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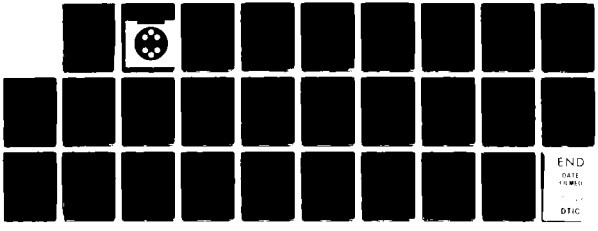

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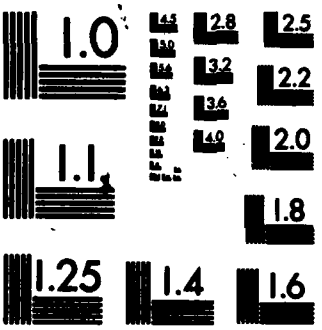

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August 1982 LCDS Report #82-23

STABILITY IN LINEAR DELAY EQUATIONS[+]

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August 1982

[+]This research has been supported in part by the Air Force Office of Scientific Research under contract #[redacted]-AFOSR 81-0198, in part by the National Science Foundation under contract #MCS 79-05774-05 and in part by the U.S. Army Research Office under contract #ARO-DAAG-29-79-C-0161.

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Jack K. Hale, Ettore F. Infante

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ABSTRACT

For linear autonomous differential difference equations of retarded or neutral type, necessary and sufficient conditions are given for the zero solution to stable (hyperbolic) for all values of the delays.

1. Introduction

This paper is devoted to the study of the effect of the delays on the asympotic behavior of the solutions of linear retarded and neutral differential difference equations. A special case of the retarded equations considered is

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{N} A_k x(t-r_k) \tag{1.1}$$

where $X \in \mathbb{R}^n$, each A_k is an $n \times n$ matrix and each $r_k > 0$, $k = 1,2,\ldots,N$. It is known that the asymptotic behavior of the solutions is determined from the solutions of the characteristic equation,

$$f(\lambda,r,A) \stackrel{\text{def}}{=} \det[\lambda I - A_0 - \sum_{k=1}^{N} A_k e^{-\lambda r_k}] = 0, \tag{1.2}$$

Let $\sigma(r,A)$ be the supremum of the real parts of the λ satisfying (1.2). It is well-known that $\sigma(r,A) < 0$ implies the zero solution of (1.1) is uniformly asymptotically stable (see, for example, [4]).

Because the supremum $\sigma(r,A)$ is attained at some specific value of λ satisfying (1.2) and the function $f(\lambda,r,A)$ is continuous in r,A, it follows that $\sigma(r,A)$ is continuous in r,A. Therefore, the property of being asymptotically stable at some point r^0,A^0 is preserved under small perturbations in r,A from r^0,A^0.

Our primary objective is to give conditions on the coefficients A in (1.1) which will ensure that Eq. (1.1) is asymptotically stable for all delays $r = (r_1, \ldots r_N)$ with $r_k > 0$, $k = 1,2, \ldots, N$; that is, we want to characterize those values of A such that $\sigma(r,A) < 0$ for all $r_k > 0$, $k = 1,2, .., N$. Some aspects of this problem have been previously discussed by Zivotovski [7], Datko [3], Repin [5], Silkowskii [6], Cooke and Ferreira [2].

If we call the set of such A the stable cone S for (1.1), then one of our results states that $A \in S$ if and only if

$$\text{(i)} \quad \det \sum_{k=0}^{N} A_k \neq 0$$

$$\text{(ii)} \quad \det[iy - A_0 - \sum_{k=1}^{N} A_k s_1^{r_1} \cdots s_N^{r_N}] \neq 0 \quad \text{for all}$$

$$y \in \mathbb{R},\ y \neq 0,\ s_j \in C,\ |s_j| = 1,\ j = 1, 2, \ldots, N.$$

These conditions also imply that the spectrum of $\sum_{k=0}^{N} A_k$ and the spectrum of A_0 lie in the left half plane.

If the equation (1.1) is a scalar equation, then the above conditions for A to be in S simplify to $\sum_{k=0}^{N} A_k < 0$, $\sum_{k=1}^{N} |A_k| \leq |A_0|$. This latter result was obtained by Zivotovski [7].

In the applications, it is *not* always true that the delays r_k vary independently of each other. For example, with three delays, r_1, r_2, r_3, one may have $r_1 = s_1$, $r_2 = s_2$, $r_3 = s_1 + s_2$ for some positive numbers s_1, s_2. In this case, the stable cone can be larger than the one obtained before. We also give a characterization of the stable cone in this case.

Finally, the results are extended to the much more complicated case of a neutral differential differente equation

$$\frac{d}{dt}[x(t) - \sum_{k=1}^{N} B_k x(t-r_k)] = A_0 + \sum_{k=1}^{N} A_k x(t-r_k)$$

The basic difficulty here arises from the fact that the asymptotic behavior of the solutions of the difference equation

$$x(t) - \sum_{k=1}^{N} B_k\, x(t - r_k) = 0$$

depends in a very complicated way upon the delays r_k. The results for this

case are contained in Section 5 and rely heavily upon Avellar and Hale [1].

2. General results for retarded equations.

Suppose $\mathbb{R}^+ = [0,\infty)$, $\mathbb{R} = (-\infty,\infty)$, $r = (r_1, \ldots, r_M) \in (\mathbb{R}^+)^M$, $\gamma_k = (\gamma_{k1}, \ldots, \gamma_{kM})$, $\gamma_{kj} \geq 0$ integers, $\gamma_k \neq 0$, $\gamma_k \cdot r = \sum_{j=1}^{M} \gamma_{kj} r_j$, $k=1,2, \ldots, N$, $j=1,2, \ldots, M$, and consider the retarded differential difference equation

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{N} A_k x(t - \gamma_k \cdot r) \tag{2.1}$$

where $x \in \mathbb{R}^n$ and each A_k, $k=0,1, \ldots, N$ is an $n \times n$ real constant matrix.

The characteristic function for Eq. (2.1) is

$$f(\lambda,r,A) = \det[\lambda I - A_0 - \sum_{k=1}^{N} A_k e^{-\lambda \gamma_k \cdot r}] \tag{2.2}$$

where $A = (A_0, A_1, \ldots, A_N) \in \mathbb{R}^{n^2(N+1)}$.

Definition 2.1. System (2.1) is said to hyperbolic at (r,A) if $f(\lambda,r,A) = 0$ implies $\operatorname{Re} \lambda \neq 0$. System (2.1) is said to be asymptotically stable at (r,A) if $f(\lambda,r,A) = 0$ implies $\operatorname{Re} \lambda < 0$.

The delays in Eq. (2.1) are the constants $\gamma_k \cdot r$, $k=1,2, \ldots, N$. They are not independent and are determined by the vector $r = (r_1,\ldots,r_M) \in (\mathbb{R}^+)^M$. For example, if $M = 2$, $N = 3$, $r = (r_1,r_2)$, $\gamma_1 = (1,0)$, $\gamma_2 = (0,1)$, $\gamma_3 = (1,1)$, then the delays are $r_1, r_2, r_1 + r_2$.

Our objective is to determine conditions on the coefficients A in (1.1) to ensure stability (or hyperbolicity) for all values of $r \in (\mathbb{R}^+)^M$. This means, in particular, that, for a given r^0, we must have stability (or hyperbolicity) for all αr^0 with $\alpha > 0$. By letting $t \to \alpha t$ in (1.1), this means that, if A^0 ensures stability (or hyperbolicity) for all $r \in (\mathbb{R}^+)^M$, then αA^0 also ensures this for every $\alpha > 0$; that is, the set of such A's is a cone. We

formalize these ideas in the following

Definition 2.2. For a given $r \in (\mathbb{R}^+)^M$, the ray γ_r through r is the set $\{\alpha\, r \in (\mathbb{R}^+)^M: \alpha \geq 0\}$. For a given $r^o \in (\mathbb{R}^+)^M$, the hyperbolic cone at r^o, designated by H_{r^o}, is defined by

$$H_{r^o} = \{A \in \mathbb{R}^{n^2(N+1)}: \text{Eq.(2.1) is hyperbolic at } (r,A) \text{ for every } r \in \gamma_{r^o}\}$$

The hyperbolic cone H is defined by

$$H = \cap\left(H_r: r \in (\mathbb{R}^+)^M\right)$$

For a given $r^o \in (\mathbb{R}^+)^M$, the asymptotically stable cone at r^o, designated by S_{r^o}, is defined by

$$S_{r^o} = \{A \in \mathbb{R}^{n^2(N+1)}: \text{Eq. (2.1) is asymptotically stable at } (r,A) \text{ for every } r \in \gamma_{r^o}\}$$

The asymptotically stable cone S is defined by

$$S = \cap_r\left(S_r: r \in (\mathbb{R}^+)^M\right)$$

In the following, the notation $\mathrm{Re}\lambda(A)$ for a matrix A designates the set consisting of the real parts of the eigenvalues of A. As a preliminary for the classification of H_r, S_r, we have the following elementary result, a form of which was proved by Datko [3].

Theorem 2.3. $A \in H_r$ [or S_r] if and only if

(i) $\mathrm{Re}\,\lambda\left(\sum_{k=0}^{N} A_k\right) \neq 0$ [or $\mathrm{Re}\,\lambda\left(\sum_{k=0}^{N} A_k\right) < 0$]

(ii) $f(iy, \alpha r, A) \neq 0$ for all $y \in \mathbb{R}$, $y \neq 0$, $\alpha \geq 0$.

Proof: Let $s_\alpha = \max\{\mathrm{Re}\,\lambda \le 0 : f(\lambda, \alpha r, A) = 0\}$, $u_\alpha = \min\{\mathrm{Re}\,\lambda \ge 0 : f(\lambda, \alpha r, A) = 0\}$ with $s_\alpha = -\infty$, $u_\alpha = +\infty$ if the corresponding set is empty. The numbers s_α, u_α are continuous in α. Condition (i) implies $s_0 < 0$, $u_0 > 0$. If $A \notin H_r$, then there is an $\alpha_0 > 0$ such that either $s_{\alpha_0} = 0$ or $u_{\alpha_0} = 0$. This contradicts (ii). Thus, (i),(ii) imply $A \in H_r$. The converse is obvious.

For the more difficult equations with distributed delays, Cooke and Ferreira [2] have obtained nontrivial results in the spirit of Theorem 2.3.

Theorem 2.4. $A \in H$ if and only if

(H_1) $\det \sum_{k=0}^{N} A_k \neq 0$

(H_2) $\det [iy - A_0 - \sum_{k=1}^{N} A_k s_1^{\gamma_{k1}} \cdots s_M^{\gamma_{km}}] \neq 0$ for all $y \in \mathbb{R}$, $y \neq 0$

$s_j \in C$, $|s_j| = 1$, $j = 1,2, \ldots, M$.

$A \in S$ if and only if $(H_2)'$ and $\mathrm{Re}\lambda(\sum_{k=0}^{N} A_k) < 0$.

Proof. Suppose $A \in H$. Then (H_1) is satisfied. If (H_2) is not satisfied at (y,s), choose $y \neq 0\theta_k$, so that $-y\theta_k > 0$ and $s_k = \exp(i\theta_k)$ for all k. With $r_k = -\theta_k/y$, we have $f(iy, r, A) = 0$ which contradicts the fact that $A \in H$.

Conversely, suppose $(H_1),(H_2)$ are satisfied and $A \notin H$. Then there is an $\alpha > 0$, $y \in \mathbb{R}$, $y \neq 0$, $r^0 \in (\mathbb{R}^+)^M$ such that $f(iy, \alpha r^0, A) = 0$. Since this contradicts (H_2), we have proved the first part of the theorem.

The condition $\mathrm{Re}\lambda(\sum_{k=0}^{N} A_k) < 0$ is equivalent to saying that Eq.(2.1) is asymptotically stable for $r = 0$. Thus, the last statement in the theorem is true.

To obtain other characterizations of H,S, we need the following lemma. We are grateful to John Mallet-Paret for assistance in the statements and proofs of the next two results.

Lemma 2.5. If

$$P(\lambda, s_1, \ldots, s_M) = \det\left[\lambda I - A_0 - \sum_{k=1}^{N} A_k s_1^{\gamma_{k1}} \cdots s_M^{\gamma_{kM}}\right], \tag{2.3}$$

then the hypothesis (H_2) implies

$$P(iy, s_1, \ldots, s_M) \neq 0 \text{ for } y \in \mathbb{R},\ |s_j| < 1,\ j = 1,2, \ldots, M. \tag{2.4}$$

Proof: Fix s_j^0, $|s_j^0| = 1$, $j = 1,2, \ldots, M$ and consider the function

$$Q(\lambda, \alpha) \overset{\text{def}}{=} P(\lambda, \alpha s_1^0, \ldots, \alpha s_M^0).$$

Designate the zeros of this equation by $\alpha(\lambda)$. Then $\alpha(\lambda)$ is meromorphic, defined on some Reimann surface over the λ-plane and $\lim_{|\lambda| \to \infty} |\alpha(\lambda)| = \infty$. Consider the curve in the α-plane defined by $\alpha(iy)$, $y \in \mathbb{R}$. Hypothesis (H_2) implies that $|\alpha(iy)| > 1$ if $y \neq 0$. Thus, $|\alpha(iy)| \geq 1$ for all y. Thus, $Q(iy, \alpha) = P(iy, \alpha s_1^0, \ldots, \alpha s_M^0) \neq 0$ for $|\alpha| < 1$, $y \in \mathbb{R}$ and all $s_j^0, |s_j^0| = 1$, $j = 1,2, \ldots, M$. This proves the lemma.

By taking each $s_j = 0$, $j = 1,2, \ldots, M$, in Lemma 2.5, we obtain

Corollary 2.6. Hypothesis (H_2) implies $\operatorname{Re} \lambda (A_0) \neq 0$.

Corollary 2.7. If $A \in H$, then $\operatorname{Re} \lambda \left(\sum_{k=0}^{N} A_k\right) \neq 0$, $\operatorname{Re} \lambda (A_0) \neq 0$ and the matrices $\sum_{k=0}^{N} A_k$ and A_0 have the same number of eigenvalues with positive and negative real parts.

If $A \in S$, then $\operatorname{Re} \lambda (A_0) < 0$, $\operatorname{Re} \lambda \left(\sum_{k=0}^{N} A_k\right) < 0$.

Proof: Let $P(\lambda, s)$ be defined by Relation (2.3). Let $Q(\lambda, s) = P(\lambda, s, \ldots, s)$. From Lemma 2.5 and Relation (2.4), $Q(iy, s) \neq 0$ for $0 \leq s < 1, y \in \mathbb{R}$. Hypothesis (H_1) implies $Q(iy, 1) \neq 0$ for all $y \in \mathbb{R}$. Thus, $Q(\cdot, s)$ has no roots on the

imaginary axis for any $s \in [0,1]$. Since $Q(\lambda,0) = \det [\lambda I - A_0]$, $Q(\lambda,1) = \det [\lambda I - \sum_{k=0}^{N} A_k]$, it follows that $\operatorname{Re} \lambda(A_0) \neq 0$, $\operatorname{Re} \lambda (\sum_{k=0}^{N} A_k) \neq 0$. Since the zeros of $Q(\lambda,s) = 0$ are continuous in s, the result follows immediately.

The verification of Hypothesis (H_2) is extremely difficult. For the case of an n^{th} order scalar equation and independent delays, this hypothesis can be written in a more convenient form.

Following Zivotovskii, consider the scalar equation

$$y^{(n)}(t) + \sum_{j=1}^{n} a_{j0} y^{(n-j)}(t) + \sum_{k=1}^{N} \sum_{j=1}^{n} a_{jk} y^{(n-j)}(t - \omega_k) = 0 \tag{2.5}$$

where $\omega_k \geq 0$, $a_{jk} \in \mathbb{R}$ for all j,k. We can now state the following generalization of the results in Zivotovskii [7].

Theorem 2.8. Let $a = (a_{jk},\ j = 1,2,\ldots,n,\ k=0,1,\ldots,N)$

$$p_0(\lambda,a) = \lambda^n + \sum_{j=1}^{n} a_{j0} \lambda^{n-j}$$

$$p_k(\lambda,a) = \sum_{j=1}^{n} a_{jk} \lambda^{n-j}$$

Then $a \in H$ if and only if

$$\sum_{j=0}^{N} a_{nj} \neq 0 \tag{2.6}$$

$$|p_0(iy,a)| > \sum_{k=1}^{N} |p_k(iy,a)| \quad \text{for all } y \neq 0,\ y \in \mathbb{R} \tag{2.7}$$

and then necessarily $a_{n0} \neq 0$.

The vector $a \in S$ if and only if (2.6),(2.7) are satisfied and

$$\operatorname{Re} \lambda < 0 \text{ if } p_0(\lambda,a) = 0. \tag{2.8}$$

Proof: Transform the equation to an equivalent system

$$\dot{x} = A_0 x(t) + \sum_{k=1}^{N} A_r x(t - \omega_k)$$

where $x = (y, y^{(1)}, \ldots, y^{(n-1)})$. The characteristic function is given by

$$f(\lambda, r, a) = p_0(\lambda, a) + \sum_{k=1}^{N} p_k(\lambda, a) e^{-\lambda \omega_k}.$$

Hypothesis (H_1) is equivalent to $f(0,r,a) \neq 0$ which is (2.6). Hypothesis (H_2) is equivalent to

$$p_0(iy,a) + \sum_{k=1}^{N} p_k(iy,a)\, s_k \neq 0 \quad \forall\, y \in \mathbb{R},\ y \neq 0,\ |s_k| = 1,$$

and this is equivalent to (2.7). The last statement follows from Corollary 2.7 since the characteristic function for A_0 is $p_0(\lambda,a)$. This proves the theorem.

We also can generalize Zivotovskii's result to the case where the delays are dependent, but it cannot be stated in such a simple fashion. The proof is the same as before.

Theorem 2.9. Consider again Eq. (2.5) with $\omega_j = \gamma_j \cdot r$, $r \in (\mathbb{R}^+)^M$, $\gamma_j = (\gamma_{j1}, \ldots, \gamma_{jM})$, γ_{jk} nonnegative integers, $\gamma_j \neq 0$. With $p_0(\lambda,a)$, $p_k(\lambda,a)$ defined as in Theorem 2.8, the vector $a \in H$ if and only if Relation (2.6) and

$$(2.9) \qquad p_0(iy,a) + \sum_{k=1}^{N} p_k(iy,a)\, s_1^{\gamma_{k1}} \ldots s_M^{\gamma_{kM}} \neq 0 \quad \forall y \in \mathbb{R},\ y \neq 0,\ |s_j| = 1, \quad j = 1,2,\ldots,M$$

are satisfied.

The vector $a \in S$ if and only if relations (2.6),(2.9),(2.8) are satisfied.

For the case of one delay in Eq. (2.1) one can obtain an equivalent formulation of the cones H,S following an idea of Repin [5] via the following lemma.

Lemma 2.10. If A_0, A_1, are real $n \times n$ matrices, then the statement

(2.10) For every $\mu \in \mathbb{R}$, $\mu \neq 0$, the solutions of the equation

$$\det \begin{bmatrix} \lambda A_0 + A_1 & -\lambda\mu I \\ \lambda\mu I & \lambda A_0 + A_1 \end{bmatrix} = 0$$

satisfy $|\lambda| < 1$

is equivalent to the statement

(2.11) For every $y \in \mathbb{R}$, $|\alpha| < 1$, $\det[iy - A_0 - \alpha A_1] \neq 0$

Proof: If $\lambda \neq 0$, $\mu \neq 0$, then the equation in (2.10) is equivalent to

$$\det \begin{bmatrix} 0 & -\lambda\mu I + (\lambda A_0 + A_1)^2 \frac{1}{-\lambda\mu} \\ \lambda\mu I & \lambda A_0 + A_1 \end{bmatrix} = 0$$

which is equivalent to

$$\det [\lambda^2\mu^2 I + (\lambda A_0 + A_1)^2] = 0$$

which is equivalent to

$$\det [-\lambda(iy) + (\lambda A_0 + A_1)] = 0$$

for $y \in \mathbb{R}$, $y \neq 0$. If $|\lambda| < 1$ when this is satisfied, then this is equivalent to saying that

$$\det [iy - A_0 - \alpha A_1] = 0, \; y \in \mathbb{R}, \; y \neq 0$$

implies $|\alpha| > 1$. Thus, the solutions of this equation for all $y \in \mathbb{R}$ satisfy $|\alpha| \geq 1$. Thus, we obtain (2.10) is equivalent to (2.11) and the lemma is proved.

Corollary 2.11. For the n-dimensional system

$$\dot{x}(t) = A_0x(t) + A_1x(t-r)$$

$A = (A_0,A_1) \in H$ if and only if (2.10) and

$$\det(A_0+A_1) \neq 0 \tag{2.12}$$

are satisfied. $A \in S$ if and only if these conditions and $\operatorname{Re}\lambda(A_0) < 0$.

Proof: This is an immediate consequence of Lemmas 2.10,2.5 and Theorem 2.4.

3. First order scalar equations. For first order scalar equations,

$$\dot{x}(t) = a_0x(t) + \sum_{k=1}^{N} a_kx(t-\gamma_k\cdot r) \tag{3.1}$$

where $a_j \in \mathbb{R}$, $j = 0,1,\ldots,N$, the characterization of the hyperbolic and asymptotically stable cones can be specified in terms of properties of the solutions of the difference equation

$$a_0y(t) + \sum_{k=1}^{N} a_ky(t-\gamma_k\cdot r) = 0, \tag{3.2}$$

It is the purpose of this section to obtain such a characterization.

Let $a = (a_0,a_1,\ldots,a_k)$,

$$\begin{aligned} \alpha(\theta,a) &= a_0 + \sum_{k=1}^{N} a_k \cos\gamma_k\cdot\theta \\ \beta(\theta,a) &= \sum_{k=1}^{N} a_k \sin\gamma_k\cdot\theta \quad , \quad \theta \in \mathbb{R}^N \end{aligned} \tag{3.3}$$

The characteristic equation for the difference equation (3.2) is

$$g(\lambda,a,r) \overset{\text{def}}{=} a_0 + \sum_{k=1}^{N} a_ke^{-\lambda\gamma_k\cdot r} \tag{3.4}$$

The functions $\alpha(\theta,a)$, $\beta(\theta,a)$ are related to the function $g(\lambda,a,r)$ by the relation

$$g(iy,a,r) = \alpha(-yr,a) - i\beta(yr,a) \tag{3.5}$$

The main result of this section is the following

Theorem 3.1. For the scalar equation (3.1), $a \in H$ if and only if

(H_3) $\sum_{k=0}^{N} a_k \neq 0$

(H_4) For each $\theta \in \mathbb{R}^M$, either $\alpha(\theta,a) \neq 0$ or simultaneously, $\alpha(\theta,a) = 0$, $\beta(\theta,a) = 0$. If (H_4) is satisfied then $a_0 \neq 0$.

The vector $a \in S$ if and only if (H_3), (H_4) and $a_0 < 0$.

Proof: For $n = 1$, the condition (H_1) in Theorem 2.4 is equivalent to (H_3) and (H_2) is the same as

$$(\alpha(\theta,a),\ y - \beta(\theta,a)) \neq 0 \quad \text{for all} \quad y \in \mathbb{R},\ y \neq 0,\ \theta \in \mathbb{R}^M$$

which is equivalent to (H_4). Corollary 2.7 implies $a_0 \neq 0$. The statement about S is also a consequence of Corollary 2.7. This completes the proof of the theorem.

An immediate consequence of Theorem 3.1 is the following result of Silkowski [6].

Corollary 3.2 Suppose the components of r are rational and define the functions

$$\gamma(y) = a_0 + \sum_{k=1}^{N} a_k \cos \gamma_k \cdot r\, y$$

$$\delta(y) = \sum_{k=1}^{N} a_k \sin \gamma_k \cdot r\, y$$

Then $a \in H_r$ if and only if

(i) $\sum_{k=1}^{N} a_k \neq 0$

(ii) For each $y \in \mathbb{R}$, $y \neq 0$, either $\gamma(y) \neq 0$ or $\gamma(y) = 0$, $\delta(y) = 0$.

Furthermore, $a \in S_r$ if and only if (i),(ii) and $a_0 < 0$. Finally, condition (ii) is equivalent to

$$\text{Re } \lambda \leq 0 \text{ if } a_0 + \sum_{k=1}^{N} a_k e^{-\lambda \gamma_k \cdot r} = 0.$$

Proof: Since the components of r are rational, we may assume the equation has only one independent delay. The result is then a special case of Theorem 3.1.

It is interesting to state Corollary 3.2 in terms of properties of zeros of polynomials. If the components of r are rational, we can write

$$a_0 + \sum_{k=1}^{N} a_k \cos \gamma_k \cdot r \; y = h(\cos y)$$

(3.8)

$$\sum_{k=1}^{N} a_k \sin \gamma_k \cdot r \; y = (\sin y)\, g(\cos y)$$

where h,g are polynomials.

Corollary 3.3. With h,g as in relation (3.8), the statement

(i) $h(1) \neq 0$ and, for every $\eta \in [0,1)$ for which $h(\eta) = 0$, it follows that $g(\eta) = 0$

is equivalent to the statement

(ii) $h(1) \neq 0$, $a_0 \neq 0$, $\text{Re } \lambda \leq 0$ if $a_0 + \sum_{k=1}^{N} a_k e^{-\lambda \gamma_k \cdot r} = 0$

Proof: This is a restatement of Corollary 3.1.

Corollary 3.4. For the scalar equation

$$\dot{x}(t) = a_0 x(t) + \sum_{k=1}^{N} a_k x(t-r_k)$$

we have $a \in H$ if and only if $\sum_{k=0}^{N} a_k \neq 0$, $\sum_{k=1}^{N} |a_k| \leq |a_0|$.

The vector $a \in S$ if and only if $a \in H$ and $a_0 < 0$.

Proof: Theorem 3.1 implies $a \in H$ if and only if $a_0 \neq 0$, $\sum_{k=0}^{N} a_k \neq 0$ and (H_4). Hypothesis (H_4) is equivalent to either

$$\alpha(\theta,a) = a_0 + \sum_{k=1}^{N} a_k \cos \theta_k \neq 0$$

or, simultaneously,

$$\alpha(\theta,a) = 0$$

$$\beta(\theta,a) = \sum_{k=1}^{N} a_k \sin \theta_k = 0.$$

The latter relation implies $\sum_{k=1}^{N} |a_k| \leq |a_0|$. Conversely, if $\sum_{k=1}^{N} |a_k| \leq |a_0|$, then $\alpha(\theta,a) = 0$ for some θ implies $\sum_{k=1}^{N} |a_k| = |a_0|$ and each component θ_k of θ is 0 or π. But this implies $\beta(\theta,a) = 0$. The last assertion about S is also a consequence of Theorem 3.1.

Corollary 3.5. For the scalar equation

$$\dot{x}(t) = a_0 x(t) + a_1 x(t-r_1) + a_2 x(t-r_2) + a_3 x(t-r_1-r_2)$$

we have $a \in H$ if and only if

(i) $a_0 + a_1 + a_2 + a_3 \neq 0$

(ii) $1 + \frac{a_1}{a_0} > \left| \frac{a_2}{a_0} + \frac{a_3}{a_0} \right|, \quad 0 \leq -a_1/a_0 < 1$

$$1 - \frac{a_1}{a_0} > \left| \frac{a_2}{a_0} - \frac{a_3}{a_0} \right|, \quad 0 \leq a_1/a_0 < 1$$

(iii) either $a_0 + a_1 = a_2 + a_3$ or $a_0 - a_1 = a_2 - a_3$.

The vector $a \in S$ if and only if $a \in H$ and $a_0 < 0$.

Proof: Theorem 3.1 implies that $a \in H$ if and only if $a_0 \neq 0$, (i) and (H_4) is satisfied; that is, if $b_j = a_j / a_0$, and

$$\alpha(\theta,a) = a_0 [1 + b_1 \cos \theta_1 + b_2 \cos \theta_2 + b_3 \cos (\theta_1 + \theta_2)],$$

$$\beta(\theta,a) = a_0 [b_1 \sin \theta_1 + b_2 \sin \theta_2 + b_3 \sin (\theta_1 + \theta_2)],$$

then, if there is a θ such that $\alpha(\theta,a) = 0$, then $\beta(\theta,a) = 0$. But, these θ are precisely the ones for which $\alpha(\theta,a) + i\beta(\theta,a) = 0$; that is,

$$1 + b_1 e^{i\theta_1} = -e^{i\theta_2}(b_2 + b_3 e^{i\theta_1}).$$

To have a solution of this equation, one must have θ_1, θ_2 take on the values 0 or π. Checking separately the cases $\theta_1 = 0$, $\theta_1 = \pi$, one observes that, for any b_1, one has a solution of this equation if and only if either

$$1 + b_1 = b_2 + b_3 \tag{3.9}$$

or

$$1 - b_1 = b_2 - b_3 \tag{3.10}$$

If we choose a value of $b = (b_1, b_2, b_3)$ which does not satisfy either of these inequalities and, if (H_4) is satisfied, then we must have $\alpha(\theta,a) \neq 0$ for all $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$. The relation $\alpha(\theta,a) \neq 0$ for all $\theta \in \mathbb{R}^2$ is equivalent to

$$\text{Re}\,[1 + b_1 e^{i\theta_1}] + \text{Re}\, e^{i\theta_2}[b_2 + e^{i\theta_1}] \neq 0$$

for all $\theta \in \mathbb{R}^2$. This implies $|b_1| \leq 1$. If $|b_1| \leq 1$, then it is easily observed that

$$\text{if} \quad 0 \leq b_1 < 1, \quad \text{then} \quad 1 - b_1 > |b_2 - b_3|,$$

(3.11)

$$\text{if} \quad 0 \leq -b_1 < 1, \quad \text{then} \quad 1 + b_1 > |b_2 + b_3|.$$

Relations (3.9), (3.10) and (3.11) are precisley the relations in (ii). Thus, (H_4) implies (ii). The converse is a straightforward reversal of the argument. This proves the corollary.

If there are three independent delays in the equation in Corollary 3.5, the condition (ii) would be replaced by $|a_1| + |a_2| + |a_3| \leq |a_0|$ which is a more restrictive condition on the coefficients than the one for only two independent delays.

4. Some examples. In this section, we give some examples illustrating the application of the results of Section 2 to equations of order ≥ 2. These examples will also show that the results in Section 3 do not generalize to systems; that is, one cannot reduce the discussion of the hyperbolic and stable cones to the discussion of properties of difference equations.

Example 4.1. Consider the system

$$\dot{x}(t) = B[x(t) - \mu x(t - r)] \tag{4.1}$$

where B is a 2x2 matrix with $\mathrm{Re}\lambda(B) < 0$, μ is a scalar, $|\mu| < 1$. We want to determine conditions on B, μ so that the matrices $(B, -\mu B) \in S$, the asymptotically stable cone.

By using the Jordan normal form for B, one sees that Theorem 2.4 asserts that $(B,-\mu B) \in S$ if and only if

$$iy - \lambda(B)(1 + \mu e^{i\theta}) \neq 0 \quad \text{for all} \quad y \in \mathbb{R},\ y \neq 0,\ \theta \in \mathbb{R},$$

for every eigenvalue $\lambda(B)$ of B. If $\lambda(B)$ is real, this relation is always satisfied. However, if $\lambda(B)$ is complex, this may not be true. In fact,

$$1 + \mu e^{i\theta} = (1 + 2\mu \cos\theta + \mu^2)^{1/2} \exp i\,\zeta(\mu,\theta)$$

where

$$\tan^{-1} \zeta(\mu,\theta) = (\mu \sin\theta)/(1 + \mu \cos\theta)$$

and $0 \leq |\zeta(\mu,\theta)| \leq \pi$, $\zeta(\mu,0) = 0$. If $\zeta_0(\mu) = \max_\theta |\zeta(\mu,\theta)|$, then $\zeta_0(\mu) \geq \pi/2$ if $0 \leq \mu < 1$, $\zeta_0(\mu) \leq \pi/2$ if $-1 < \mu \leq 0$.

We can now assert that $(B,-\mu B) \in S$ if and only if

$$\frac{\pi}{2} < \arg \lambda(B) + \zeta_0(\mu) < \pi$$

for all eigenvalues $\lambda(B)$ of B. This clearly puts a restriction on the eigenvalues of B and μ with the restrictions being more severe for $\mu > 0$ than for $\mu < 0$.

This example shows that the results in Section 3 cannot be generalized to systems. In fact, the zero solution of the difference equation

$$By(t) - \mu By(t-r) = 0$$

is asymptotically stable for every μ, $|\mu| < 1$, and $(B,-\mu B)$ may not belong

to S for every B,μ.

Example 4.2. Consider the equation

$$\ddot{x}(t) + a_0\dot{x}(t) + a_1\dot{x}(t-r) + a_2x(t) + a_3x(t-r) = 0 \tag{4.2}$$

along with the characteristic equation

$$\lambda^2 + \lambda\,(a_0 + a_1e^{-\lambda r}) + a_2+a_3e^{-\lambda r} = 0.$$

If $a = (a_0,a_1,a_2,a_3)$, then Theorem 2.8 implies that $a \in H$, the hyperbolic cone, if and only if

$$a_2 + a_3 \neq 0 \tag{4.3}$$

$$|P(iy)| > |Q(iy)| \quad \text{for all} \quad y \neq 0,\ y \in \mathbb{R} \tag{4.4}$$

$$P(\lambda) = \lambda^2 + a_0\lambda + a_2$$

$$Q(\lambda) = a_1\lambda + a_3$$

The condition (4.4) is equivalent to

$$f(y^2) \overset{\text{def}}{=} (y^2-a_2)^2 + (a_0^2-a_1^2)y^2-a_3^2 > 0.$$

for all $y \in \mathbb{R}$, or, equivalently, $f(r) > 0$ for $r > 0$. It is easy to show the quadratic function $f(r) > 0$ for $r > 0$ if and only if

$$\begin{aligned} &a_0^2 - a_1^2 - 2a_2 \geq 0 \ \text{ implies } \ |a_2| > |a_3| \\ &a_0^2 - a_1^2 - 2a_2 < 0 \ \text{ implies } \ a_2^2 > a_3^2 + \tfrac{1}{4}(a_0^2-a_1^2-2a_2)^2 \end{aligned} \tag{4.5}$$

Thus, $a \in H$ if and only if (4.3),(4.5) are satisfied.

Theorem 2.8 also implies that $a \in S$, the asymptotically stable cone, if and only if (4.3),(4.5) and $\mathrm{Re}\ \lambda < 0$ if $\lambda^2 + a_0\lambda + a_2 = 0$; that is,

$$a_0 > 0,\ a_2 > 0. \tag{4.6}$$

If Eq. (4.2) is transformed to a system of order two

$$\dot{x}(t) = A_0x(t) + A_1x(t-r)$$

then the zero solution of the difference equation

$$A_0y(t) + A_1y(t-r) = 0$$

is asymptotically stable if $|a_3| < |a_2|$ which does not imply anything about H.

5. Neutral equations. In this section, we generalize the results of Section 2 and 3 to neutral differential difference equations

$$\frac{d}{dt}\left[x(t) - \sum_{k=1}^{N} B_k x(t-\gamma_k r)\right] = A_0 x(t) + \sum_{k=1}^{N} A_k x(t-\gamma_k r) \tag{5.1}$$

where $x \in \mathbb{R}^n$, each A_0, A_k, B_k, $k = 1,2,\ldots,N$, is an $n \times n$ constant matrix and the γ_k, r are the same as before. The characteristic function for Eq. (5.1) is

$$g(\lambda,r,A,B) = \det\left[\lambda\left(I - \sum_{k=1}^{N} B_k e^{-\lambda\gamma_k r}\right) - A_0 - \sum_{k=1}^{N} A_k e^{-\lambda\gamma_k r}\right] \tag{5.2}$$

where $A = (A_0, A_1, \ldots, A_N)$, $B = (B_1, \ldots, B_N)$.

Definition 5.1. System (5.1) is said to be hyperbolic at (r,A,B) if there is a $\delta > 0$ such that $\{\mathrm{Re}\lambda : g(\lambda,r,A,B) = 0\} \cap [-\delta, \delta] = \emptyset$.

System (5.1) is said to be (uniformly) asymptotically stable at (r,A,B) if it

it hyperbolic at (r,A,B) and $\{\mathrm{Re}\lambda: g(\lambda,r,A,B) = 0\} \cap [-\delta,\infty) = \emptyset$.

Definition 5.2. For a given $r^0 \in (\mathbb{R}^+)^M$, the hyperbolic cone at r^0, designated by H_{r^0}, is defined as

$H_{r^0} = \{(A,B) \in \mathbb{R}^{n^2(N+1)} \times \mathbb{R}^{n^2 N}$ Eq.(5.1) is hyperbolic at (r,A,B) for every $r = \alpha r^0$, $\alpha \geq 0\}$

The hyperbolic cone H is defined by

$$H = \cap \{H_r : r \in (\mathbb{R}^+)^M\}$$

For a given $r^0 \in (\mathbb{R}^+)^M$, the asymptotically stable cone at r^0, designated by S_{r^0} is the set of $(A,B) \in H_{r^0}$ such that (r,A,B) is asymptotically stable for every $r = \alpha r^0$, $\alpha \geq 0$. The asymptotically stable cone S is defined by

$$S = \cap_r \{S_r : r \in (\mathbb{R}^+)^M\}$$

The set H_r is not really a cone in (A,B) space. In fact, if $(A,B) \in H_r$, then $(\alpha A,B) \in H_r$ for every $\alpha \geq 0$, but $(\alpha A, \alpha B)$ will generally not be. The reason for this is that, if $r \mapsto \alpha r$, $t \mapsto \alpha t$, then the new equation has coefficients $(\alpha A,B)$. In spite of this fact, we retain the term cone for H_r, but it should be remembered that the property of being a cone holds only in the A variable.

Our objective is to give a classification of the hyperbolic and asymptotically stable cones. This problem is much more difficult than the corresponding one for the retarded equation in Section 2 because the set of real parts of the zeros of the characteristic function

$$\Delta(\lambda,r,B) = \det\left[I - \sum_{k=1}^{N} B_k e^{-\lambda \gamma_k \cdot r}\right] \tag{5.3}$$

of the difference equation

$$y(t) - \sum_{k=1}^{N} B_k y(t-\gamma_k \cdot r) = 0 \tag{5.4}$$

does not depend continuously on r.

For the difference equation (5.4), we need the definitions analgous to Definitions 5.1 and 5.2 for Eq. (5.1).

Definition 5.3. System (5.4) is said to be hyperbolic at (r,B) if there is a $\delta > 0$ such that $\{\mathrm{Re}\lambda : e(\lambda,r,B) = 0\} \cap [-\delta,\delta] = \emptyset$. System (5.4) is said to be (uniformly) asymptotically stable at (r,B) if it is hyperbolic and $\{\mathrm{Re}\lambda : e(\lambda,r,B) = 0\} \cap [-\delta,\infty) = \emptyset$.

If Equation (5.4) is hyperbolic (asymptotically stable) at (r,B), then it is hyperbolic (asymptotically stable) at $(\alpha r,B)$ for every $\alpha > 0$. If we assume

$$\det(I - \sum_{k=0}^{N} B_k) \neq 0, \tag{5.5}$$

then it is also hyperbolic at $\alpha = 0$ because $\{\mathrm{Re}\lambda : e(\lambda,0,B) = 0\}$ is empty. Thus, with (5.5), if Eq. (5.4) is hyperbolic at (r^0,B) it is hyperbolic for every $(\alpha r,B)$, $\alpha \geq 0$. This means there is no reason to use the concept of hyperbolic cone at r^0 for Eq. (5.4). It becomes only necessary to discuss whether or not hyperbolic is or is not preserved under variations in r.

Throughout this section, we assume (5.5) is always satisfied.

Definition 5.4. Eq. (5.4) is said to be hyperbolic locally at (r^0,B) if there is a neighborhood $U(r^0)$ of r^0 such that Eq. (5.4) is hyperbolic at (r,B) for every $r \in U(r^0)$. Eq. (5.4) is said to be hyperbolic globally at B if it is hyperbolic at (r,B) for every $r \in (R^+)^N$. The hyperbolic set for Eq. (5.4) is the set $\{B \in R^{n^2N} : \text{Eq. (5.4) is hyperbolic at every } (r,B), r \in (R^+)^N\}$

Similar definitions are made for asymptotically stable locally at (r^0,B), asymptotically stable globally at B and the asymptotically stable set for Eq. (5.4).

In Definition 5.4, no mention is made of the variation of the concept of hyperbolic with respect to variations in the coefficients B. The reason for this is that if Eq. (5.4) is hyperbolic at (r^0,B^0), then there is a neighborhood $V(B^0)$ such that Eq. (5.4) is hyperbolic at (r^0,B) for every $B \in V(B^0)$ (see Avellar and Hale [1]).

We need the following fundamental result from [1].

Lemma 5.1. Fix $B \in \mathbb{R}^{n^2N}$. The following statements are equivalent:

(i) There is an $r^0 \in (\mathbb{R}^+)^M$ with rationally independent components such that Eq. (5.4) is hyperbolic at (r^0,B).

(ii) There is an $r^0 \in (\mathbb{R}^+)^M$ such that Eq. (5.4) is hyperbolic locally at (r^0,B).

(iii) Eq. (5.4) is hyperbolic globally at B.

(iv) $1 \notin I(r,B) \overset{\text{def}}{=} \{|\mu(\theta)| : \det\left[\mu I - \sum_{k=1}^{N} B_k e^{i\gamma_k\cdot\theta}\right] = 0, \theta \in \mathbb{R}^M\}$

(v) If

$$E(\rho,\theta,r,B) = \det\left[I - \sum_{k=1}^{N} B_k e^{-\rho\gamma_k\cdot r} e^{i\gamma_k\cdot\theta}\right], \quad \theta \in \mathbb{R}^M, \ \rho \in \mathbb{R}$$

and $\Gamma(r,B) = \{\rho: \exists\ \theta \text{ with } E(\rho,\theta,r,B) = 0\}$, then $0 \notin \Gamma(r,B)$.

The same result holds with hyperbolic replaced by asymptotically stable and (iv),(v) are replaced by

(iv)' $I(r,B) \subseteq [0,1)$

(v)' $\Gamma(r,B) \cap [0,\infty) = \emptyset$.

With $g(\lambda,r,A,B)$ as in Relation (5.2), define

$$a^-(r,A,B) = \sup \{Re\lambda \leq 0 : g(\lambda,r,A,B) = 0\}$$

(5.6)

$$a^+(r,A,B) = \inf \{Re\lambda \geq 0 : g(\lambda,r,A,B) = 0\}$$

and define $a^-(r,A,B) = -\infty$, $a^+(r,A,B) = +\infty$ if the corresponding set is empty.

We need the following result which is stated without proof. The proof uses some special properties of characteristic functions which we have not used before. The reader can supply the details following ideas from [4, Ch. 12].

Lemma 5.2. If Eq. (5.1) is hyperbolic (asymptotically stable) at (r,A,B), then the difference equation (5.4) is hyperbolic (asymptotically stable) at (r,B). If Eq. (5.4) is hyperbolic at (r,A,B), then $a^-(\alpha r,A,B)$, $a^+(\alpha r,A,B)$ are continuous in α for $\alpha > 0$.

One can now generalize the results of Section 2 to the neutral equation (5.1) (see also Datko [3].)

Theorem 5.3. $(A,B) \in H_r$ (or S_r) if and only if

(i) $$Re\lambda \left[(I - \sum_{k=1}^{N} B_k)^{-1} \sum_{k=0}^{N} A_k \right] \neq 0 \quad (\text{or} < 0)$$

(ii) $g(iy,\alpha r,A,B) \neq 0$ for all $y \in \mathbb{R}$, $y \neq 0$, $\alpha \geq 0$.

Proof: The proof is the same as the proof of Theorem 2.1, making use of Lemma 5.1.

Theorem 5.4. If

$$(5.7)\quad P(\lambda,s_1,\ldots,s_M,A,B) = \det\left[\lambda-(I-\sum_{k=1}^{N} B_k s_1^{\gamma_{k1}}\ldots s_M^{\gamma_{km}})^{-1}(A_0+\sum_{k=1}^{N} A_k s_1^{\gamma_{k1}}\ldots s_M^{\gamma_{kM}})\right]$$

then $(A,B) \in H$ if and only if

(H_1) Eq. (5.4) is hyperbolic globally at B

(H_2) $P(0,1,\ldots,1,A,B) \neq 0$

(H_3) $P(iy,s_1,\ldots,s_M,A,B) \neq 0$

for all $y \in \mathbb{R}$, $y \neq 0$, $|s_j| = 1$, $j = 1,2,\ldots,M$.

The pair $(A,B) \in S$ if and only if $(H_1),(H_3)$ and $\operatorname{Re}\lambda\left[(I-\sum_{k=1}^{N} B_k)^{-1}\sum_{k=0}^{N} A_k\right] < 0$.

Proof: Suppose $(A,B) \in H$. Then Lemma 5.2 implies (H_1). Lemma 5.1, part (iv) implies there are constants $\delta > 0$, $\eta > 0$ such that $|\det(I-\sum_{k=1}^{N} B_k s_1^{\gamma_{k1}}\ldots s_M^{\gamma_{kM}})| \geq \delta$ for $||s_j|-1| < \eta$, $j = 1,2,\ldots,M$. Therefore, P in Relation (5.7) is well defined for $||s_j|-1| < \eta$. Theorem 5.3 implies (H_2). If (H_3) is not satisfied for some $(y,s_1,\ldots,s_M)$, $y \neq 0$, $|s_k| = 1$, choose $\theta_k \in \mathbb{R}$ so that $s_k = \exp(i\theta_k)$, $-y\,\theta_k > 0$, $k = 1,2,\ldots,M$. If $r_k = -\theta_k/y$, then $g(iy,r,A,B) = 0$, which contradicts the fact that $(A,B) \in H$. Thus, $(A,B) \in H$ implies (H_1)-(H_3).

Conversely, suppose (H_1)-(H_3) and $(A,B) \notin H$. Then there is an $\alpha > 0$, $y \in \mathbb{R}$, $y \neq 0$, $r^0 \in (\mathbb{R}^+)^M$ such that $g(iy, \alpha r^0,A,B) = 0$. This contradicts (H_3) and proves the first part of the theorem.

The condition $\operatorname{Re}\lambda\,[(I-\sum_{k=1}^{N} B_k)^{-1}\sum_{k=0}^{N} A_k] < 0$ is equivalent to saying that Eq. (5.1) is asymptotically stable at $r = 0$. Thus, the last statement in the theorem is true.

Our next objective is to obtain an alternative characterization of H,S in terms of $\mathrm{Re}\lambda(A_0)$ as in Section 2.

Lemma 5.5. If Eq. (5.4) is hyperbolic globally at B, then, for any $s_1^0,\ldots,s_M^0$, $|s_j^0| = 1$, $j = 1,2,\ldots,M$, Hypothesis (H_3) implies either

$$P(iy,\alpha s_1^0,\ldots,\alpha s_M^0) \neq 0 \quad \text{for} \quad y \in \mathbb{R}, \quad |\alpha| < |$$

or

$$P(iy,\alpha s_1^0,\ldots,\alpha s_M^0) \neq 0 \quad \text{for} \quad y \in \mathbb{R}, \quad |\alpha| > |$$

If Eq. (5.4) is asymptotically stable globally at B, only the first alternative holds.

Proof: From the hypotheses on Eq. (5.4), there is a $\delta_1 > 0$ such that for any $0 < \delta < \delta_1$, there is an $\eta > 0$ such that

$$\left|\det\left(I-\sum_{k=1}^{N} B_k s_1^{\gamma_{k1}} \ldots s_M^{\gamma_{kM}}\right)\right| \geq \eta$$

for $1-\delta \leq |s_j| \leq 1+\delta$, $j = 1,2,\ldots,M$. Fix s_j^0, $|s_j^0| = 1$, $j = 1,2,\ldots,M$ and define $Q(\lambda,\alpha) = P(\lambda,\alpha s_1^0,\ldots,\alpha s_M^0)$ for $\alpha \in \mathbb{C}$, $\lambda \in \mathbb{C}$. Designate the zeros of this function by $\alpha(\lambda)$. Then $\alpha(\lambda)$ is meromorphic, defined on some Reimann surface over the λ-plane. As $|\lambda| \to \infty$, one must have

$$\det\left(I - \sum_{k=1}^{N} B_k [\alpha(\lambda)s_1^0]^{\gamma_{k1}} \ldots [\alpha(\lambda)s_M^0]^{\gamma_{kM}}\right) \to 0.$$

Thus, for large $|\lambda|$,

$$\alpha(\lambda) \in \{\rho \in \mathbb{C}: |\rho| \leq 1-\delta_1\} \cup \{\rho \in \mathbb{C}: |\rho| \geq 1+\delta_1\}$$

Consider the curve in the α-plane defined by $\alpha(iy)$, $y \in \mathbb{R}$.

For y very large and positive, suppose $\alpha(iy) \in \{ \rho \in \mathbb{C}: |\rho| \geq 1 + \delta_1 \}$. Then Hypothesis (H_3) implies that $|\alpha(iy)| > 1$ if $y \neq 0$. Thus, $|\alpha(iy)| \geq 1$ for all y. Consequently, $Q(iy,\alpha) = P(iy,\alpha s_1^0, \ldots, \alpha s_M^0) \neq 0$ for $|\alpha|<1$, $y \in \mathbb{R}$. If $\alpha(iy) \in \{ \rho \in \mathbb{C}: |\rho| \leq 1 - \delta_1 \}$ for large positive y, then Hypothesis (H_3) implies $|\alpha(iy)| < 1$ for all $y \neq 0$. Thus, $|\alpha(iy)| \leq 1$ for all y and $Q(iy,\alpha s^0) \neq 0$ for all $|\alpha| > 1$. This proves the lemma.

Corollary 5.6. If $(A,B) \in H$, then

$$\mathrm{Re}\lambda[(I - \sum_{k=1}^{N} B_k)^{-1} \sum_{k=0}^{N} A_k] \neq 0.$$

If $(A,B) \in S$, then $\mathrm{Re}\lambda(A_0) < 0$, $\mathrm{Re}\lambda[(I - \sum_{k=1}^{N} B_k)^{-1} \sum_{k=0}^{N} A_k] < 0$.

Proof: Let $Q(\lambda,s) = P(\lambda, s, \ldots, s)$. For real s, Lemma 5.5 implies either

$$Q(iy,s) \neq 0 \quad \text{for} \quad 0 \leq s < 1, \quad y \in \mathbb{R}$$

or

$$Q(iy,s) \neq 0 \quad \text{for} \quad 1 < s, \quad y \in \mathbb{R}$$

Hypothesis (H_2) implies $Q(iy,1) \neq 0$ for all $y \in \mathbb{R}$. Thus, $Q(\cdot,s)$ has no roots on the imaginary axis either for $0 \leq s \leq 1$ or for $1 \leq s$. Since

$$Q(\lambda,1) = \det[\lambda I - (I - \sum_{k=1}^{N} B_k)^{-1} \sum_{k=0}^{N} A_k],$$

it follows that $\mathrm{Re}\lambda[(I - \sum_{k=1}^{N} B_k)^{-1} \sum_{k=0}^{N} A_k] \neq 0$. This proves the first part of the corollary.

The last part follows from the last statement in Lemma 5.5 and the proof of Corollary 2.7.

Remark 5.7. One can have Eq. (5.4) asymptotically stable globally at B and have $(A,B) \in H$, $(A,B) \notin S$. In fact, consider the equation

$$\frac{d}{dt}[x(t) - cx(t-r)] = ax(t) \tag{5.8}$$

with c real, $0 < c < 1$, $a > 0$. The characteristic function is

$$\lambda(1-ce^{-\lambda r}) - a$$

which is $\neq 0$ for $\lambda = iy$, $y \in \mathbb{R}$. Also, there is a real $\lambda > 0$ such that $\lambda(1-c\exp-\lambda r) = a$.

Let us now consider in more detail the scalar n^{th}-order neutral equation

$$y^{(n)}(t) - \sum_{k=1}^{N} b_k y^{(n)}(t-\omega_k) - \sum_{j=1}^{n} a_{jo} y^{(n-j)}(t) - \sum_{k=1}^{N} \sum_{j=1}^{n} a_{jk} y^{(n-j)}(t-\omega_k) = 0 \tag{5.9}$$

where $\omega_k \geq 0$, $b_k \in \mathbb{R}$, $a_{jk} \in \mathbb{R}$ for all j,k. Let $b = (b_1, \ldots, b_N)$, $a = (a_{jk}, j = 1, \ldots, n, k = 0, 1, \ldots, N)$, $\omega = (\omega_1, \ldots, \omega_N)$. If we write this equation as a system of first order equations,

$$\frac{d}{dt}[x(t) - \sum_{k=1}^{N} B_k x(t-\omega_k)] = A_0 x(t) + \sum_{k=1}^{N} A_k x(t-\omega_k),$$

for $x = (y, y^{(1)}, \ldots, y^{(n-1)})$, we have

$$B_k = (b_{ij}^{(k)}), \quad b_{nn}^{(k)} = b_k, \; b_{ij}^{(k)} = 0 \text{ otherwise}$$

$$A_0 = \begin{bmatrix} 0 & 1 & 0 \ldots 0 \\ 0 & 0 & 1 \ldots 0 \\ - & - & - \quad - \\ 0 & 0 & 0 \ldots 1 \\ a_{no} & a_{n-1,0} & a_{n-2,0} \quad a_{10} \end{bmatrix}$$

$$A_k = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ - & - & - & - \\ 0 & 0 & \cdots & 0 \\ a_{nk} & a_{n-1,k} & \cdots & a_{1k} \end{bmatrix}, \quad k = 1,2, \ldots, N.$$

The characteristic function is

$$g(\lambda,\omega,a,b) = p_0(\lambda, a) - \sum_{k=1}^{N} p_k(\lambda,a,b)e^{-\lambda\omega_k} \tag{5.10}$$

$$p_0(\lambda,a) = \det[\lambda - A_0] = \lambda^n - \sum_{j=1}^{n} a_{j0}\lambda^{n-j}$$

$$p_k(\lambda,a,b) = b_k\lambda^n + \sum_{j=1}^{n} a_{jk}\lambda^{n-j}$$

The corresponding difference equation is

$$y(t) - \sum_{k=1}^{N} b_k y(t-\omega_k) = 0 \tag{5.11}$$

with characteristic function

$$e(\lambda,\omega,b) = 1 - \sum_{k=1}^{N} b_k e^{-\lambda\omega_k} \tag{5.12}$$

One can now prove the following generalization of the result of Zivotovskii [7].

Theorem 5.8. For Eq. (5.9), $(a,b) \in H$ if and only if

(5.13) Eq. (5.11) is hyperbolic

(5.14) $\sum_{k=0}^{N} a_k \neq 0$

(5.15) $|p_0(iy,a)| > \sum_{k=1}^{N} |p_k(iy,a,b)|$ for all $y \in \mathbb{R}$, $y \neq 0$.

The vector $(a,b) \in S$ if and only if (5.14), (5.15), and

(5.16) $$\sum_{k=1}^{N} |b_k| < 1, \quad \mathrm{Re}\lambda < 0 \ \underline{\text{if}} \ p_0(\lambda,a) = 0.$$

Proof: From Theorem 5.4, condition (5.14) is equivalent to Hypothesis (H_2). Hypothesis (H_3) is equivalent to

$$p_0(iy,a,b) + \sum_{k=1}^{N} p_k(iy,a,b)\, s_k \neq 0$$

for all $y \in \mathbb{R}$, $y \neq 0$, $|s_k| = 1$, $k = 1,2, \ldots, N$. But this clearly is equivalent to (5.15). Thus, the first part of the theorem is true.

If $(a,b) \in S$, then Lemma 5.2 implies Eq. 5.11 is asymptotically stable at b. Part (v) of Lemma 5.1 implies this is equivalent to $\sum_{k=1}^{N} |b_k| < 1$.

Corollary 5.6 implies $\mathrm{Re}\lambda < 0$ if $p_0(\lambda,a) = 0$. This proves the theorem.

We can generalize Theorem 5.8 to the case where the delays are dependent, but the result cannot be stated in such a simple fashion. The proof is the same as before.

Theorem 5.9. Consider again Eq. (5.9) with $\omega_k = \gamma_k \cdot r$, $r \in (\mathbb{R}^+)^M$, $\gamma_k \neq 0$, $\gamma_k = (\gamma_{k1}, \ldots, \gamma_{kM})$, $\gamma_{kj} \geq 0$ integer. With $p_0(\lambda,a)$, $p_k(\lambda,a,b)$ defined as in (5.10), the vector $(a,b) \in H$ if and only if (5.13)(5.14) in Theorem 5.8 are satisfied and

(5.17) $$p_0(iy,a,b) + \sum_{k=1}^{N} p_k(iy,a,b) s_1^{\gamma_{k1}} \cdots s_M^{\gamma_{kM}} \neq 0$$

for all $y \in \mathbb{R}$, $y \neq 0$, $|s_k| = 1$, $k = 1,2, \ldots, M$.

The vector $(a,b) \in S$ if and only if (5.14), (5.17) and (5.16) are satisfied.

6. Scalar neutral equations. For first order scalar equations

(6.1) $$\frac{d}{dt}[x(t) - \sum_{k=1}^{N} b_k x(t-\gamma_k \cdot r)] = a_0 x(t) + \sum_{k=1}^{N} a_k x(t-\gamma_k \cdot r)$$

where $a_0, a_k, b_k \in \mathbb{R}$, $k = 1,2, \ldots, N$, the characterization of the hyperbolic and asymptotically stable cones can be specified in terms of the properties of the solutions of the difference equations

$$y(t) - \sum_{k=1}^{N} b_k y(t - \gamma_k \cdot r) = 0 \tag{6.2}$$

$$a_0 z(t) - \sum_{k=1}^{N} a_k z(t - \gamma_k \cdot r) = 0 \tag{6.3}$$

It is the purpose of this section to obtain such a characterization which generalizes the results of Section 3.

As in Section 3, let $b = (b_1, \ldots, b_N)$, $a = (a_0, a_1, \ldots, a_N)$,

$$\alpha(\theta,a) = a_0 + \sum_{k=1}^{N} a_k \cos \gamma_k \cdot \theta$$

$$\beta(\theta,a) = \sum_{k=1}^{N} a_k \sin \gamma_k \cdot \theta \tag{6.4}$$

Exactly as in the proof of Theorem 3.1, one uses Theorem 5.9 to obtain the following result.

Theorem 6.1 . For Eq. (6.1), $(a,b) \in H$ if and only if

(6.5) Eq. (6.2) is hyperbolic globally at b.

$$\sum_{k=0}^{N} a_k \neq 0 \tag{6.6}$$

(6.7) Either $\alpha(\theta,a) \neq 0$ or $\alpha(\theta,a) = 0$, $\beta(\theta,a) = 0$.

and then necessarily $a_0 \neq 0$. The vector $(a,b) \in S$ if and only if

$$\sum_{k=1}^{N} |b_k| < 1, \quad a_0 < 0, \tag{6.8}$$

and (6.6),(6.7) are satisfied.

For the equation

$$(6.9)\quad \frac{d}{dt}\left[x(t) - \sum_{k=1}^{N} b_k x(t-r_k)\right] = a_0 x(t) + \sum_{k=1}^{N} a_k x(t-r_k)$$

the above theorem has a very simple interpretation for the case of stability globally in the delays. In fact, Eq. (6.9) is stable globally in the delays if and only if

$$(6.10)\quad \begin{aligned} &\sum_{k=1}^{N} |b_k| < 1, \\ &\sum_{k=0}^{N} a_k < 0, \\ &\sum_{k=1}^{N} |a_k| \leq |a_0| . \end{aligned}$$

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