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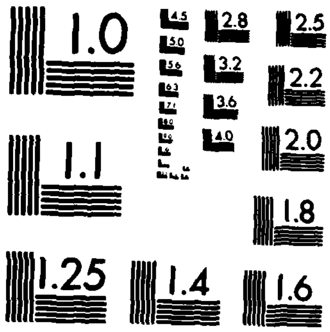

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ITEM #20, CONTINUED:

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by

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ABSTRACT

A Poisson stream of arrival rate λ_I and service time distribution $A_I(x)$ has preempt resume priority over a second stream of rate λ_{II} and distribution $A_{II}(x)$. Abundant theoretical results exist for this system, but severe numerical difficulties have made many descriptive distributions unavailable. Moreover, the distribution of total time in system of low priority customers has not been discussed theoretically. It is shown that the waiting time sequences of such customers before first entry into service is a Lindley process modified by replacement. This leads to the total time distribution needed. A variety of descriptive distributions, transient and stationary, is obtained numerically via the Laguerre transform method.

Key words: Priority queue, Preempt/resume discipline, Modified Lindley process, total time in system, Numerical distributions

§0. Introduction

A priority queue describes two M/G/1 service systems interacting through a common server. The first system, to be called system I, has a Poisson input stream of customers of intensity λ_I, and independent service times with c.d.f. $A_I(x)$. Similarly, system II has intensity λ_{II} and c.d.f. $A_{II}(x)$. All interarrival times and service times are independent of each other. A substantial literature dating back to the fifties treats the interaction of the two customer streams when system I customers have preemptive priority over system II customers. Abundant results have been obtained giving first and second moments of interest and some asymptotic results (see, e.g., Gaver [1], Heathcote [3], Jaiswal [4], Keilson [5,8,9], Miller [18] and Prabhu [19]). The distribution of random variates of interest required for system design have not been available, however, because of Laplace transform difficulties. Many asymptotic results such as heavy traffic approximations for waiting times have been flawed by lack of error bounds and disturbing relaxation time problems.

An algorithmic procedure has been needed providing accurate numerical distributions for effective service times, busy periods, and ergodic waiting times. The Laguerre transform method introduced by Keilson and Nunn [11], Keilson, Nunn and Sumita [12] and studied further by Sumita [21] provides a framework for evaluating multiple convolutions and other continuum operations. Many of the distributions required have been obtained previously thereby [11,12,13,14,20,21]. For the total time in system of low priority customers in the system of interest, however, new probabilistic analysis has been needed before the Laguerre procedure is applicable. This analysis and related asymptotic results are the focus of this paper. The Laguerre transform will serve only as a tool, but one whose power will be made evident hopefully through the results.

A recent paper by the authors, "The Depletion Time for M/G/1 Systems and a Related Limit Theorem" [14], discusses single server M/G/1 systems with many classes of customers and complex order of service disciplines. That paper provides a survival function bound for the time in system of any customer at ergodicity, giving rise to a robust (but non-exponential) limit theoretic bound for heavy traffic. In the present paper, the significance of the service time distributions of competing classes and of their traffic intensities is emphasized.

In Section 1, the system studied is described and the main results are summarized. A modified Lindley process is derived in Section 2, which represents the waiting time until first entry into service of the k-th low priority customer. In Section 3, the Laplace transforms of stationary distributions of interest are given, and related heavy traffic limit theorems are established. A final section is devoted to numerical examples. All ergodic and transient distributions described in the previous sections are evaluated numerically using the Laguerre transform method.

§1. The system of interest and main results

We consider the two customer streams described in Section 0 with arrival intensities λ_I and λ_{II} and service times T_I and T_{II} with c.d.f.'s $A_I(x)$ and $A_{II}(x)$. We assume that T_I and T_{II} have finite second moments and that the ergodicity condition $\rho_S = \rho_I + \rho_{II} < 1$ holds, where $\rho_I = \lambda_I E[T_I]$ and $\rho_{II} = \lambda_{II} E[T_{II}]$. The corresponding transforms are denoted by $\alpha_I(w) = E[e^{-wT_I}]$ and $\alpha_{II}(w) = E[e^{-wT_{II}}]$. A class I customer evicts any class II customer from the service facility. When that class I customer and all subsequent class I customers with overlapping presence complete service, the evicted class II customer resumes service. The queue discipline of class II is FIFO. The queue discipline in class I is irrelevant for class II service delay provided the busy period for class I customers is undisturbed.

A tool we shall appeal to frequently is that of effective service time distribution. Suppose a random service time T in the absence of interruptions has c.d.f. A(x). Interruptions Δ with c.d.f. B(x) arrive in a Poisson stream of rate λ. Let $\alpha(w) = E[e^{-wT}]$ and $\beta(w) = E[e^{-w\Delta}]$. Then it has been shown [4, 5] that the elapsed time T^{eff} until service is completed has c.d.f. $A^{eff}(x)$ with $\alpha^{eff}(w) = E[e^{-wT^{eff}}]$ given by

$$(1.1) \qquad \alpha^{eff}(w) = \alpha(w + \lambda - \lambda\beta(w)) \quad .$$

In particular, for the preempt-resume system described, the effective service time transform $\alpha_{II}^{eff}(w)$ of class II customers is given by

$$(1.2) \qquad \alpha_{II}^{eff}(w) = \alpha_{II}(w + \lambda_I - \lambda_I \sigma_{BPI}(w)) \quad ,$$

since class I customers ignore all class II customers and see their own M/G/1 system. Here $\sigma_{BPI}(w)$ is the server busy period transform for class I M/G/1 systems. This transform $\sigma_{BPI}(w)$ satisfies the classical Takács equation [22]

$$(1.3) \qquad \sigma_{BPI}(w) = \alpha_I(w + \lambda_I - \lambda_I\sigma_{BPI}(w)) \quad .$$

Of related interest is the stationary waiting time distribution for class I and class II customers. That for class I is given by the familiar Pollaczek-Khintchine distribution $F_{PKI}(x)$ [16] with transform

$$(1.4) \qquad \phi_{PKI}(w) = E[e^{-wW_{PKI}}] = \frac{1 - \rho_I}{1 - \rho_I\{\frac{1 - \alpha_I(w)}{wE[T_I]}\}}$$

where W_{PKI} is the stationary waiting time for class I. For the class II customers, the discussion of waiting time is much more difficult. Such a customer has a waiting time before first entry into service, and may be evicted repeatedly by streams of overlapping class I customers. For the class II customers, the total time in system rather than the waiting time is needed. Nevertheless, the waiting time before first entry into service provides a stepping stone to the time in system. Let W_k be the time until first entry into service of the k-th class II customer. It will be shown in Section 2 that the sequence W_k has a Lindley process-like structure in that one has

$$(1.5) \qquad W_{k+1} = \begin{cases} W_k + \xi_{k+1} & \text{if } W_k + \xi_{k+1} \geq 0 \ , \\ T_{X0} & \text{if } W_k + \xi_{k+1} < 0 \ . \end{cases}$$

Here $\xi_{k+1} = S_k^{eff} - \Delta_{k+1}$ where S_k^{eff} is the effective service time of the k-th class II customer, i.i.d. with transform given in (1.2), and Δ_{k+1} is the interarrival time between the k-th and the (k+1)-st class II customers. The variate T_{X0} is the first passage time of the server backlog process $B_I(t)$ of class I M/G/1 system from $B_I(0+) = X$ to zero [14]. The variate X is given, as we will see in Section 2, by $X \overset{d}{=} B_I(\frac{1}{\lambda_{II}} E)$ with E being the exponential variate of the unit mean. The process W_k may be called a Lindley process [17] modified by replacement. An algorithmic procedure will be given for evaluating the sequence of distributions of W_k recursively via (1.5) based on the Laguerre transform method, and in turn to the stationary distribution of W_k. The distributions of W_k are of separate interest in that they display the approach to ergodicity and provides relaxation time information numerically.

Alternatively, it will be shown that the stationary waiting time W_{II} before first entry into service of the class II customers is given by

$$(1.6) \qquad \phi_{II}(w) = \phi_{PKS}(w + \lambda_I - \lambda_I \sigma_{BPI}(w))$$

where $\phi_{PKS}(w) = E[e^{-wW_{PKS}}]$ and W_{PKS} is the stationary waiting time seen by all customers in the system when there is no priority and the service discipline is FIFO. A comparison of (1.6) with (1.1) then gives rise to the following formal statement.

(1.7) *At stationarity a class II customer experiences, before its first entry into service, the system P-K delay for $\lambda_S = \lambda_I + \lambda_{II}$ and $A_S(x) = [\lambda_I A_I(x) + \lambda_{II} A_{II}(x)]/\lambda_S$ modified by interruptions at Poisson rate λ_I with duration T_{BPI}.*

§2. The modified Lindley process with replacement for W_k

In this section we establish the recursion relations (1.5) for the waiting times W_k of the class II customers described in Section 1. The total time in system of the k-th class II customer is then obtained using W_k as a stepping stone.

For notational convenience, we denote the k-th class II customer by C_k. Let C_k arrive at the system at time τ_k. We note that the interarrival times

$$(2.1) \qquad \Delta_{k+1} = \tau_{k+1} - \tau_k$$

are i.i.d. and exponentially distributed with parameter λ_{II}. Let C_k first receive service at time τ_k^* and leave the system at time τ_k^{**} so that

$$(2.2) \qquad \tau_k^* = \tau_k + W_k \; ; \quad \tau_k^{**} = \tau_k^* + S_k^{eff} = \tau_k + W_k + S_k^{eff} \quad .$$

Here S_k^{eff}, the effective service time of C_k, is i.i.d. with transform $\alpha_{II}^{eff}(w) = E[e^{-wS^{eff}}]$ given by (1.2). We note that W_k and S_k^{eff} are independent and that the total time in system, U_k, of C_k is given by

$$(2.3) \qquad U_k = \tau_k^{**} - \tau_k = W_k + S_k^{eff} \quad .$$

Since the distribution of S_k^{eff} is known, the distribution of W_k leads to that of U_k.

Suppose that is ready present when C_k leaves the system at τ_k^{**}. This is equivalent to saying that $\Delta_{k+1} \le U_k = W_k + S_k^{eff}$. Hence W_{k+1} is given by (cf. Figure 2.1a),

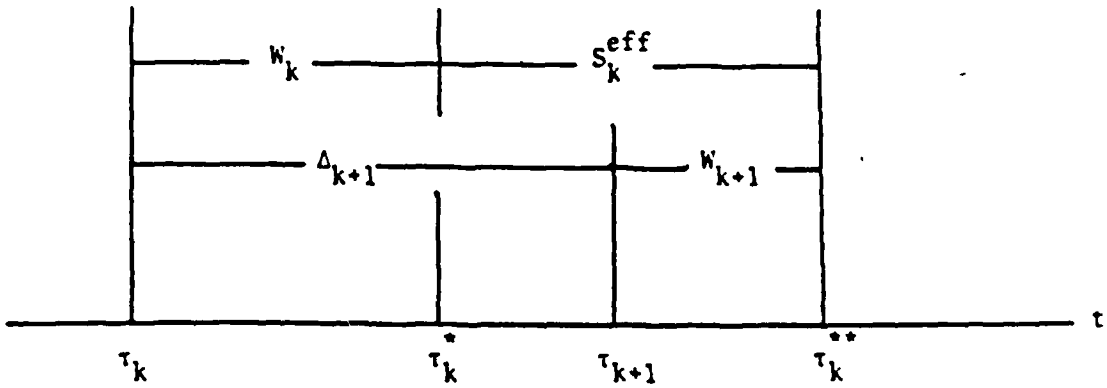

Figure 2.1.a. $\Delta_{k+1} \leq W_k + S_k^{eff}$

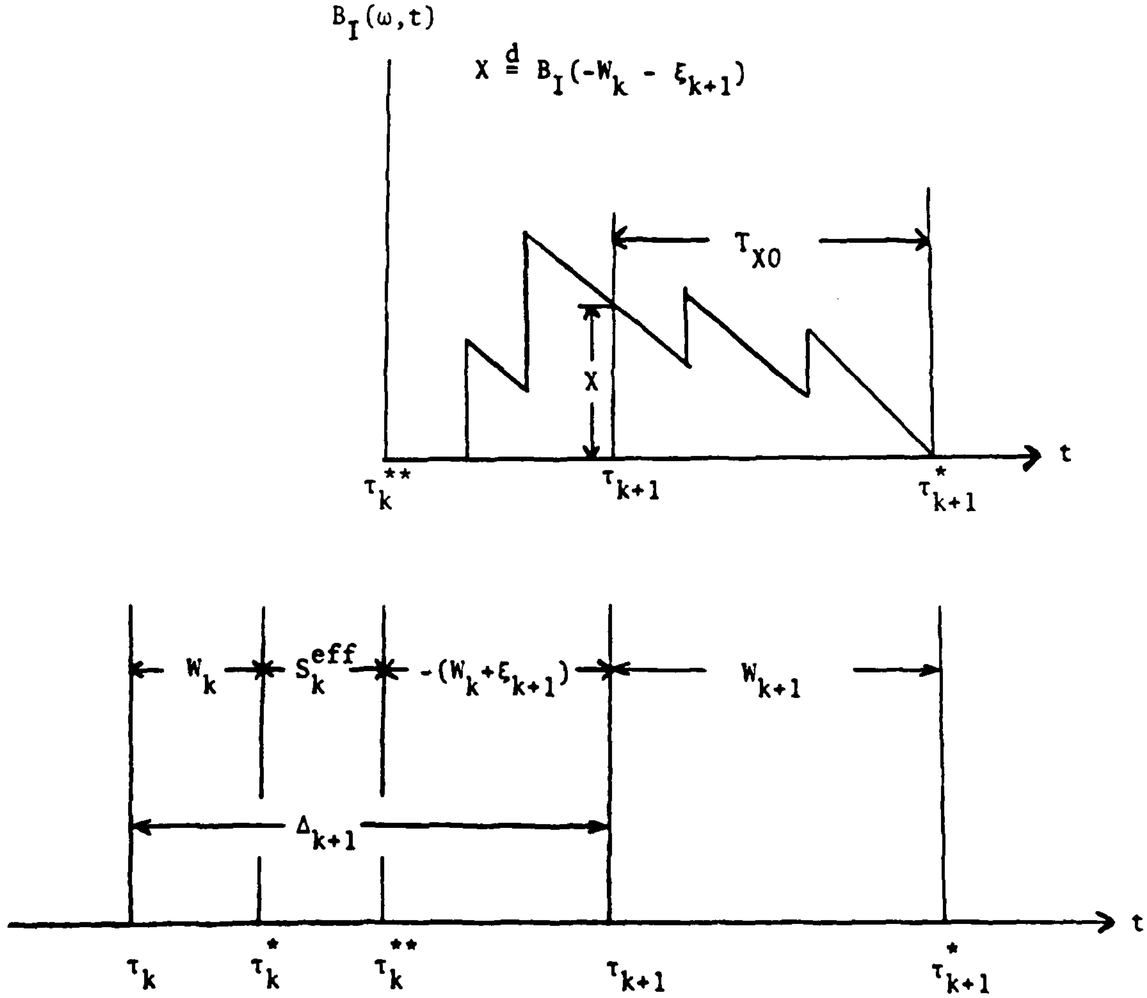

Figure 2.1.b. $\Delta_{k+1} > W_k + S_k^{eff}$

(2.4) $\quad W_{k+1} = W_k + \xi_{k+1} \quad \text{if} \quad W_k + \xi_{k+1} \geq 0 \quad ,$

where

(2.5) $\quad \xi_{k+1} = S_k^{eff} - \Delta_{k+1} \quad .$

We note from (2.2) and (2.4) that the first service entry epoch of C_{k+1}, when $W_k + \xi_{k+1} \geq 0$, is

(2.6) $\quad \tau_{k+1}^{*} = \tau_{k+1} + W_{k+1} = \tau_k + W_k + S_k^{eff} = \tau_k^{**} \quad , \quad W_k + \xi_{k+1} > 0 \quad ,$

since no class I customers are present at departures of class II customers.

We now suppose that C_{k+1} has not arrived yet when C_k leaves the system (cf. Figure 2.1b). This means that the server becomes idle at the departure of C_k, and C_{k+1} arrives at the system after a delay of $- W_k - \xi_{k+1}$ subsequent to the departure of C_k. Let $B_I(t)$ be the server backlog process of class I M/G/1 system when $B_I(0) = 0$. When C_{k+1} arrives at the system, the server has backlog $B_I(-W_k - \xi_{k+1})$. While the server works on his backlog, other class I customers may arrive who also precede C_{k+1}. Hence, when $W_k + \xi_{k+1} < 0$, the waiting time of C_{k+1} is the first passage time T_{X0} of $B_I(t)$ from its initial load X to zero where $X \stackrel{d}{=} B_I(-W_k - \xi_{k+1})$. For notational convenience, we define $T_{00} \stackrel{d}{=} 0$.

The negative support of the variate $W_k + \xi_{k+1} = W_k + S_k^{eff} - \Delta_{k+1}$ is contributed only by Δ_{k+1}, which is exponentially distributed. Hence the p.d.f. of $W_k + \xi_{k+1}$ on the negative real axis is of the form $p_k^- \lambda_{II} e^{\lambda_{II} x} U(-x)$ with $p_k^- = P[W_k + \xi_{k+1} < 0]$ where $U(x) = 1$, $x \geq 0$, and $U(x) = 0$, $x < 0$. Hence when $W_k + \xi_{k+1} < 0$, one has

$$(2.7) \qquad X \overset{d}{=} B_I(-W_k - \xi_{k+1}) \overset{d}{=} B_I(\frac{1}{\lambda_{II}} E) \ , \quad W_k + \xi_{k+1} < 0 \ ,$$

where E is the exponential variate of mean one. From this we have the following theorem.

Theorem 2.1

Let W_k be the waiting time before first entry into service of the k-th class II customer. Then

$$(2.8) \qquad W_{k+1} = \begin{cases} W_k + \xi_{k+1} & \text{if} \quad W_k + \xi_{k+1} \geq 0 \ , \\ T_{X0} & \text{if} \quad W_k + \xi_{k+1} < 0 \ , \end{cases}$$

where $\xi_{k+1} = S_k^{eff} - \Delta_{k+1}$ and T_{X0} is the first passage time of the server backlog process $B_I(t)$ for class I M/G/1 system from X to zero. The distribution of X is given by (2.7).

The process W_{k+1} in (2.8) may be called a modified Lindley process in that the homogeneous process $W_k + \xi_{k+1}$ is modified by a retaining boundary at the origin with replacement at independently chosen T_{X0}. We note that the Laplace transform $\beta_X(w) = E[e^{-wX}]$ of X in (2.7) can be written as

$$(2.9) \qquad \beta_X(w) = E[e^{-wB_I(\frac{1}{\lambda_{II}} E)}] = \lambda_{II} \int_0^\infty e^{-\lambda_{II} t} E[e^{-wB_I(t)}]dt \ .$$

The double Laplace transform $\int_0^\infty e^{-st} E[e^{-wB_I(t)}]dt$ is given [14, 22] by

$$(2.10) \qquad \int_0^\infty e^{-st} E[e^{-wB_I(t)}]dt = \frac{1 - w\varepsilon_I(s)}{s + \lambda_I\{1 - \alpha_I(w)\} - w} \ .$$

Here $\varepsilon_I(s) = \int_0^\infty e^{-st} E_I(t)dt$ where $E_I(t) = P[B_I(t) = 0]$. From Equations (2.9) and (2.10), one then concludes that

$$(2.11) \qquad \beta_X(w) = \frac{\lambda_{II}\{1 - w\varepsilon_I(\lambda_{II})\}}{\lambda_{II} + \lambda_I\{1 - \alpha_I(w)\} - w} \quad .$$

The following theorem can now be readily shown.

Theorem 2.2

Let $\sigma_{X0}(w) = E[e^{-wT_{X0}}]$ where T_{X0} is given in Theorem 2.1. Then $\sigma_{X0}(w) = (\lambda_{II}/(\lambda_{II} - w))[1 - \varepsilon_I(\lambda_{II})\{w + \lambda_I - \lambda_I\sigma_{BPI}(w)\}]$.

Proof

As shown in (2.1) of [14], one has

$$\sigma_{X0}(w) = \beta_X(w + \lambda_I - \lambda_I\sigma_{BPI}(w))$$

$$= \frac{\lambda_{II}[1 - \varepsilon_I(\lambda_{II})\{w + \lambda_I - \lambda_I\sigma_{BPI}(w)\}]}{\lambda_{II} + \lambda_I\{1 - \alpha_I(w + \lambda_I - \lambda_I\sigma_{BPI}(w))\} - \{w + \lambda_I - \lambda_I\sigma_{BPI}(w)\}} \quad .$$

The theorem then follows from the Takács equation (1.3). □

One easily finds that $\sigma_{X0}(w) \to \lambda_{II}\varepsilon_I(\lambda_{II})$ as $w \to +\infty$ so that T_{X0} has mass $\lambda_{II}\varepsilon_I(\lambda_{II})$ at the origin. Let $r_{X0}(x)$ be the probability density of T_{X0} on the positive real axis and define $\gamma_{X0}(w) = \int_0^\infty e^{-wx} r_{X0}(x)dx$. One then has $\gamma_{X0}(w) = \sigma_{X0}(w) - \lambda_{II}\varepsilon_I(\lambda_{II})$ and the next corollary is immediate from Theorem 2.2.

Corollary 2.3

(a) $P[T_{X0} = 0] = P[X = 0] = \lambda_{II}\varepsilon_I(\lambda_{II})$.

(b) $\gamma_{X0}(w) = \dfrac{\lambda_{II}}{\lambda_{II} - w}[1 - \lambda_S\varepsilon_I(\lambda_{II}) + \lambda_I\varepsilon_I(\lambda_{II})\sigma_{BPI}(w)]$.

As we will see in Section 4, Corollary 2.3 plays a key role for evaluating the distribution of W_k recursively via the Laguerre transform method. The numerical value of $\varepsilon_I(\lambda_{II})$ is therefore needed.

Theorem 2.4

Let $h(s,w) = s + \lambda_I\{1 - \alpha_I(w)\} - w$. For each $s > 0$, $h(s,w)$ is strictly monotone decreasing in w, $w \geq 0$, and has a unique zero at $w_0(s) = 1/\varepsilon_I(s) > 0$.

Proof

We note that $h(s,w) = (s + \lambda_I - w)\{1 - \frac{\lambda_I}{s + \lambda_I - w}\alpha_I(w)\}$. Then from Rouche's theorem, $h(s,w)$ has only one zero. This unique zero is attained on the positive real axis when $s > 0$, since

$$\frac{\partial}{\partial w} h(s,w) = -1 - \lambda_I \frac{d}{dw}\alpha_I(w) \leq -(1 - \rho_I) < 0$$

and $h(s,0) = s$ and $h(s,w) \to -\infty$ as $w \to \infty$. Hence $h(s,w)$ attains zero at $w_0(s) > 0$. It is known that $\varepsilon_I(s) = [s + \lambda_I - \lambda_I\sigma_{BPI}(s)]^{-1}$. One then sees, from the Takács equation (1.3), that

$$h(s, \frac{1}{\varepsilon_I(s)}) = s + \lambda_I - \lambda_I\sigma_I(\frac{1}{\varepsilon_I(s)}) - \frac{1}{\varepsilon_I(s)}$$

$$= s + \lambda_I - \lambda_I\sigma_{BPI}(s) - s - \lambda_I + \lambda_I\sigma_{BPI}(s) = 0 \quad .$$

Hence $w_0(s) = \frac{1}{\varepsilon_I(s)}$, proving the theorem. □

Since $h(\lambda_{II}, w)$ is strictly monotone decreasing in w, $w \geq 0$, one can numerically evaluate $\varepsilon_I(\lambda_{II}) = 1/w_0(\lambda_{II})$ straightforwardly.

Remark 2.5

We note that the numerator of (2.10) vanishes at $w_0(s) = \frac{1}{\varepsilon_I(s)}$. Hence $\int_0^\infty e^{-st}E[e^{-wB_I(t)}]dt$ is regular for $Re(w) \geq 0$ when $s > 0$, and the transform in (2.10) has only nonnegative support. This, in turn, implies that $\gamma_{X0}(w)$ in Corollary 2.3 has nonnegative support only, as required.

It has been shown (cf. [10], p. 233, item (d)) that the ordinary Lindley waiting time process is stochastically monotone. In particular, for the sequence of the ordinary waiting times W_k^L, one has $W_{k+1}^L \succ W_k^L$, $k = 0,1,2,\ldots$, i.e., $P[W_{k+1}^L > x] \geq P[W_k^L > x]$, when $W_0^L = 0$. The waiting times may then be said to be sequentially monotone. The modified Lindley process (2.8) is also sequentially monotone when $W_0 = 0$, as we prove next.

Theorem 2.6

Let W_k be defined as in (2.8) with $W_0 = 0$. Then W_k are sequentially monotone, i.e., $W_{k+1} \succ W_k$, $k = 0,1,\ldots$.

Proof

It is clear that $W_1 \succ W_0 = 0$. Suppose $W_k \succ W_{k-1}$. Let $V_k = [W_k + \xi_{k+1}]^+$ where $[X]^+ = \max\{0,X\}$ and define the survival functions $\bar{F}_{W,k}(x) = P[W_k > x]$ and $\bar{F}_{V,k}(k) = P[V_k > x]$. We note that $W_k \succ W_{k-1}$ implies $V_k \succ V_{k-1}$ (cf. [10]). From (2.8) one sees that

$$(2.12) \qquad \bar{F}_{W,k+1}(x) = \bar{F}_{V,k}(x) + \bar{F}_{V,k}(0+)\cdot\bar{R}_{X0}(x) \ , \qquad x > 0 \ .$$

Here $\bar{R}_{X0}(x) = P[T_{X0} > x]$ is the survival function of the replacement distribution for the modified Lindley process. From the induction hypothesis, one has $\bar{F}_{V,k}(x) \geq \bar{F}_{V,k-1}(x)$, $x \geq 0$, so that $\bar{F}_{W,k+1}(x) \geq \bar{F}_{W,k}(x)$, $x \geq 0$, proving the theorem. □

Corollary 2.7

(a) Let $E_k = P[W_k = 0]$. Then $E_k = \bar{F}_{V,k}(0+)\cdot\lambda_{II}\varepsilon_I(\lambda_{II})$ is monotonically decreasing in k.

(b) $E[W_k]$ is monotonically increasing in k.

Proof

Part (a) follows from Corollary 2.3, Theorem 2.6, and (2.12). Part (b) is immediate from Theorem 2.6. □

§3. The ergodic distribution of total time in system for low priority customers; its heavy traffic approximation

As shown in Section 2, the sequence W_k of time spent before first entry into service of the k-th class II customer is a modified Lindley process with replacement. For the ordinary Lindley process, when the virtual value $W_k + \xi_{k+1}$ is negative, replacement is at zero. For our case replacement distribution has support on the positive continuum as well as at zero and the standard Wiener-Hopf methods must be altered. The compensation method introduced earlier in [6, 7] and presented in simplified form in [2], provides a quick analysis. One sees that the required c.d.f. $F_{WII}(x)$ of W_k at ergodicity is given by

$$(3.1)\qquad F_{WII}(x) = \int_{-\infty}^{\infty} C(x-y)\,dG_{\infty}^{H}(y) \quad .$$

Here $G_{\infty}^{H}(x)$ is the ergodic green distribution of the underlying homogeneous process $W_{k+1}^{H} = W_k^{H} + \xi_{k+1}$, and $C(x)$ is the c.d.f. of the compensation. In transform notation (3.1) becomes

$$(3.2)\qquad \phi_{WII}(w) = X(w)\gamma_{\infty}^{H}(w)$$

where

$$(3.3)\qquad \gamma_{\infty}^{H}(w) = \frac{1}{1 - \frac{\lambda_{II}}{\lambda_{II} - w} \cdot \alpha_{II}^{eff}(w)}$$

and

$$(3.4)\qquad X(w) = K[\sigma_{X0}(w) - \frac{\lambda_{II}}{\lambda_{II} - w}] \quad .$$

In (3.3) and (3.4), $\alpha_{II}^{eff}(w)$ and $\sigma_{X0}(w)$ are described in (1.2) and Theorem 2.2, respectively. K is a normalization constant.

Consider next the M/G/1 system for which $\lambda_S = \lambda_I + \lambda_{II}$ and $A_S(x) = \frac{\lambda_I}{\lambda_S} A_I(x) + \frac{\lambda_{II}}{\lambda_S} A_{II}(x)$ describing the service stream seen by the server when the two classes are given equal priority. If the service discipline is FIFO, the Pollaczek-Khintchine transform for the stationary waiting time is

$$(3.5) \qquad \omega_{PKS}(w) = \frac{1 - \rho_S}{1 - \rho_S \frac{1 - \alpha_S(w)}{wE[T_S]}} .$$

The statement in (1.6) may now be given formally.

<u>Theorem 3.1</u>

Let $\rho_S = \lambda_S E[T_S] = \rho_I + \rho_{II} < 1$. Then the ergodic distribution of the time until first entry into service for class II customers with FIFO discipline has the transform

$$(3.6) \qquad \phi_{WII}(w) = \omega_{PKS}(w + \lambda_I - \lambda_I \sigma_{BPI}(w)) .$$

<u>Proof</u>

Let $Z(w) = w + \lambda_I - \lambda_I \sigma_{BPI}(w)$. From Theorem 2.2 and (3.4), one has $X(w) = -K_1 Z(w)/(\lambda_{II} - w)$. From (1.2) and (3.3), one also has $\gamma_\infty^H(w) = [1 - \frac{\lambda_{II}}{\lambda_{II} - w} \cdot \alpha_{II}(Z(w))]^{-1}$. Hence from (3.2), one sees that

$$\phi_{WII}(w) = X(w)\gamma_\infty^H(w) = \frac{K_1 Z(w)}{w - \lambda_{II}\{1 - \alpha_{II}(Z(w))\}} .$$

Since $w = Z(w) - \lambda_I\{1 - \sigma_{BPI}(w)\} = Z(w) - \lambda_I\{1 - \alpha_I(Z(w))\}$, one then has

$$\phi_{WII}(w) = \frac{K_1 Z(w)}{Z(w) - \lambda_I\{1 - \alpha_I(Z(w))\} - \lambda_{II}\{1 - \alpha_{II}(Z(w))\}} .$$

Dividing both the numerator and the denominator by $Z(w)$, the above equation leads to

$$\phi_{WII}(w) = \frac{K_1}{1 - \rho_S \frac{1 - \alpha_S(Z(w))}{Z(w)E[T_S]}}$$

where $\alpha_S(w) = \frac{\lambda_I}{\lambda_S}\alpha_I(w) + \frac{\lambda_{II}}{\lambda_S}\alpha_{II}(w)$. From $\phi_{WII}(0+) = 1$, one easily finds that $K_1 = 1 - \rho_S$ and the theorem follows from (3.5). □

From Theorem 3.1 and (2.3), the transform of the time in system of class II customers at ergodicity is immediate. One has:

<u>Theorem 3.2</u>

Let $\rho_S < 1$ and let $U_{II} = W_{II} + S_{II}^{eff}$ be the total time in system of class II customers at ergodicity. If $\phi_{UII}(w) = E[e^{-U_{II}w}]$, then

$$\phi_{UII}(w) = \omega_{PKS}(w + \lambda_I - \lambda_I\sigma_{BPI}(w))\alpha_{II}(w + \lambda_I - \lambda_I\sigma_{BPI}(w)) . \tag{3.7}$$

The transform results given in (1.2), (1.3), (3.5), Theorem 3.1 and Theorem 3.2 lead to the following relations between the first moment of various variates of interest.

(3.8a) $E[T_S] = E[T_I] + E[T_{II}]$.

(3.8b) $E[T_{BPI}] = \dfrac{E[T_I]}{1 - \rho_I}$.

(3.8c) $E[S_{II}^{eff}] = E[T_{II}](1 + \lambda_I E[T_{BPI}]) = \dfrac{E[T_{II}]}{1 - \rho_I}$.

(3.8d) $E[W_{PKS}] = \dfrac{\rho_S}{2(1 - \rho_S)} \cdot \dfrac{E[T_S^2]}{E[T_S]}$.

(3.8e) $E[W_{II}] = E[W_{PKS}](1 + \lambda_I E[T_{BPI}]) = \dfrac{E[W_{PKS}]}{1 - \rho_I}$.

(3.8f) $E[U_{II}] = E[W_{II}] + E[T_{II}^{eff}] = \dfrac{E[W_{PKS}] + E[T_{II}]}{1 - \rho_I}$.

The heavy traffic limit theorem for class II customers can now be given.

Theorem 3.3

Let the competing service times T_I, T_{II} have finite second moments. Let $(\lambda_{Ij}, \lambda_{IIj})$ be a sequence of arrival rates for which $\lambda_{I_j} = K\, \lambda_{II_j}$, $K > 0$, and let $\rho_S \to 1-$. Then both $W_{II}/E[W_{II}]$ and $U_{II}/E[U_{II}]$ converge in distribution to the exponential variate with mean one.

Proof

When λ_{I_j} and λ_{II_j} are in fixed ratio, the system service time T_S has the transform $\alpha_S(w) = (\lambda_{I_j}/\lambda_{S_j})\alpha_I(w) + (\lambda_{II_j}/\lambda_{S_j})\alpha_{II}(w)$ so that the distribution of T_S does not change with j. From Theorem 3.1, (3.5) and (3.8e), one has

$$(3.9)\qquad \phi_{WII}\left(\frac{w}{\bar{W}_{II}}\right) = \frac{1-\rho_S}{1-\rho_S\alpha_S^*\left(\frac{w}{\bar{W}_{II}}\left\{1+\frac{\rho_I}{1-\rho_I}\sigma_{BPI}^*\left(\frac{w}{\bar{W}_{II}}\right)\right\}\right)}$$

where $\phi_X^*(w) = \{1-\phi_X(w)\}/wE[X]$ and $E[W_{II}] = \bar{W}_{II}$. Further, from (3.8) and (3.9),

$$(3.10)\qquad \phi_{WII}\left(\frac{w}{\bar{W}_{II}}\right) = (1-\rho_S)\left[1-\rho_S\sigma_S^*\left(\frac{w}{\bar{W}_{PKS}}\left\{1-\rho_I\left(1-\sigma_{BPI}^*\left(\frac{w}{\bar{W}_{II}}\right)\right)\right\}\right)\right]^{-1} .$$

One may then employ the Taylor expansion with remainder of both $\alpha^*(w)$ and $\sigma_{BPI}^*(w)$ out to the linear term in w, and proceed classically invoking the continuity theorem for characteristic functions to find

$$(3.11)\qquad \phi_{WII}\left(\frac{w}{\bar{W}_{II}}\right) \to \frac{1}{1+w}, \quad \text{as } \rho_S \to 1- ,$$

demonstrating the convergence of $W_{II}/E[W_{II}]$. The convergence of $U_{II}/E[U_{II}]$ is immediate since the effective service time is bounded stochastically from above. □

Remark 3.4

One could also inquire about the limiting behavior when λ_{I_j} and λ_{II_j} are not in fixed ratio. For application of heavy traffic approximation, however, one has specified values of λ_I and λ_{II} with $\rho_S = 1-\varepsilon$ for $\varepsilon > 0$, small. The approach path of λ_{I_j} and λ_{II_j} to the values λ_I and λ_{II} is then irrelevant.

§4. Evaluation of the distributions of interest via the Laguerre transform

In this section the distributions of interest described in the previous sections are evaluated numerically via the Laguerre transform. The reader is referred to [11,12,21] for the underlying theory of the Laguerre transform. All figures are given at the end of this section. The following example is considered:

System I: High priority class

$$\lambda_I = \frac{1}{4}\ , \ \bar{A}_I(x) = P[T_I > x] = \frac{1}{3}[e^{-\frac{1}{2}x} + e^{-x} + e^{-3x}]$$

$$E[T_I] = \frac{10}{9}\ , \quad \rho_I = \lambda_I E[T_I] = 0.278 \quad .$$

System II: Low priority class

$$\lambda_{II} = \frac{1}{2}\ , \quad \bar{A}_{II}(x) = P[T_{II} > x] = \frac{1}{2}[e^{-x} + e^{-2x}]$$

$$E[T_{II}] = \frac{3}{4}\ , \quad \rho_{II} = \lambda_{II} E[T_{II}] = 0.375 \quad .$$

Total system

$$\lambda_S = \lambda_I + \lambda_{II} = \frac{3}{4}\ , \quad \bar{A}_S(x) = P[T_S > x] = \frac{1}{3}\,\bar{A}_I(x) + \frac{2}{3}\,\bar{A}_{II}(x)$$

$$E[T_S] = \frac{1}{3}\,E[T_I] + \frac{2}{3}\,E[T_{II}] = 0.870\ , \quad \rho_S = \rho_I + \rho_{II} = 0.653 \quad .$$

(A) Ergodic distributions of W_{II} and U_{II}

From the transform results (1.2), (1.3) and (3.5), one obtains easily the corresponding (generalized) probability densities as given below.

(4.1) $$s_{BPI}(x) = \sum_{n=0}^{\infty} e^{-\lambda_I x} \frac{(\lambda_I x)^n}{(n+1)!} a_I^{(n+1)}(x) \quad .$$

(4.2) $$s_{II}^{eff}(x) = \sum_{n=0}^{\infty} \{e^{-\lambda_I x} \frac{(\lambda_I x)^n}{n!} a_{II}(x)\} * s_{BPI}^{(n)}(x) \quad .$$

(4.3) $$f_{PKS}(x) = (1 - \rho_S)\delta(x) + f_{PKS}^{+}(x)$$

where

(4.4) $$f_{PKS}^{+}(x) = (1 - \rho_S) \sum_{n=1}^{\infty} \rho_S^n \{\frac{1}{E[T_S]} \bar{A}_S(x)\}^{(n)} \quad .$$

Here $a^{(n)}(x)$ is the n-fold convolution of $a(x)$ with itself. The asterisk also denotes convolution and $\delta(x)$ is the delta function.

Similarly from Theorem 3.1 and Theorem 3.2, the probability densities $f_{WII}(x)$ and $f_{UII}(x)$ take the forms

(4.5) $$f_{WII}(x) = (1 - \rho_S)\delta(x) + f_{WII}^{+}(x)$$

where

(4.6) $$f_{WII}^{+}(x) = \sum_{n=0}^{\infty} \{e^{-\lambda_I x} \frac{(\lambda_I x)^n}{n!} f_{PKS}^{+}(x)\} * s_{BPI}^{(n)}(x)$$

and

(4.7) $$f_{UII}(x) = (1 - \rho_S) s_{II}^{eff}(x) + f_{WII}^{+}(x) * s_{II}^{eff}(x) \quad .$$

The Laguerre transform enables one to evaluate systematically all of these probability densities which heretofore have been behind "the

Laplacian curtain". The Fourier-Laguerre sharp coefficients of $a_I(x)$ and $a_{II}(x)$ are readily obtained analytically. Using the relevant operational properties of the transform, Equations (4.1) through (4.7) lead to the coefficients of each density. They, in turn, can be converted to the coefficients of the corresponding survival functions, thereby bypassing numerical integration. The final inversion from the Laguerre coefficients to function values is straightforward. In Figure 4.1, the survival functions $\bar{S}_{BPI}(x) = P[T_{BPI} > x]$ and $\bar{S}_{II}^{eff}(x) = P[S_{II}^{eff} > x]$ are plotted. Figure 4.2 depicts the survival functions $\bar{F}_{PKS}(x) = P[W_{PKS} > x]$, $\bar{F}_{WII}(x) = P[W_{II} > x]$ and $\bar{F}_{UII}(x) = P[U_{II} > x]$. We note that both W_{PKS} and W_{II} have mass $1 - \rho_S$ at the origin.

(B) <u>Modified Lindley process with replacement</u>

It has been seen in Theorem 2.1 that the waiting time before first entry into service, W_k, of the k-th class II customer follows the modified Lindley process given in (2.8), i.e.,

$$(4.8) \qquad W_{k+1} = \begin{cases} W_k + \xi_{k+1} & \text{if} \quad W_k + \xi_{k+1} \geq 0 \ , \\ T_{X0} & \text{if} \quad W_k + \xi_{k+1} < 0 \ , \end{cases}$$

where $\xi_{k+1} = S_k^{eff} - \Delta_{k+1}$ and the transform $\sigma_{X_0}(w) = E[e^{-T_{X_0}w}]$ is given in Theorem 2.2. As shown in (2.3), the total time spend in the system U_k by the k-th class II customer is then given by

$$(4.9) \qquad U_k = W_k + S_k^{eff} \ .$$

We now show how these transient distributions can be evaluated via the Laguerre transform.

Let a(x) be the p.d.f. of the i.i.d. random variates ξ_k. The Laguerre sharp coefficients $(a_n^{\#})_{-\infty}^{\infty}$ of a(x) are easily obtained from those corresponding to S_k^{eff} and Δ_{k+1}. From Corollary 2.3, the variate T_{X0} has mass $R_0 = \lambda_{II}\varepsilon_I(\lambda_{II})$ at the origin, which can be calculated using Theorem 2.4. In our example, $R_0 = 0.848$. The probability density $r_{X0}(x)$ of T_{X0} on the positive real axis has the Laplace transform $\gamma_{X0}(w)$ given in Corollary 2.3. The Laguerre sharp coefficients $(r_n^{\#})_0^{\infty}$ of $r_{X0}(x)$ are then obtained, using Corollary 2.3(b), from those corresponding to T_{BPI} and Δ_k. Let $E_k = P[W_k = 0]$ and let $f_k(x)$ be the probability density of W_k on $(0,\infty)$ so that $E_k + \int_0^{\infty} f_k(x)dx = 1$. Assuming that E_k and the Laguerre sharp coefficients $(f_n^{\#}(k))_0^{\infty}$ of $f_k(x)$ are known, we next establish an algorithm for obtaining E_{k+1} and $(f_n^{\#}(k+1))_0^{\infty}$ in terms of $(a_n^{\#})_{-\infty}^{\infty}$, R_0, $(r_n^{\#})_0^{\infty}$, E_k and $(f_n^{\#}(k))_0^{\infty}$.

Let $f_{k+1}^H(x)$ be the probability density of $W_k + \xi_{k+1}$. One then has

$$(4.10) \qquad f_{k+1}^H(x) = E_k a(x) + f_k(x) * a(x) \quad , \quad -\infty < x < \infty \ .$$

The associated Laguerre sharp coefficients $(f_n^{H\#}(k+1))_{-\infty}^{\infty}$ of $f_{k+1}^H(x)$ are then given by

$$(4.11) \qquad f_n^{H\#}(k+1) = E_k a_n^{\#} + \sum_{m=0}^{\infty} a_{n-m}^{\#} f_m^{\#}(k) \quad , \quad -\infty < n < \infty \ .$$

Let $f_{k+1}^+(x) = f_{k+1}^H(x)U(x)$ where $U(x) = 0$, $x < 0$ and $U(x) = 1$, $x \geq 0$. The coefficients $(f_n^{+\#}(k+1))_0^{\infty}$ of $f_{k+1}^+(x)$ are found from

(4.12) $$f_0^{+\#}(k+1) = -\sum_{n=1}^{\infty} f_n^{H\#}(k+1) \; ; \quad f_n^{+\#}(k+1) = f_n^{H\#}(k+1) \, , \quad n \geq 1 \; .$$

Let $p_{k+1}^- = P[W_k + \xi_{k+1} < 0]$. Then $p_{k+1}^- = \int_{-\infty}^{0} f_{k+1}^{H}(x)dx = 1 - \int_0^{\infty} f_{k+1}^{+}(x)dx$ so that

(4.13) $$p_{k+1}^- = 1 + 2\sum_{n=0}^{\infty} f_{2n+1}^{+\#}(k+1) \; .$$

From (4.8), one finally has

(4.14) $$E_{k+1} = p_{k+1}^- R_0$$

and

(4.15) $$f_n^{\#}(k+1) = f_n^{+\#}(k+1) + p_{k+1}^- g_n^{\#} \, , \quad n \geq 0 \; .$$

Equations (4.11) through (4.15) enable one to calculate E_{k+1} and $(f_n^{\#}(k+1))_0^{\infty}$ recursively for $k = 0,1,2,\ldots$, starting with $W_0 \overset{d}{=} 0$ (i.e., $E_0 = 1$, $f_0^{\#}(0) = 1$ and $f_n^{\#}(0) = 0$, $n \geq 1$). The coefficients $(f_{Un}^{\#}(k))_0^{\infty}$ corresponding to U_k are given from (4.10) by

(4.16) $$f_{Un}^{\#}(k) = E_k s_n^{eff\#} + \sum_{m=0}^{n} f_{n-m}^{\#}(k) s_m^{eff\#}$$

where $(s_n^{eff\#})_0^{\infty}$ are the Laguerre sharp coefficients of $s_{II}^{eff}(x)$.

In Figure 4.3, the survival functions $\bar{F}_{Wk}(x) = P[W_k > x]$ are plotted for $k = 1,2,3,4,5,10,20,30,40,50$ and $0 \leq x \leq 10$. The absolute difference between $\bar{F}_{W50}(x)$ and its ergodic survival function $\bar{F}_{II}(x) = \int_x^{\infty} f_{WII}(y)dy$ obtained in (A) is bounded by 10^{-6} for $0 \leq x \leq 10$, using the first 150 Laguerre coefficients. This assures numerical stability and accuracy of the Laguerre transform procedure. We note that the stochastic monotonicity of W_k in k given in Theorem 2.6 can be observed in Figure 4.3.

Figure 4.4 and Figure 4.5 show the convergence of E_k to $1 - \bar{\rho}_S$ and that of $E[W_k]$ to $E[W_{II}]$ as $k \to \infty$, respectively. Both E_k and $E[W_k]$ are calculated using the Laguerre sharp coefficients. It has been shown [15, 21] that the Laguerre sharp norm defined by

$$(4.17) \qquad ||f||_2^\# = \sqrt{\sum f_n^{\#2}}$$

provides a distance between any two distributions. In Figure 4.6 this Laguerre sharp norm distance between W_k and W_{II} for $1 \leq k \leq 50$ is exhibited thereby quantifying the rate of approach to ergodicity. These distances also provide convenient stopping criterion for the computation. One can see that for $k \geq 25$ the distance is bounded by 0.01. Finally, in Figure 4.7 the survival functions $\bar{F}_{Uk}(x) = P[U_k > x]$ are plotted for $k = 1,2,3,4,5,10,20,30,40,50$ and $0 \leq x \leq 10$. All computations were done on a DEC 10 computer in a timesharing mode using APL as the programming language. Relevant formulae were usually coded in a straightforward way using the first 150 Laguerre coefficients. The results displayed here were typically obtained with CPU time in seconds.

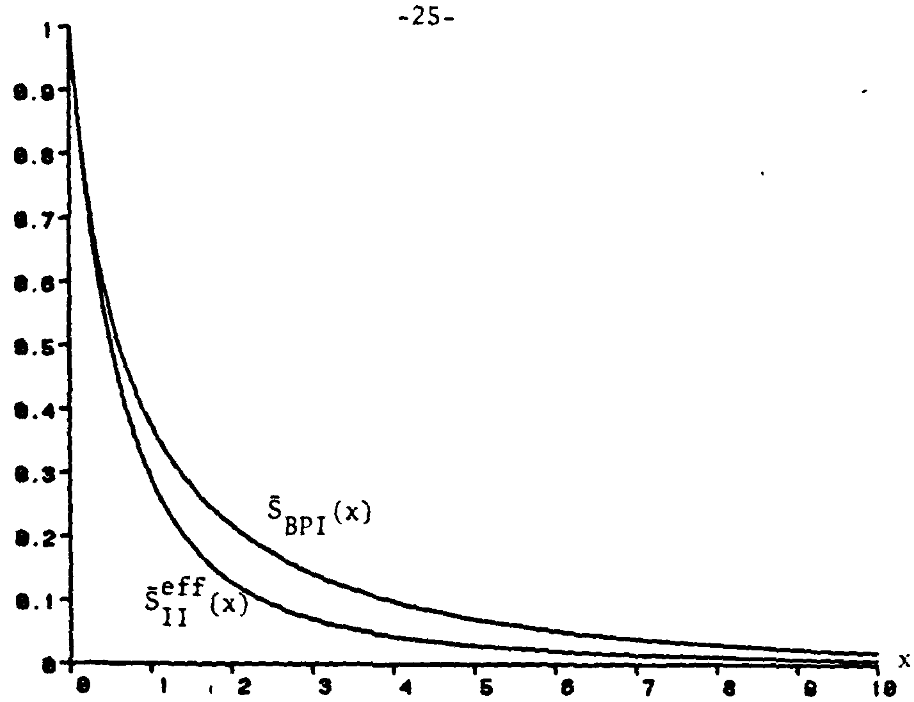

Figure 4.1. The survival functions of S^{eff}_{II} and T_{BPI}

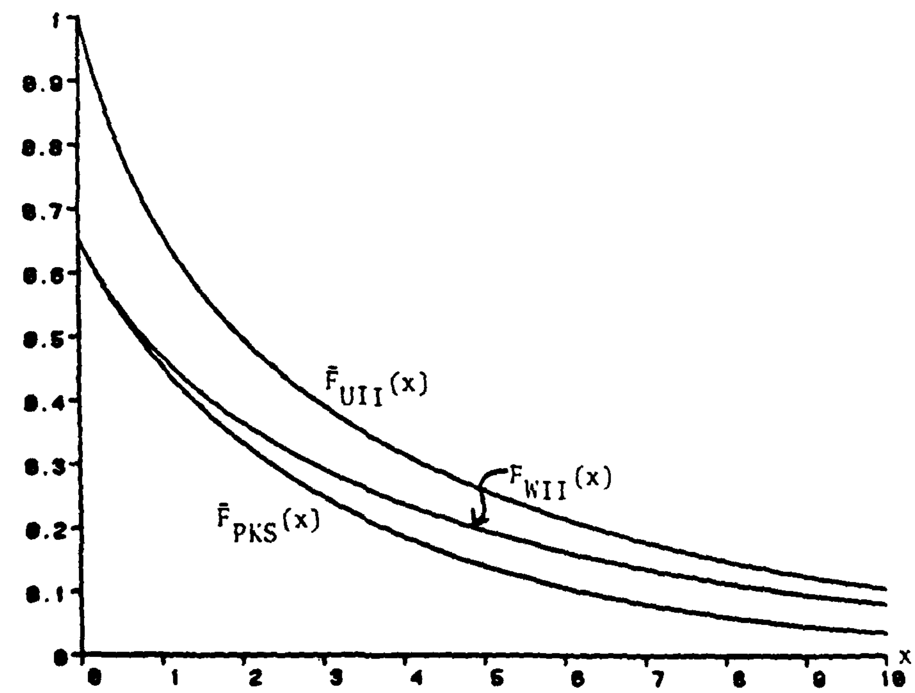

Figure 4.2. The survival functions of W_{PKS}, W_{II} and U_{II}

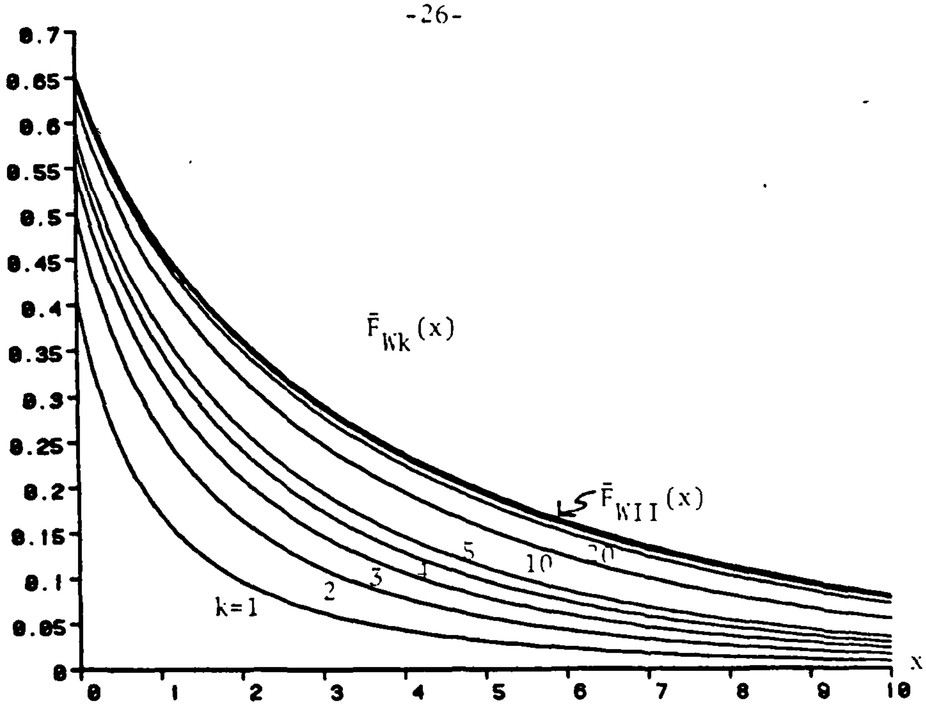

Figure 4.3. The survival functions of W_k

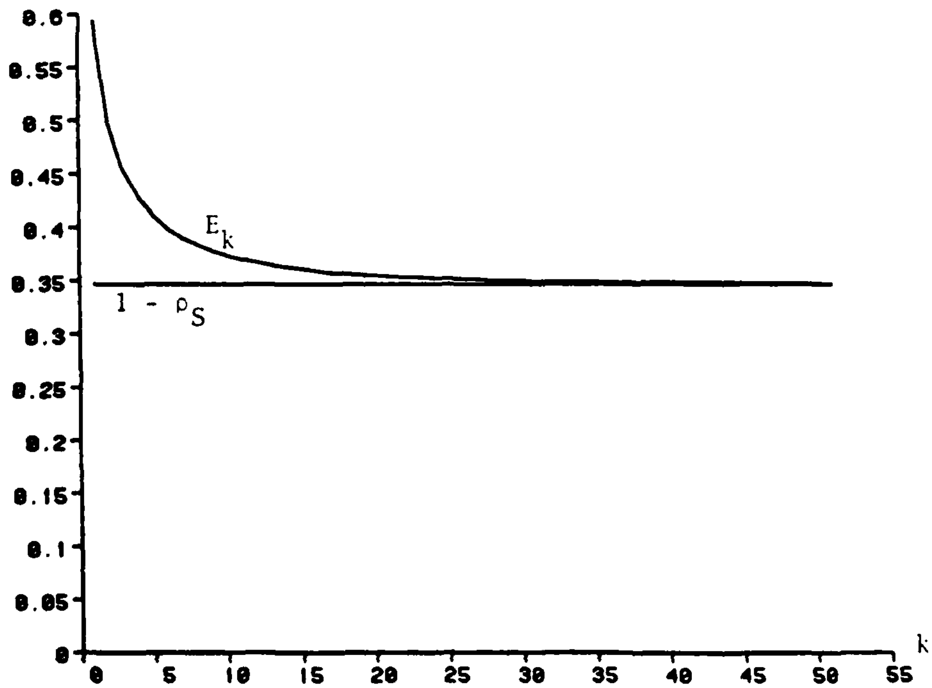

Figure 4.4. The values of E_k

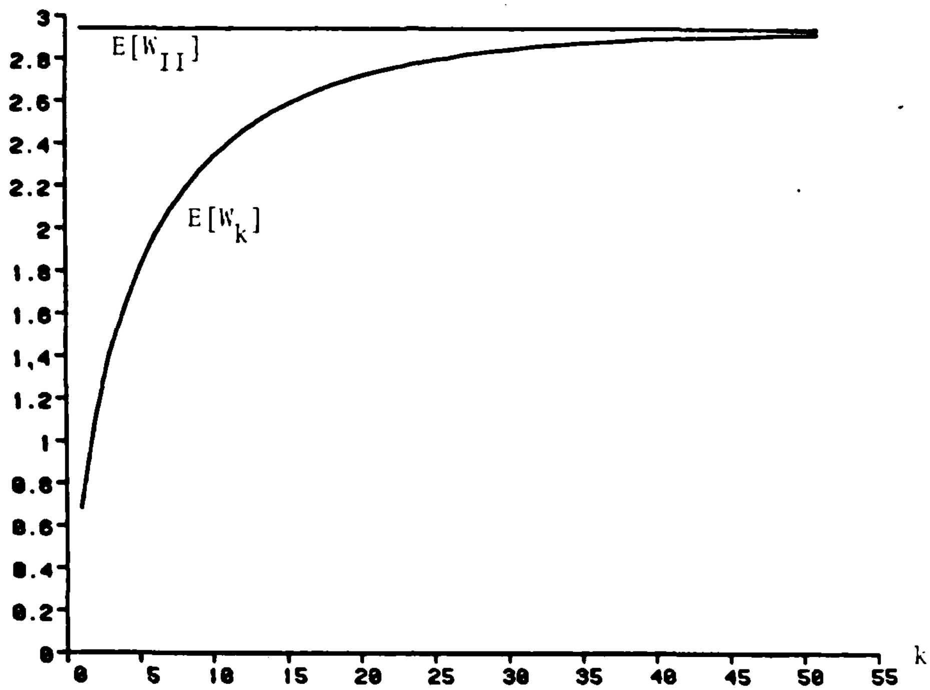

Figure 4.5. The values of $E[W_k]$

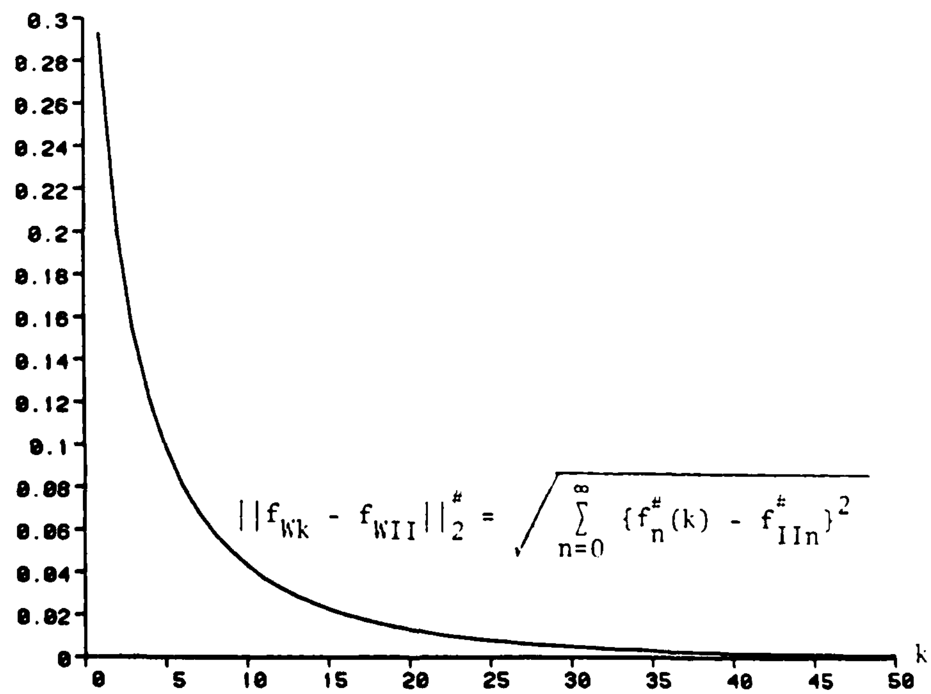

Figure 4.6. The Laguerre sharp norm distance between W_k and W_{II}

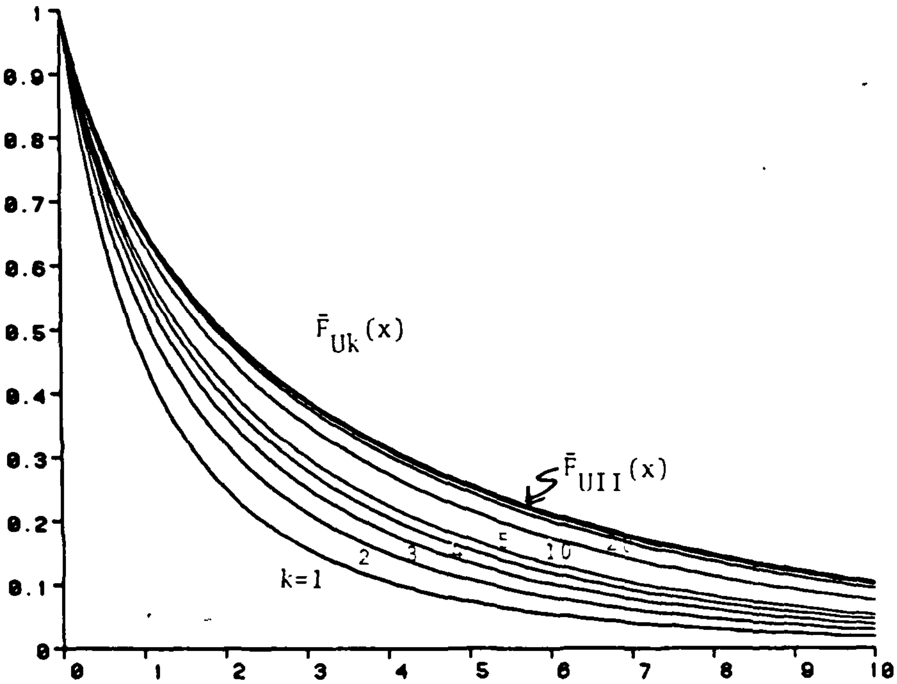

Figure 4.7. The survival functions of U_k

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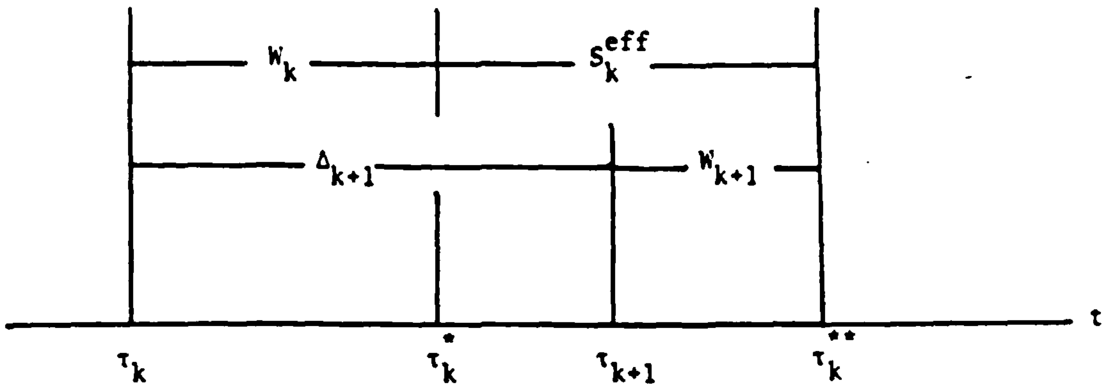

Figure 2.1.a. $\Delta_{k+1} \leq W_k + S_k^{eff}$

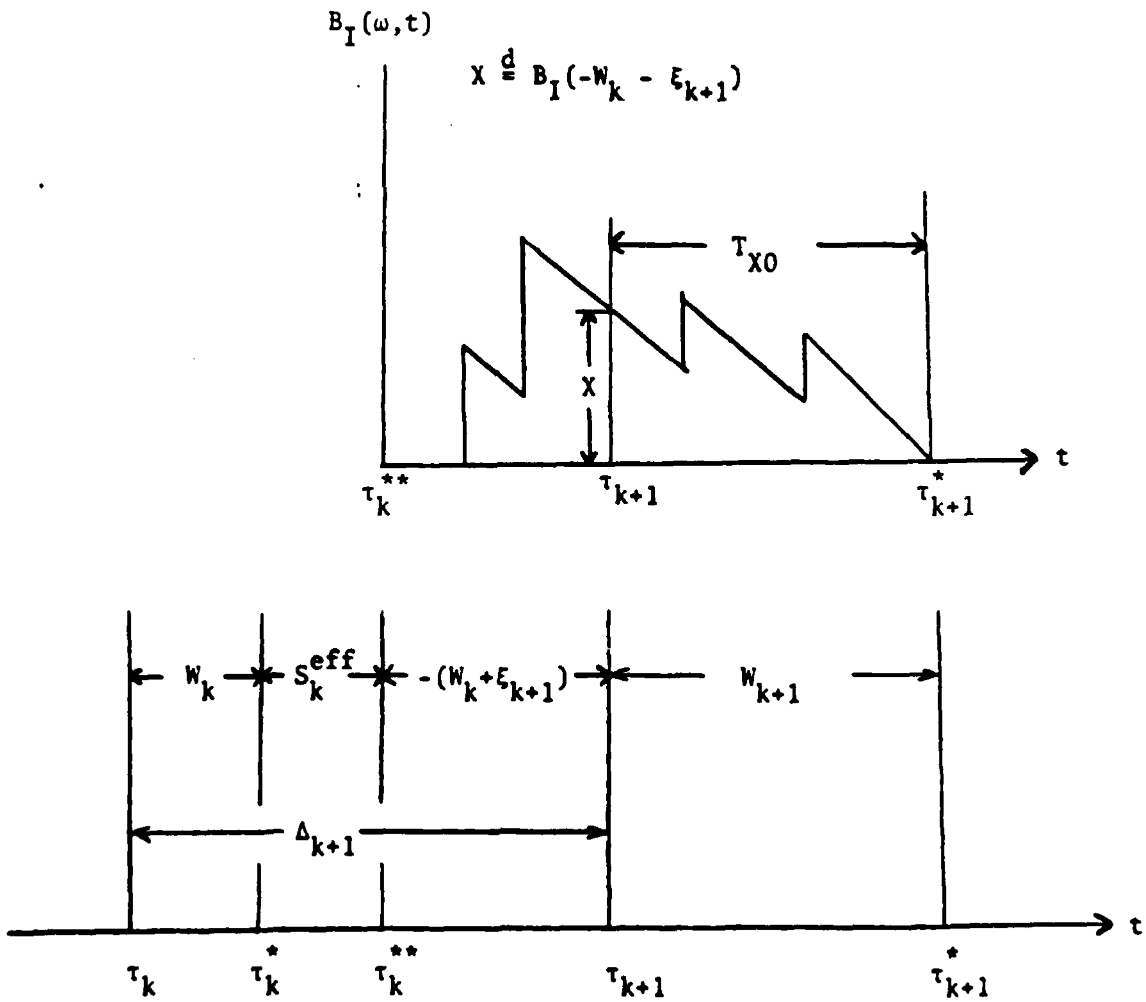

Figure 2.1.b. $\Delta_{k+1} > W_k + S_k^{eff}$

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