i-1-c)\Delta)]$$

$$+ p_{i+1}H((i+1-c)\Delta) + q_{i+1}[H((i+1+c)\Delta)-H((i+1-c)\Delta)], \qquad (3.2)$$

and taking $x = i\Delta$ yields

$$q_i = p_{i-1}[H(i\Delta)-H((i-1+c)\Delta)]^+ + (p_{i-1}-q_{i-1})[H(\min(i\Delta,(i-1+c)\Delta))-H((i-1-c)\Delta)]^+$$

$$+ p_{i+1}H(\min(i\Delta,(i+1-c)\Delta)) + q_{i+1}[H(\min(i\Delta,(i+1+c)\Delta))-H((i+1-c)\Delta)]^+ . \qquad (3.3)$$

The recursions (3.2 and 3) define an infinite system of homogeneous linear equations in $\{p_i,q_i\}_{i=-\infty}^{\infty}$. In addition, since the DDM system and input are symmetric, the output is symmetric, i.e.

$$p_{-i} = p_i , \qquad (3.4)$$

and since the input has density,

$$q_{-i} = \lim_n P(Y_n = -i\Delta,\ X_{n+1} \le -i\Delta) = \lim_n P(Y_n = i\Delta,\ X_{n+1} \ge i\Delta) = p_i - q_i . \qquad (3.5)$$

The infinite homogeneous system given by (3.2-5) is supplemented by the following non-homogeneous normalization requirements. Due to the periodicity of the output and the limiting stationary distribution

$$\sum_{i\ \mathrm{even}} p_i = \tfrac{1}{2} = \sum_{i\ \mathrm{odd}} p_i \ , \tag{3.6}$$

and so from (3.5),

$$\sum_{i\ \mathrm{even}} q_i = \tfrac{1}{4} = \sum_{i\ \mathrm{odd}} q_i \ . \tag{3.6.1}$$

The system given by (3.2-6) is solved by the Method of Reduction (see the Appendix), and we recover the distribution F_i^* using (3.1).

2. <u>The asymptotic (mean square) sample point error</u>.

As noted by Fine [ ],

$$F_e(x) \overset{\Delta}{=} \lim_n P(e_n \le x) = \lim_n P(X_n - Y_n \le x)$$
$$= \lim_n \sum_i P(Y_n = i\Delta,\ X_n \le x + i\Delta) = \sum_i F_i(x+i\Delta) \ , \tag{3.7}$$

where $F_i(x) \overset{\Delta}{=} \lim_n P(Y_n = i\Delta,\ X_n \le x)$. The distributions $\{F_i(\cdot)\}_{i=-\infty}^{\infty}$ are derived in terms of the distributions $\{F_i^*(\cdot)\}_{i=-\infty}$ by

$$F_i(x) = \lim_n [P(Y_{n-1} = (i-1)\Delta,\ b_n = 1,\ X_n \le x) + P(Y_{n-1} = (i+1)\Delta,\ b_n = -1,\ X_n \le x)]$$

$$= F_{i-1}^*(x)[1-H(i-1+c)\Delta)] + [F_{i-1}^*(x) - q_{i-1}]^+ \ [H((i-1+c)\Delta) - H((i-1-c)\Delta)]$$

$$+ F_{i+1}^*(x)H((i+1-c)\Delta) + F_{i+1}^*(\min(x,(i+1)\Delta))[H((i-1+c)\Delta) - H((i-1-c)\Delta)] \ . \tag{3.8}$$

Since F_i^* has density f_i^*, it follows from (3.8) that F_i has density f_i, which for $c \ge 1$ (which turns out to be the region of greatest interest) is given by

$$f_i(x) = \begin{cases} f_{i-1}^*(x)H((1-i-c)\Delta) + f_{i+1}^*(x)H((i+1+c)\Delta), & x<(i-1)\Delta, \\ f_{i-1}^*(x)H((1-i+c)\Delta) + f_{i+1}^*(x)H((i+1+c)\Delta), & (i-1)\Delta \le x<(i+1)\Delta \\ f_{i-1}^*(x)H((i-1+c)\Delta) + f_{i+1}^*(x)H((i+1-c)\Delta), & (i+1)\Delta \le x. \end{cases}$$

Substituting the expressions for f_i^* we find $f_i(x) = a_i(x)h(x)$, where the step functions $a_i(x)$ are given in Gerr [7].

Having found f_i it is straightforward to derive the asymptotic sample point mean square error. Letting e denote the limit in distribution of $e_n \overset{\Delta}{=} X_n - Y_n$, from (3.7) we derive

$$\text{MSE(SP)} = \int_{-\infty}^{\infty} x^2 f_e(x)dx = \sum_i \int_{-\infty}^{\infty} x^2 f_i(x+i\Delta)dx = \sum_i \int_{-\infty}^{\infty} (x-i\Delta)^2 f_i(x)dx$$

$$= 1-2\Delta \sum_i i \int_{-\infty}^{\infty} x f_i(x)dx + \Delta^2 \sum_i i^2 p_i \ . \qquad (3.9)$$

B. Numerical results and discussion.

Recall that the infinite system (3.2-6) is solved by the Method of Reduction. This involves setting $p_i=0=q_i$ for $|i| > I$, solving the resulting finite system exactly, and taking as the solution for the infinite system the limit as $I\to\infty$ of these "approximating solutions." The Theory of Majorants, developed by Kantorovich and Krylov [11], and presented in the Appendix, assures us that the Method of Reduction results in solution of the system encountered here.

We calculate MSE(SP) when the i.i.d. input is standard normal. Regarding the Method of Reduction, for all Δ and c, the value of MSE(SP) computed when $I = 4$ agreed with the $I = 5$ value to about four significant figures. In Table I we exhibit the output level probabilities p_i, $i=0,\ldots,5$ for DDM and DM with optimal parameter settings. Note that these probabilities decrease rapidly, facilitating the numerical technique. In addition, our computations for DDM with large c (i.e. DM) agree with Fine's [5] for DM to nearly four significant figures.

In Figure 2 we plot MSE(SP) as a function of c for Δ = .6, .75 and .9. Additional computations have shown that $(\Delta,c)_{opt}$ = (.745, 2.29) is the pair of values at which MSE(SP) is minimized and equals .5691. Fine [5] found that for DM, Δ_{opt} = .77, giving MSE(SP) = .6402. Thus, the reduction in MSE(SP) of optimal DDM versus optimal DM is about 11.1%. Note that the change in the value of the optimal Δ is quite small. However, recalling how the size of Δ determines the trade-off between round-off and slope overload, it is not surprising that the optimal Δ is smaller for DDM than for DM. The additional DDM loop is designed to mitigate the effect of slope overloads, which previously could only be done through the use of larger Δ.

The shape of the curves in Figure 2 is typical of the response of DDM for the inputs considered. However, for i.i.d. input, the performance of DDM is uncharacteristically poor when c is small. This may be rationalized as follows: When c=0, Y_n tracks X_{n+1}, but MSE(SP) is measured relative to X_n, which is independent of X_{n+1}. We expect that when the input is (more) positively correlated, smaller settings of c will result in greater reductions in MSE(SP).

Fine [5] has pointed out $1-2/\pi \approx .3634$ is a lower bound for MSE(SP) of a predictive feedback/quantization scheme that has a fixed two-level quantizer when the input is i.i.d. standard normal, and that this lower bound is attained only by PCM with quantizer output levels $\pm (2/\pi)^{\frac{1}{2}} \approx \pm .798$. This is not surprising; PCM is non-differential, and so may have independent output values, but DM is differential, hence its output values are necessarily correlated. However, this lower bound does not apply to systems that use delay, such as DDM, or perform "tree searches" (cf. Chan and Anderson [3]), i.e. are interpolative.

In Figure 3 we plot MSE(SP) as a function of Δ for c = 1.5, 2.25, 3., and 8. (which corresponds to c = ∞, i.e. DM). The relative lack of dependence of MSE(SP) on Δ for DM of i.i.d. standard normal input was noted by Fine [5], and we see that a similar result holds for DDM. In addition, we find the improvement of DDM with near optimal c (=2.25) over DM to be almost uniform with respect to Δ in its range of interest.

For the input considered here, the increments $I_n \overset{\Delta}{=} X_n - X_{n-1}$ have $E(I_n^2) = 2$ and $E(I_{n+1} I_n) = E(-X_n^2) = -1$, so that adjacent increments have correlation -1/2. When adjacent increments are more positively correlated, slope overloads should be more prevalent and we would expect even greater improvement in the performance of DDM relative to DM, and for lower settings of c as well.

IV. DDM OF WIENER PROCESS INPUT

The analysis of the response of DDM (with perfect integration) to Wiener process input parallels that of Janardhanan [10] for DPCM with matched integration of stationary first-order Gauss-Markov input. In Subsection A the limiting distribution of the predictive error is shown to satisfy an integral equation obtained by conditioning on the preceding step, and the asymptotic sample point and time-averaged mean square errors are expressed in terms of this limiting distribution. In Subsection B we present computational results.

The input X_t, $t \geq 0$, is the standard Wiener process. The increments $x_n \triangleq X_n - X_{n-1}$, $n=1,2,\ldots$, are i.i.d. $N(0,T)$, and we let ϕ_T denote their density.

A. Analysis

From Subsection II.B we have that the distribution $F_n^*(x) \triangleq P(e_n^* \leq x)$ has density f_n^* and converges weakly to a distribution F^* having density f^*. f^* is shown to satisfy an (CKE) integral equation (4.2), which is solved by Galerkin's Method (see the Appendix). The joint density of e^* and e, the limits in distribution of e_n^* and $e_n \triangleq X_n - Y_n$, respectively, is then expressed in terms of f^* in (4.3). This yields the limiting distribution of $e_n(t) \triangleq X_{nT+t} - Y_n$, $t \in [0,T)$, (4.10), enabling the calculation of the asymptotic predictive, sample point, and time-averaged mean-square-errors (4.13, 14, and 16).

1. The density f^* of e^*.

Since the I_n's are independent, from (2.9) we have that e_{n-1}^* and I_{n+1} are independent. Thus, using (2.7) we derive

$$\begin{aligned} F_n^*(x) &= P(e_{n-1}^* \leq x+\Delta-I_{n+1},\ b_n=1) + P(e_{n-1}^* \leq x-\Delta-I_{n+1},\ b_n=-1) \\ &= \int_{-\infty}^{\infty} \{[F_{n-1}^*(x+\Delta-u) - F_{n-1}^*(c\Delta-u)]^+ \\ &\quad + [F_{n-1}^*(\min(x+\Delta-u, c\Delta-u)) - F_{n-1}^*(\max(0,-c\Delta-u))]^+ + F_{n-1}^*(\min(x-\Delta-u,-c\Delta-u)) \\ &\quad + [F_{n-1}^*(\min(0,c\Delta-u,x-\Delta-u)) - F_{n-1}^*(-c\Delta-u)]^+\}\phi_T(u)\,du \,. \end{aligned} \tag{4.1}$$

Since F^* is the stationary distribution for e_n^* it satisfies the recursion (4.1). Substitution, differentiation, and a change of variables yields

$$f^*(x) = \{[1(x>(c-1)\Delta) \int_{-\infty}^{\infty} + 1(-(c+1)\Delta \le x \le (c-1)\Delta) \int_0^{\infty}]\phi_T(x+\Delta-v) \quad (4.2)$$

$$+ [1(x<-(c-1)\Delta) \int_{-\infty}^{\infty} + 1(-(c-1)\Delta \le x \le (c+1)\Delta) \int_{-\infty}^{0}]\phi_T(x-\Delta-v)\}f^*(v)dv.$$

As f^* is a density, it also satisfies

$$\int f^* = 1 \text{ and } f^* \ge 0 \ . \quad (4.2.1,2)$$

From (4.2) we see that since f^* is integrable, it is bounded, and hence square integrable, i.e. $f^* \epsilon L_2(\text{Leb})$. In Subsection B of the Appendix we show that the integral operator A given by the right hand side of (4.2) is bounded and linear on $L_2(\text{Leb})$. Thus f^* is an eigenfunction of A having eigenvalue one. We now show that the manifold of integrable solutions to $f=Af$ is one dimensional.

Theorem 3. Suppose f is integrable and satisfies $f=Af$. Then $f=rf^*$, where $r=\int f$.

Proof: Examining the kernel of A, we see that since f is integrable, it is bounded and continuous, except possibly at the four points $\pm (c\pm1)\Delta$. Let f^+ and f^- be the positive and negative parts of f and set $p=\int f^+$, $q=\int f^-$ and $r=\int f$. Then $f=f^+-f^-$ and $p-q=r$. In Section III.B it was shown that $T^n F_0 = F_n^* \Rightarrow F^*$ for any initial distribution F_0. Thus for all bounded continuous functions h, $\int hA^n f^+ \to p\int hf^*$ and $\int hAf^- \to q\int hf^*$, so that $\int hA^n f \to r\int hf^*$. But $A^n f = A^{n-1}Af = A^{n-1}f = f$ for all n. Hence for all bounded continuous functions h,

$$\int hf = r\int hf^* \ ,$$

and since f and f^* integrable and continuous except possibly at four points, $f = rf^*$ a.e. □

The unique integrable solution to (4.2) is found by Galerkin's Method (see the Appendix).

2. The joint density of e and e*.

Let $F_n(u,v) \triangleq P(e_n \le u,\ e_n^* \le v)$. By a sequence of steps similar to those used at the beginning of the previous section we derive

$$F_n(u,v) = \int_{-\infty}^{\infty} \{[F_{n-1}^*(\min(u+\Delta, v+\Delta-w)) - F_{n-1}^*(c\Delta-w)]^+$$

$$+ [F_{n-1}^*(\min(c\Delta-w, u+\Delta, v+\Delta-w)) - F_{n-1}^*(\max(0,-c\Delta-w))]^+$$

$$+ F_{n-1}^*(\min(-c\Delta-w, u-\Delta, v-\Delta-w))$$

$$+ [F_{n-1}^*(\min(0, c\Delta-w, u-\Delta, v-\Delta-w)) + F_{n-1}^*(-c\Delta-w)]^+\}\phi_T(w)dw.$$

From (2.7 and 9) and II.B.3 we have that $F_n^* \Rightarrow F^*$ that has density f*. As F_n^* is uniformly bounded (by one) and ϕ_T is integrable, it follows that F_n converges weakly to a distribution F having density f. The joint density f of e and e* is then given in terms of the density f* of e* by

$$f(u,v) = \{f^*(u+\Delta)[1(v>(c-1)\Delta)+1(-(c+1)\Delta \le v \le (c-1)\Delta)1(u>-\Delta)] \tag{4.3}$$

$$+f^*(u-\Delta)[1(v<-(c-1)\Delta)+1(-(c-1)\Delta \le v \le (c+1)\Delta)1(u \le \Delta)]\}\cdot\phi_T(v-u).$$

The density f(·) of e is the marginal

$$f(u) = \int_{-\infty}^{\infty} f(u,v)dv. \tag{4.4}$$

3. The sample point and time-averaged mean square error.

Having derived the joint density of e and e*, we may now calculate the time-averaged mean-square-error

$$MSE(TA) \triangleq \frac{1}{T}\int_0^T E(e^2(t))dt ,$$

where e(t) is the weak limit of $e_n(t) \triangleq X_{nT+t} - Y_n$.

Let $F_n(x;t) \triangleq P(e_n(t) \le x)$. Then

$$F_n(x;t) = \iint P(e_n(t) \le x \mid e_n = u,\ e_n^* = v)dF_n(u,v) . \tag{4.5}$$

However,

$$P(e_n(t) \le x \mid e_n = u, e_n^* = v) = \sum_i P(e_n(t) \le x \mid e_n = u, e_n^* = v, Y_n = i\Delta)P(Y_n = i\Delta), \tag{4.6}$$

and

$$P(e_n(t)\le x|e_n=u,e_n^*=v,Y_n=i\Delta) = P(X_{nT+1}\le x+i\Delta|X_{nT}=u+i\Delta,X_{(n+1)T}=v+i\Delta,Y_n=i\Delta). \quad (4.7)$$

$$= P(X_{nT+t}\le x+i\Delta|X_{nT}=u+i\Delta,X_{(n+1)T}=v+i\Delta) ,$$

since Y_n is a functional of $X_{(n+1)T},\ldots, X_0$.

Let $B_n(t) \triangleq X_{nT+t} - [(1-\frac{t}{T})X_{nT} + \frac{t}{T}X_{(n+1)T}]$ for $t\epsilon[0,T]$ and $n=0,1,2,\ldots$. Clearly, for all n, B_n is a zero-mean Gaussian process with $B_n(0) = 0 = B_n(T)$. Furthermore, it is easily shown that $E(B_n(s)B_n(t)) = s(1-\frac{t}{T})$ for $0\le s\le t\le T$. Thus for all n, $B_n(t)$, $t\epsilon[0,T]$, is distributed as a Brownian Bridge on $[0,T]$, which we will denote by $B_T(t)$. In addition, since $E[B_n(t)X_{nT}] = 0 = E[B_n(t)X_{(n+1)T}]$ for all n and $t\epsilon[0,T]$, $B_n(t)$ is independent of both X_{nT} and $X_{(n+1)T}$. This gives us

$$P(X_{nT+t}\le x+i\Delta|X_{nT}=u+i\Delta,X_{(n+1)T}=v+i\Delta)$$

$$= P(B_n(t)\le x+i\Delta-[(1-\frac{t}{T})(u+i\Delta)+\frac{t}{T}(v+i\Delta)]|X_{nT}=u+i\Delta,X_{(n+1)T}=v+i\Delta)$$

$$= P(B_n(t)\le x-[(1-\frac{t}{T})u+\frac{t}{T}v]) = P(B_T(t)\le x-[(1-\frac{t}{T})u+\frac{t}{T}v]), \quad (4.8)$$

and combining (4.6-8) that

$$P(e_n(t)\le x|e_n=u,e_n^*=u) = P(B_T(t)\le x-[(1-\frac{t}{T})u+\frac{t}{T}v]) . \quad (4.9)$$

The distribution on the right hand side of (4.9) is bounded, continuous in u and v, and absolutely continuous in x with respect to Lebesgue measure. Thus, we may substitute (4.9) into (4.5) , pass to the weak limit on n, and differentiate with respect to x to obtain the density $f(\cdot;t)$ of $e(t)$;

$$f(x;t) = \iint f_t(x-[(1-\frac{t}{T})u+\frac{t}{T}v])dF(u,v) , \quad (4.10)$$

where f_t is the density of $B_T(t)$. This enables us to calculate

$$E(e^2(t)) = \iiint x^2 f_t(x-[(1-\frac{t}{T})u+\frac{t}{T}v])dxdF(u,v)$$

$$= \iiint (x+(1-\frac{t}{T})u+\frac{t}{T}v)^2 f_t(x)dxdF(u,v)$$

$$= E[B_T^2(t)] + (1-\frac{t}{T})^2 E(e^2) + 2\frac{t}{T}(1-\frac{t}{T})E(ee^*) + (\frac{t}{T})^2 E(e^{*2}) \qquad (4.11)$$

with $E[B_T^2(t)] = t(1-\frac{t}{T})$. Finally, substituting (4.11) into the definition for MSE(TA), we perform the integrations over time to obtain

$$\text{MSE(TA)} = \frac{1}{3}[E(e^2) + E(ee^*) + E(e^{*2})] + \frac{T}{6} . \qquad (4.12)$$

For DM we see that $e^* \triangleq e + I$, where I is N(0,T) independent of e. Thus for DDM with $c = \infty$,

$$\text{MSE(TA)} = \frac{1}{3}[E(e^2) + E[e(e+I)] + E[(e+I)^2]] + \frac{T}{6} = E(e^2) + \frac{T}{2},$$

which is Equation (11) in Masry and Cambanis [14]

Using Subsections A.1 and A.2, where the densities of e and e* as well as their joint density were derived, it is a straightforward task to calculate MSE(SP) = $E(e^2)$, MSE(PR) = $E(e^{*2})$ and $E(ee^*)$. The simplest calculation is for MSE(PR). By definition,

$$\text{MSE(PR)} = \int_{-\infty}^{\infty} x^2 f^*(x)dx . \qquad (4.13)$$

From (4.3 and 4) we have

$$\text{MSE(SP)} = \int_{-\infty}^{\infty} u^2 f(u)du = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^2 f(u,v)dvdu$$

$$= \text{MSE(PR)} + \Delta^2 + 4\Delta[\int_{-\infty}^{\infty}\int_{-\infty}^{-c\Delta} + \int_{-\infty}^{0}\int_{-c\Delta}^{c\Delta}]u\ f^*(u)\phi_T(v-u)dvdu$$

$$= \text{MSE(PR)} + \Delta^2 - 4\Delta\int_0^{\infty}[2\Phi(\frac{c\Delta+u}{T^{\frac{1}{2}}}) - 1]uf^*(u)du , \qquad (4.14)$$

where Φ is the cumulative standard normal distribution function. From (4.3) we have

$$E(ee^*) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} uv\ f(u,v)dvdu$$

$$= \text{MSE(PR)} + \Delta^2 + 2\Delta\{\int_{-\infty}^{\infty}\int_{-\infty}^{-c\Delta} + \int_{-\infty}^{0}\int_{-c\Delta}^{c\Delta}\}(u+v)f^*(u)\phi_T(v-u)dvdu$$

$$= \text{MSE(SP)} - 4\Delta\int_0^{\infty}\phi(\frac{c\Delta+u}{T^{\frac{1}{2}}})f^*(u)du . \qquad (4.15)$$

Substituting (4.14) and (4.15) in (4.12) yields

$$\mathrm{MSE(TA)} = \mathrm{MSE(PR)} + \frac{2\Delta^2}{3} + \frac{T}{6} + 2\Delta\left[\int_{-\infty}^{\infty}\int_{-\infty}^{-c\Delta} + \int_{-\infty}^{0}\int_{-c\Delta}^{c\Delta}\right](u + \frac{v}{3}) f^*(u)\phi_T(v-u)\,dv\,du$$

$$= \mathrm{MSE(PR)} + \frac{2\Delta^2}{3} + \frac{T}{6} - \frac{4\Delta}{3}\int_0^{\infty}\{\phi(\frac{c\Delta+u}{T^{\frac{1}{2}}}) + 2u[2\Phi(\frac{c\Delta+u}{T^{\frac{1}{2}}})-1]\} f^*(u)\,du. \qquad (4.16)$$

B. Results and Discussion

Using (4.13-16) we compute MSE(SP) and MSE(TA). Due to the stationarity of the increments of the Wiener input and the inclusion of Δ in the (new) DDM overload loop, MSE(SP)/T and MSE(TA)/T are both functions of $\Delta=\Delta_0 T^{\frac{1}{2}}$ and c, but $\mathrm{MSE}(\Delta_0 T^{\frac{1}{2}},c)/T$ does not depend on T. Thus, it suffices to carry out the computations for T=1 only. The values obtained for the MSEs upon solving (4.2) by Galerkin's Method (cf. the Appendix regarding the use of Projection Methods and the Method of Reduction) converged to four significant figures when N=5.

In Figures 4 and 5 we plot MSE(SP)/T and MSE(TA)/T, respectively, as functions of c for $\Delta = .75T^{\frac{1}{2}}$, $.9T^{\frac{1}{2}}$ and $1.05T^{\frac{1}{2}}$. Masry and Cambanis [14] found that for DM of Wiener input, $\min_\Delta$MSE(SP) = .585T and $\min_\Delta$MSE(TA) = 1.085T, with $\Delta_{opt} = 1.05T^{\frac{1}{2}}$ for both. Our calculations with c large agree with theirs to about three significant figures. For DDM of Wiener input, MSE(SP) attains its minimum of .4844T for $(\Delta,c)_{opt} = (.941T^{\frac{1}{2}}, 1.17)$, while MSE(TA) takes its minimum value of .6698T for $(\Delta,c)_{opt} = (.909T^{\frac{1}{2}}, .296)$. These values represent 17.1% and 35.4% reductions, respectively, from DM. Note that the conjecture made at the close of Section III has, in this instance, been justified. With the same rationale, we expect that the improvement obtained by using DDM instead of DM for a (first-order) Gauss-Markov input will be greater than that obtained for an i.i.d. input but less than that obtained here.

Finally, in Figure 6 we plot MSE(TA)/T as a function of $\Delta_0=\Delta T^{-\frac{1}{2}}$ for c=0, .3, .6, and 5. Once more we find that for a good choice of c, the improvement of DDM over DM is nearly uniform in Δ.

V. DDM OF STATIONARY FIRST-ORDER GAUSS-MARKOV INPUT

The analysis of the response of DDM to stationary first-order Gauss-Markov input parallels that of Slepian [15] for DM of stationary Gaussian inputs that have rational spectral density. As in our study of DDM of i.i.d. input, we derive the limiting joint distribution of input and output. This is done by solving the system of integral equations obtained by conditioning on the preceding output value. We then compute the asymptotic sample point and time-averaged mean-square-errors.

The input X_t, $t \geq 0$, is taken to be the stationary first-order Gauss-Markov process that has mean zero and covariance function $R(s) = E(X_t X_{t+s}) = e^{-|s|}$. Throughout, we let $\rho \stackrel{\Delta}{=} e^{-T}$.

A. Analysis

1. The asymptotic distribution of (Y_n, X_{n+1})

We first show that the limiting joint distributions of $(Y_n = i\Delta, X_{n+1})$, $i=0,\pm 1,\pm 2,\ldots$, exist weakly and have densities which satisfy a (CKE) system of integral equations (5.4), which is solved by Galerkin's Method (see the Appendix).

Let $F_i^{(n)}(x) \stackrel{\Delta}{=} P(Y_n = i\Delta, X_{n+1} \leq x)$. By conditioning backwards we obtain

$$\begin{aligned} F_i^{(n)}(x) &= P(Y_{n-1}=(i-1)\Delta, b_n=1, X_{n+1}\leq x) + P(Y_{n-1}=(i+1)\Delta, b_n=-1, X_{n+1}\leq x) \\ &= P(Y_{n-1}=(i-1)\Delta, X_{n+1}>(i-1+c)\Delta, X_{n+1}\leq x) \\ &+ P(Y_{n-1}=(i-1)\Delta, (i-1-c)\Delta\leq X_{n+1}\leq (i-1+c)\Delta, X_n>(i-1)\Delta, X_{n+1}\leq x) \\ &+ P(Y_{n-1}=(i+1)\Delta, X_{n+1}<(i+1-c)\Delta, X_{n+1}\leq x) \\ &+ P(Y_{n-1}=(i+1)\Delta, (i+1-c)\Delta\leq X_{n+1}\leq (i+1+c)\Delta, X_n\leq (i+1)\Delta, X_{n+1}\leq x). \end{aligned} \tag{5.1}$$

All terms on the right hand side of (5.1) are of the following form:

$$\begin{aligned} P(Y_{n-1}=k\Delta, X_n\in A, X_{n+1}\in B, X_{n+1}\leq x) &= P((X_0,\ldots,X_n)\in E_k, X_n\in A, X_{n+1}\in B, X_{n+1}\leq x) \\ &= \int_{-\infty}^{x} 1_B(x_{n+1}) \int_{E_k} 1_A(x_n)\phi_{n+1}(x_0,\ldots,x_{n+1})\,dx_0\ldots dx_n\, dx_{n+1}, \end{aligned}$$

where $E_k = g_{n-1}^{-1}(\{k\Delta\}), g_{n-1}(X_0,\ldots,X_n) = Y_{n-1}$, and ϕ_{n+1} is the multivariate density of $(X_0,\ldots,X_{n+1})$. Thus $F_i^{(n)}$ has density $f_i^{(n)}$. Returning to (5.1), we have

$$\begin{aligned} F_i^{(n)}(x) = &\int_{-\infty}^{\infty} f_{i-1}^{(n-1)}(u)P(X_{n+1}>(i-1+c)\Delta, X_{n+1}\le x|Y_{n-1}=(i-1)\Delta, X_n=u)du \\ &+ \int_{(i-1)\Delta}^{\infty} f_{i-1}^{(n-1)}(u)P((i-1-c)\Delta\le X_{n+1}\le(i-1+c)\Delta, X_{n+1}\le x|Y_{n-1}=(i-1)\Delta, X_n=u)du \\ &+ \int_{-\infty}^{\infty} f_{i+1}^{(n-1)}(u)P(X_{n+1}<(i+1-c)\Delta, X_{n+1}\le x|Y_{n-1}=(i+1)\Delta, X_n=u)du \qquad (5.2) \\ &+ \int_{-\infty}^{(i+1)\Delta} f_{i+1}^{(n-1)}(u)P((i+1-c)\Delta\le X_{n+1}\le(i+1+c)\Delta, X_{n+1}\le x|Y_{n-1}=(i+1)\Delta, X_n=u)du. \end{aligned}$$

Because $f_i^{(n)} \le \sum_i f_i^{(n)} = \phi$, the standard normal density, $f_i^{(n)}(x) \le (2\pi)^{-\frac{1}{2}}$ for all n and x. In Subsection II.A it was shown that $F_i^{(n)} \Rightarrow F_i$. Thus, the following lemma implies that F_i has density f_i.

Lemma 3. Suppose for all n, F_n has density f_n such that $f_n \le M < \infty$ a.e. (Leb). If $F_n \Rightarrow F$, then F has density f and $f \le M$ a.e. (Leb).

Proof: Let A be any open set. Then for all n, $F_n(A) \le M$ Leb (A), and therefore $F(A) \le M$ Leb (A). Hence F is absolutely continuous with respect to Lebesgue measure, and its density $f \le M$ a.e. (Leb). □

Because $Y_{n-1} = g_{n-1}(X_0,\ldots,X_n)$ and X_n is Markovian, we have (a.e)

$$P(X_{n+1}\in A, X_{n+1}\le x|X_n=u, Y_{n-1}=k\Delta) = P(X_{n+1}\in A, X_{n+1}\le x|X_n=u) = \int_{-\infty}^{x} 1_A(v)\phi_T(v|u)du \;, \qquad (5.3)$$

where ϕ_T denotes the conditional density of X_{t+T} given X_t. This is a bounded (by one) and continuous function of u, and an absolutely continuous function of x. Hence, substituting (5.3) into (5.2), passing to the limit in n, and differentiating with respect to x, we obtain the following system of integral equations for the asymptotic joint densities $\{f_i\}$ a.e.; for $i=0, \pm 1, \pm 2,\ldots$,

$$\begin{aligned} f_i(x) = \{&1_{A_i}(x)\int_{-\infty}^{\infty} f_{i-1}(u) + 1_{B_i}(x)\int_{(i-1)\Delta}^{\infty} f_{i-1}(u) \qquad (5.4) \\ &+ 1_{C_i}(x)\int_{-\infty}^{\infty} f_{i+1}(u) + 1_{D_i}(x)\int_{-\infty}^{(i+1)\Delta} f_{i+1}(u)\}\phi_T(x|u)du, \end{aligned}$$

where

$$A_i \triangleq ((i-1+c)\Delta,\infty),\ B_i \triangleq [(i-1-c)\Delta,\ (i-1+c)\Delta]\ ,$$

$$C_i \triangleq (-\infty,(i+1-c)\Delta),\ D_i \triangleq [(i+1-c)\Delta,(i+1+c)\Delta]\ .$$

Note that setting $c = \infty$ reduces (5.4) to Eq. (23) in Slepian [15]. Of course, in addition to (5.4) we have the boundary condition

$$\sum_i f_i = \phi \quad \text{a.e.} \tag{5.4.1}$$

Let $\tilde{f} \triangleq (\ldots,f_{-1},f_0,f_1,\ldots)^T$. Since $0 \le f_i \le \sum_i f_i$, from (5.4.1) we have that $\tilde{f} \in \ell_2(L_2(\phi^{-1}))$. In Subsection C of the Appendix we show that the system of integral operators $\tilde{A}$ given by the right hand side of (5.4) is bounded and linear on $\ell_2(L_2(\phi^{-1}))$, and that $\tilde{f}$ is the unique solution to $\tilde{g} = A\tilde{g}$ subject to $\sum_i g_i = \phi$. $\tilde{f}$ is found by Galerkin's Method (see the Appendix).

2. The (mean square) sample point error.

Having found the limiting distribution of (Y_n,X_{n+1}), we derive in (5.5) the limiting distribution of (Y_n,X_n), which enables us to calculate the asymptotic mean square sample point error (5.6).

Let $G_i^{(n)}(x) \triangleq P(Y_n=i\Delta,X_n \le x)$. Conditioning backwards yields

$$\begin{aligned} G_i^{(n)}(x) = \int_{-\infty}^{x} du\{&f_{i-1}^{(n-1)}(u)[P(X_{n+1}\in A_i|X_n=u) + 1(u>(i-1)\Delta)P(X_{n+1}\in B_i|X_n=u)] \\ &+ f_{i+1}^{(n-1)}[P(X_{n+1}\in C_i|X_n=u) + 1(u\le(i+1)\Delta)P(X_{n+1}\in D_i|X_n=u)]\} \end{aligned}$$

Since $G_i^{(n)} \Rightarrow G_i$ that has density g_i, we have

$$\begin{aligned} g_i(x) &= f_{i-1}(x)[P(X_{n+1}\in A_i|X_n=x) + 1(x>(i-1)\Delta)P(X_{n+1}\in B_i|X_n=x)] \\ &\quad + f_{i+1}(x)[P(X_{n+1}\in C_i|X_n=x) + 1(x\le(i+1)\Delta)P(X_{n+1}\in D_i|X_n=x)] \\ &= \{f_{i-1}(x)[\textstyle\int_{A_i} + 1(x>(i-1)\Delta)\int_{B_i}] \\ &\quad + f_{i+1}(x)[\textstyle\int_{C_i} + 1(x\le(i+1)\Delta)\int_{D_i}]\}\phi_T(y|x)dy\ , \end{aligned} \tag{5.5}$$

and just as in (3.7) and (3.9),

$$\mathrm{MSE(SP)} = 1-2\Delta\sum_i i\int_{-\infty}^{\infty} x\, g_i(x)dx + \Delta^2\sum_i i^2 p_i \ . \tag{5.6}$$

3. The time-averaged mean square error.

We calculate the time-averaged mean square error

$$\mathrm{MSE(TA)} \triangleq \frac{1}{T}\int_0^T E[e^2(t)]dt \ ,$$

where $e(t)$ is the limit in distribution of $e_n(t) \triangleq X_{nT+t}-Y_{nT}$, in much the same way as Slepian [ ]. Its expression is given in (5.10).

For $t\in(0,T)$, let $F_i^{(n)}(x;t) \triangleq P(Y_n=i\Delta, X_{nT+t}\le x)$. We show first that $F_i^{(n)}(\cdot;t)$ has weak limit $F_i(\cdot;t)$, and that all of these distributions have densities.

The same steps used in the derivation of (5.2) yield

$$\begin{aligned} F_i^{(n)}(x;t) &= \int_{-\infty}^{\infty} f_{i-1}^{(n-1)}(u)P(X_{n+1}\in A_i, X_{nT+t}\le x|Y_{n-1}=(i-1)\Delta,\ X_n=u)du \\ &+ \int_{(i-1)\Delta}^{\infty} f_{i-1}^{(n-1)}(u)P(X_{n+1}\in B_i, X_{nT+t}\le x|Y_{n-1}=(i-1)\Delta, X_n=u)du \\ &+ \int_{-\infty}^{\infty} f_{i+1}^{(n-1)}(u)P(X_{n+1}\in C_i, X_{nT+t}\le x|Y_{n-1}=(i+1)\Delta, X_n=u)du \\ &+ \int_{-\infty}^{(i+1)\Delta} f_{i+1}^{(n-1)}(u)P(X_{n+1}\in D_i, X_{nT+t}\le x|Y_{n-1}=(i+1)\Delta, X_n=u)du. \end{aligned} \tag{5.7}$$

Recalling that $Y_{n-1} = g_{n-1}(X_0,\ldots,X_n)$, and X_t, $t\ge 0$, is first-order Markovian, we have (a.e.)

$$\begin{aligned} P(X_{n+1}\in E, X_{nT+t}\le x|Y_{n-1}=k\Delta, X_n=u) &= P(X_{n+1}\in E, X_{nT+t}\le x|X_n=u) \\ &= \int_E\int_{-\infty}^{x} \phi_{T-t}(w|v)\phi_t(v|u)dvdw. \end{aligned} \tag{5.8}$$

Since $F_i^{(n)} \Rightarrow F_i$ that has density f_i, from (5.7 and 8) we get

$$\begin{aligned} f_i(x;t) = \{&[\int_{-\infty}^{\infty}\int_{A_i} + \int_{(i-1)\Delta}^{\infty}\int_{B_i}]f_{i-1}(u) + [\int_{-\infty}^{\infty}\int_{C_i} + \int_{-\infty}^{(i+1)\Delta}\int_{D_i}]f_{i+1}(u)\} \\ &\phi_{T-t}(w|x)\phi_t(x|u)dwdu, \end{aligned} \tag{5.9}$$

and so similar to (5.6) we have

$$\mathrm{MSE(TA)} = \frac{1}{T}\sum_i \int_0^T\int_{-\infty}^{\infty} (x-i\Delta)^2\, f_i(x;t)\,dx\,dt$$

$$= 1+\Delta^2 \sum_i i^2 p_i - \frac{2\Delta}{T}\sum_i i\int_0^T\int_{-\infty}^{\infty} x f_i(x;t)\,dx\,dt. \quad (5.10)$$

B. Numerical Results and Discussion

The numerical analysis (cf. the Appendix regarding the use of the Method of Reduction) is carried through for T=.51, .36 and .22 (ρ=.6, .7 and .8). The values for the MSEs converged to five, four, and three significant figures, respectively, when N=10, 13 and 15 respectively. In every case, the value of I ceased to have an effect when $(I-1)\Delta > 3.5$, i.e. when the largest output level exceeded 3.5.

In Figures 7 and 8 we graph MSE(SP) and MSE(TA), respectively, as functions of c for T=.51 with Δ= .45, .55, and .65, and for T=.36 with Δ=.45, .525, and .6. For large c, our calculations for MSE(TA) appear in good agreement with Slepian's [15]. The two sets of plots in each figure are very similar both to each other and to their corresponding Wiener input plots. This is encouraging in that it points to a certain uniformity in the performance of DDM for positively correlated signals.

In Tables II and III we display the minimum values of MSE(SP) and MSE(TA), respectively, with the (optimal) parameter setting(s) that yield these minima, for both DM and DDM. We find that MSE(SP) is reduced about 17%, while MSE(TA) is reduced about 37%, by using optimal DDM instead of optimal DM. In every case, the optimal value of Δ is smaller for DDM than for DM. Note that these results are quite similar to those found for a Wiener input.

VI. CONLCUDING DISCUSSION

A. <u>The effects of perfect integration and step-function reconstruction on mean-square-error and input-output synchronization.</u>

Recall that when $c=\infty$, $Y_n^{DDM} = Y_n^{DM}$, while when $c=0$, $Y_n^{DDM} = Y_{n+1}^{DM}$, so that the (c=0)-DDM output is just a T-shift of the DM output <u>for any choice of reconstruction</u> Y_t based on $\{Y_n\}$. This gives us

$$\begin{aligned} MSE(TA;(c=0)\text{-}DDM) &= \text{"}\lim_n\text{"} \frac{1}{T}\int_0^T E[(X_{nT+t}-Y_{nT+t}^{DDM})^2]dt \\ &= \text{"}\lim_n\text{"} \frac{1}{T}\int_0^T E[(X_{nT+t}-Y_{(n+1)T+t}^{DM})^2]dt \\ &= MSE(TA;\ T\text{-shifted DM}), \end{aligned}$$

where $\text{"}\lim\text{"}_n\, a_n = \lim_n \frac{1}{2}(a_n+a_{n+1})$ since Y_n has period two. Hence, from Figures 5 and 8 we see that for DM with perfect integration and step function reconstruction $Y_t=Y_{nT}$ for $t\in[nT,(n+1)T)$, about one third of MSE(TA) is due to a lag in the tracking, rather than a distortion of the shape, of the signal. This has an important implication regarding the use of the (time-averaged) mean square error criterion. When one is mainly interested in preserving the shape of the input and associates no loss to small time lags, such as in speech or image transmission, a more appropriate measure of system performance may be

$$MSE(TA;I\text{-shifted}) = \min_{s\in I}\ \text{"}\lim_n\text{"} \frac{1}{T}\int_0^T E[(X_{nT+t}-Y_{nT+t+s})^2]dt,$$

where I is an interval that corresponds to the time shifts deemed acceptable.

In Figures 9 and 10 we graph MSE(SP) and MSE(TA), respectively for DM and T-shifted DM (i.e. DDM with $c=\infty$ and $c=0$) with perfect integration and, in the time-averaged case, step-function reconstruction, of stationary first-order Gauss-Markov input. We see that while MSE(SP; T-shifted DM) < MSE(SP;DM) only when Δ is small (so small that Y_n^{DM} cannot "keep up" with X_n), MSE(TA; T-shifted DM) < MSE(TA;DM) uniformly in Δ, i.e. Y_t^{DM} lags behind X_t equally for all Δ. Thus, the lag in the tracking of X_t by Y_t^{DM} here is mainly a result of the use of the step-function reconstruction.

It is clear that MSE(SP) does not depend on the reconstruction and measures only what may be called the "encoding-decoding error" of the system, while MSE(TA) measures both the encoding-decoding error and the "reconstruction error." In general, the reconstruction cannot compensate for deficiencies in encoding-decoding. Specifically, given different "modulations" Y_n, Z_n of X_n, n=0,1,2,..., and using the same reconstruction for both Y_t and Z_t, if $MSE_Y(SP) < MSE_Z(SP)$, we expect that $MSE_Y(TA) < MSE_Z(TA)$. However, one expects that, in most cases, MSE(TA)-MSE(SP), though positive, can be made small by using more sophisticated reconstructions, which have higher complexity and require greater memory. For example, for a Wiener input W_t, $t \geq 0$, suppose we are given the perfect modulation $Y_n = W_{nT}$, n=0,1,2,..., the step-function reconstruction $Y_{nT+t}^{SF} = Y_n$, and the linearly interpolated reconstruction $Y_{nT+t}^{LI} = (1-\frac{t}{T})Y_n + \frac{t}{T}Y_{n+1}$, for $t \in [0,T)$. Although in this case MSE(SP) = 0, it is easily verified that MSE(TA:SF) = T/2 and MSE(TA:LI) = T/6. When the performance of the reconstruction is of secondary interest to that of the modulator, MSE(SP) is the more appropriate criterion.

B. Unification of results.

The inputs analyzed in Sections III-V have samples X_n, n=0,1,2,..., that may be viewed as generated by the first-order autoregressive sequence $X_{n+1} = \rho X_n + \alpha I_{n+1}$, where $0 \leq \rho \leq 1$, $0 < \alpha \leq 1$, and $\{I_n\}$ are i.i.d. N(0,1). In Section III we have $\rho=0$, $\alpha=1$; in Section IV we have $\rho=1$, $\alpha=1$; in Section V we have $0<\rho<1$, $\alpha = (1-\rho^2)^{\frac{1}{2}}$. In Figure 11 we plot, for DM and DDM with perfect integration, $\alpha^{-2} \min_{\Delta}$ MSE(SP) and $\alpha^{-2} \min_{(\Delta,c)}$MSE(SP), respectively, as functions of ρ. Dividing by α^2 "standardizes" the MSE with respect to the increment power. The optimal "standardized" performance of both DM and DDM is seen to vary only slightly with ρ, and the improvement attainable by using DDM instead of DM to be nearly uniform in ρ.

Arnstein [1] has analyzed DM with matched integration when $\alpha=1$, and his plot of $\min_\Delta$MSE(SP) as a function of ρ in Figure 4.2 is included inFigure 11. We see that for DM of first order autoregressive inputs, the loss incurred by using the perfect instead of the matched integrator is less than 10% when $\rho > .8$, and vanishes as $\rho \to 1$. In addition, we note the following: for DDM with matched integration, when $\rho = 0$, $\alpha = 1$, $\min_{(\Delta,c)}$ MSE(SP) = .3634 (at $(\Delta,c)=(.798,\infty)$; cf. Fine [5]), and when $\rho=1$, $\alpha=1$, $\min_{(\Delta,c)}$ MSE(SP) = .4844 (cf. Subsection IV.C). For DDM with matched integration, we expect that a plot of $\alpha^{-2}\min_{(\Delta,c)}$ MSE(SP) versus ρ will resemble Arnstein's [1;Figure 4.2] matched integration DM curve and have .3634 and .4844 (instead of .585) as limits at zero and one, respectively. A conjecture for this curve is also included in Figure 11.

In Figure 12 we graph Δ_{opt}/α (with respect to MSE(SP) as a function of ρ for DM and DDM with perfect integration. Note that Δ_{opt}/α varies only slightly with $\rho = (1-\alpha^2)^{1/2}$, and thus also with α; it is encouraging that the optimal Δ has a linear relation with the increment scale. The reduction in Δ_{opt}/α for DDM from DM increases monotonically from 3% to 10% as ρ goes from zero to one. Comparing the DM curve with Arnstein's [1;Figure 4.1] plot of Δ_{opt} versus ρ for DM with matched integration when $\alpha=1$, included in Figure 12, reveals that for $0<<\rho<1$ the matched integrator also permits the use of slightly smaller Δ. This is because with the matched integrator the random variables to be (binary) quantized, i.e. the predictive errors, are in general smaller, and because the matched integrator itself compensates for the more common type of slope overloads, those during which $|X_t| \to 0$. Using the matched integrator in DDM should produce a similar reduction.

In Figure 13 we graph c_{opt} (with respect to MSE(SP)) as a function of ρ. As expected, the more positive the correlation of the samples, the more useful the new DDM overload loop, and hence the smaller the optimal value of c. It is interesting that the plot is highly linear.

APPENDIX. THE SOLUTION OF INFINITE LINEAR SYSTEMS

In each of our analyses, we encounter the problem of solving an infinite linear system. In our first analysis, we show that the joint distributions $\{F_i^*(\cdot)\}$ may be expressed in terms of the constants $\{p_i, q_i\}$ and the input distribution $H(\cdot)$ (cf. (3.1)). We then derive an infinite system of homogeneous linear equations in these constants (cf. (3.2-5)) and a normalizing "boundary" condition (cf. (3.6)). This infinite linear system is solved by the Method of Reduction. This involves setting $p_i = 0 = q_i$ for all $i > I$, solving the resulting finite system, and taking as the solution for the infinite system the limit of the finite "approximating" solution as $I \uparrow \infty$. (See Kantorovich and Krylov [11] for more on the Method of Reduction.) In the first subsection below we show why this may be done.

In our last two analyses, the desired (joint) density is shown to satisfy a (system of) linear integral operator equation(s) of the second kind (cf. (4.2) and (5.4)) as well as a normalizing boundary equation (cf. (4.2.1) and (5.4.1)), and to belong to a Hilbert space. Henceforth we let f denote the desired (joint) density;

$$g = Ag \tag{A.1}$$

represent the (system of) integral operator equation(s) ((4.2) or (5.4)) for which, subject to a given normalization, f is the unique solution; H be the Hilbert space to which f belongs; and $\{e_n\}$ be a complete and orthonormal system in H. Since $f \in H$, it has Fourier expansion

$$f(\cdot) = \sum f_n e_n(\cdot) \ , \tag{A.2}$$

where the Fourier coefficients are given by

$$f_n = \langle f, e_n \rangle_H \ .$$

Substituting (A.2) into (A.1) and taking inner products (in H) on both sides of the resulting equation yields an infinite system of linear equations in the $\{f_n\}$, which is solved by the Method of Reduction. This procedure is known as Galerkin's Method, one of many so-called Projection Methods for the solution of (integral) operator equations. (For more on these methods see Krasnoselskii et.al. [13] and Zabreyko et.al [16].) In Subsections B and C below we show that the (system of) integral operator(s) A is bounded on H. It then follows (cf. Gerr [7] and Gerr and Cambanis [6]) that Galerkin's Method may be used.

A. Majorant Systems and the Method of Reduction

We first note several properties of the infinite system of linear equations obtained in the analysis of DDM of i.i.d. input. It is composed of an infinite system of homogeneous linear equations, given by (3.2 and 3), in the unknowns $\{p_i, q_i\}_{i=-\infty}^{\infty}$, supplemented by the non-homogeneous normalization (3.6). Thus it has the following form:

$$x_i = \sum_j a_{ij} x_j, \qquad i = 0, \pm 1, \pm 2, \ldots, \tag{A.3.1}$$

$$\sum_i x_i = b , \tag{A.3.2}$$

where $b = \sum_i (p_i + q_i) = 1.5$. Using (3.5) appropriately in (3.2 and 3) results in $0 \le a_{ij} \le 1$ for all i,j, with $a_{ij} \ne 0$ for at most four values of j for all i. As a shorthand we write $\tilde{x} = A\tilde{x}$ for (A.3.1) and define

$\tilde{p} = (\ldots, p_{-1}, q_{-1}, p_0, q_0, p_1, q_1, \ldots)^T$.

<u>Theorem A.1</u>. If $\tilde{z} \in \ell_1$ satisfies (A.3.1), then $\tilde{z} = (1.5)^{-1} (\sum_i z_i)\tilde{p}$. Thus (A.1) has unique solution in ℓ_1.

<u>Proof</u>: Let K denote the positive cone in ℓ_1, i.e. $K \triangleq \{x \in \ell_1;\ x_i \ge 0 \text{ all } i\}$, and let $y \in K$. In Section II.C. it was shown that $F_1^{(n)} \Rightarrow F_i$ from any initial state and so, in particular, $p_i^{(n)} \underset{n}{\to} p_i$ and $q_i^{(n)} \underset{n}{\to} q_i$. Thus $A^n y \underset{n}{\to} (1.5)^{-1} \|\tilde{y}\| \tilde{p}$

pointwise. As $\tilde{z} \in \ell_1$ may be written $\tilde{z} = \tilde{z}_1 - \tilde{z}_2$, where $\tilde{z}_1, \tilde{z}_2 \in K$, we have that $A^n\tilde{z} \to (1.5)^{-1}(\sum_i z_i)\tilde{p}$ pointwise, with $|\sum_i z_i| \le \|\tilde{z}\| < \infty$. Since $A\tilde{z} = \tilde{z}$, $A^n\tilde{z} = \tilde{z}$ for all n, hence $\tilde{z} = (1.5)^{-1}(\sum_i z_i)\tilde{p}$. □

This implies that there is a single linear dependency in A. If we sum the equations (A.3.1) over i we obtain the trivial identity $\sum_i x_i = \sum_i x_i$. Thus the system

$$x_i = \sum_j a_{ij} X_j, \qquad i = \pm 1, \pm 2, \ldots, \tag{A.4.1}$$

$$\sum_i x_i = 1, \tag{A.4.2}$$

has $\tilde{p}$ as its unique solution. It now follows that the system

$$x_i = \sum_j a_{ij} X_j, \quad i = \pm 1, \pm 2, \ldots, \tag{A.5.1}$$

$$x_0 = 1,$$

has as its unique solution $\tilde{q} = \tilde{p}/p_0$ (having identified x_0 with $p_0 > 0$). In Theorem A.2 below we show that $\tilde{q}$ may be found by the Method of Reduction so that finally $\tilde{p} = p_0\tilde{q} = 1.5\|\tilde{q}\|^{-1}\tilde{q}$.

Theorem A.2. The system (A.5) can be solved by the Method of Reduction.

Proof: In general, an infinite system of linear equations in the infinite set of unknowns $\{x_i\}_{i=-\infty}^{\infty}$ may be written as

$$x_i = \sum_j a_{ij} x_j + b_0, \qquad i=0, \pm 1, \pm 2, \ldots . \tag{A.6}$$

The system of equations

$$X_i = \sum_j A_{ij} X_{ij} + B_j, \qquad i=0, \pm 1, \pm 2, \ldots,$$

is "Marjorant for the system (A.6)" when for all i and j, $|a_{ij}| \le A_{ij}$ and $|b_i| \le B_i$. The system (A.5) is its own majorant, and by Theorem A.1, has unique solution $\tilde{q} \in K$, the positive ℓ_1 cone. The result then follows from Theorems I-IV on pages 20-26 in Kantorovich and Krylov [11]. □

B. DDM of Wiener Process Input

We prove below that the integral operator A given by the right hand side of (4.2) is bounded on L_2(Leb). To prove this it suffices to show that the integral operator B defined by

$$[Bf](x) = 1(x>(c-1)\Delta)\int_{-\infty}^{\infty}\phi_T(x+\Delta-u)f(u)du \quad (A.7)$$

is bounded linear operator on L_2(Leb), the other terms being treated similarly.

Clearly, B is linear. From (A.7) we have

$$\begin{aligned} ||Bf||^2 &= \int_{-\infty}^{\infty}[Bf]^2 \\ &\le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_T(x+\Delta-u)\phi_T(x+\Delta-v)|f(u)f(v)|dudvdx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_{2T}(u-v)|f(u)f(v)|dudv \\ &\le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_{2T}(u-v)f^2(v)dudv \\ &= \int_{-\infty}^{\infty}f^2 = ||f||^2 . \end{aligned} \quad (A.8)$$

Hence B is bounded, with $||B|| \le 1$.

It is easily shown that B is not compact on L_2(Leb). Similar to (A.8) we have $||Bf||^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(\frac{u+v}{2} - c\Delta)\phi_{2T}(u-v)f(u)f(v)dudv$. Putting $f_n(x) = 1(n \le x \le n+1)$, we see that although $f_n \overset{w}{\to} 0$ in L_2(Leb),

$$\begin{aligned} ||Bf_n||^2 &= \int_n^{n+1}\int_n^{n+1}\Phi(\frac{u+v}{2} - c\Delta)\phi_{2T}(u-v)dudv \\ &\underset{n}{\to} \int_0^1\int_0^1\phi_{2T}(u-v)dudv > 0 , \end{aligned}$$

and so $Bf_n \not\to 0$ in L_2(Leb).

As a final consequence of this subsection we have that the integral operator D defined by

$$[Df](x) = \int_{-\infty}^{\infty} \phi_T(x-u) f(u) du$$

is bounded, with $\| D \| \le 1$, but not compact, on L_2(Leb).

C. DDM of Stationary First-order Gauss-Markov Input

We prove below that the system of integral operators $\tilde{A}$ given by the right hand side of (5.4) is a bounded linear operator on $H = \ell_2(L_2(\phi^{-1}))$. Note that in this case, since every $\tilde{g} \in H$ is integrable, the fact that $\tilde{f} \in H$ is the unique integrable solution to (5.4) implies that 1 is a simple eigenvalue of $\tilde{A}$.

Let $\tilde{g} \in H$. Using Mehler's Formula

$$\phi_T(x|u) = \phi(x) \sum_{n=0}^{\infty} \rho^n H_n(x) H_n(u) \quad ,$$

where $H_n(\cdot)$ is the Hermite polynomial of order n, from (5.4) we have

$$\begin{aligned}
\| \tilde{A}\tilde{g} \|_H^2 &= \sum_{i=-\infty}^{\infty} \| (\tilde{A}\tilde{g})_i \|_{L_2(\phi^{-1})}^2 \\
&\le 8 \sum_{i=-\infty}^{\infty} \iiint \{ |g_i(u) g_i(v)| + |g_{i+1}(u) g_{i-1}(v)| \} \phi_T(x|u) \phi_T(x|v) \phi^{-1}(x) \, dudvdx \\
&= 8 \sum_{i=-\infty}^{\infty} \sum_{n=0}^{\infty} \rho^{2n} \iint \{ |g_i(u) g_i(v)| + |g_{i+1}(u) g_{i-1}(v)| \} H_n(u) H_n(v) \, dudv \\
&= 8 \sum_{i=-\infty}^{\infty} \sum_{n=0}^{\infty} \rho^{2n} [\langle |g_i|, H_n\phi \rangle^2 + \langle |g_{i+1}|, H_n\phi \rangle \langle |g_{i-1}|, H_n\phi \rangle] \\
&\le 16 \, \| \tilde{g} \|_H^2 \quad .
\end{aligned}$$

where the inner product $\langle \cdot , \cdot \rangle$ is in $L_2(\phi^{-1})$. Thus $\tilde{A}$ is a bounded linear operator on $\ell_2(L_2(\phi^{-1}))$.

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TABLE I

Output Level Probabilities

	DDM w/$(\Delta,c)_{opt}$	DM w/Δ_{opt}
p_0	.383750	.384036
p_1	.246477	.246382
p_2	.058091	.057994
p_3	.003522	.003618
p_4	.000034	.000038
p_5	.00000003	.00000004

TABLE II

minMSE(SP) and Optimal Parameters of DM and DDM

	DM		DDM	
f	minMSE(SP)	Δ_{opt}	minMSE(SP)	$(\Delta,c)_{opt}$
.6	.357	.72	.296	(.64,1.57)
.7	.287	.66	.237	(.59,1.47)
.8	.201	.57	.166	(.52,1.37)

TABLE III

minMSE(TA) and Optimal Parameters of DM and DDM

	DM		DDM	
f	minMSE(SP)	Δ_{opt}	minMSE(TA)	$(\Delta,c)_{opt}$
.6	.675	.57	.430	(.56,.42)
.7	.545	.57	.338	(.54,.37)
.8	.380	.53		

Figure 1

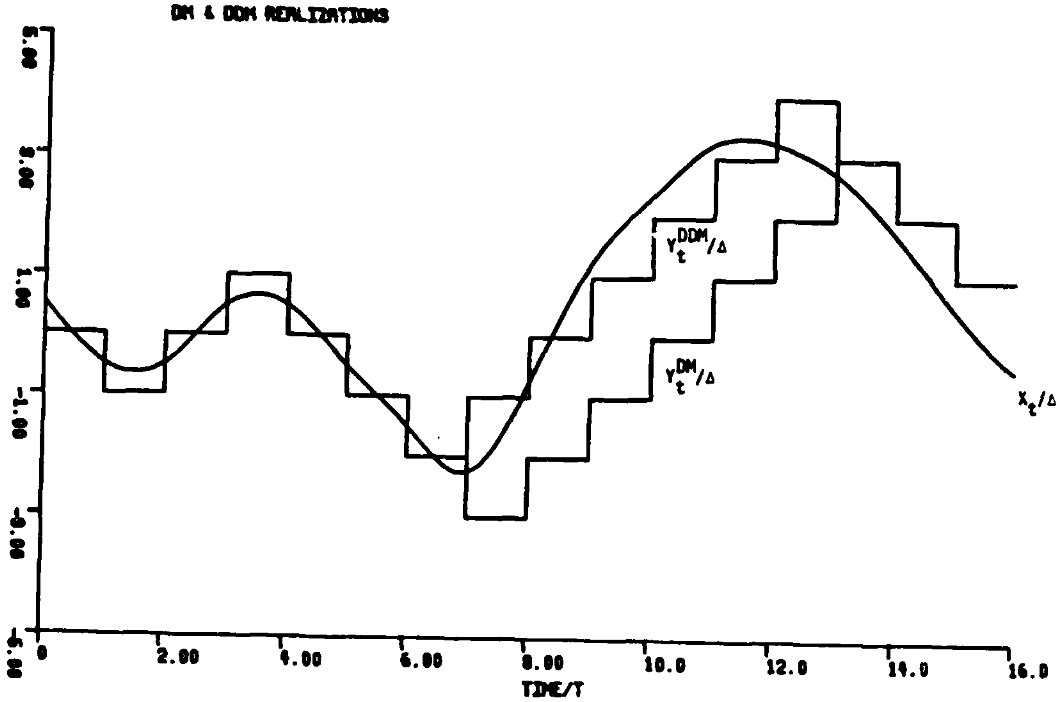
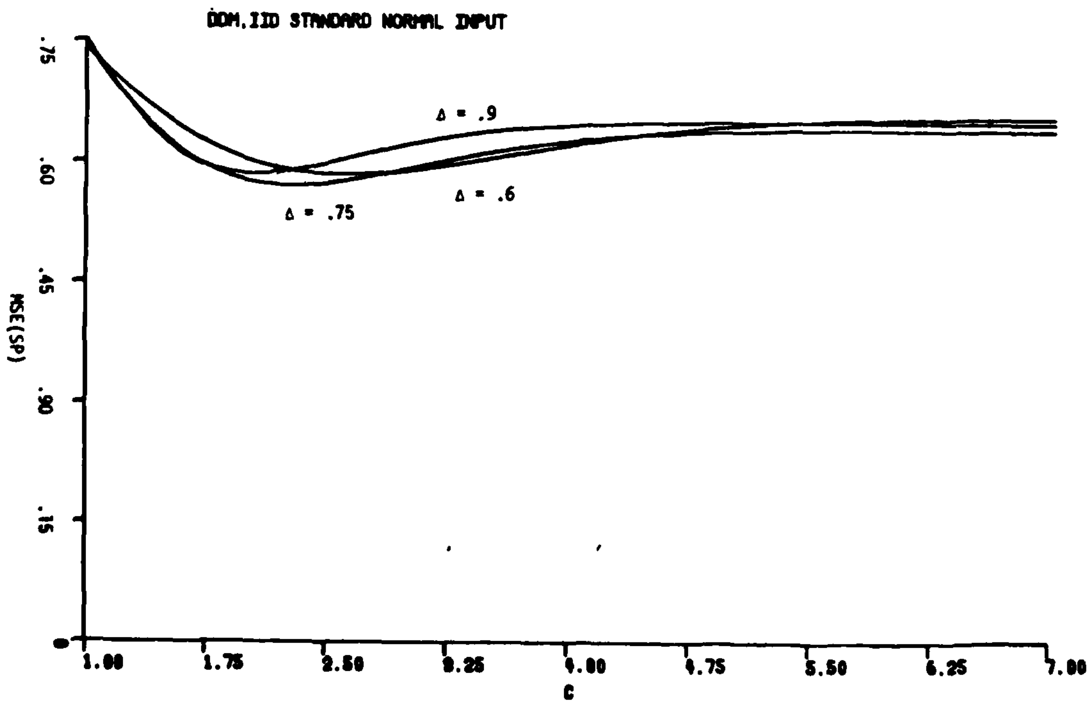


Figure 2



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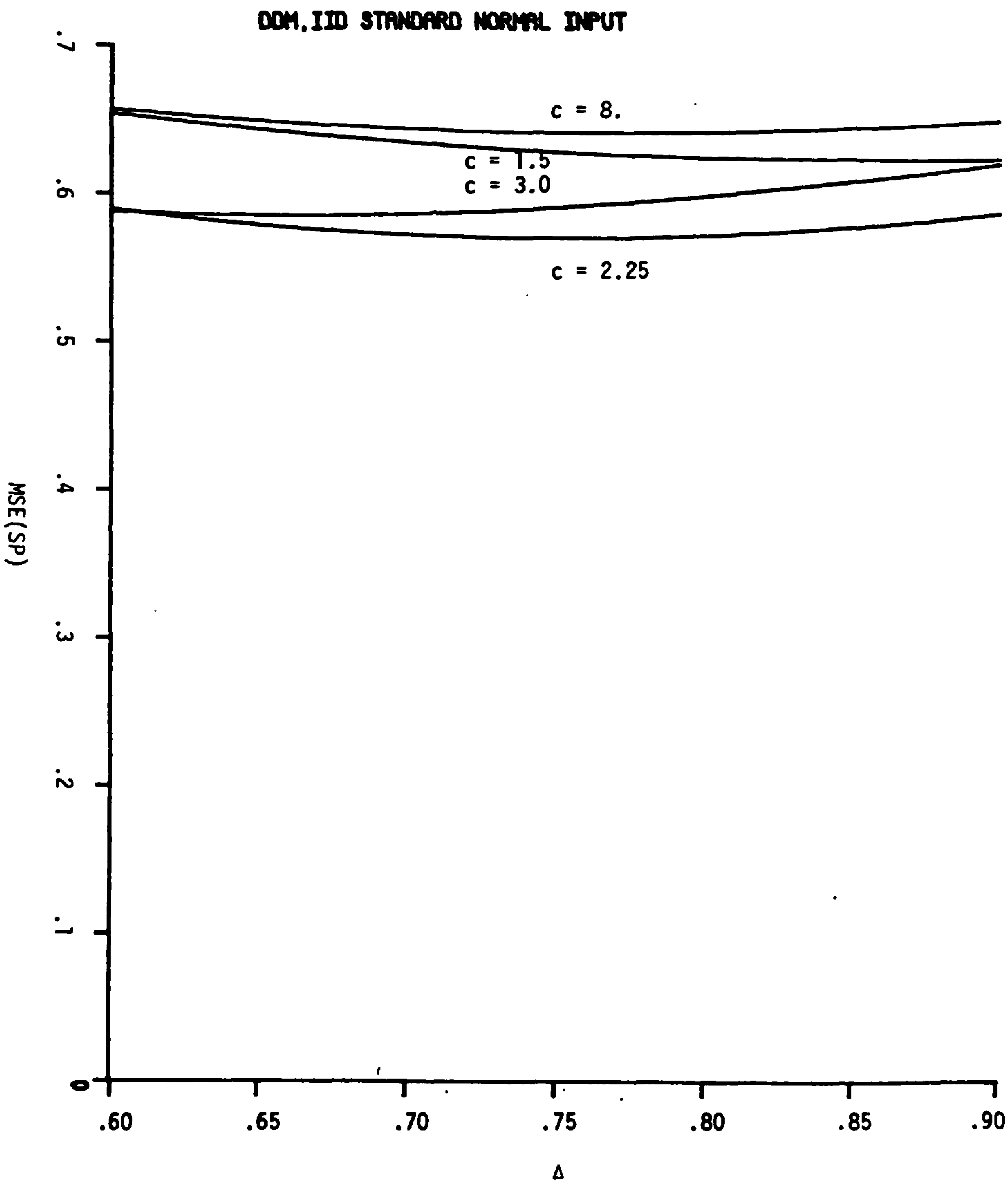
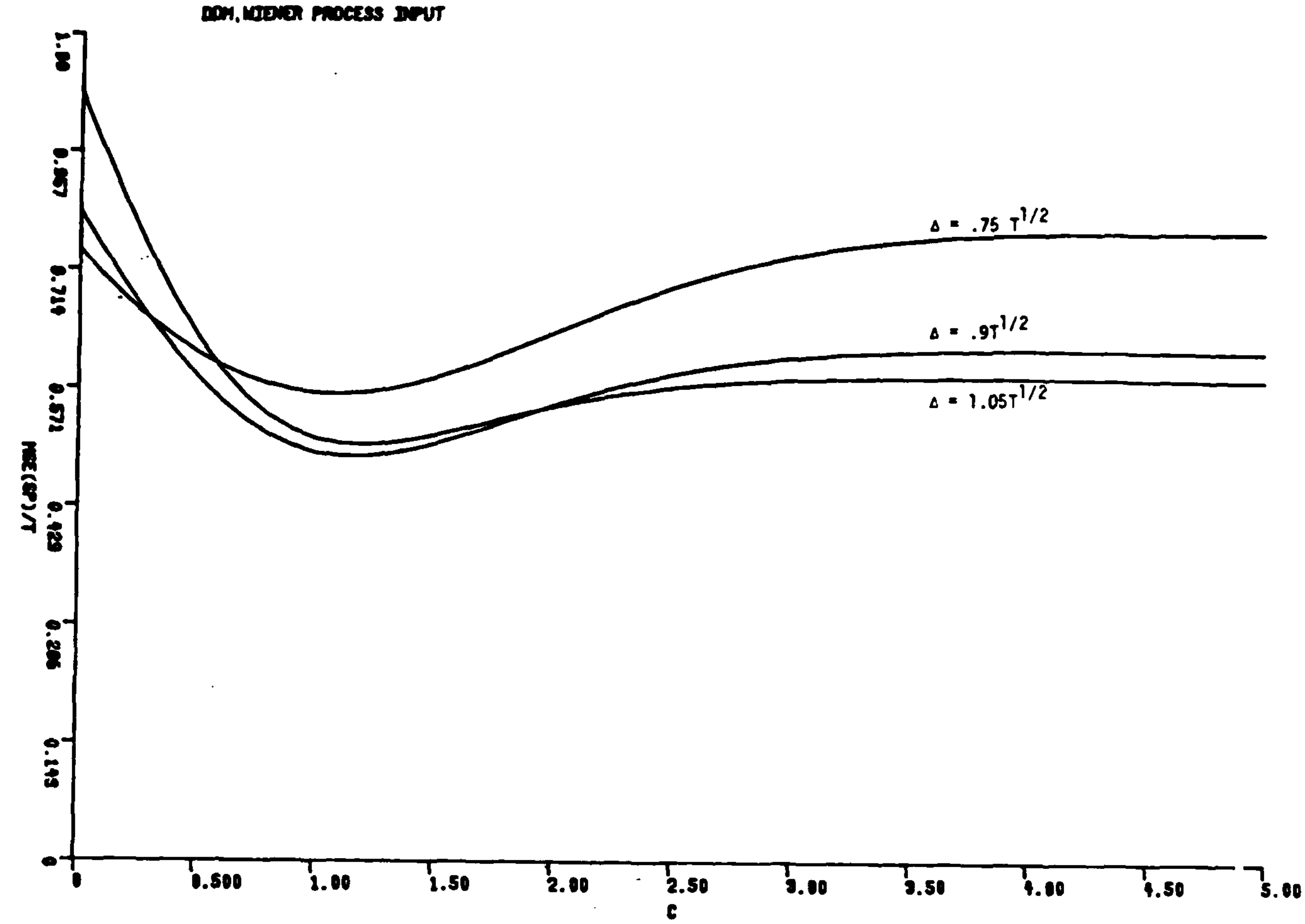


DDM, IID STANDARD NORMAL INPUT
c = 8.
c = 1.5
c = 3.0
c = 2.25
MSE(SP)
.7
.6
.5
.4
.3
.2
.1
0
.60
.65
.70
.75
.80
.85
.90
Δ

Figure 4



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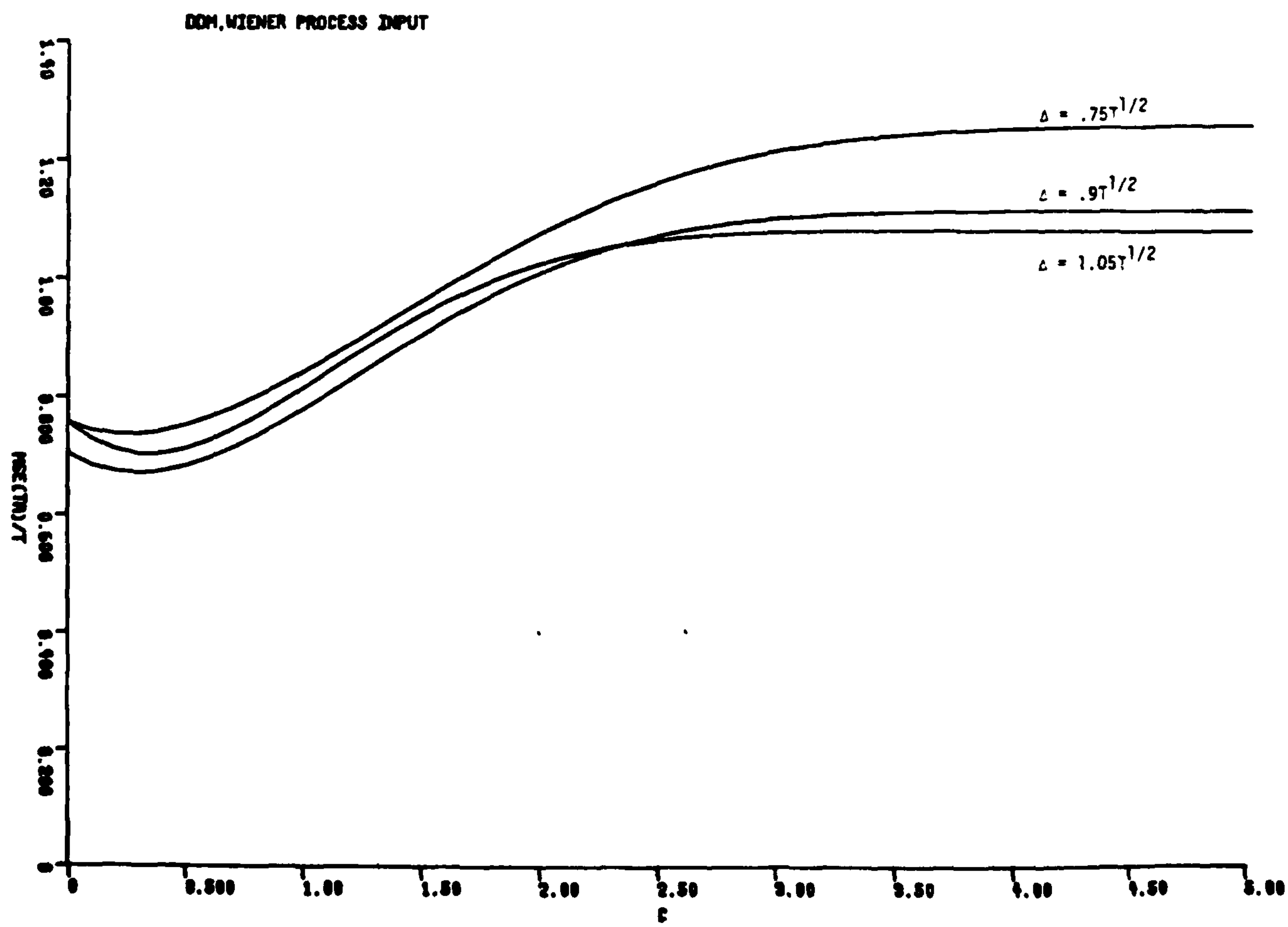

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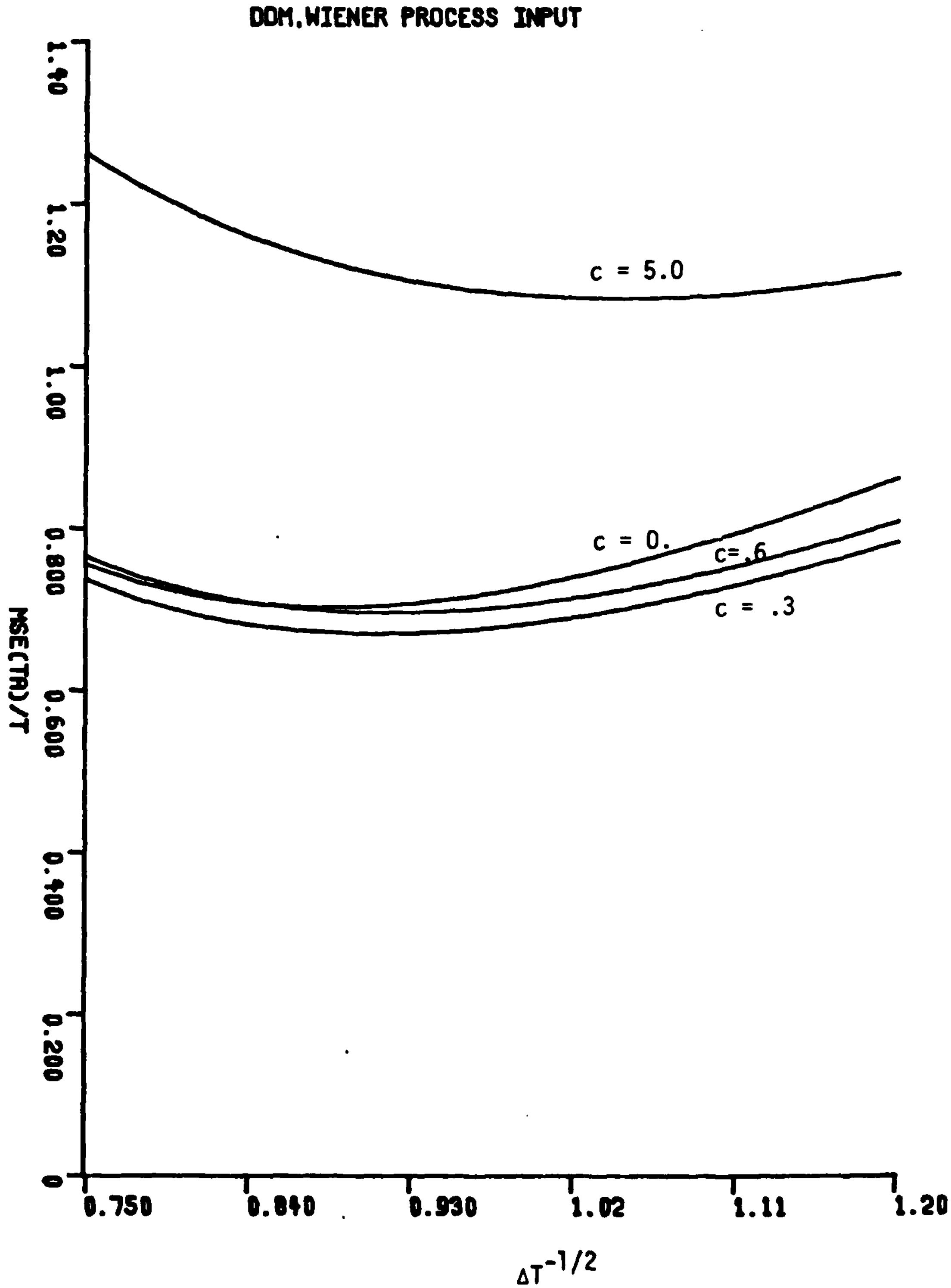

DDM.WIENER PROCESS INPUT
1.40
1.20
1.00
0.800
0.600
0.400
0.200
0
MSE(TR)/T
c = 5.0
c = 0.
c=.6
c = .3
0.750
0.840
0.930
1.02
1.11
1.20
ΔT-1/2

Figure 7

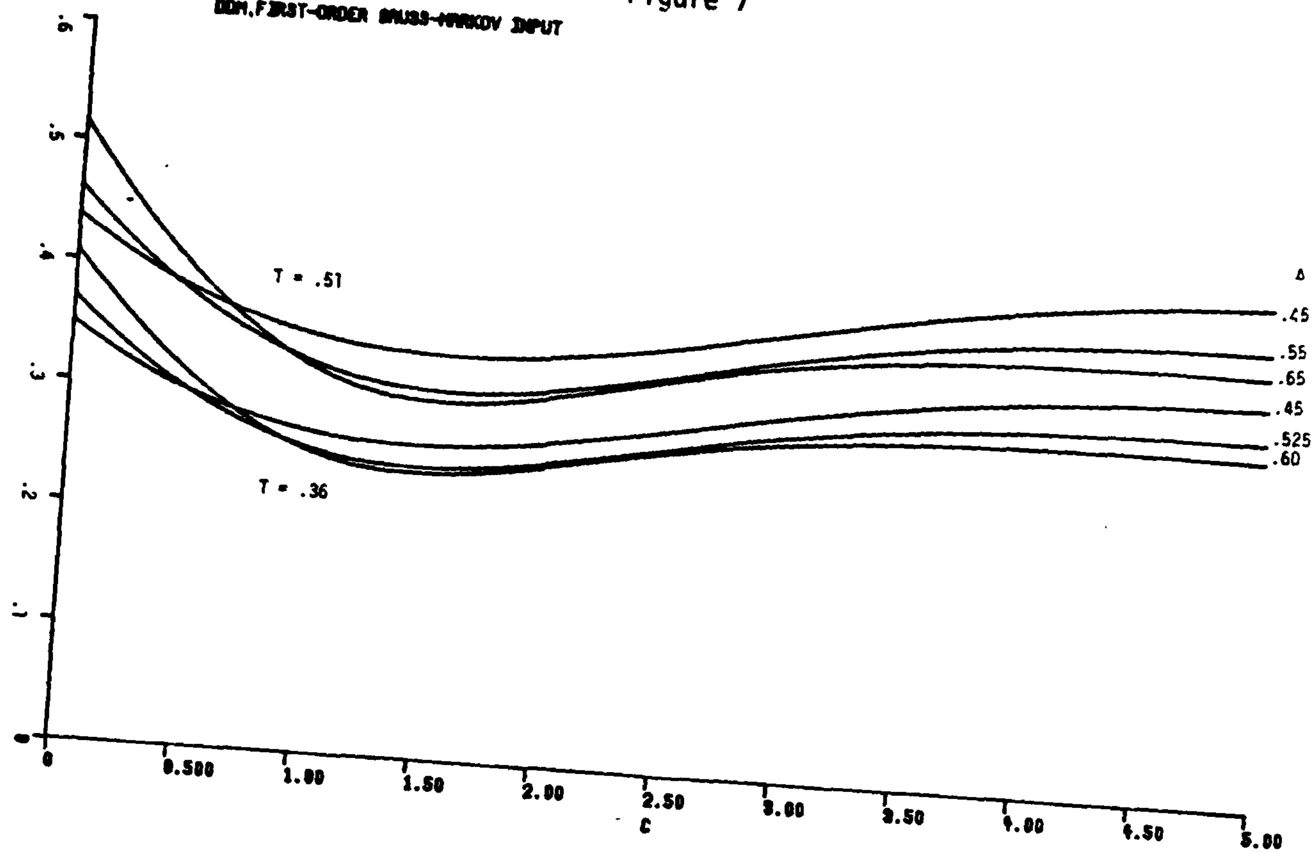


Figure 8

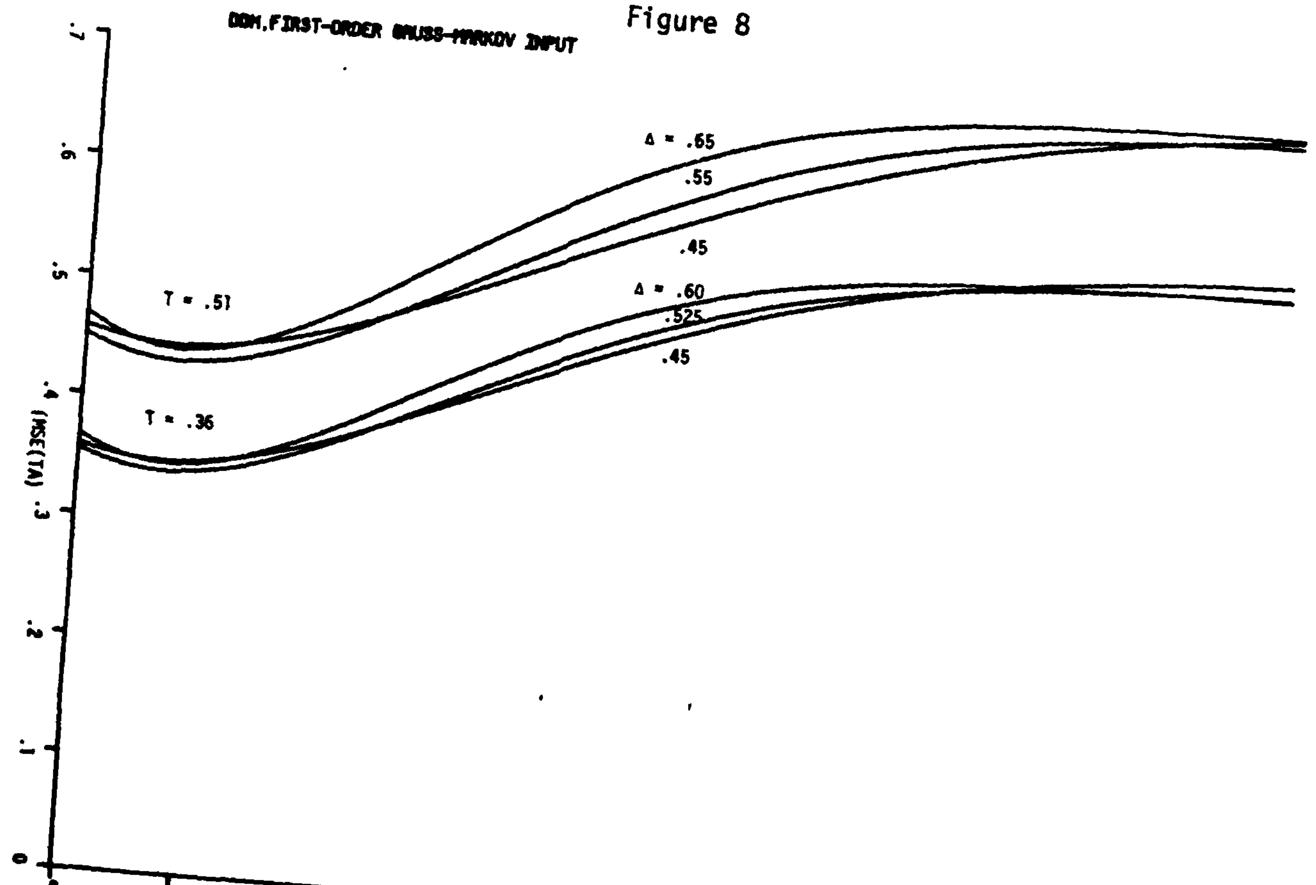


Figure 9

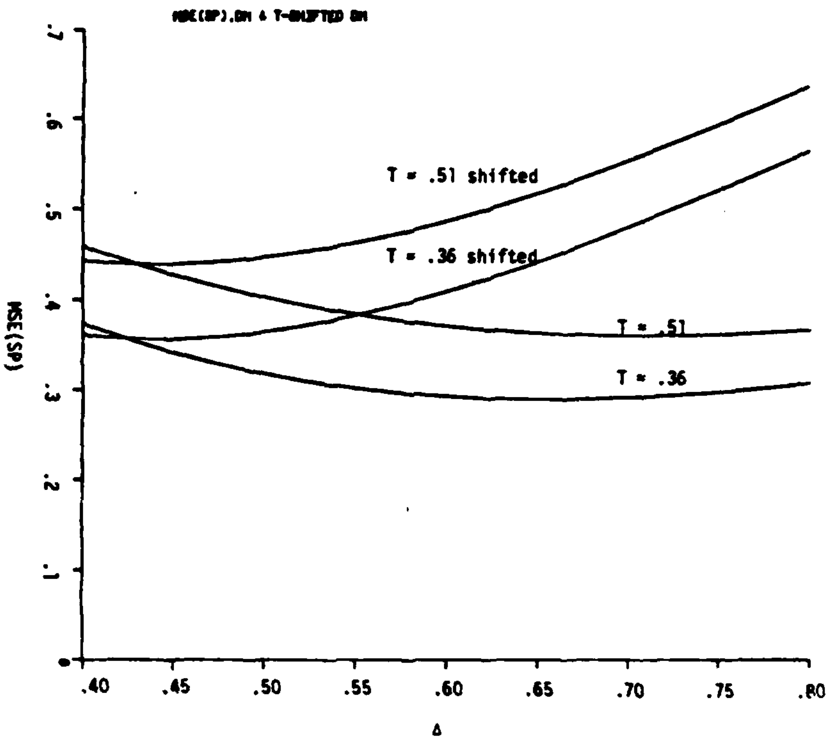


Figure 10

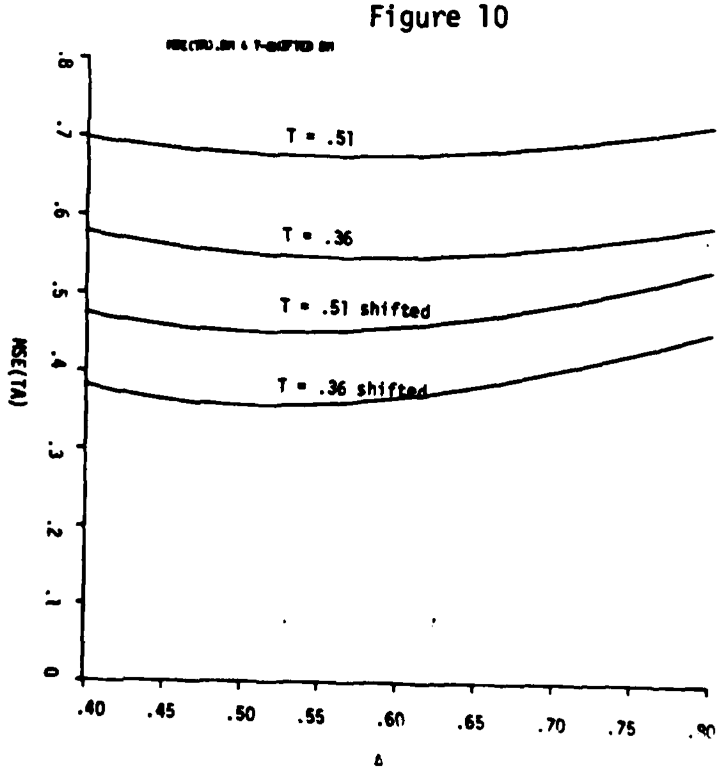


Figure 11

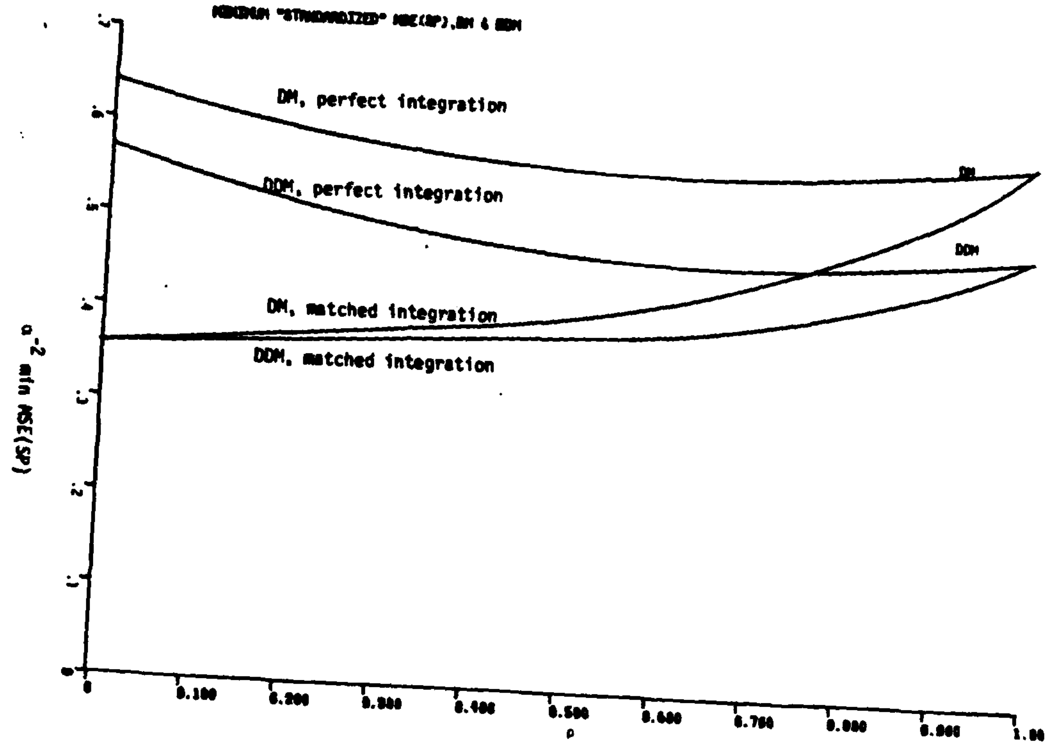


Figure 12

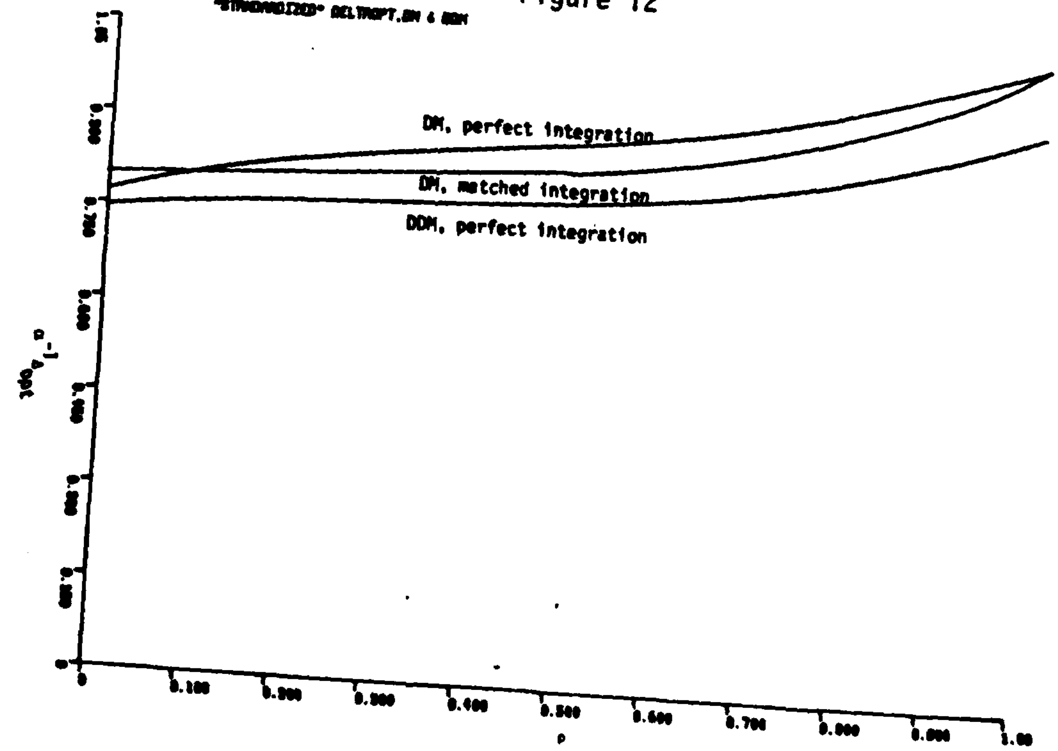
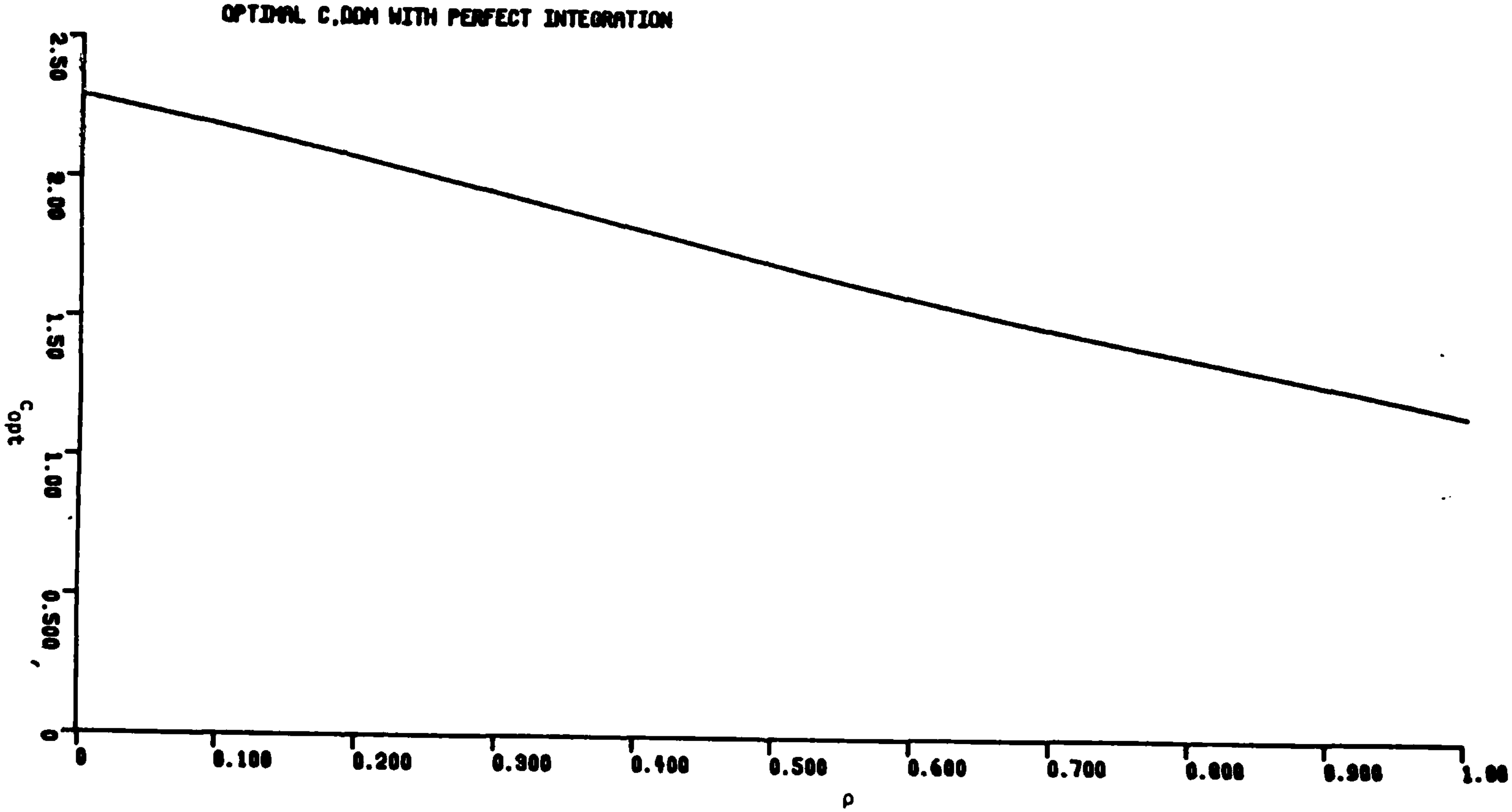


Figure 13



OPTIMAL C.DOM WITH PERFECT INTEGRATION
2.50
2.00
1.50
1.00
0.500
0
C_{opt}
0
0.100
0.200
0.300
0.400
0.500
0.600
0.700
0.800
0.900
1.00
ρ

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18. SUPPLEMENTARY NOTES

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Delayed delta modulation, stochastic stability, discrete time i.i.d. input, Wiener input, Gauss-Markov input, sample point mean square error, time averaged mean square error.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Delayed Delta Modulation (DDM) uses a second feedback loop in addition to the standard DM loop. While the standard loop compares the current predictive estimate of the input to the current sample, the new loop compares it to the upcoming sample so as to detect and anticipate slope overloading. Since this future sample must be available before the present output is determined and the estimate updated, delay is introduced at the encoding.

The performance of DDM with perfect integration and step-function reconstruction is analyzed for each of three inputs. In every case, the stochastic

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stability of the system is established. For a discrete time i.i.d. input, the (limiting) joint distribution of input and output is derived, and the (asymptotic) mean square sample point error MSE(SP) is computed when the input is Gaussian. For a Wiener input, the joint distribution of the sample point and predictive errors is derived, and MSE(SP) and the time-averaged MSE (MSE(TA)) are computed. For a stationary first-order Gauss-Markov input, the joint distribution of input and output is derived, and MSE(SP) and MSE(TA) computed. Graphs of the MSE's illustrate the improvement attainable by using DDM instead of DM. With optimal setting of parameters, MSE(SP) (MSE(TA)) is reduced about 15% (35%).

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