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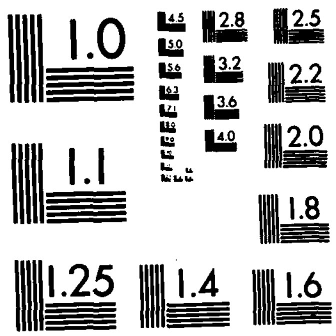

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RECURSIVE RATIONAL CHOICE

by

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ABSTRACT

The concept of a choice function, characterized by means of a set-valued mapping on restricted families of subsets of a space of alternatives is employed in an essential way in the theory of consumer choice in mathematical economics to construct demand correspondences (Mukherji [1977], Richter [1966], Sonnenschein [1971] and Uzawa [1956]). A concomitant consideration of such a function, arising out of Arrow's seminal considerations of social choice (Arrow, [1963]), is the extent to which a choice function may be considered rational. This problem has been treated extensively by Richter [1971]. However, a further consideration of rationality has been developed by Kramer [1974] in the consideration of whether or not a decisive choice function that is regular rational in the sense of Richter [1971] when defined on subsets of a denumerably infinite domain of alternatives, can be realized in principle by means of a device of artificial intelligence.

It is the purpose of the present study to indicate the means by which Kramer's results may be generalized to considerations of stronger computing devices than the finite state automata considered in Kramer's approach, and to domains of alternatives having the cardinality of the continuum. The means we employ in the approach makes use of the theory of recursive functions in the context of Church's Thesis. The result, which we consider as a preliminary result to a more general research program, shows that a choice function that is rational in the sense of Richter (not necessarily regular) when defined on a restricted family of subsets of a continuum of alternatives, when recursively represented by a partial predicate on equivalence classes of approximations

by rational numbers, is recursively unsolvable. By way of Church's Thesis, therefore, such a function cannot be realized by means of a very general class of effectively computable procedures. An additional consequence that can be derived from the result of recursive unsolvability of rational choice in this setting is the placement of a minimal bound on the amount of computational complexity entailed by effective realizations of rational choice. However, the principal interpretation of the result, in our present framework, is that a distinction must be placed between what is meant by a recursive representation of rational choice, and a recursive realization of that representation by effectively computable procedures.

RECURSIVE RATIONAL CHOICE*

by

Alain A. Lewis**

I. Introduction

The suggestion that certain axiomatic structures found in economic theory can, and should be, perhaps, subjected to metamathematical considerations not surprisingly enough is found in the treatise of Von Neuman/Morgenstern, The Theory of Games and Economic Behavior, (Princeton University Press [1944]), in their discussion of axiomatic formulations of n-person extensive games, Chapter II, Sections 8.4.1 and 10.2. However, very little attention has been paid to the metamathematical consideration of the mathematical systems that are employed in mathematical economics. Notable exceptions in this regard are the works of Aumann and Wesley [1936]. In particular, it would seem worthwhile to consider that branch of metamathematics termed the theory of elementary formal systems which in turn would be useful in characterizing those mathematical concepts employed in economic theory that are constructive in a procedural sense. In the area of programming, economic theory has dealt with the notion of algorithm somewhat extensively in computational procedures for arriving at equilibrium prices. Notable in this regard is the work of Scarf [1974] and an excellent survey of the current state of combinatorial optimization, found in the article by Klee [1980]. But very little research has been done on more general items of whether, say, preference structures defined on arbitrary infinite domains are constructible[1/]. In other words, in addition to assuming that

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a decision making entity acts as though it made rational choices on compact metrizable budget spaces, if we were to assume within the theory that these choice made were, in principle, realizable, which we will take to mean that the choice are computable in principle, under what conditions would the preference structure that represents this choice be constructive?

Specifically, we observe that the concept of a rational choice structure can be considered on at least three levels:

(a) the preference level

(b) the choice level

(c) the computational or constructive level.

In relegating the first two distinctions as being secondary, our primary interest will be on the third consideration. That is, we wish to consider whether or not a rational choice function can be realized as a computational procedure. In contrast to the leading result given by Kramer [1974], which is negative on the item of realizable rational choice in terms of a computational procedure that is equivalent to a Mealy automaton, we provide the means to extend his result by way of Richter's framework of rational choice [1971], to the stronger context of computability and therefore realizability, in the sense of a Turing machine. The means of the extension employs a variant of Church's Thesis, in the context of recursive set theory. This approach, to make a further distinction, in contrast to the paper by Campbell [1978], which treats in specific fashion, the item of realization of choice functions defined on finite sets of alternatives only, enables us to address the issue at the level where the space of alternatives is of infinite cardinality, as found in the traditional treatment of Paretian utility as set forth by Debreu [7] .

II. Rational Choice Functions

Typically, one means by a preference structure on a set of alternatives, X , the set X together with a binary relation on X x X . That is, a relation, $\geq$, such that

$$\geq : X \times X \rightarrow \{0,1\}$$

and such that for χ_1, $\chi_2 \in X$, $\chi_1 \geq \chi_2$ assigns 1 or 0 depending on whether χ_1 is or is not at least as good as χ_2 . One usually requires that a preference relation satisfies the following axioms :

(1) $\forall \chi \in \chi[(\chi \geq \chi)]$

(2) $\forall \chi,y,z \in \chi\Big[[(\chi \geq y \cdot\wedge\cdot (y \geq z)] \rightarrow (\chi \geq z)\Big]$

(3) $\forall \chi,y \in \chi[(\chi \geq y) \cdot\vee\cdot (y \geq z)]$

We will restrict our attention initially to sets of alternatives, X, of denumerably infinite cardinality having the usual economic interpretation of contingent alternatives indexed by time into the future. The following result is well known in such a setting:

II.1 <u>Theorem</u>: If the set X is denumerable, then there exists a real-valued function f on X such that if $\geq$ is a preference relation for

for χ_1, $\chi_2 \in X$, $\chi_1 \gtrsim \chi_2$ if and only if $f(\chi_1) \geq f(\chi_2)$. Pf: cf. P. Fishburn, Utility for Decisionmaking, Chapter II, J. Wiley & Sons, [1969].

The function f is said to represent the preference relation $\gtrsim$. By convention, if it should occur that for χ_1, $\chi_2 \in X$, $\chi_1 \gtrsim \chi_2$ and $\chi_2 \gtrsim \chi_1$, then for a representation f of $\gtrsim$ one has $f(\chi_1) = f(\chi_2)$ and we say that χ_1 and χ_2 are indifferent under $\gtrsim$ and write this as $\chi_1 \sim \chi_2$. We will in later usage require that the quotient space under $\sim$, $\chi/\sim$, also be of denumerably infinite cardinality.

By a choice function, we will mean a set valued mapping with the following features:

(a) $C: S \to S$ for $S \subsetneq P(X)$

(b) $\forall A \in S[C(A) \subseteq A]$

A choice function then is defined on a subcollection of subsets of the power set of X, taking values in the power set of X, such that the range of C on an element of its domain is a subset of that element.

An extremely common means of defining a choice function for a given preference structure $(\chi, \gtrsim)$, with representation f, is as follows:

$$C(A) = \{\chi \in A : \forall y \in A\,[f(y) \not> f(\chi)]\}$$

The interpretation of which is, for a given subset A, an element of the subfamily S, the choice function specifies the order maximal elements of A with respect to the representation f of the preference. This formulation will be of particular interest to our inquiry in subsequent sections.

T. Bergstrom, in Journal of Economic Theory, Vol. 10, No. III, [1975] pp. 403-404, has shown that a choice function as we have defined it, is nonempty on compact subsets A in X, under the appropriate topology, when the preference order is acyclic, and the lower contour sets:

$$P_{\chi}^{-1} = \{y \in X : \chi \gtrsim y\}$$

are pointwise open in the relative topology of X, where the relation $\gtrsim$ is derived as $(\gtrsim \,.\!\wedge\!.\, \not\lesssim)$. Similar topological conditions in the context of recursive sets will be required in later investigations.[2/]

The framework of Richter [1971] gives us the interpretation of rational choice that follows. In this framework, the subfamily S is taken to mean a collection of budget sets in the traditional economic setting. However, in general, that specific interpretation is not required. A preference $\gtrsim$ rationalizes choice on X in the sense given by Richter ([1971],p.31) if for $A \in S$, $\mathbf{C}(A) = \{\chi \in A : \forall y \in A\ [\chi \gtrsim y]\}$. The emphasis of the attribute, rational, is then placed upon the manner in which choices are selected from subsets of X. This usage is in contradistinction to that employed to characterize rational preferences, which are usually typified as being not intransitive. This form of distinction is an extremely useful one, for one can then inquire, in the manner that Richter successfully accomplishes, into those varieties of preferences that generate rational choices in the above sense. As it turns out this will be a useful distinction for our purposes, as we will ultimately be concerned with the means for representing choice functions that are rational in the above sense within the framework of an elementary formal system -- not the preferences that generate them. Noteworthy, is that Richter's framework is quite weak, in the sense that no specific assumptions are made concerning the relation $\geq$ per se in the definition of a rational choice function. However, to inquire when a preference exists that will rationalize a given choice function is not at all superfluous, for simple examples show that irrational choices in the sense of Richter exists, i.e., choices that cannot be rationalized by a preference relatic

II.2 Theorem (Richter [1971]): There exists an irrational choice.

Proof: Let $X = \{a,b,c\}$. Consider $A_1 = X$ and $A_2 = \{a,b\}$, and suppose $C(A_1) = \{b\}$ and $C(A_2) = \{a\}$.

If $\succsim$ were a preference relation that rationalizes C , then since $b \in C(A_1)$ then $b \succsim \chi$ for any $\chi \in X$. In particular $b \succsim \chi$ for $\chi \in A_2 \subseteq X$. But then $b \in C(A_2)$ which is false.

Q.E.D.

This example violates a postulate for rational choice given by H. Uzawa in "A Note on Preference and Axioms of Choice," Annals of the Institute of Statistical Mathematics, Vol. 8, [1956] pp. 36-40.[3/] if A, B $\in$ S, and $A \subseteq B$, then if C is rational, then we should have $A - C(A) \subseteq B - C(B)$, i.e., any element not chosen from a subset of alternatives remains unchosen when the subset is enlarged by more alternatives.

Richter provides us with those conditions that serve to sufficiently characterize rational choice in terms of two axioms (Richter [1971] pp. 33-35):

The V-Axiom: $\chi V(C)y \iff \exists A \in S[\chi \in C(A) \cdot\wedge\cdot y \in A]$

The W-Axiom: $\chi W(C)y \iff \exists\{z_j\}_{j=1}^{n} \subseteq S[\chi V(C)z_1, \ldots, z_b V(C)y]$

The following two theorems proven by Richter ([1971], Theorems II and V) provide an important qualitative distinction between two possible meanings that can be given to rational choice in the above sense.

II.3 <u>Theorem</u> (Richter [1971]): If a choice function C on a set X is such that for any $A \in S$,

$$C(A) = \{\chi : \chi \in A \cdot\wedge\cdot \forall y \in A[\chi V(C)y]\}$$

then C is reflexive-rational.

II.4 <u>Theorem</u> (Richter [1971]): If a choice function C on a set X is such that for any $A \in S$,

$$C(A) = \{\chi : \chi \in A \cdot\wedge\cdot \forall y \in A[\chi W(C)y]\}$$

then C is transitive-rational.

A further qualitative distinction among the possible meanings of rational choice (cf. Richter [1966] can be obtained in terms of the following:

II.5 <u>Theorem</u> (Richter [1966]): A choice function C on a set X is regular-rational if and only if for any $A \in S$,

$$\chi, y \in A \left[\left[[\chi \in C(A) \cdot\wedge\cdot [yW\chi]\right] \rightarrow y \in C(A)\right] .$$

This latter theorem is given in terms of Richter's Congruence Axiom. A regular-rational choice function is rationalized by a regular preference relation, where a regular preference is reflexive, transitive, and total.

An additional concept that will be of interest to our own inquiry is the concept of representable choice. A choice function $C : S \to S$ is representable in the sense of Richter if there exists a function $f: X \to R_+$ such that for all $A \in S$, $C(A) = \{\chi \in A : \forall y \in A[f(\chi) \geq f(y)]\}$. The proposition on page 48 of Richter [23], gives necessary and sufficient conditions in terms of an augmentation of the W-Axiom for a choice function to be representable. The W-Axiom is shown to be a necessary condition (Richter [1971] p. 46) for a choice function to be representable in that context.

One observes, importantly, that within the framework of Richter, the consideration of a rational choice function is not necessarily restricted to considerations of the transitivity (or lack of) of an underlying preference, seemingly the predominant considerations in earlier inquiries into the theory of choice generated by Arrow's seminal work [1963]. What we wish to turn our attention to, however, is yet another level of discourse of rational choice, that of mathematical constructibility. That is, does the notion of a rational choice function as given in the framework of Richter have a constructive realization? What we are asking is, in addition to assuming that a decision entity acted as though it were making rational choices among a set of alternatives, as a not unreasonable paradigm of human behavior, if one were to assume within the theory that these choices were made in a manner that were realizable in principle, under what conditions would the procedure of choice be realizable by effectively computable means? The result that we provide would seem to limit the circumstances under which one can give a favorable response to such a question.

III. Constructive Representations of Rational Choice Functions

Constructive mathematics is often referred to as algorithmic mathematics, the history of which dates back to at least the time of Euclid and the name for which derives from the 9th century mathematician Al-Khurvarizmi of Islam (Kleene [1965]).

By a constructive mathematical concept, one means a concept that is the result of a process of construction which is realizable along the following lines:

(1) One assumes a clearly defined, fixed collection of primitive objects.

(2) One assumes an unambiguous list of rules for forming new objects from previously constructed ones. These are the admissible steps in the process.

(3) One assumes that the process of construction is carried out in discrete time units.

Perhaps the most widely employed mathematical structure in constructive mathematics is what is known as an elementary formal system. The use of the term, "formal", is to denote in the manner of Smullyam's <u>Theory of Formal Systems</u>, Annals of Mathematics Studies, No.47, Princeton University Press, [1961], finitary objects and discretized procedures, for the decidability of proofs and provability. We give a brief characterization as follows for the purpose of general background.

A few preliminary definitions will be required:

Definition I: By an alphabet one means an ordered finite set of primitive symbols. Denoted as "K".

Definition II: By a string is meant a finite line or sequence of elements in K and we say that the string is in K.

Definition III: For a set $\{\chi_j\}_{j=1}^{n} \subseteq K$, let $(\chi_i, \ldots \chi_n)$ be the string formed from $\{\chi_j\}_{j=1}^{n}$. Then the length of the string is n.

Definition IV: Let χ and Y be strings in K then the string χY is in K and is termed the concatenation of χ and Y, which goes by $(\chi_i, \ldots \chi_n, Y_i, \ldots, Y_m)$ and is of length $n = m$.

By an elementary formal system over an alphabet K, (ξ), is meant

(1) The alphabet K.

(2) An alphabet of symbols, V, the variables.

(3) An alphabet of symbols, P, the predicates: each of finite degree.

(4) A pair of symbols, ($\rightarrow$, ,) called implication and punctuation.

(5) A finite sequence, $A_1, \ldots, A_n$ of wffs. termed the axioms of (ξ). A wff. of ξ is an expression of the form $P\, t_i, \ldots, t_m$ for $t_1, \ldots, t_m$ terms or $F_1 \rightarrow F_2 \ldots \rightarrow F_n$ where each F_j has the form $Pt_1, \ldots, t_m$.

By a <u>provable string</u> *of* (ξ) *one means a string* χ *that is either*

(a) an instance of the axioms by way of substitution, or

(b) derivably by a finite sequence of applications of substitutions of the axioms and the deduction rule of modus ponens.

If χ is provable in (ξ), $\vdash_{\xi}\chi$ is the symbol used to indicate this. It can be shown that (Smullyan <u>op cit</u>):

III.1 <u>Theorem</u>: The theorems of (ξ) are precisely the provable strings of (ξ).

Let $\tilde{K}$ denote the set of all finite strings in K. Let the term attribute denote a set in $\tilde{K}$ or a member of P. Then let α be an attribute of (ξ) and $W \subseteq K$. We say that α <u>represents</u> W in (ξ) if and only if

$$\forall \chi \in \tilde{K} \{\chi \in W \iff \vdash_{\xi} \alpha\ (\chi)\}$$

As an example of an elementary formal system consider the following: Suppose we take K = {1}. Then a string of length n is simply $\sum_{j=1}^{n} 1 = n$. Suppose we next wish to represent the set of even numbers. Let E be the predicate "is even" and let X be a variable in V. As axioms allow:

(1) E11

(2) $E\chi \to E\chi 11$

III.2 <u>Theorem</u>: $\forall \chi \in V \{\vdash_{\xi} E\chi \iff \chi \text{ is even}\}$. Then E <u>represents</u> the set of even numbers in the above sense.

Proof: Enumerate the even numbers in order, $e_1, \ldots, e_n, \ldots$ Then $c_1 = 11$ and by the 1st Axiom E11 is even. Suppose Ee_n for some n. Then by the 2nd Axiom $Ee_n \rightarrow Ee_n 11$. By modus ponens Ee_n, $Ee_n \rightarrow Ee_n 11$ yields $Ee_n 11$. Arbitrary even numbers are then provable in (ξ). Any provable instance of E over (ξ) is hereditarily true by properties of modus ponens. However, $E\chi$ is true means that $E\chi$ is even. Q.E.D.

One of the more important formal systems historically was devloped by Alonzo Church in the 1930s at Princeton, the λ-calculus [1941], which may be viewed as an elementary formal system of number theoretic predicates.[4/] The reknown Church's Thesis asserts that those mathematical concepts that one views as being constructive are provable in the context of the λ-calculus, i.e., the λ-calculus represents the constructive functions. Significantly, Kleene and subsequently Turing showed respectively that the λ-recursive functions of Gödel and the computable functions of Turing in the sense of being the output of an automaton, are also provable within Church's λ-calculus (Kleene [1936], Kleene [1965] and Turing [1936].) Since no categorical definition of effectively computable can be derived, different notions having been set forth by the mathematicians, Kalmar and Pêter ([1965] and Journal of Symbolic Logic) differing slightly from that found in Church's Thesis, one regards Church's assertion as a thesis rather than a theorem. However, the ensuing equivalence of the constructive mathematical systems of Gödel, Kleene, and Turing provide ineluctable evidence that the recursive functions are, perhaps, the most general notion accessible by human endeavors. Rogers [1967], Ch.I refers to this as the evidence for the Basic Result, by which is meant the equivalences to the λ-calculus mentioned above. On this latter point the reader is also referred to Putnam [1973].

The following are useful facts (cf. Kleene [1950]) that we will make reference to subsequently. The expression $\lambda\chi[\phi(\chi)]$ denotes the <u>partial function</u> $<\chi,y>$ which gives the value y when χ takes an integer value, by way of $\phi(\chi)$. The domain of the partial function so defined is a subset of the natural numbers, $D(\phi) \subseteq \mathbb{N}$, as is its range, i.e. $R(\phi) \subseteq \mathbb{N}$, and thus may be considered as a number theoretic predicate.

The class of <u>primitive recursive functions</u> is the smallest class Ω of functions such that:

(a) The <u>constant functions</u> $\lambda\chi_1 \dots \chi_K[m]$ for $1 \leq K$, $0 \leq m$, are in Ω.

(b) The <u>successor function</u> $\lambda\chi[\chi + 1]$ is in Ω.

(c) The <u>identity functions</u> $\lambda\chi_1 \dots \chi_K[\chi_i]$ are in Ω.

(d) If f is a function of K variables in Ω, and if $g_1, g_2, \dots, g_k$ are functions of m variables in Ω then the following function is in Ω:

$$\lambda\chi_1 \dots \chi_m [f(g_1(\chi_1, \dots, \chi_m), \dots, g_K(\chi_1, \dots, \chi_m))] .$$

Functions derived from the composition of functions in Ω are in Ω.

(e) If h is a function of $K + 1$ variables in Ω, and g is a function of $K - 1$ variables in Ω, then the unique function f of K variables satisfying

$$f(0, \chi_2, \ldots, \chi_K) = g(\chi_2, \ldots, \chi_K)$$

$$f(y + 1, \chi_2, \ldots, \chi_K) = h(y, \chi_2, \ldots, \chi_K), \chi_2, \ldots, \chi_K)$$

is in Ω , for $1 \leq K$. Functions derived from primitive induction in Ω are in Ω .

The class of recursive functions are generated by the schema:

$$f(\ ) = \ y \, P(\ , y) \quad .$$

The expression $\mu y\{P(\chi, y)\}$ is read: "the least y such that $P(\chi, y)$ holds", where μ is Gödel's mu-operator,[5] and $P(\chi, y)$ is a primitive recursive function obtained by the procedures given above. The general recursive functions are obtained if $P(\chi, y)$ is regular, i.e. $\forall\chi\exists y P(\chi, y)$. If $P(\chi, y)$ is not necessarily regular, then only the partial recursive functions are obtained.

Further basic notions that relate to the above, that follow from Church's Thesis,[6] and which we will subsequently make use of are:

(i) Recursive Functions $\subsetneq$ Partial Recursive Functions.

(ii) Partial Recursive Function $\cong$ A Turing Machine That May Not Halt.

(iii) Recursive Function $\cong$ A Turing Machine That Always Halts.

(iv) Recursively Enumerable Set $\cong$ A Set Whose Characteristic Function is Partially Recursive.

(v) Recursive Set $\cong$ A Set Whose Characteristic Function is Recursive.

(vi) A Recursive Set is such that both it and its complement are Recursively Enumerable.

(vii) A Set is Recursively Enumerable if it is the range/domain of a Partial Recursive Function.

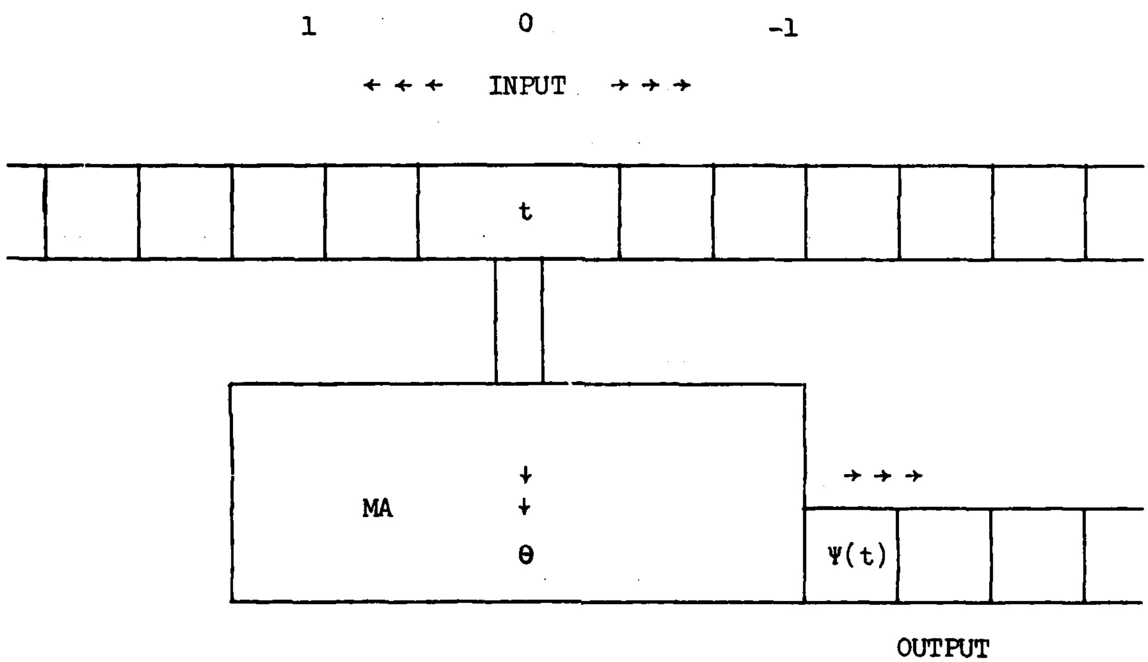
With this background in place, we may now turn to the consideration of whether of not a rational choice function, in the sense of Richter, admits of a constructive representation within a formal system that we shall interpret in the form of an automaton (cf. Starke, [1972]), that is of the Mealy variety. We provide this result as a preliminary item to the more general setting of a Turing machine treated subsequently.

Imagine an idealized computing device that had k components, each of which can obtain m states, where m and k are finite integers. Let the description of the machine's computing process be given by the following:

$$MA = < s, t, \theta(s), \Psi(t), m = \{1, 0, -1\} >$$

The machine can be viewed as a kind of scanning device that looks at symbols on a tape and then signifies an output. The description, MA, for Mealy automation (cf. Starke [1972],Ch.I), provides the following kind of rule: If in state s, and if the input symbol t is scanned, then go to the state $\theta(s)$, and signify the output $\Psi(t)$, and then move the input tape either right one space, -1; left one space, 1; or leave the tape where it is, 0.

Diagramatically, we can visualize MA as:



Let there be two finite languages (not necessarily distinct) L_I and L_0 corresponding to an input language, and an output language, respectively. L_I contains distinguished elements $\{\Delta, \xi\}$ that are used to indicate when distinguished segments of the input tape are begun, and terminated. In the language L_0 there is the symbol Λ to indicate the null output. In the manner of section I, we may construct on each language the elementary formal system $\xi(L_I)$ and $\xi(L_0)$. One can then view the input tape as comprised of strings of wff.s in $\xi(L_I)$, while the output tape can be viewed as strings of wff.s in $\xi(L_0)$. It is then permissible to view the automaton MA as a composite formal system with components $\xi(L_I)$ and $\xi(L_0)$.

Consider now the quotient space $X/\sim$, of indifference classes of a set of infinite alternatives. Select then from $2^{X/\sim}$ the class of all finite sets, and call it F. Let us make the assumption that $X/\sim$ is denumerably infinite. If we assume that any finite set in F has a well-formed representation as a string in $\xi(L_I)$, then from the denumerability of F, we may form the input tape $T(F)$ that encodes the members of F as wff.s in $\xi(L_I)$.

Let us denote the totality of states for the machine MA as S. Then $||S|| = k^m$ and is finite, for k the number of components in MA. For a given string χ on $T(F)$, the function:

$$\tau_\chi = S \times \{-1, 1\} \rightarrow [S \times \{-1, 1\}] \cup \{0\}$$

specifies the transition rule of the machine MA with respect to χ. For the pair, (s,-1), the machine is in state s, at the right most symbol of χ. If the machine runs the string χ to the right and goes to state $\theta(s) = s'$, then set the value of the function $\tau_\chi(s, -1) = (\theta(s), -1)$.

One sees readily that the total number of such functions for strings on $T(F)$ if $Q = (2k^m + 1)^{2k^m} = ||R(\tau)||^{||D(\tau)||}$, for $R(\tau) =$ the range of τ and $D(\tau) =$ the domain of τ. Then, the sets $\{T_{\tau_{\chi_j}}\}_{j=1}^{Q}$ give a partition of T(F) such that if $\chi, y \in T_{\tau_{\chi_j}}$, then $\tau_\chi = \tau_y$, i.e., each $T_{\tau_{\chi_j}}$ is an equivalence class of strings that generate the same transition function, τ_{χ_j}.

Before presenting the result that follows, we need to formalize precisely what it means for a Mealy automaton to realize a function.

Definition V: Allow $f : I \to K$ to be a partial function defined on arbitrary sets I and K. A Mealy automaton given as $MA = \langle s, t, \theta(s), \Psi(t), m = \{1,0,-1\}\rangle$ is said to realize f if and only if:

(1) for every $i \in D(f) \subseteq I$ there is a unique wff. in $\xi(L_I)$ that formally represents i.

(2) for every $k \in R(f) \subseteq K$ for which there is an $i \in D(f) \subseteq I$ such that $f(i) = k$, there is a unieque wff. in $\xi(L_0)$ that formally represents k.

(3) if t is the wff. in $\xi(L_I)$ that formally represents an $i \in D(f) \subseteq I$, then $\Psi(t)$ is the unique wff. in $\xi(L_0)$ that formally represents that $k \in R(f) \subseteq K$ for which $f(i) = k$.

The following result is due to Kramer [1974].

III.3 Theorem: Let $X/\sim$ be denumerably infinite and let $\gtrsim$ be a reflexive total preference on χ that rationalizes the choice function $C(A) = \{\chi \in A : \forall y \in A[\chi \geq y]\}$ for $A \in F$, then the formal system $MA = \langle S, \Theta, \Psi, m, \xi(L_I), \xi(L_0)\rangle$ cannot realize $C(A)$ in the sense of Definition V on F .

Proof: Suppose MA realized the choice function $C(A)$ when $A \in F$, then A would be encoded by a string $\chi_{a_1}, \ldots \chi_{a_m}$ in $\xi(L_I)$ and appear on $T(F)$, the input tape. Then we require that $\Psi(\chi_{a_1}, \ldots \chi_{a_m})$ be a wff. in (L_0). Clearly, the reflexivity of $\gtrsim$ requires that if $\chi_1 \neq \chi_2$ in $\xi(L_I)$ then $\Psi(\chi_1) \neq \Psi(\chi_2)$ in $\xi(L_0)$.

It can be demonstrated that for an input tape segment $\chi_1 \ldots \chi_n$ such that $\chi_j \in T_{\tau_{\chi_i}}$, if the machine accepts the segment scanning to the right and printing $\Psi(\chi_1 \ldots \chi_n)$, then

$$\ln(\Psi(\chi_1 \ldots \chi_n)) = \sum_{j=1}^{n} f_j(\chi_j)$$

where $\ln(\cdot)$ is the length of the output string $\Psi(\chi_1 \ldots \chi_n)$ and each of the functions are such that $f_j : T_{\tau_{\chi_i}} \to \mathbb{N}$ (cf. Kramer [1974], pp.48-49). Since the cardinality of inputs on $T(F)$ is that of the natural numbers, and since $\{T_{\tau_{\chi_i}}\}_{i=1}^{Q}$ partitions $T(F)$, at least one member of $\{T_{\tau_{\chi_i}}\}_{i=1}^{Q}$, $T_{\tau_{\chi_0}}$, must contain infinitely many substrings, representing distinct members of $F \subseteq 2^{X/\sim}$. The set $P = \{\chi_{01}, \chi_{02}, \chi_{03}, \ldots \chi_{0n}, \ldots\}$ can then be formed in terms of distinguished singletons, one each from the members of $T_{\tau_{\chi}}$.

One sees readily that all sets of the form $\{\alpha_i\} \cup \{\alpha_i\}$ or $\{\alpha\ \} \cup \{\alpha_i\}$ must yield output strings of identical length, when represented as $\chi_{0i}\chi_{0j}$ or $\chi_{0j}\chi_{0i}$ in P, i.e., $\ln(\Psi(\chi_{0i}, \chi_{0j})) = \ln(\Psi(\chi_{0j}, \chi_{0i}))$. Clearly, for distinct α_i, $\alpha_j \in X/\sim$, either $\alpha_j \gtrsim \alpha_i$ or $\alpha_i \gtrsim \alpha_j$, and if $A = \{\alpha_i\} \cup \{\alpha_i\}$, then, we have $C(A) = \alpha_i$. Then it follows that $\ln(\Psi(\chi_{0i}\chi_{0j})) = \ln(\Psi(\chi_{0j}\chi_{0i})) = L$, where $L = \ln(\Psi(\chi_{0i}))$ or $\ln(\Psi(\chi_{0j}))$ depending on whether $\alpha_i \gtrsim \alpha_j$ or $\alpha_j \gtrsim \alpha_i$ respectively.

However, the output alphabet is finite for $\xi(L_0)$, say of cardinality B. Then the number of distinct strings in $\xi(L_0)$ of length L is bounded sharply by $(B)^L$.

Then, for q sufficiently large, say $q > (B)^L$, if i^*, $j^* > q$, then if χ_{oi}^* and χ_{oj}^* represent α_i^* and α_j^* respectively (α_i^*, $\alpha_j^* \in X$), since both i^* and j^* are in excess of q, $\ln(\Psi(\chi_{oi}^*)) = \ln(\Psi(\chi_{oj}^*))$ must imply that $\Psi(\chi_{oi}^*) = \Psi(\chi_{oj}^*))$, and therefore if MA were to realize $C(A)$ for $A = \{\alpha_j^*\}$ or $\{\alpha_i^*\}$, $C(\alpha_j^*) = C(\alpha_i^*)$. But, as $\chi_{oj}^* \neq \chi_{oi}^*$ only if $\alpha_j^* \neq \alpha_i^*$, by the preference structure, no common choice is possible. Then, if we take realization to mean constructive in the sense of the computing process of a Mealy automaton,7/ MA cannot realize $C(A)$. Q.E.D.

The implications of the above result seem less than definitive with respect to the general issue of computational choice, and this is so for at least two reasons. First, as is well known, Mealy automata are somewhat limited in their comparative ability to compute, when compared with more general varieties of computing devices, i.e., Turing machines. For details see the discussion given in Hopcroft and Ullman [1979] Ch.III. Second, the theorem is in terms of reflexive-total forms of rationalizations of a choice function, and it would seem desirable to enlarge the scope of reference of the theorem in this regard in terms of Richter's weaker framework of rational choice, requiring less restriction on the underlying preference relation. We provide, in the following sections, what we feel is an approach that addresses these two items in the context of recursive sets. An additional advantage to the framework we will develop is its capability to consider the issue of the constructive representation of rational choice functions defined on families of subsets of the continuum, which is the traditional setting for the problem of consumer choice in economic theory (Debreu, [1959], Ch.IV). This capability is obtained by means of the concept of a recursive metric space, which in its construction replaces the input and output languages of the Mealy automaton with sets of a notation formed on the natural numbers. The Turing framework of computability, of which the Mealy automaton is a special instance,[8/] is then obtained by means of Church's Thesis by consideration of recursive functions on the notation.

IV. Recursive Metric Spaces

Our goal will be to consider the item of consumer choice in the traditional setting, where the space of alternatives, X, is taken to mean a compact, convex subset of $\mathbb{R}^n$.[9/] In order to apply Church's Thesis to what we will define subsequently as a recursive rational choice function, it will be necessary to endow the space of alternatives with sufficient recursive structure to render the relevant mathematical structures number theoretic in the sense described previously. To obtain this structure, we shall employ the concepts of recursive real numbers and notation systems as developed by Rice [1954] and Moschovakis [1965]. We make use of Moschovakis' terminology and framework. Thebasic terminology of recursive function theory that will be used can be found in Kleene [1950] or Rogers [1967].

It can be shown in the manner of Post and Kleene [1954], that there exists a set of primitive recursive functions (cf. Section II):

$$\text{sign}: \mathbb{N} \to \{0,1\}$$

$$\text{den}: \mathbb{N} \to \mathbb{N}$$

$$\text{num}: \mathbb{N} \to \mathbb{N}$$

such that the following mapping is a one-to-one correspondence of the set of natural numbers, $\mathbb{N}$, onto the set, Q, of fractions in lowest terms expressed as:

$$r_\chi : \mathbb{N} \to Q$$

where

$$r(\chi) = (-1)^{\text{sign}(\chi)} \text{num}(\chi)/_{\text{den}(\chi)}$$

Then, for each real number $\alpha \in \mathbb{R}$, there is at least one number theoretic function

$$f : \mathbb{N} \to \mathbb{N}$$

such that

$$\forall \chi, y \in \mathbb{N} \; (|r(f(\chi)) - r(f(\chi + y))| < 2^{-\chi})$$

and

$$\alpha = \lim_{\chi \to \infty} r(f(\chi))$$

In this manner, the real numbers are expressed as limits of particular Cauchy sequences of rational numbers. In a somewhat straightforward fashion, one may obtain an appropriate constructive analogue by requiring the function f to be a general recursive function. Then we may consider a real number α to be a recursive real number if there exists a general recursive function $f : \mathbb{N} \to \mathbb{N}$ for which the above expressions in terms r_χ are satisfied. An alternative, but equivalent, definition of recursive real numbers can be found in Rice [1954] and Robinson (Journal of Symbolic Logic, Vol.16).

By means of Kleene's Normal Form Theorem (Kleene [1950] p.288) for each partial recursive function $f(\chi_1, \ldots, \chi_n)$ of $n \geq 1$ numerical values, there is a natural number $n(f)$ called the Gödel number[10/] of the function $f(\chi_1, \ldots, \chi_n)$ such that the following expression is true:[11/]

$$f(\chi_1, \ldots, \chi_n) \simeq U(\mu y T_n(n(f), \chi_1, \ldots, \chi_n, y)) \simeq \{n(f)\} (\chi_1, \ldots, \chi_n)$$

for $U(y)$ a specific primitive recursive function and $\forall n \in \mathbb{N}$, $T_n(n(f)), \chi_1, \ldots, \chi_n, y)$ is a specific primitive recursive predicate as employed by Kleene ([1950], p.281).

If the function $f(\chi)$ is general recursive instead of partial recursive, then one may replace the relation $\simeq$, which meeans "equal when defined," with the usual relation of equality, =, in the normal form expression. Then for each recursive function, in particular for the general recursive functions, there is a Gödel number, by way of Kleene's Normal Form Theorem, that can be associated with the function, namely $n(f)$.

If the general recursive function $f(\chi)$ with Gödel number $n(f)$ determines the recursive real number α, in the sense that we stipulated, we shall term $n(f)$ an R-index of α in the manner of Moschovakis [16] and denote $\alpha = \alpha_{n(f)}$. The set $\mathbb{N}(R)$ of natural numbers that are R-indices of recursive real numbers can be characterized as follows:

$$\begin{aligned} n(f) \in \mathbb{N}(R) \iff & \forall\chi\exists Z T_1(n(f), \chi, Z) \wedge \\ & \forall\chi\forall y\forall Z\forall t\ [T_1(n(f), \chi, Z) \wedge \\ & T_1(n(f), \chi + y, t) \Rightarrow \\ & |r(U(Z)) - r(U(t))| < 2^{-\chi}\ ] \end{aligned}$$

The function $U(\chi)$ is the same as that used in the Normal Form Theorem, and the predicate T_1 is a monadic version of the primitive predicate T_n used in Kleene's Theorem as referenced above.

Then each element $n(f) \in \mathbb{N}(R)$ is seen to determine a real number $\alpha_{n(f)}$. However, the correspondence cannot be one-to-one for the reason that different Gödel numbers may determine the same function and it may occur that different functions may determine the same real number.

We arrive then at a natural equivalence relation induced on $\mathbb{N}(R)$ given as $\sim_{\mathbb{N}(R)}$, for which

$$f \sim_{\mathbb{N}(R)} g \iff \alpha_f = \alpha_g$$

which simply says that members of $\mathbb{N}(R)$ are taken to be equivalent, if, and only if, they determine the same real number.

Then the ordered pair, $(\mathbb{N}(R), \sim_{\mathbb{N}(R)})$ can be considered as a form of <u>notation</u> for the set of recursive real numbers. The abstraction to a <u>notation system</u> seen as an ordered pair $(T, \sim_T)$ for $T \subseteq \mathbb{N}$ and $\sim_T$ an equivalence relation on T, is immediate and is due essentially to Moschovakis in this form (cf. Moschovakis [1965], p.43).

An alternative definition and approach to notation systems is found in Shapiro [1956] who regards a notation as a function

$$\zeta : \mathbb{N} \to T$$

where $Rng(\zeta)$ is elementary inductive, i.e., can be endowed with a recursion theoretic structure, and $DOM(\zeta) \subsetneq \mathbb{N}$ so that ζ may in fact be partial. Shapiro's approach is consistent with Moschovakis' framework as can be seen by allowing ζ to take the form:

$$\zeta = (I_{\mathbb{N}} \circ t) : \mathbb{R} \to \mathbb{N}$$

where $I_{\mathbb{N}}$ is the identity function on the natural numbers and t is the function that associates the recursive real numbers with their Gödel number representations.

The extension of the concept of a notation system as we have developed it thus far, from recursive elements of $\mathbb{R}$ to recursive elements of $\mathbb{R}^n$ can now be easily constructed as follows. Let the notation system $(\mathbb{N}(R^n), \sim_{\mathbb{N}(R^n)})$ be constructed in the following manner from n-tuples of R-indices of recursive real numbers:

$$\mathbb{N}(R^n) = \{<n(f_1), \ldots, n(f_n)> : n(f_1), \ldots, n(f_n) \in \mathbb{N}(R)\}$$

where

$$<n(f_1), \ldots, n(f_n)> = \prod_{j=o}^{n-1} P_j^{n(f_{j+1})}$$

for $\{P_o, \ldots, P_{n-1}\}$ the set of initial prime numbers commencing with $P_o = 2$. The equivalence relation $\sim_{\mathbb{N}(R^n)}$ is defined by means of the following expression:

$$<n(f)> \sim_{\mathbb{N}(R^n)} <n(g)> \iff \{[<n(f)> \in \mathbb{N}(R^n), <n(g)> \in \mathbb{N}(R^n)] \cdot \wedge \cdot \forall_{j \leq n}[n(f_j) \sim_{\mathbb{N}(R)} n(g_j)]\}$$

By making use of the above framework, we may now proceed to endow $\mathbb{R}^n$, a subset of which we will employ as the domain of feasible alternatives, with sufficient recursive structure for the problem of a rational choice function.

By a <u>recursive metric space</u>, in the manner os Moschovakis [17], we will mean a notation system, $(T, \sim_T)$, together with a binary recursive operator $D: T \to \mathbb{R}$ such that:

(i) $\forall \alpha, \beta \in T[(D(\alpha,\beta) = 0) \Leftrightarrow (\alpha = \beta)]$

(ii) $\forall \alpha, \beta \in T[D(\alpha,\beta) = D(\beta,\alpha)]$

(iii) $\forall \alpha, \beta, \gamma \in T[D(\alpha,\gamma) \leq D(\alpha,\beta) + D(\beta,\gamma)]$

If now we consider the notation systems developed earlier for $\mathbb{R}$ and $\mathbb{R}^n$, namely $(\mathbb{N}(R), \sim_{\mathbb{N}(R)})$ and $(\mathbb{N}(R^n), \sim_{\mathbb{N}(R^n)})$, respectively, we can form the recursive metric spaces:

$$M(\mathbb{R}) = \langle(\mathbb{N}(R), \sim_{\mathbb{N}(R)}), D_{\mathbb{R}}\rangle$$

and

$$M(\mathbb{R}^n) = \langle(\mathbb{N}(R^n), \sim_{\mathbb{N}(R^n)}), D_{\mathbb{R}^n}\rangle$$

by taking the operators defined in each case to be

$$D_{\mathbb{R}} = |o| \text{ and } D_{\mathbb{R}^n} = [\sum_{j\leq n} (|o|_j)^2]^{1/2}$$

so that the following obtains in $M(R)$ and $M(R^n)$:

$$\forall n(f), n(g) \in \mathbb{N}(R)\ D_{\mathbb{R}}(n(f), n(g)) = |n(f) - n(g)|$$

$$\forall \langle n(f)\rangle, \langle n(g)\rangle \in \mathbb{N}(R^n)\ D_{\mathbb{R}^n}(\langle n(f)\rangle, \langle n(g)\rangle) = [\sum_{j\leq n} |n(f_j) - n(g_j)|^2]^{1/2}$$

It can be shown that both the operators $D_{\mathbb{R}}$ and $D_{\mathbb{R}^n}$ are in fact recursive on $\mathbb{N}$ with restrictions partially recursive on $\mathbb{N}(R)$ and $\mathbb{N}(R^n)$ respectively.

For the case of the real line $\mathbb{R}$, and therefore for finitely many copies of it $\mathbb{R}^n$, the recursive metric spaces $M(\mathbb{R})$ and $M(\mathbb{R}^n)$ can be given a concrete representation by topological means. The representation is obtained in terms of Rice's original definition of recursive real numbers which, we will recall is equivalent to the one we have employed earlier in our development of notation systems.

A sequence $\{n_j\}_{j=1}^{\infty} \subseteq \mathbb{N}$ of integers is said to be dyadic if the following conditions are met:

(i) $\forall_j \geq 0 \; [n_j = 0 \vee n_j = 1]$

(ii) $\exists K \forall_j > K \; [n_j = 0]$

(iii) $\exists e \in \mathbb{N} \forall_j \geqq 0 \; [n_j = U(\mu y T_1(e,j,y))]$

The number α is a recursive real number in the original sense of Rice [1954] if there exists an integer e which is a Gödel number for a dyadic sequence such that:

$$\alpha = \sum_{j=0}^{\infty} U(\mu y T_1(e,j,y)) 2^{-j}$$

The following proposition, as we show, enables the straightforward construction of $M(\mathbb{R})$ and $M(\mathbb{R}^n)$:

IV.1 Proposition: If $\chi \in Q$ for Q the set of rational numbers, and thus $\chi = c/d$ for some c, $d \in \mathbb{N}$, the function $\xi(c,d,j) = \{\chi_j\}$ is a partial recursive function, for $\{\chi_j\}$ the j^{th} digit in the dyadic expansion of χ.

Proof: Case [4] Sec.3 pp.16-18 in the manner suggested by Rice [1954].

By way of the proposition, the rationals, Q, are in fact, recursive real numbers, and we may denote be $\mathbb{N}(Q)$ and $\mathbb{N}(Q^n)$ those sets of R-indices and n-tuples of R-indices of Q and Q^n respectively, from the Gödel numbers of the recursive predicates used to generate the dyadic sequences that correspond to the rational numbers. Making use of the equivalence relations $\sim_{\mathbb{N}(Q)}$, and $\sim_{\mathbb{N}(Q^n)}$, one could effectively discretize $\mathbb{R}$ and $\mathbb{R}^n$ by means of induced equivalence classes, i.e. members of $\mathbb{R}$ or $\mathbb{R}^n$ would be equivalent in $\mathbb{N}(Q)$ or $\mathbb{N}(Q^n)$ respectively if they have the same rational numbers in a sufficiently close fixed approximation; in which case their Gödel numbers would be the same. This can be performed by defining two elements in $\mathbb{R}$ to be equivalent under rational approximation if there exists a rational number sufficiently close to both elements for finitely many places of the initial segments of their decimal expansions, or alternatively, whose difference is less than 10^{-K} for $K \in \mathbb{N}$, sufficiently large and fixed. This procedure does in fact yield recursive metric spaces which we can denote as

$$QM(\mathbb{R}) = <(\mathbb{N}(Q), \sim_{\mathbb{N}(Q)}), D_Q>$$

and

$$QM(\mathbb{R}^n) = <(\mathbb{N}(Q^n), \sim_{\mathbb{N}(Q^n)}), D_{Q^n}>$$

for D_Q and D_{Q^n} the restrictions of $D_{\mathbb{R}}$ and $D_{\mathbb{R}^n}$ to $QM(\mathbb{R})$ and $QM(\mathbb{R}^n)$.

Unfortunately, the spaces $QM(\mathbb{R})$ and $QM(\mathbb{R}^n)$ do not yield interesting enough topological structures for many of the operations of real analysis that one would desire to have in a recursive setting. More specifically, one observes that the natural topology[12/] on QM(IR) that one can generate by means of a basis of spheres, having rational radius S(β, K), formed from the metric D_Q as follows

$$S(\beta,K) = \{\alpha \in QM(\mathbb{R}) : D_Q(\beta,\alpha) < 2^{-K}\}$$

for $\alpha \in QM(\mathbb{R})$ and $K \in \mathbb{N}$ yields a topology that is separable, and thus, second countable. But, as $\|Q\| = |\sim|_0$ and thus $\|QM(\mathbb{R})\| = |\sim|_0$, from the effective isomorphism of QM(IR) to IN, we can express QM(IR) as $\bigcup_{j \in \mathbb{N}} \{\alpha_j\}$, the union of one element sets, each containing no non-degenerate sphere of positive radius, i.e. a sphere of radius 2^{-K} for some $K \in \mathbb{N}$. By the well known category theorem of Baire for metric spaces,[13/] QM(IR) cannot be a complete metric space. More importantly for our purposes, QM(IR) is not recursively complete.[14/] Since IR is a complete metric space,[15/] it seems not unreasonable to desire that that property be retained in a suitable recursive representation of IR. To accomplish this, by familiar techniques of analysis,[16/] we may take as the desired recursive metric space, the recursive completion[17/] of QM(IR), which from the simple observation that Q is dense in IR,[18/] will yield precisely the space M(IR) of Moschovakis [1965] for the recursive real line. By way of the above construction, it is an additional observation that not only is M(IR) recursively complete, but recursively separable and recursively connected[19/]. Thus, we preserve in recursive analogue, important topological features of IR that can be made use of in the recursive setting for rational choice.

V. Recursive Rational Choice

In the present section, we will presently employ the concept of a recursive metric space, as developed in the preceding section, to characterize Richter's [1971] framework of rational choice in a recursive setting. From this setting, based on the recursive real numbers that correspond to rational approximations of a subset of $\mathbb{R}^n_+$, we obtain the notion of a recursive rational choice function. Following this development, we demonstrate the principal result of the paper.

Begin by considering a set X which is compact and convex in $\mathbb{R}^n_+$, which we take in the usual sense to mean the space of alternatives for the problem of consumer's choice. Let us denote by R(X) that subset of the recursive metric space $M(\mathbb{R}^n)$ which we characterize as:

$$R(X) = rcl\{\alpha \in QM(\mathbb{R}^n) : \exists x \in X \cdot\hat{}\cdot\ t(x) \in \alpha\}$$

for $t: \mathbb{R}^n \to \mathbb{N}(Q^n)$ the function that associates n-tuples of real numbers with Gödel numbers of a notation derived from a fixed approximation by memebers of Q^n, and where rcl denotes the recursive closure in the natural topology induced by the metric on $QM(\mathbb{R}^n)$. Consider next the following two items:

<u>Definition I</u>: $\mathbb{F}_R = \{A \in P(R(X))$ and A is recursive$\}$

<u>Definition II</u>: $C: \mathbb{F}_R \to P(R(X))$ and $\forall A \in \mathbb{F}_R\,[C(A) \subseteq A]$

We will term the pair $\langle R(X), \mathbb{F}_R \rangle$ a <u>recursive space of alternatives</u> and the set function C a <u>recursive choice</u> on $\langle R(X), \mathbb{F}_R \rangle$. The elements of the collection of alternatives $\mathbb{F}_R$, being recursive are thus effectively computable, and a recursive choice is defined on effectively calculable subsets of the space of alternatives taking as values, subsets of the elements of its domain. We will say that a recursive choice on $\langle R(X), \mathbb{F}_R \rangle$ is <u>recursive rational</u> if the following two items exist:

<u>Definition III</u>: $\gtrsim \; : R(X) \times R(X) \rightarrow \{1,0\}$

<u>Definition IV</u>: $f : \; R(X) \rightarrow \mathbb{N}$ such that:

(a) f is a potentially partial recursive function

(b) $\forall \alpha, \beta \in R(X)[(\alpha \gtrsim \beta) \rightarrow f(\alpha) \geqq f(\beta)]$

(c) $\forall A \in \mathbb{F}_R \, [C(A) = \{\alpha : \forall \beta \in A(f(\alpha) \gtrsim f(\beta))\}]$

From the above two items, a recursive choice on $\langle R(X), \mathbb{F}_R \rangle$ os recursive rational if there exists a binary relation on pairs in $R(X)$, and a function that is potentially partial recursive, i.e. a function that can be extended to a partial recursive function, defined on R-indices in $R(X)$ with range in $\mathbb{N}$, that preserves the order induced by $\gtrsim$, (which may in fact be partial), such that the values of C on members of $\mathbb{F}_R$ are the order maximal elements, in accordance with f, on the domain of C. Thus, the notion of a recursive rational choice includes both the notions that the choice is <u>rational</u> in the sense of Richter [1971], that the choice is described in terms of the order maximal elements of a binary relation, and that the relation is <u>representable</u>, again in Richter's sense [1971], by means of a function that can be extended to be partially recursive.

The following lemma is provided as a prerequisite to the main result and has the straightforward interpretation that a recursive rational choice function preserves recursiveness from its domain or alternatively, that its co-domain, per fixed element in its domain, is effectively computable as a recursive set. A consequence of the lemma given in the discussion that follows is that a recursive rational choice function enables an effectively computable representation of rational choice by means of Turing machines.

V.I Lemma: Allow C to be a recursive rational choice on $\langle R(X), \mathbb{F}_R \rangle$ then for any A in the collection $\mathbb{F}_R$, $C(A) \in \mathbb{F}_R$.

Proof: Let A be a recursive subset of $R(X)$, and consider $C(A)$. If it should occur that $\sim\exists\alpha \in A[\forall\beta \in A(f(\alpha) \geq f(\beta))]$ for f, a potentially partial recursive function, then $C(A) = \phi$ and the result is entirely trivial. Then assume $C(A) \neq \phi$ and allow $m(A) = \{f(\alpha) \in \mathbb{N}: f(\alpha) = \max_A f\}$. Then $C(A) = f^{-1}(m(A) \cap A$, and we see that the lemma follows from the fact that f is potentially partially recursive. For then there exists an extension of f, $\hat{f}$, such that $\hat{f}$ is in fact partially recursive. By definition, being the maximal value of f over A, m(A) is trivially recursive and therefore, since $\hat{f}$ is partially recursive by well known facts (Rogers [1967] Sec.5.3, Th.VII) $\hat{f}^{-1}(m(A))$ is recursive. Then by a further elementary fact of recursive sets (Rogers [1967], Sec.5.5, Th.XIV)[20], since A is recursive and $f \subseteq \hat{f}$, the result comes from the expression of $C(A)$ as the intersection of two recursive sets. Q.E.D.

The interpretation of the above lemma can be enlarged upon as follows. Consider the co-domain of a recursive rational choice on $<R(X), \mathbb{F}_R>$, $\{C(\mathbb{F}_{R_j})\}_{j \in \mathbb{N}}$, when its domain is restricted to a sub-family of $\mathbb{F}_R$, $\{\mathbb{F}_{R_j}\}_{j \in \mathbb{N}}$, which one can effectively enumerate by means of the natural numbers. Then, by the result of the lemma, since for each $j \in \mathbb{N}$, $\mathbb{F}_{R_j} \in \mathbb{F}_R$, $C(\mathbb{F}_{R_j})$ is a recursive set of n-tuples of R-indices in $\mathbb{F}_{R_j}$, and therefore $\{C(\mathbb{F}_{R_j})\}_{j \in \mathbb{N}}$ comprises a sub-family of $\mathbb{F}_R$ which one can effectively enumerate by means of the natural numbers. Since $\mathbb{N}$ is a recursively enumerable set (by item iii on page 14 it is the range of the identity mapping which is primitive recursive) both $\{\mathbb{F}_{R_j}\}_{j \in \mathbb{N}}$ and $\{C(\mathbb{F}_{R_j})\}_{j \in \mathbb{N}}$ may be regarded as recursively enumerable families of elements in $\mathbb{F}_R$. What this means in turn is that given an effective listing of sets of alternatives that are themselves effectively computable sets comprised in turn of n-tuples of R-indices of effectively computable numbers, i.e. recursive real numbers, a recursive rational choice will generate an effective listing of the choices made from the given collection of alternatives. Moreover, the choices themselves will be effectively computable sets of n-tuples of R-indi[illegible] of effectively computable numbers.

If we return briefly to the discussion of Church's Thesis given in section II, the above feature of a recursive rational choice function yields a further interpretation in terms of equivalences (ii) (iii) and (v) of Church's Thesis given on pages 13 and 14 of section II, which we restate:

(ii) A Partial Recursive Function is equivalent to a Turing machine that sometimes halts, i.e. when it is defined.

(iii) A Recursive Function is equivalent to a Turing machine that always halts.

(v) A Recursive Set is equivalent to a set whose characteristic function is recursive.

Then in terms of the equivalences, if we substitute the attribute recursive by the notion of effectively computable by a Turing machine, the paradigm of choice behavior that recursive rational choice describes is one that in principle is comprised of mathematical concepts and cperations that can be performed by an ideal device of artificial intelligence:[21/]

(a) Elements in $R(X)$ are machine computable in principle by means of the recursive real numbers.

(b) Elements in $\mathbb{F}_R$ are machine computable in principle being recursive sets of elements in $R(X)$.

(c) Preferences on $R(X) \times R(X)$ are machine computable in principle by means of a representation that can be extended to be partially recursive.[22/]

(d) Choices made from members of $\mathbb{F}_R$ are machine computable in principle since by Lemma IV.1 such choice are themselves elements of $\mathbb{F}_R$.

(e) By Lemma V.1 if we can effectively list by machine computable procedures sets of alternatives that are each individually machine computable, then we can effectively list[23/] by machine computable procedures, the choices made from the sets of alternatives, which by item (d) will be themselves machine computable in principle.

We may in fact, take items (a) - (e) as a suitable definition for the paradigm of rational choice to be <u>recursively representable</u> on a recursive space of alternatives.

<u>Definition V</u>: A choice function $C : \mathbb{F}_R \to P(R(X))$, on a recursive space of alternatives $\langle R(X), \mathbb{F}_R \rangle$, is recursively representable if items (a) - (e) are met.

V.2 <u>Proposition</u>: A recursive rational choice on a recursive space of alternatives is recursively representable.

<u>Proof</u>: By construction of $\langle R(X), \mathbb{F}_R \rangle$ and Lemma IV.1 in application when required. Q.E.D.

By an obvious comparison, we see that recursiveness and hence, machine computable in principle, replaces, in the number theoretic setting, the concept of representability derived in terms of the wff.s of the elementary formal systems discussed in section II. However, an analagous distinction must be made between the assertion that a mathematical construction can obtain a recursive representation and, as we shall see, the somewhat stronger assertion that a given recursive representation of a mathematical construction has a <u>recursive realization</u>. To say that a recursive realization can obtain is to say that there is a computational procedure, in the present context, a Turing machine, that will actually perform the mathematical operation of the entire model, given that the operations that mathematically describe the model can themselves be represented by computational procedures. To make this latter notion precise, we require the definition that follows, of recursive solvability, which we will in turn employ to define recursive realizability.

Definition VI: A set A, of natural numbers, is said to be recursively solvable if and only if there is a general recursive function $\Phi : \mathbb{N} \to \mathbb{N}$ such that:

$$\phi(n) = 1 \quad \text{if} \quad n \in A$$

$$\phi(n) = 0 \quad \text{if} \quad n \notin A$$

If A is not recursively solvable, then A is said to be recursively unsolvable.

By a further application of Church's Thesis contained in equivalence (v) if a set is recursively solvable, then it is obviously a recursive set, thus having a characteristic function that is a recursive set. Alternatively phrased, we may say that a set is recursively solvable if it is possible to determine by machine computable procedures the membership of the set unambiguously, recalling equivalence (iii) above. We can now obtain a definition of the recursive realization of rational choice in terms of the solvability of its graph.

Definition VII: The graph of a recursive rational choice function C, denoted as graph (C), in the domain $\{\mathbb{F}_{R_j}\}_{j \in \mathbb{N}} \subseteq \mathbb{F}_R$, and co-domain

$\langle \mathbb{F}_{R_j} \rangle\}_{j \in \mathbb{N}}$ is the collection of pairs $\langle \mathbb{F}_{R_j}, C(\mathbb{F}_{R_j})\rangle$ indexed by $j \in \mathbb{N}$, viewed as a subset of $P(M(\mathbb{R}^n)) \times P(M(\mathbb{R}^n))$, the product space of subsets of $M(\mathbb{R}^n)$.[24] We will say that graph (C) has full domain if for some $K \in \mathbb{N}$, and all pairs $i \neq j > K$, $\mathbb{F}_{R_j} \Delta \mathbb{F}_{R_j} \neq \phi$.[25]

<u>Definition VIII</u>: A recursive choice on $\langle R(X), \mathbb{F}_R\rangle$ that is also recursive rational is said to be recursively realizable if and only if for any choice of full domain $\{\mathbb{F}_{R_j}\}_{j \in \mathbb{N}} \subseteq \mathbb{F}_R$, graph (C) is a recursive set of the product space of subsets of $M(\mathbb{R}^n)$, $P(M(\mathbb{R}^m) \times P(M(\mathbb{R}^m))$.

The meaning of Definition VIII is simply that from among the possible members of graph (C), which we may view as being pairs of sequences in $\mathbb{F}_R \times \mathbb{F}_R$, and thus having components that are effectively computable, it is possible to unambiguously describe the membership of graph (C) by effectively computable means. By way of Definition VI, Definition VIII yields, by way of this last observation, an immediate proposition.

V.3 <u>Proposition</u>: A recursive choice on $\langle R(X), \mathbb{F}_R\rangle$ that is recursive rational is recursively realizable if and only if for any choice of full domain $\{\mathbb{F}_{R_j}\}_{j \in \mathbb{N}} \subseteq \mathbb{F}_R$, the graph of C is recursively solvable.[26]

The principal result of the paper, to which we now turn, serves to demonstrate the distinction, within our recursive setting, between the notion of <u>effectively computable representation</u> of rational choice, and the somewhat stronger notion of an <u>effectively computable realization</u>[27]

of rational choice as set forth in definitions VI, VII, and VIII. The means we employ to demonstrate this distinction, is to prove that for any "reasonable choice" of domain,[28/] a non-trivial[29/] recursive rational choice has an unsolvable graph, and thus by Proposition IV.2 cannot be recursively realized. The argument of the proof can be summarized as follows. We first show that statements about n-tuples of R-indices in $M(\mathbb{R}^n)$ can be reduced to statements about single R-indices in $M(\mathbb{R})$ by a suitable choice of notation system. In the new notation system, for which subsets of $M(\mathbb{R}^n)$ may be regarded as subsets of $M(\mathbb{R})$, we then show that there does not exist a predicate $\Psi : \mathbb{N} \to \mathbb{N}$. The restriction of which Ψ/S, for S equal to the set of R-indices of $\{C(\mathbb{F}_{R_j})\}_{j \in \mathbb{N}}$ that comprise the co-domain of graph (C) for choice of full domain $\{\mathbb{F}_{R_j}\}_{j \in \mathbb{N}} \subseteq \mathbb{F}_R$, can have a graph that is itself recursive. The non-recursiveness of graph (Ψ/S) is shown to follow from the fact that by means of the Kleene-Mostowski classification of subsets of the natural numbers, graph (Ψ/S) is not a $\Sigma_o - \pi_o$ set.[30/]

It then follows that if graph (Ψ/S) is not $\Sigma_o - \pi_o$, then there can be no recursive function realizing graph (Ψ/S) where the recursive function $\Phi : \mathbb{N} \to \mathbb{N}$ realizes graph (Ψ/S) if and only if:

$$\begin{cases} \Phi(n) = 1 & \text{if } n \in \text{graph } (\Psi/S) \\ \Phi(n) = 0 & \text{if } n \notin \text{graph } (\Psi/S) \end{cases}$$

By Definition V, therefore, graph (Ψ/S) is recursively unsolvable, which in turn violates a necessary condition that graph (C) be recursively solvable.

It is worth remarking that this procedure of proof thus presumes that the decision for whether an element of graph (Ψ/S) is, in fact, an element of $\mathbb{N}$, has been solved by way of the notation system, and our concern is then for a given element of $\mathbb{N}$; is there an effective procedure to determine whether or not Ψ/S is satisfied by that element.

The following is a statement of the theorem we next proceed to demonstrate.

V.4 Theorem: Allow $<R(X), \mathbb{F}_R>$ to be a recursive space of alternatives derived from the recursive metric space of $\mathbb{R}^n$, $M(\mathbb{R}^n)$, for $R(X)$ the recursive representation of a compact, convex subset of $\mathbb{R}^n_+$. Let $C : \mathbb{F}_R \to \mathbb{F}_R$ be a non-trivial recursive rational choice on $<R(X), \mathbb{F}_R>$ and select from the class of sequences $(\mathbb{F}_R)^{\mathbb{N}}$, any element $\{\mathbb{F}_{R_j}\}_{j\in\mathbb{N}} \subseteq \mathbb{F}_R$ that comprises a full domain for graph $(C) \subseteq \mathbb{F}_R \times \mathbb{F}_R$. Then, per fixed selection of $\{\mathbb{F}_{R_j}\}_{j\in\mathbb{N}}$, (1) the co-domain of graph (C) is non-recursive, which implies (2) graph (C) is recursively unsolvable and therefore (3) the choice function C cannot be recursively realized.

Proof: We begin by making the observation that it is possible to make a straight forward correspondence from the notation system $(\mathbb{N}(R^n), \sim_{\mathbb{N}(R^n)})$ which we have employed to construct the recursive metric space $M(\mathbb{R}^n)$ to a notation system $(\hat{\mathbb{N}}(R), \sim_{\hat{\mathbb{N}}(R^n)})$ having the same equivalence relation $\sim_{\mathbb{N}(R^n)}$, and where $\hat{\mathbb{N}}(R)$ is the set of R-indices for the recursive real numbers under an appropriate choice of Gödel numbering. Before stating this as a proposition however, let us remark that the technique of Gödel numberings employed to assign natural numbers to the partial recursive functions is actually a method to encode the defining equations for the function arithemtically by means of the products of prime numbers raised to exponential powers of odd integers (as explained in Kleene [1950] p.206, p.284 and pp.223-231, in particular). Furthermore, Gödel numberings are not unique (as explained in Rogers [1967] sections 7.8 and 1.10) and those properties that are known to be recursively invariant, such as unsolvability, hold under all admissible codings (Rogers [1967], p.95).

V.4.1 <u>Proposition</u>: For an appropriate choice of Gödel numbering, there exists a notation $\hat{\mathbb{N}}(\mathbb{R})$ of R-indices for the recursive real numbers for which $\mathbb{N}(R^n) \subseteq \hat{\mathbb{N}}(R)$.

<u>Proof</u>: In the prime factorization representation (Kleene [1950], p.230) rename the n^{th} prime as 2 for n the dimension of $\mathbb{R}^n_+$ and for choice of Gödel numbering, take that enumeration of the initial primes co-finite with the initial segment $\{P_o, \ldots P_{n-1}\}$, with the n^{th} prime renamed. Generate next, a coding for which the assignment of the R-indices of the set of numbers $\{P_o, \ldots P_{n-1}\}$ is the value of the prime it indices. By Proposition IV.1 this can be performed by virtue of the fact that the prime numbers are elements in Q . Let $\hat{\mathbb{N}}(R)$ denote the entire set of R-indices generated in the choice of Gödel numbering. Then from the fact that there exists partial recursive functions to determine the algebraic operations of a field (Moschovakis [1965] Lemma 4) and that constructive proofs exist for the equivalence:

$$\lim_{n\to\infty} (\alpha_n \times \beta_n) = \lim_{n\to\infty} \alpha_n \times \lim_{n\to\infty} \beta_n$$

by way of Rice [1954] Theorem 4, a notation system for the recursive real numbers is closed under multiplication. Define next, for $n(f)$ an n-tuple of elements of $\hat{\mathbb{N}}(R)$, the mapping $\gamma(n(f))$ derived from the construction of $\mathbb{N}(R^n)$ given as

$$\mathbb{N}(R^n) = \{<n(f_1),\ldots, n(f_n)> \; : \; n(f_1),\ldots, n(f_n) \in \hat{\mathbb{N}}(R)\}$$

by

$$\gamma(n(f)) = <n(f_1),\ldots, n(f_n)> = \prod_{j=0}^{n-1} P_j^{n(f_{j+1})} \quad .$$

From the choice of notation, we see immediately that $\gamma : (\hat{\mathbb{N}}(R))^n \to \hat{\mathbb{N}}(R)$ is injective, i.e. one to one and therefore that $\mathbb{N}(R^n) \subseteq \hat{\mathbb{N}}(R)$ in a well defined manner. Q.E.D.

The effect of the proposition is that we may consider n tuples of R-indices in $M(\mathbb{R}^n)$ as R-indices in the notation $\hat{\mathbb{N}}(R)$ of the metric space:

$$\hat{M}(\mathbb{R}) = <(\hat{\mathbb{N}}(R), \sim_{\hat{\mathbb{N}}(R)}), D_R>$$

which is in fact a recursive metric space of $\mathbb{R}$ 31/.

By virtue of the mapping g, for each point in $M(\mathbb{R}^n)$, there is a unique point in $\hat{M}(\mathbb{R})$ that corresponds to its image under g. Therefore, sets in $M(R^n)$ will correspond to sets in $\hat{M}(\mathbb{R})$ in a well defined manner. In particular, the set $\hat{R}(X)$ and the members in the class $\mathbb{F}_R$

will have well defined images in $\hat{M}(\mathbb{R})$[32/], i.e. for $\mathbb{F}_j \in \mathbb{F}_R$,
$\gamma(\mathbb{F}_j) = \{\alpha \in \hat{M}(\mathbb{R}) : \exists \beta \in \mathbb{F}_j \subseteq M(\mathbb{R}^n) \therefore \gamma(\beta) = \alpha\}$ and etc.

We will require the following two definitions in terms of items given in Definitions I-III of Appendix I, before the statement of the next lemma.

Definition IX: A relation is potentially K-enumerable or potentially anti-K-enumerable, if it has an extension[33/] which is K-enumerable or anti-K-enumerable.[34/]

Definition X: A relation is potentially-partially-K-enumerable or potentially-partially-anti-K-enumerable if it can be extended to a relation that is partially-K-enumerable or partially-anti-K-enumerable.[35/]

We will not introduce the necessary formalism to provide a proof of the next lemma, which can be obtained by means of a reformulation of Shapiro's Extension Theorem ([1956] Theorem I.6), which one can view as a means to extend the Kleene-Mostowski Hierarchy in Appendix I to the domain of the partial recursive predicates.

V.4.2 Lemma: Allow $\bigcup_{j \in \mathbb{N}} P(\mathbb{N}^j)$ to denote the set of all relations on $\mathbb{N}$, and assume that $\Gamma \in \bigcup_{j \in \mathbb{N}} P(\mathbb{N}^j)$ is the restriction of some $\Psi \in \bigcup_{j \in \mathbb{N}} P(\mathbb{N}^j)$ where Ψ is potentially partially Σ_K or π_K. Then Γ is the restriction of some $\Phi \in \bigcup_{j \in \mathbb{N}} P(\mathbb{N}^j)$ where Φ is Σ_K or π_K.

To commence the proof of assertion (1) of the theorem, select from $(\mathbb{F}_R)^{\mathbb{N}}$ an element $\{\mathbb{F}_{R_j}\}_{j\in\mathbb{N}}$ for which $\exists K \in \mathbb{N}$ s.t. $\forall ij > K$ $\mathbb{F}_{R_j} \Delta \mathbb{F}_{R_i} \neq \phi$, and consider the image of graph (C) under the mapping g which consists of a pair of sequences $<\{\gamma(\mathbb{F}_{R_j})\}_{j\in\mathbb{N}}, \{\gamma(C(\mathbb{F}_{R_j}))\}_{j\in\mathbb{N}}>$ in the product space : $P(\hat{M}(R)) \times P(\hat{M}(R))$. By means of Theorem VIII of Rogers [1967] (Sec. 5.3 p.65), the elements of the sequence $\{\gamma(C(\mathbb{F}_{R_j}))\}_{j\in\mathbb{N}}$ forming the co-domain of the image of graph (C) under the mapping g are recursive sets of R-indices in $\hat{M}(\mathbb{R})$[36/] by virtue that each $\mathbb{F}_{R_j} \in \mathbb{F}_R$. From the fact that C is non-trivial and thus on $\{\mathbb{F}_{R_j}\}_{j\in\mathbb{N}}$, which was chosen full, $\{C(\mathbb{F}_{R_j})\}_{j\in\mathbb{N}}$ is not the null sequence; neither is $\{\gamma(C(\mathbb{F}_{R_j}))\}_{j\in\mathbb{N}}$, therefore. Again, from the fullness of the domain, for $j > K$, we obtain $\bigcup_{j>K} \gamma(C(\mathbb{F}_{R_j})) \neq \phi$, and we can choose from the indices in excess of K, a finite subcollection $\{j_1,\ldots j_m\}$ for which $\bigcup_{i=1}^{m} \gamma(C(\mathbb{F}_{R_{ji}})) \neq \phi$. Furthermore, per choice of subcollection, since the class of recursive sets is closed under finite unions (Theorem XIV of Rogers [1967] Sec. 5.5 p.68), $\bigcup_{i=1}^{m} \gamma(C(\mathbb{F}_{R_{ji}}))$ is itself a recursive set of R-indices in $\hat{M}(\mathbb{R})$[37/].

We require the use of the following lemma.

V.4.3 Lemma: The image, under the mapping g of the codomain of a non-trivial recursive rational choice with full domain, is contained in a bounded interval in $\hat{M}(\mathbb{R})$.

<u>Proof</u>: From the fact that the original space of alternatives X is compact and convex in $\mathbb{R}^n_+$, and therefore bounded, elements in $R(X)$ are then component-wise bounded by an element in $M(\mathbb{R}^n)$, say the element $(n(c_1),\ldots, n(c_n))$ for c_i = constant, $i = 1,\ldots, n$, in the order induced on $M(R^n)$ by the strong vectorial order on $\mathbb{R}^n$, i.e. for χ, $y \in \mathbb{R}^n$, $\chi \leqq y$ if and only if $\chi_i \leqq y_i$ for all $i = 1,\ldots, n$. Then in the notation $(\hat{\mathbb{N}}(R), \sim_{\hat{\mathbb{N}}(R)})$, the image of $R(X)$ is bounded by the element $\overset{n-1}{\underset{j=0}{\pi}} P_j^{n(c_{j+1})}$ in a recursively induced order on the notation agreeing with the natural order on $\mathbb{R}$.[38/] We may then regard $\gamma(R(X))$ as a set of R-indices of recursive real numbers in the interval $[0, R_K]$ where R_K is the recursive real number whose equivalence class is named by $\overset{n-1}{\underset{j=0}{\pi}} P_j^{n(c_{j+1})}$ in $\hat{\mathbb{N}}(R)$. The lemma follows from the trivial observation that for any $\mathbb{F}_j \in \mathbb{F}_R$, $C(\mathbb{F}_j) \subseteq R(X)$ and therefore, $\gamma(C(\mathbb{F}_j)) \subseteq \gamma(R(X))$. Q.E.D.

<u>Definition XI</u>: Allow S to be a collection of recursive real numbers, and define the relation $I_S : \mathbb{N} \to \{1,0\}$ as:

$I_S(\alpha) = 1$ iff. α is an R-index of $\hat{M}(\mathbb{R})$ for some $\chi \in S$,

$I_S(\alpha) = 0$ otherwise,

for subsets $\{S_1, \ldots, S_\kappa\}$ of S we can define

$$I \bigcup_{j=1}^{\kappa} S_j = \bigvee_{j=1}^{\kappa} IS_j \text{ , and similarly for}$$

$$I \bigcap_{j=1}^{\kappa} S_j = \bigwedge_{j=1}^{\kappa} IS_j \ .$$

<u>Definition XII</u>: A set of recursive real numbers, S, is said to be effectively indicated in $\hat{M}(\mathbb{R})$ if

(1) $\forall \alpha \in \mathbb{N} \quad I_S(\alpha) \neq 0$.

(2) The set $T = \{\alpha \in \hat{M}(R) : I_S(\alpha) = 0\}$ is such that

(i) $\forall \alpha \in \mathbb{N} \ I_{\bar{S} \cap T}(\alpha) \neq 0$,

(ii) $I^{-1}_{S \cap T}(1) \subseteq I^{-1}_{(\bar{S}+\{q\}) \cap T}$ for q any positive recursive real number.

What the definition says is that if we can effectively indicate a set of recursive real numbers in $\hat{M}(\mathbb{R})$, we can first effectively identify a non-empty set of R-indices of the set in $\hat{M}(\mathbb{R})$, for which there is also the means to determine the indices of its complement unambiguously, where the indication of the complement is, in a sense, "well contained", in that the indication is not affected by positive translations. The somewhat ad hoc flavor of definition XII can be dispelled by the realization that among the sets of recursive real numbers that are effectively indicated are the intervals, $[0, R_K]$, for R_K a positive recursive real number and from which the following proposition derives by straight forward verification of the definition.

V.4.4 Proposition: Let S be a set of recursive real numbers of the form: $[0, R_K)$ or $[0, R_K]$ for R_K a recursive real number in $\mathbb{R}_+ - \{0\}$. Then S is effectively indicated in $\hat{M}(\mathbb{R})$ if and only if, in the notation $(\hat{\mathbb{N}}(R), \sim_{\mathbb{N}(R)})$ the intervals $[0, \alpha(R_K))$ or $[0, \alpha(R_K)]$ are effectively indicated in $\hat{M}(\mathbb{R})$.

Proof: By Proposition IV.1, the rationals are recursive real numbers, from which for $\alpha(\chi)$ and R-index of a rational element of S, $I_S(\alpha(\chi)) = 1$. The verification of items 2(i) and 2(ii) for the set T is obtained by selecting a rational element y in the complement of S in $\mathbb{R}_+ - \{0\}$ whose R-index, $\alpha(y)$, satisfies $I_{\bar{S} \cap T}(\alpha(y)) = 1$ on the subspace of $\hat{M}(\mathbb{R})$, $\hat{M}(\mathbb{R}_+)$. This establishes the sufficiency of the proposition.

To obtain the necessity of the proposition it will suffice to show that the recursive real numbers are order isomorphic to their indices. To this end, we may observe that, in the manner of Lemma 5 of Moschovakis [1967], p.57, the partial recursive function defined as

$$\text{less}\ (\alpha(\chi),\ \alpha(y)) \equiv \chi < y$$

for χ and y recursive real numbers, and $\alpha(\chi)$ and $\alpha(y)$ their respective R-indices in $(\hat{\mathbb{N}}(R), \sim_{\hat{\mathbb{N}}(R)})$ generates a recursive order isomorphism by means of the scheme:

$$\text{less}\,(\alpha(\chi),\,\alpha(y)) \simeq \mu t[T_1(\alpha(\chi),\,(t)_o,\,(t)_1) \wedge T_1(\alpha(y),\,(t)_o,\,(t)_2) \wedge r(\cup((t)_2) - r(\cup(t)_1) > 2^{-(t)_o+1}]$$

where $r(\cup((t)))$ is the same function we have employed in section IV, p.23, to characterize the R-indices. The function $\text{less}(\alpha(\chi),\,\alpha(y))$ can be viewed as an algorithm when applied to R-indices of recursive real numbers, and will terminate if and only if $\chi < y$ in the natural order on $\mathbb{R}$. Q.E.D.

It now follows that per choice of finite subcollection $\{j_1, \ldots, j_m\}$ of indices for which $j_i > K$ $i = 1, \ldots, m$ and thus $\bigcup_{i=1}^{m} \gamma(C(\mathbb{F}_{ji})) \neq \phi$, from the recursiveness of each $\gamma(C(\mathbb{F}_{ji}))$ as a set of R-indices in $\hat{M}(\mathbb{R})$, $\bigcup_{i=1}^{m} \gamma(C(\mathbb{F}_{ji})) \subseteq [0, \alpha(R_K)]$, by Lemma V.4.3, where $\alpha(R_K)$ is the R-index of the bound derived for $\gamma(R(\chi))$. Then by Proposition V.4.4, $\bigcup_{i=1}^{m} \gamma(C(\mathbb{F}_{ji}))$ is a recursive subset of an effectively indicated interval in $\hat{M}(\mathbb{R})$. We show next, that this leads to a contradiction by way of the following concepts:

Definition XIII: Allow Φ_1 and Φ_2 to be elements of $\bigcup_{j \in \mathbb{N}} P(\mathbb{N}^j)$ such that Φ_1 is n-*ary* and Φ_2 is m-*ary*. Then Φ_1 is strongly reducible to Φ_2, written $\Phi_1 << \Phi_2$ if there are partial recursive n-*ary* functions $f_1, \ldots, f_n$ for which $\Phi_1 \quad \lambda(\chi_1, \ldots, \chi_n) \quad \Phi_2(f_1(\chi_1, \ldots, \chi_n), \ldots, f_m(\chi_1, \ldots, \chi_n))$[39].

<u>Definition XIV</u>: In terms of the Kleene-Mostowski Hierarchy[40/], a relation Φ is strictly π_K if Φ itself is π_K and for any π_K relation Λ is such that $\Lambda << \Phi$.

To obtain item (1) of the theorem, assume that the co-domain of graph $(\mathcal{C})$ was in fact recursive, and then that by way of the coding function $\gamma : \mathbb{N}^n \to \mathbb{N}$, $\bigcup_{j \in \mathbb{N}} \gamma(C(\mathbb{F}_{R_j}))$ were recursive. Consider next graph $(\bigcup_{i=1}^{m} \gamma(C(\mathbb{F}_{R_j})))$ which is identical to the function I_S defined in Definition XI with $S = \bigcup_{i=1}^{m} \gamma(C(\mathbb{F}_{R_{ji}}))$. Then since graph $(\bigcup_{i=1}^{m} \gamma(C(\mathbb{F}_{R_{ji}})) \subset$ graph $(\bigcup_{j \in \mathbb{N}} \gamma(C(\mathbb{F}_{R_j})))$, one sees that graph $(\bigcup_{i=1}^{m} \gamma(C(\mathbb{F}_{R_{ji}})))$ is a restriction of graph $(\bigcup_{j \in \mathbb{N}} \gamma(C(\mathbb{F}_{R_j})))$. Since $\bigcup_{i=1}^{m} \gamma(C(\mathbb{F}_{R_{ji}})) \subseteq [0,\alpha(R_K)]$ and by choice of $\{j_1,\ldots, j_m\}$, $\bigcup_{i=1}^{m} \gamma(C(\mathbb{F}_{R_{ji}})) \neq \phi$, $\bigcup_{i=1}^{m} \gamma(C(\mathbb{F}_{R_{ji}}))$ is effectively indicated by the effective indication of $[0,\alpha(R_K)]$ given by Proposition V.4.4 by way of items 2(i) and 2(ii) of Definition XII and the fact that $\mathbb{N} - [0,\alpha(R_K)] \subseteq \mathbb{N} - \bigcup_{i=1}^{m} \gamma(C(\mathbb{F}_{R_{ji}}))$. For $S = \bigcup_{i=1}^{m} \gamma(C(\mathbb{F}_{R_{ji}}))$, if it were true that graph $(\bigcup_{i=1}^{m} \gamma(C(\mathbb{F}_{R_{ji}}))$ were recursive, then the function I_S would be recursive, as its characteristic function from equivalence (v) on p.13 of section II; but as in this instance, I_S is effectively indicated, by Theorem II.10 and Theorem II.2 of Shapiro, [1956] pp.291-294 since I_S is a recursive set of natural numbers, the relation that portrays I_S on $\mathbb{N} \times \{1,0\}$ must be strictly Σ_2. It follows, then, by Kleene's Hierarchy Theorem ([1950], pp.283-284), that the function I_S cannot be Σ_1 within the Kleene-Mostowski Hierarchy of Appendix I, and thus I_S cannot be Σ_o-π_o by Post's Theorem ([1954], pp.283), by means of which it would have to be $\Sigma_1 \cap \pi_1$. Further, if I_S is strictly Σ_2, then graph $(\bigcup_{i=1}^{m} \gamma(C(\mathbb{F}_{R_{ji}})))$ is not potentially recursive, and hence if it cannot be extended to be recursive, by Lemma V.2 it cannot be the restriction of any recursive graph, and in particular, of a recursive graph $(\bigcup_{j\in\mathbb{N}} \gamma(C(\mathbb{F}_{R_{ji}})))$. But in that case, if graph $(\bigcup_{j\in\mathbb{N}} \gamma(C(\mathbb{F}_{R_j})))$ is not recursive, since graph $(\bigcup_{j\in\mathbb{N}} \gamma(C(\mathbb{F}_{R_j}))) = I_S$ with $S = \bigcup_{j\in\mathbb{N}} \gamma(C(\mathbb{F}_{R_j}))$ is the characteristic function of $\bigcup_{j\in\mathbb{N}} \gamma(C(\mathbb{F}_{R_j}))$, then $\bigcup_{j\in\mathbb{N}} \gamma(C(\mathbb{F}_{R_j}))$ is not recursive. The latter item is in contradiction to the assumption that the co-domain of C is recursive, establishing item (1) of the theorem.

We can now obtain item (2) of the theorem by observing that if the co-domain of graph (C) is not recursive, and thus not $\Sigma_o - \pi_o$ in the hierarchy, it cannot be potentially $\Sigma_o - \pi_o$. Then by another application of Lemma V.2, if we make the assumption that graph (C) is recursive, a contradiction arises from the simple fact that the co-domain of graph (C) is a restriction of graph (C).

Item (3) of the Theorem follows from item (2) by way of Proposition V.3.

This concludes the proof of Theorem V.4. Q.E.D.

VI. Further Discussion and a Specific Example.

Before turning our attention to a specific example, a few comments on the interpretation of the result thet we have just demonstrated are in order: (1) The first item to mention is that if rationality is constrained by effective computability, within the frameowrk of recursive functions, as a representation of effective computability, the notion of realization is stronger than the notion of representation in the sense that the latter does not imply the former. Which is to say, that an effective listing of effectively computable sets of alternatives, as can be obtained from a recursive rational choice function does not imply an effectively computable procedure, for the correspondence between the two listings, viewed as the graph of the choice function. This item can be given a further interpretation in an economic context by means of considering the traditional concept of a demand correspondence in a recursive setting. Alternatively, we can express this item in terms of types of computing devices. A recursive rational choice function on a recursive space of alternatives, in a sense[41/], represents a collection of types of computing devices: (i) There is a computing device to generate elements of the space of alternatives, viewed as recursive real numbers; (ii) There is a computing device to generate sets of alternatives in the space, viewed as recursive subsets of the space; (iii) There is a computing device to generate choices from the sets of alternatives, viewed as a recursive order-preserving function of preferences defined on the space of alternatives; (iv) There is a computing device to generate sets of chosen alternatives, viewed as recursive subsets of sets of alternatives; (v) There is a computing device to effectively list sets of alternatives, viewed as the domain of the graph of a recursive rational choice function; (vi) There is a computing device to effectively list sets of chosen alternatives, viewed as the co-domain of the graph of a recursive rational choice function. Types (i)-(vi) of computing devices comprise, by way of Definition V of Section V , a recursive representation of rational choice. However, the theorem we have just demonstrated says that there cannot be a computing device, that when presented with the information of the devices (i)-(vi), will unambiguously, and correctly perform the task of associating, for a given listing by the devices (v) and (vi), recursive sets of alternatives with the recursive sets chosen from them. This is the meaning of the theorem's result that the graph of a recursive rational choice function is not recursively solvable. (2) The second item worth mentioning is that the definition we have employed of recursive realization is by no means absolute, although it seems perhaps the most natural one to consider on intuitive grounds. Another version of recursive realization might reasonably consider that the existence of a machine to perform a correct association of sets of alternatives, in an effective listing, with the sets of choices from those alternatives, in an effective listing, as basic to a recursive representation of natural choice and from this, attempt to arrive at those recursive structures that give rise to the representation in this sense. This version would then consider a Turing machine realization in the sense of Proposition V.3 , as a recursive representation of rational choice, and the existence of the requisite class of relations, not necessarily recursive, and not necessarily rational in Richter's sense, that are implied by a representation in this sense as a recursive realization. A version of recursive realization in this sense would then be relative to the complexity of the required computation by the machine to represent rational choice, differing relations being required by differing levels of complexity[42/]. (3) The second item brings us to a third consideration of the interpretation of the result. It would be an incorrect inference from the result of the theorem to say that a recursive rational choice function as a paradigm of economic behavior does not exist. The first reason being the discussion provided in the second item above. The second reason is that we may, and should, view the theorem as a statement of the amount of information mathematically contained in recursively enumerable families of recursive sets of alternatives as being insufficient to effectively determine a proper correspondence to a recursive enumeration of the choices from those alternatives by the class of computational procedures representable by Turing machines. This is not to say that within the notation of the space of alternatives provided by the natural numbers that there may be in fact other sets of natural numbers that do in fact provide enough mathematical information for that task to be performed. However, this consideration leads us to topics found within the theory of the relative solvability of recursive structures, which we discuss in a forthcoming paper[43/].

The example we now provide illustrates the significance of Theorem V.4 in the context of a rational choice function definition:

(1) $X = \mathbb{N}$

(2) $\gtrsim : X \times X \to \{1,0\}$ such that

for $n_1, n_2 \in X$, $n_1 \gtrsim n_2 \iff n_1 - n_2 > 0$

(3) $f : X \to \mathbb{N}$ for f = the identity function $\mathbb{N}$

(4) $\mathbb{F}_R = \{A \in P(\mathbb{N})$ and A is finite $\}$

(5) $C: \mathbb{F}_R \to P(\mathbb{N})$ such that

$\forall A \in \mathbb{F}_R \quad C(A) = \{n \in A : \forall m \in A \; (f(n) \geqq f(m))\}$

The above items have the following interpretation. The set X we take as the space of alternatives, which is simply the set of natural numbers. The binary relation $\gtrsim$ we take as a preference order and is merely the natural strict order on the natural numbers. The function f we take as an order preserving representation of the preference order $\gtrsim$. The set $\mathbb{F}_R$ is the collection of feasible subsets of alternatives, and is merely the class of all finite subsets of $\mathbb{N}$. The function $C: \mathbb{F}_R \to P(\mathbb{N})$ we term a choice on $< X, \mathbb{F}_R >$ and is defined to select the order maximal element from a member of the family $\mathbb{F}_R$. Then, from the fact that any finite set is recursive (Rogers [1967], 5.1, p.57), and the fact that the identity function on the natural numbers is primitive recursive (section III item (c) p.13), and thus partial recursive, the interpretation of items (1) - (5) of our present example is seen to satisfy Df.s I - IV of Section V. If we then regard $\mathbb{N}$ as its own notation,[44] in a sense, it becomes its own recursive metric space, and items (1) - (5) then comprise a recursive rational choice on $<X, \mathbb{F}_R>$. Let us now examine the recursive realization of the example. We first check to see that the example is recursively representable.

Choose from $\mathbb{F}_R$ in this case the following class of subsets:

$$\left\{ \{1\}, \ \{1,2\}, \ \{1,2,3\}, \ \{1,2,3,4\}, \ \ldots, \{1,\ldots, n\}, \ldots \right\}$$

which is simply the listing, in order of occurence, of the co-final segments of $\mathbb{N}$ which we will denote as $\{\mathbb{F}_{R_j}\}_{j \in \mathbb{N}}$. Clearly, for any $j \in \mathbb{N}$, we know without question that $\mathbb{F}_{R_j} = \{1,\ldots,j\}$ and it can be said that the class $\{\mathbb{F}_{R_j}\}_{j \in \mathbb{N}}$ is determined effectively by its listing on $\mathbb{N}$, or that it is effectively listed. Additionally, it can be seen that for all distinct pairs of indices i, j members of $\{\mathbb{F}_{R_j}\}_{j \in \mathbb{N}}$ are such that $\mathbb{F}_{R_j} \Delta \mathbb{F}_{R_i} \neq \phi$, and thus $\{\mathbb{F}_{R_j}\}_{j \in \mathbb{N}}$ comprises a full domain in the sense of Df. VII of Section V, for which the corresponding class of elements chosen becomes

$$\left\{ C(\{1\}), \ C(\{1,2\}), \ C(\{1,2,3\}), \ C(\{1,2,3,4\}), \ \ldots, \ C(\{1,\ldots,n\}) \right\}$$

which is easily seen to be simply the listing, in order of occurence, of the singleton sets of $\mathbb{N}$. Then, for any $j \in \mathbb{N}$, we know without question that for $\mathbb{F}_{R_j} = \{1,\ldots j\}$, $C(\mathbb{F}_{R_j}) = \{j\}$ by definition of the choice function. An inspection of the foregoing then reveals that items (a) - (e) of Df. V of Section V are satisfied and that therefore the choice function C of the example is recursively representable.

Consider next the graph of the choice function, which is comprised of pairs $<\{1,\ldots j\},\{j\}>$ in $\mathbb{F}_R \times \mathbb{F}_R$ indexed by $j \in \mathbb{N}$, the co-domain of which, $\{j\}_{j\in\mathbb{N}}$ is an effective listing of singleton sets of $\mathbb{N}$. To see that the co-domain of graph C in this instance is non-recursive observe first that $\{j\}_{j\in\mathbb{N}}$ is in fact a correspondence of the form

$$\mathcal{D}^{`}: \mathbb{N} \rightarrow \mathbb{F}_R \text{ such that}$$

$$\forall j \in \mathbb{N} \quad \mathcal{D}(j) = C(\mathbb{F}_{R_j})$$

Then, by definition of the choice of domain $\{\mathbb{F}_{R_j}\}_{j\in\mathbb{N}}$ and the choice of f as the identity function on $\mathbb{N}$ to represent the preference order $\geq$, $C(\mathbb{F}_{R_j})$ has the explicit form of:

$$C(\mathbb{F}_{R_j}) = \mathit{arg}\max_j \; f|_{\mathbb{F}_{R_j}}$$

and thus that

$$\mathcal{D}(j) = \mathit{arg}\max_j \; f|_{\mathbb{F}_{R_j}}$$

which reads the value of $\mathbb{F}_{R_j}$ that maximizes the function f when restricted to the set $\mathbb{F}_{R_j}$. Obviously, by the equivalences, the co-domain of graph C in our example would be recursive if and only if the correspondence $\mathcal{D} : \mathbb{N} \rightarrow \mathbb{F}_R$ were a recursive correspondence of $j \in \mathbb{N}$. It is known, however, that $\mathcal{D}(j) = \mathit{arg}\max_j f|_{\mathbb{F}_{R_j}}$ cannot be recursive as it minorizes[45/]

every recursive correspondence[46/]. The non-recursiveness of graph (C) in this instance, and thus its non-realizability, follows from the now familiar techniques of reasoning of Section IV that revolve around the use of Lemma V.2.

What the example indicates is that even when the notion of recursive representation is restricted to an effective listing of the maximal elements of effectively listed finite subsets of the natural numbers, the computational difficulty in realizing the representation exceeds that which can be obtained by means of the recursive functions. One can, however employ the distinction between representability and realizability, obtained by the unsolvability result of Theorem V.4 to place a minimal bound on the degree of difficulty that a computational realization of rational choice would entail by means of a classification of complexity associated with relative degrees of unsolvability. We address this issue in the specific context of recursive representations of uncompensated demand correspondences in a sequel paper.[47/] It is in this last item that we sincerely believe the larger significance of recursive function theory to mathematical economics lies, and that is to provide a means of measurement of the relative difficulty that the realization of its paradigms, by effectively computable procedures, would entail.

APPENDIX I

To obtain the principal result of the paper, we will require the following classification of subsets of the natural numbers, known as the Kleene-Mostowski Hierarchy (Cf. Putnam [1973], pp.77-80 and Hermes [1965] pp.192-202). The classification is made in terms of the structure of the definitions that define the subsets. The definitions are in turn given by bounded quantified expressions involving recursive predicates. The following diagramme depicts the hierarchy, and the definitions that follow provide an interpretation.

- - - -	- - - -
$\exists x\forall y\exists z\forall w\Phi xyzw$	$\forall x\exists y\forall z\exists w\Phi xyzw$
$\exists x\forall y\exists z\Phi xyz$	$\forall x\exists y\forall z\Phi xyz$
$\exists x\forall y\Phi xy$	$\forall x\exists y\Phi xy$
$\exists x\Phi x$	$\forall x\Phi x$

Φ

Definition I:[48] A string is a non-trivial sequence of symbols "∿" and "∃", that represent logical negation and existential quantification in the first-order predicate calculus. The order of a string is the number of existential quantifiers employed in its formation. The string S is a universal quantifier if ∿S∿ is an existential quantifier.

Definition II: A string is a Kleene string if it has a representation as an alternating sequence of existential and universal quantifiers.

Definition III: A relation is K-enumerable if it has a representation of the form $S\Phi$ where Φ is a recursive relation, and S is a Kleene string beginning with an existential quantifer of order K•A relation anti-K-enumerable if it is the complement of a K-enumerable relation.

Returning to the diagram we see that the Kleene-Mostowski Hierarchy is comprised of subsets of the natural numbers that are defined in terms of K-enumerable and anti-K-enumerable relations for $K = 0,1,2,3,\ldots,$ those subsets that are both 0-enumerable and anti-0-enumerable being in the lower most compartment. Alternatively, a recursive predicate in the form of a K-enumerable relation is termed a Σ_K predicate and a recursive predicate in the form of an anti-K-enumerable relation is termed a π_K predicate. Obviously, a π_0 predicate is also Σ_0. One can then re-label the Kleene-Mostowski Hierarchy in the following form:

----	----
Σ_4 Sets	π_4 Sets
Σ_3 Sets	π_3 Sets
Σ_2 Sets	π_2 Sets
Σ_1 Sets	π_1 Sets
Σ_0-π_0 Sets	

It is a somewhat remarkable fact of the theory of recursive functions that the Kleene-Mostowski Hierarchy composed of the Σ_K and π_K sets, often called the Arithmetic Hierarchy is identical to the following classification of recursively enumerable sets and their complements:

----	-----
Sets R.E. in Sets R.E. in a[illegible].E. set	Complements of Sets R.E. in sets R.E. in an R.E. set
Sets R.E. in an R.E. set	Complements of sets R.E. in an R.E. set
R.E. sets	Complements of R.E. sets
Recursive sets	

The result that establishes the equivalence between Σ_K sets and π_K sets and relatively recursively enumerable sets and their complements (Putnam [1973] p.80), terms the Kleene-Post Representation Theorem. A discussion of relative recursiveness, i.e., what it means for a set to be recursively enumerable in a recursively enumerable set, can be found in Putnam, p.75

It is the Kleene-Post Representation Theorem and the consequent equivalence between the Arithmetic Hierarchy and relatively recursively enumerable subsets of natural numbers, that provide the key device of our principal result. In the text, we demonstrate that the recursive unsolvability of recursive rational choice follows from the fact that any predicate that describes the set of R-indices of the recursive real numbers of the image of a recursive rational choice function, when defined in a restricted sub-family of recursive sets in its domain, cannot have a recursive graph by showing that its graph belongs to a fixed place in the Arithmetic Hierarchy, away from the Σ_0-π_0 sets.

APPENDIX II

In this appendix we provide a brief description of Turing machines, and demonstrate that, within the framework of the description, finite automata can be viewed as a special instance of Turing machines.

(A) Turing Machines

Definition I: By an alphabet A, we will mean a finite set of elements called symbols which includes a distinguished symbol B, termed the blank symbol.

Definition II: A Turing machine Z over the alphabet A is a quadruple (S,m,s_o,f) when S is a finite set, s_o and f are elements of S, and $m : A \times (S - \{f\} \to A \times S \times \{1,-1,0\}$. The set S is called the set of states of Z, s_o the initial state, f the finial state, and m the transition function.

Definition III: For a given Turing machine $Z = (S,m,s_o,f)$ over an alphabet A, an instantaneous description of Z is a triplet (t,s,p) for t a finite sequence of elements of A; p positive integer not greater than the length of t, and s as an element of S. t is called the tape in Z, p the number of the scanned square, and s the state of Z.

Definition IV: For a given Turing machine $A = (S,m,s_o,f)$ over an alphabet A, the yield operation , on instantaneous descriptions of Z is defined as follows: $X(Z) \to Y(Z)$ if and only if at least one of the following obtains where a_i and b_i are in A for all positive integers i .

1. $X = (a_1 \ldots a_n, s, p)$ and $Y = (b_1 \ldots b_n, s', p')$ with $a_j = b_j$ for all $j \neq p$, $m(a_p, s) = (b_p, s', p', -p)$ and either $p < n$ or $p = p' = n$.

2. $X = (a_1 \ldots a_{n-1} a_n^p, s, n)$ and $Y = (a_1 \ldots a_{n-1}\ b_p^\beta, s', n+1)$ where $m(a_n, s) = (b_n, s', 1)$.

3. $X = (a_1 \ldots a_{n-1} a_n, s, n)$ and $Y = (a_1 \ldots a_{n-1}\ b_n, s', n-1)$ where $m(a_n, s) = (b_n, s, -1)$ and $b_n \neq \beta$.

4. $X = (a_1 \ldots a_{n-1} a_n, s, n)$ and $Y = (a_1 \ldots a_{n-1}, s', n-1)$ where $m(a_n, s) = (\beta, s', -1)$.

Definition V: A computation by a Turing machine Z over an alphabet A is a finite sequence $X_1, \ldots, X_q$ of instantaneous descriptions of A such that for all $i = 1, \ldots, q - 1$, $X_i(Z) \to X_{i+1}(Z)$ and for a finite sequence of elements t of A and some integer p, $X_q = (t, f, p)$. We then say that X_1 begins the computation and that X_q is the resultant of X_1.

Definition VI: Given a subset D of $\mathbb{F} = \bigcup_{j \in \mathbb{N}} A^j$, for $\mathbb{F}$ the set of all finite sequences of elements of the alphabet A, the function $\Phi = D \to \mathbb{F}$ if said to be computed by the Turing machine Z over the alphabet A if the following conditions hold. For each $t \in D$ there is a computation by Z beginning with $(t, s_o, 1)$ such that the resultant of $(t, s_o, 1)$ is $(\Phi(t), f, p)$ for some integer p.

Definition VII: Let $\mathbb{F}_o$ denote $\bigcup_{j \in \mathbb{N}} \{0,1\}^j$, the collection of all finite sequences of elements from the two element set $\{0,1\}$.

The Turing machine over the alphabet $\{0,1,\beta\}$ is said to compute the function f from n-tuples of non-negative integers to non-negative integers if it computes the function $\tilde{f} : D_o \rightarrow \mathbb{F}_o$ when

1. D_o is the set of strings of the form:

$$\bar{n}_1 \beta \bar{n}_2 \beta \ldots \beta \bar{n}_n \quad \text{for} \quad (n_1, \ldots n_n) \in \text{Dom}(f)$$

2. $\tilde{f}(\bar{n}_1 \beta \ldots \beta \bar{n}_n)$ is defined as $\overline{f(n_1, \ldots, n_n)}$ where $\bar{n}$ for $n \in \mathbb{N}$ denotes the binary encoding of the natural number n.

(B) Finite Automata

Definition VIII: For a given Turing machine over an alphabet A, the $\mathbb{F}$-yield operation $\underset{\mathbb{F}}{\rightarrow}$ between instantaneous descriptions of Z is defined as follows: $X(Z) \underset{\mathbb{F}}{\rightarrow} Y(Z)$ if and only if at least one of the following conditions obtains where a_i and b_i are in A for a postitive integer i.

1. $X = (a_n a_{n-1} \ldots a_1 a_o, s, p)$ and $Y = (b_n b_{n-1} \ldots b_1 b_o, s', p+1)$ for $p < n$, $b_j = a_j$ for all $j \neq p$, and also that $m(a_p, s) = (b_p, f, -1)$.

2. $X = (a_n a_{n-1} \ldots a_1 a_o, s, p)$ and $Y = (b_n b_{n-1} \ldots b_1 a_o, f, n+1)$ with $m(a_n, s) = (b_n, f, -1)$.

3. $X = (a_n a_{n-1} \ldots a_1 a_o, s, n)$ and $Y = (\beta b_n a_{n-1} \ldots a_1 a_o, s', n+1)$ with $m(a_n, s) = (b_n, s', -1)$ and $s' \neq f$.

Definition IX: An $\mathbb{F}$ computation by a Turing machine Z over the alphabet A is a finite sequence $X_1 \ldots X_z$ of instantaneous descriptions of Z such that for all $i = 1, \ldots, q - 1$ $X(Z) \underset{\mathbb{F}}{\rightarrow} Y(Z)$ and $X = (t, s_o, 0)$ and $S_q = (t', f, \ell(t'))$ where t and t' and finite sequences of elements in A and $\ell(t')$ denotes the length of the sequence t'. We say that X_1 begins the computation and that X_q is the resultant of X_1.

Definition X: For a fixed subset D of $\mathbb{F} = \bigcup_{j \in \mathbb{N}} A^j$, the function $\Phi : D \rightarrow \mathbb{F}$ is said to be computed by the Turing machine Z viewed as a finite automaton over A if the following conditions hold. For each $t \in D$ there is an $\mathbb{F}$-computation by Z beginning with $(t, s_o, 0)$ and the resultant of $(t, s_o, 0)$ is $(\Phi(t), f, \Phi(t)))$.

A computation by a finite state machine always begins on the right-most square of the input tape and proceeds by moving one square to the left at each stage of its computation. The tape of the finite state machine can be extended indefinitely, however unlike the Turing machine, finite state machines cannot add or take away blank squares. .he ability to "print" and "erase" is the major distinction between the two forms as can be seen from a comparison of the conditions that define their respective computation processes.

A further difference between the two forms of Turing machine is that the class of functions computable by a finite state machine is restricted by the length of input tapes. The result below taken from the paper by Ritchie demonstrates this restriction, which was actually employed in the result of Kramer discussed in section II of the present paper.

Theorem (Ritchie op cit p.164): If Φ is a function computed by a finite automaton Z with K non-final states then, for each argument t is the domain of Φ, the length $\Phi(t)$ is at most $K + \ell(t)$ where $\ell(t)$ is the length of the tape t.

Proof: Since A is finite, after Z has read all of t it proceeds to move left reading blank squares. However, if it, (Z), enters the same state twice it cycles and will then fall into an infinite loop. Since K is the number of distinct non-final states of A and t is in the domain of Φ , Z must enter the final state f within K steps after reading the last symbol of t. Therefore, the length of $\Phi(t)$ is at most K plus the length of t. Q.E.D.

A discussion of further limitations of finite state machines can be found in the article by C.C. Elgot, "Decision Problems of Finite Automata Design and Related Arithmetics," Transactions AMS, 98, [1961] pp.21-51.

It should further be observed that the comparative strength in computing capability obtained by Turing machines, relative to that of finite automata, serves to distinguish our approach from the works of Futio[50/] and Gottinger[51/], which, like Kramer's approach, consider the item of rationality in decision making for social decision rules as representable by finite state machines. The issue of complexity in their approach is defined in terms of Krohn-Rhodes decomposition theory[52/]. Within our framework of Turing computability, obtained by means of Church's Thesis, complexity takes the form of degrees of unsolvability[53/].

APPENDIX III

We present in this appendix a brief summary of some of the important structural features of the recursive metric space $M(\mathbb{R}) = \langle \mathbb{N}(R) \sim_{\mathbb{N}(R)}), \mathcal{D}_R \rangle$ defined in Section III. The terminology is that of Moschovakis [1965], wherein can be found proofs of the propositions.

Definition I: A sequence $\{\alpha_j\}_{j \in \mathbb{N}}$ for each $\alpha_j \in M(\mathbb{R})$ is said to be recursive if there is a general recursive function $f : \mathbb{N} \to \mathbb{N}$ such that for all $j \in \mathbb{N}$, $f(j) \in \mathbb{N}(\mathbb{R})$ and $\alpha_j = [f(j)]$, where $[f(j)]$ is an equivalence class under $\sim_{\mathbb{N}(R)}$. The Gödel number of f, $n(f)$ is said to index the sequence.

Definition II: A sequence $\{\alpha_j\}_{j \in \mathbb{N}}$ for each $\alpha_j \in M(\mathbb{R})$ is said be recursively Cauchy if there is a general recursive function $g : \mathbb{N} \to \mathbb{N}$ such that for all j, $K \in \mathbb{N}$ $\mathcal{D}_R(\alpha_{g(j)}, \alpha_{g(j)+K}) < 2^{-j}$. The function g is called a Cauchy criterion for the sequence $\{\alpha_j\}_{j \in \mathbb{N}}$ and the Gödel number of g, $n(g)$ is termed a criterion index for the sequence.

A typical property of $\mathbb{R}$ that one would wish $M(\mathbb{R})$ to preserve in recursive analogue is that it is complete. We state the fact that $M(\mathbb{R})$ has such a property in terms of the following item.

Property A: A recursive metric space is said to have Property A if there is a partial recursive function $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ called a convergence function, such that if $n(f)$ is an index of a recursive sequence with a criterion index $n(g)$, and if there is an α such that

$\alpha = \lim_{j \to \infty} \alpha(j)$, then $h(n(f), n(g))$ is well defined as an element of the notation for the metric space and $\alpha = [h(n(f), n(g))]$.

Definition III: If a recursive metric space satisfies Property A and if every recursively Cauchy sequence has a limit, it is said to be recursively complete.

Proposition I: The recursive metric space $M(\mathbb{R})$ is recursively complete.

Another feature of $\mathbb{R}$ that one would desire $M(\mathbb{R})$ to possess is that $\mathbb{R}$ is separable. That $M(\mathbb{R})$ is in fact separable can be verified immediately by the constructions of $QM(\mathbb{R})$ from R-indices of the rational numbers which by Proposition III are recursive real numbers. Further, since the rationals can be made isomorphic to $\mathbb{N}$, they form a recursively enumerable subset of $M(\mathbb{R})$.

Definition IV: A recursive metric space is recursively separable if there is a recursively enumerable subset of the space that is dense.

Proposition II: $M(\mathbb{R})$ is recursively separable.

Definition V: A listable predicate of n-tuples of R-indices in $M(\mathbb{R})$ is a predicate $P : (\mathbb{N}(R))^n \to \{1,0\}$ for which there is a partial recursive function $f : \mathbb{N}^n \to \{1,0\}$ for which it is true that

$$f(n_1,\ldots,n_n) = 1 \text{ if and only if } P(n(f_1),\ldots,n(f_n)) = 1 \quad .$$

Proposition III: For a fixed $\alpha_o \in M(\mathbb{R})$ and for any $K \in \mathbb{N}$, the open sphere, $S(\alpha_o, K)$ with center α_o and radius 2^{-K} defined as:

$$S(\alpha_o, K) = \{\beta \in M(\mathbb{R}) : \mathcal{D}_R(\alpha_o, \beta) < 2^{-K}\}$$

is a listable subset of $M(\mathbb{R})$.

We next obtain by way of Proposition III the fact that $M(\mathbb{R})$ is connected in the natural topology on $M(\mathbb{R})$ induced by the metric $\mathcal{D}_R$ with the spheres $S(\alpha_o, K)$ as a basis.

Proposition IV:: $M(\mathbb{R})$ is connected in the natural topology.

Proof: The open sets in the natural topology on $M(\mathbb{R})$ are taken as the recursive union of spheres, i.e. an open set has the form:

$$0 = \bigcup_{j, f(j)} S([f(j)], g(j))$$

for $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ partial recursive. The functions f and g are said to index 0.

By Theorem 2 of Moschovakis [1965], one observes that if 0 is an open set in a recursive metric space satisfying Propoerty A, then its complement is recursively closed, i.e. contains the recursive limits of its recursive sequences. In particular, this is true of $M(\mathbb{R})$ since it has Propoerty A. To see that $M(\mathbb{R})$ is connected, we show that no proper subset of $M(\mathbb{R})$ is both recursively open and recursively closed.

Let $\mathbb{F}$ be a proper recursively closed subset of $M(\mathbb{R})$. We show that $\mathbb{F}$ is not recursively open. Choose $\alpha_x \notin \mathbb{F}$ and $\alpha_y \notin \mathbb{F}$

which can be done since $\phi \neq \mathbb{F} \neq M(\mathbb{R})$. Assume that $\alpha_x > \alpha_y$ in the order on $M(\mathbb{R})$ induced by the order on $\mathbb{R}$ and define $\alpha_z = \sup \{\alpha_t \in \mathbb{F} : \alpha_t < \alpha_x\}$. Then it is true that $\alpha_x \geqq \alpha_z \geqq \alpha_y$. Then $\alpha_z \in \mathbb{F}$, because $\mathbb{F}$ is recursively closed and for any $S(\alpha_z, K)$, $(S(\alpha_z, K) \cap \mathbb{F}) \neq \phi$. But if $\alpha_z \in \mathbb{F}$, then $\alpha_x > \alpha_z$, but then, any $S((\alpha_x - \alpha_z) / \alpha_2, K)$ with K sufficiently large is such that it is true that $(S((\alpha_x - \alpha_z) / \alpha_2, K) \cap \mathbb{F}) = \phi$ and since the choice of α_x is arbitrary, α_z cannot be interior to $\mathbb{F}$ and so $\mathbb{F}$ cannot be recursively open. Q.E.D.

Finally, we state two results that refer to the fact that $M(\mathbb{R})$ is a Baire space in the sense that it is not the recursive union of recursively closed, nowhere dense sets of which in the classical setting yields that every denumerable subset of a perfect metric space is of the first category; a metric space being perfect if it has no isolated points which is true of $M(\mathbb{R})$.

<u>Proposition V</u>: Every recursively enumerable subset of a perfect recursive metric space, and therefore of $M(\mathbb{R})$, is of the first category.

<u>Proposition VI</u> : The complement of a recursively enumerable subset of a recursively separable, recursively complete, perfect recursive metric space, and therefore of $M(\mathbb{R})$, is recursively dense.

From Proposition VI one sees that the recursive closure of the subspace $QM(\mathbb{R})$ is in fact $M(\mathbb{R})$ from Proposition II.

Footnotes

1/ Professor Kenneth Arrow of Stanford has brought our attention to the recent work of Professor Douglas S. Bridges of the University College of Buckingham, "Preference and Utility-A Constructive Development", Discussion Paper, [1980].

2/ See Alain A. Lewis "Recursive Choice Functions" in preparation.

3/ See The additional discussion of rationality provided in Kenneth J. Arrow "Rational Choice Functions and Orderings" Economica N.S. Vol. 26, May [1959], pp. 121-127.

4/ That is, predicates that are defined on n-typles of natural numbers. Since any natural number represents a finite sum of the unit integer, number theoretic predicates are regarded as having effectively constructible domains.

5/ See Rogers [1967], p. 29.

6/ See The discussion in Rogers [1967], Ch. 1, and Putnam [1973], Sec. 1. The latter provides an intuitive discussion of Turing machines, which for present purposes may be regarded as ideal computation of the value of a given function.

7/ See Appendix II

8/ This is demonstrated in Appendix II wherein can be found a detailed comparison of the two computing processes.

9/ This is not restrictive. More abstract topological structures can be obtained by taking X to be infinite, compact, and metrizable to obtain an analogous recursive framework. Cf. Lewis, fn. p.5 op cit.

10/ Gödel numberings are discussed in Kleene ([1950], p.206 and p.289)

11/ That is, by way of a fixed y, by means of the mu-operator, we may regard the function f in terms of the number n(f) that forms a component of the domain of satisfiability of the predicate T_n.

12/ The discrete topology is an uninteresting alternative as well, as it suffers from similar deficiencies.

13/ A further observation is that QM($\mathbb{R}$) is totally disconnected in this instance as well. A discussion of Baire's Category Theorem can be found in L.M. Graves, The Theory of Functions of a Real Variable, Sec. 9, Ch XVI, Theorem 33, McGraw Hill, Second Edition, [1956].

14/ A more detailed discussion of this point is found in our Appendix III. We demonstrate there that the recursive imcompleteness of QM($\mathbb{R}$) follows from the "thin-ness" of its cardinality resulting from recursive isomorphisms to $\mathbb{N}$.

15/ The completeness of $\mathbb{R}$ is a frequently employed feature within economic theory. For example, it is employed to characterize the familiar upper and lower pre-order topologies by means of convergent sequences in preference contour sets. Cf. Debreu [1959] Propositions 1-4, and W. Neufeind's "On Continuous Utility", Journal of Economic Theory Vol.5, [1972] pp.174-176. Recursive completeness of characterizations of $\mathbb{R}$ would seem desirable to obtain constructive analogues of similar results.

16/ Cf. H.J. Kowalsky Topological Spaces, Ch.IV Df. 32.c, and Theorem 32.6, Academic Press, [1965]. The completion is of course dependent on the choice of metric, and is not topologically invariant.

17/ Or, what is the same thing the recursive closure of QM($\mathbb{R}$) consisting of itself together with the set of recursive limits of recursive sequences in QM($\mathbb{R}$).

18/ Cf. Appendix III.

19/ Cf. Appendix III.

20/ The class of recursive sets is closed under set theoretic operations.

21/ The precise description of which is in the Appendix II.

22/ That is, the representation is recursive when defined, which is in keeping with the possibility that the preferences may be partial.

23/ By which we mean that both the sets of alternatives and the respective choices from those alternatives can be indexed by $\mathbb{N}$.

24/ By Lemma V.1, the graph of C is in fact a subset $\mathbb{F}_R \times \mathbb{F}_R$.

25/ It follows easily that a full domain in $\mathbb{F}_R$ is not a null sequence of subsets, and contains infinitely many distinct members that are enumerated effectively. This assumption is not restrictive to the means of proof of the principal result and its usefulness is in terms of the economic examples we provide in that families of competitive budgets when recursively represented, typically will comprise full domains.

26/ An elementary discussion of the relationship between recursively solvable problems and effectively computable realizations is found in Putnam's article [1973], pp.70-71. A more advanced and technical discussion of recursive realizability is provided by Kleene [1950], Sec.82, p.501, and Kleene [1971].

27/ Within the equivalences of Church's Thesis, of course.

28/ That is, domains that are full in the sense of Df. VI.

29/ By non-trivial we mean that for a fixed domain $\{\mathbb{F}_{R_j}\}_{j\in\mathbb{N}} \subsetneq \mathbb{F}_R$, the co-domain of graph (C) is not the null sequence if $\{\mathbb{F}_{R_j}\}_{j\in\mathbb{N}}$ is full.

30/ The $\Sigma_o - \pi_o$ sets are precisely the sets that are recursive sets in the Kleene-Mostowski classification, which we provide in Appendix I.

31/ In fact we can show, as in the forthcoming "Recursive Choice Functions", that $\hat{M}(\mathbb{R})$ is in fact recursively isomorphic to $M(\mathbb{R})$.

32/ Furthermore, the images preserve the recursive and topological features of the sets in the domain. Cf. Appendix III.

33/ Q extends T, denoted as $T \subset Q$ if Q is defined and agrees with T whenever T is defined. When T is defined, we say that T is a restriction of Q.

34/ Or, in terms of the equivalences of Appendix I, Σ_K or π_K.

35/ The latter two terms are obtained by replacing the recursive relations in Df.III of Appendix I with partial recursive relations.

36/ The formula for γ is arithmetic rendering it a recursive function. Then the fact is obtained by noting that the characteristic function of $\mathbb{F}_{R_j}$ has the following form: $\chi_{\mathbb{F}_{R_j}} = \chi_{\gamma^{-1}\circ\gamma(\mathbb{F}_{R_j})}$.

37/ The necessity to select a finite subcollection as seen by the counter-example provided by Exercise 5-28 ▲ b of Rogers [1967], p.75.

38/ Moschovakis [1956], pp. 225-227.

39/ That is, Φ_1 is a restriction Φ_2 by way of the $f_1,\ldots,f_n$. Cf. Df. IX.

40/ Relations are classified in the hierarchy by means of their graphs.

41/ By the means of Church's Thesis.

42/ Cf. Alain A. Lewis, "Relatively Recursive Rational Choice"; forthcoming Stanford IMSSS Technical Report, November [1981] where this concept is developed.

43/ Cf. Alain A. Lewis, op cit .

44/ We could be more rigorous by showing that any notation for $\mathbb{N}$ is recursively isomorphic to $\mathbb{N}$ at the expense of more formalism.

45/ That is, if $G : \mathbb{N} \to \mathbb{F}_R^-$ were any recursive correspondence, then $\exists K \in \mathbb{N}$ such that $\forall j > K \quad \mathcal{D}(j) > G(j)$.

46/ This result is due to T. Rado "On Non-Computable Functions," Theorem 1, Bell System Technical Journal May [1962] pp. 887-884, and is proved by an inductive argument on the bound of states for a Turing machine that generates the members of a finite set. Rado's result is a straightforward means of distinguishing the class of effectively definable functions from the class of effectively computable functions.

47/ Cf. the forthcoming "Recursive Choice Functions".

48/ This terminology is due to Shapiro [1956].

49/ Robert W. Ritchie, "classes of Predictably Computable Functions," Transactions AMS, Vol. 102, [1963] pp. 139-173.

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