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COALESCENCE OF SPHEROMAKS

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ABSTRACT

The coalescence of two spheromaks is described using the quasi-ideal model of reconnection. Nonaxisymmetric effects are found to play an important role in general. Possible applications of repeated coalescence are studied, including heating the spheromak plasma, building up a large spheromak from smaller ones, and maintaining a spheromak in steady state. The analysis suggests that coalescence experiments could also provide a valuable method for studying magnetic field relaxation.

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I. INTRODUCTION

One of the important virtues that the spheromak shares with other compact torus devices is the absence of a toroidal field coil linking the plasma, which greatly simplifies the engineering. A further consequence of this geometry is the possibility of moving the plasma along the external vertical field (making possible a moving ring field reversed reactor¹). In this paper, we explore yet another consequence of this geometry, the possibility of repeated coalescence of separately formed spheromaks. Coalescence of two partially distinct spheromaks has in fact been observed on the University of Maryland experiment, PS-1.² Spheromak coalescence has potential applications to heating the spheromak plasma, to building up a large spheromak from smaller ones (thus making it easier to attain fusion conditions), to maintaining a spheromak in steady-state, and to the experimental study of magnetic field relaxation.

The astron experiment at Livermore attempted to use the repeated coalescence of e-layer pulses to produce a field-reversed e-layer.³ Coalescence of e-layer pulses was observed, although field reversal was not achieved in these experiments. Later experiments at Cornell demonstrated the merging of two electron rings below field reversal to produce a field-reversed electron ring.⁴ Numerical simulations of the astron experiment⁵ also looked into the possibility of maintaining a field reversed e-layer by repeated injection of smaller pulses. These simulations, assuming axisymmetry, came to a negative conclusion.

The merging of ion rings has been proposed as a method of

maintaining a field reversed ion ring, and axisymmetric numerical simulations are in progress.⁶

The recent interest in spheromaks has sparked several proposals concerning spheromak coalescence.^{7,8,9} In this paper, we describe spheromak coalescence using the quasi-ideal model of reconnection.^{10,11,12} We will see that there are two analytically soluble limits: one in which Kadomtsev's theory of nonlinear tearing modes¹⁰ can be applied, and one in which Taylor's relaxation theory^{13,14} is valid. In the Kadomtsev limit there is an infinite set of conserved quantities that completely determines the motion; while in the Taylor limit only one of these quantities remains invariant, the system evolving to a state of minimum energy subject to the single invariant. By solving in both limits and investigating the domains of validity of both approximations we develop a general picture of spheromak coalescence. Our analysis focuses particularly on the feasibility of maintaining a spheromak in steady state by repeated coalescence. We also calculate the heating during coalescence for some particular cases, and look at the feasibility of building large spheromaks by the repeated coalescence of smaller ones. The important role that various nonlinear, nonaxisymmetric, tearing processes can play during coalescence will suggest that spheromak coalescence can also provide a valuable experimental framework for studying these processes.

An axisymmetric coalescence is shown schematically in Fig. 1. Depending on the force exerted on the spheromaks by the externally applied magnetic field, and depending on the force due to the spheromak fields (the spheromaks attract if their toroidal currents are in the

same direction), two initially separated spheromaks can approach each other by moving along the vertical field. As the spheromaks merge, the corresponding flux surfaces reconnect. After the process has gone to completion, the two spheromaks have combined to form a single spheromak.

Coalescence has been observed on the University of Maryland spheromak experiment, PS-1.² Depending on the configuration of the bias magnetic field, the initial formation stage in this device sometimes produces two separate magnetic islands, which subsequently coalesce on a time scale short compared to the resistive decay time of the plasma currents. The coalescence involves the reconnection of a substantial fraction of the closed flux surfaces. In these experiments, the initial formation appears to be well described by an axisymmetric, resistive MHD code.¹⁵

Coalescence of tokamak plasmas is also possible in principle. Shafranov has proposed heating tokamak plasmas by initially forming a number of separate rings, inside an elongated vacuum chamber, which subsequently merge.¹⁶ His analysis assumes that the reconnection process is axisymmetric, and that the plasma is incompressible. The use of repeated coalescence to maintain the toroidal current in a tokamak plasma has also been proposed;¹⁷ however, the fact that the toroidal field coil must link both the main and refueling plasma rings raises questions concerning the practicality of repeated coalescence in the case. (Note that the changing poloidal flux through the hole of the refueling ring which initially drives its toroidal current must not also thread the main plasma ring.) The analysis of Ref. 17 also assumes axisymmetric reconnection and plasma incompressibility. We will see

that we cannot rely on purely axisymmetric reconnection to maintain the currents in a spheromak.

In Sec. II we look at purely axisymmetric reconnection, and afterwards include nonaxisymmetric effects in Secs. III and IV. Sec. III deals with two step coalescence processes consisting of an axisymmetric reconnection followed by a nonaxisymmetric tearing or double-tearing mode. We apply our analysis there to the problem of maintaining spheromak currents in steady-state, and calculate the efficiency of the refueling process. In Sec. IV we look at turbulent coalescence. This is found to be of particular interest for heating the spheromak plasma, and for building up large spheromaks from smaller ones.

II. AXISYMMETRIC RECONNECTION

We begin by looking at axisymmetric reconnection, and afterwards include nonaxisymmetric effects in Secs. III and IV. The assumption of axisymmetry is a reasonable one if the initial and final spheromak equilibria are stable to nonaxisymmetric modes. We will see that this condition is, in practice, quite restrictive.

Having assumed axisymmetry, our flux surfaces correspond to level surfaces of the poloidal flux function, $\psi(\underline{r})$. Flux surfaces which reconnect must have the same value of ψ . We use the quasi-ideal model^{10,11,12} to describe the reconnection process. The plasma is assumed to obey ideal MHD except in a narrow region about the x-point. When flux surfaces reconnect, the enclosed toroidal fluxes add. Using the fact that $\chi(\psi)$ is an invariant in ideal MHD, we get

$$\chi_f(\psi) = \chi_1(\psi) + \chi_2(\psi) \quad , \quad (1)$$

where χ denotes the toroidal flux, the subscript f corresponds to the final equilibrium state, and the subscripts 1 and 2 correspond to the initial spheromaks. Equation (1) applies, of course, only if the flux surface ψ lies inside both spheromak plasmas initially. We take the plasma, but not necessarily the plasma current, to extend to the separatrix. In general, the initial spheromaks can have different values of ψ on the magnetic axis, ψ_1 and ψ_2 . If $|\psi_1| > |\psi_2|$, then $\chi_f(\psi) = \chi_1(\psi)$ for $|\psi_2| < |\psi| < |\psi_1|$. The value of ψ on the magnetic axis after coalescence is ψ_1 .

From Eq. (1), and from the relation $q(\psi) = d\chi/d\psi$ (where q is the safety factor), it follows that

$$q_f(\psi) = q_1(\psi) + q_2(\psi) . \quad (2)$$

Either Eq. (1) or Eq. (2) can be viewed as specifying an infinite number of constants of motion for axisymmetric coalescence. The total magnetic helicity, K , used in Taylor's relaxation theory,^{13,14} is one of these invariants,

$$K \equiv \int_v A \cdot B = -2 \int \chi(\psi) d\psi = 2 \int \psi q(\psi) d\psi , \quad (3)$$

where we have used the boundary conditions $\chi = 0$ on the magnetic axis and $\psi = 0$ on the plasma surface to integrate by parts.

The poloidal current distribution can be expressed in terms of q ,

$$I(\psi) = q(\psi) d\psi/dV \langle 1/R^2 \rangle_\psi^{-1} , \quad (4)$$

where $\langle \rangle_\psi$ denotes an average over a flux surface, V is the volume enclosed by the flux surface, and R is the radius in cylindrical coordinates.¹⁸ Together with the plasma pressure, $I(\psi)$ determines the equilibrium through the Grad-Shafranov equation

$$-\nabla^* \psi = R^2 p'(\psi) + II'(\psi) ,$$

where

$$\nabla^* \equiv R \frac{\partial}{\partial R} \frac{1}{R} \frac{\partial}{\partial R} + \frac{\partial^2}{\partial z^2} .$$

Eliminating $I(\psi)$ from the Grad-Shafranov equation in terms of the known $q(\psi)$, we obtain a QDE which determines the equilibrium.¹¹ The boundary condition on ψ at infinity is given by the external fields. The plasma $\beta (= 8\pi p/B^2)$ must not be too large if the equilibrium is to be stable to Mercier modes.¹⁹⁻²¹ Since the poloidal and toroidal betas are comparable, it follows that the spheromak equilibrium will be determined by the q profile alone to a good approximation.

Now we can look more closely at the conditions under which the assumption of axisymmetry is valid. First we show that two coalescing spheromaks with decreasing q profile must have the same value of total poloidal flux to preserve axisymmetry. In Fig. 2 we show schematically what happens when the poloidal fluxes are different. The final q profile is not monotonic, and is therefore unstable to ideal modes²² and to double tearing²³ modes. Note that this argument is specific to decreasing q profiles. For initial tokamak-like q profiles, coalescence gives a q profile which is monotonic, although rapidly varying near the ψ corresponding to the smaller value of total poloidal flux.

In addition to requiring equal poloidal fluxes, we must also require that the q 's on the magnetic axis for the two coalescing spheromaks sum to less than unity, so that the final spheromak will be stable to $n = 1$ internal kink modes.²¹ If their peak value of q is too small, however, the initial spheromaks can be unstable to nonaxisymmetric ideal modes. Numerical results from the PEST code indicate that q on the magnetic axis must be at least $1/3$.²⁴

Now we turn to the question of maintaining a spheromak equilibrium in steady-state via coalescence. We will be concerned only with

maintaining the q profile and the value of the total poloidal flux. Throughout this paper we use the term "refueling" as a shorthand for the process by which we restore the spheromak currents, although we will not be concerned with restoring the plasma density. The density can be increased separately through the injection of neutral pellets. Of course we must require that coalescence not increase the plasma density of the fusion spheromak above its initial value. (We call the spheromak to be refueled the "fusion spheromak".) This places a bound on the number of particles allowed in the refueling spheromak,

$$N_2 \lesssim N_1 (\Delta\psi/\psi_0)(\tau_M/\tau_N) ,$$

where N_2 and N_1 are the numbers of particles in the refueling and fusion spheromaks respectively, ψ_0 is the total poloidal flux in the fusion spheromak, $\Delta\psi$ is the change in its poloidal flux due to coalescence, and τ_M and τ_N are, respectively, the time scales for flux diffusion and for particle loss. As we have mentioned, the plasma β must not be too large, so $q(\psi)$ alone will determine the refueling equilibrium to a good approximation.

Suppose the fusion spheromak initially has $\psi = \psi_0$ on its magnetic axis. After some time, the poloidal flux decays to a value $\psi_0 - \delta\psi_0$. To restore the initial value of poloidal flux by purely axisymmetric reconnection, the refueling spheromak must have $\psi = \psi_0$ on its magnetic axis; but we have seen that axisymmetry is not preserved unless the poloidal fluxes of the coalescing spheromaks are equal. We conclude that it is impossible to maintain a spheromak in steady-state by purely

axisymmetric coalescence. That conclusion may appear somewhat paradoxical, since we can use Eq. (2) to find a q profile for the refueling spheromak which restores the fusion spheromak to its initial state.

Again, suppose we initially have a spheromak with safety factor q_0 on the magnetic axis and total poloidal flux ψ_0 , as shown schematically in Fig. 3a. After some small time has elapsed, both these quantities will have decayed somewhat, to values $q_0 - \delta q_0$ and $\psi_0 - \delta\psi_0$, also shown in Fig. 3a. (The value of q on the magnetic axis decreases because the current profile becomes more peaked there.) To restore the initial q profile via coalescence requires a refueling spheromak whose $q(\psi)$ is the difference of the two q profiles of Fig. 3a (see Fig. 3b). The required q profile is discontinuous, and therefore clearly not physically realizable. The obvious thing to try is a smoothed version of this profile, as indicated by the dashed line in Fig. 3b. Adding this smoothed profile to the decayed q profile of the original spheromak, we obtain the $q(\psi)$ shown in Fig. 3c. The equilibrium corresponding to Fig. 3c is unstable to nonaxisymmetric modes (e.g. double tearing modes). It is clear that we will run into the same problem with any refueling spheromak having a smooth $q(\psi)$, and also if we try to use a number of different refueling spheromaks in succession.

Although the equilibrium corresponding to Fig. 3c is unstable, we would expect the resulting turbulence to be localized near $\psi = \psi_0 - \delta\psi_0$, and to result in a smoothing of $q(\psi)$. This suggests a possible two step process for restoring spheromak currents, an axisymmetric coalescence with a refueling spheromak having the smoothed $q(\psi)$ of Fig. 3b, followed by a

turbulent smoothing of the resulting q profile. When we examine the required equilibrium for the refueling spheromak, however, this does not appear to be a very attractive scheme.

We solve the Grad-Shafranov equation analytically in the limit that $\delta q \ll 1$ and $\delta\psi_0/\psi_0 \ll 1$, with $p = 0$, $I(\psi)$ given by Eq. (4), and $q(\psi)$ corresponding to the smoothed profile in Fig. 3b. Our approach is to match solutions which are valid in four different regions, as follows:

- 1) Near the magnetic axis we Taylor expand all quantities in $\psi - \psi_0$.
- 2) For the narrow region near $\psi_0 - \delta\psi_0$ with rapidly varying q we derive a jump condition.
- 3) This is the region where q is small, but the distance from the magnetic axis is still much less than the major radius of the spheromak.
- 4) Far from the magnetic axis we can use the elliptic function solution for the fields due to a circular loop of current.

Further details are given in the Appendix. We find that the equilibrium has a hollow current profile and a large flux hole (that is, a large current free region near the separatrix). The low value of q on the magnetic axis ($q = \delta q_0$ there) may make stability hard to achieve.

Using nonaxisymmetric effects to produce poloidal flux from toroidal flux, we can find considerably more attractive refueling schemes.

III. NONAXISYMMETRIC EFFECTS DUE TO SLOWLY GROWING MODES

If, having assumed that the coalescence process is purely axisymmetric, we find that the resulting spheromak is unstable to nonaxisymmetric modes, we must conclude that our assumption of axisymmetry has broken down at some point. When the unstable modes grow sufficiently slowly, however, we can still treat the coalescence as a two stage process: an axisymmetric reconnection followed by a nonaxisymmetric tearing mode. This treatment is valid if the e-folding time of the fastest growing mode is long compared to the time scale on which the axisymmetric reconnection occurs. For example, for the $n = 1$, $m = 1$ mode, which will be of particular interest to us in this section, the growth rate is small if the maximum value of q is not much larger than one. For an equilibrium having $q = 1.3$ on the magnetic axis, Gautier et al. found a growth rate of $.03 \tau_A^{-1}$, where τ_A is the Alfvén time.²¹ By comparison, in the experiment reported in Ref. 2, two islands containing about 25% of the total poloidal flux coalesced in a time not much longer than the Alfvén time. In a numerical study of the coalescence of two-dimensional islands, Pritchett and Wu found that the reconnection time for this type of forced reconnection is relatively insensitive to the value of the resistivity.²⁵

The tearing modes we will be concerned with in this section will be localized sufficiently close to the magnetic axis that helical symmetry can be assumed. The flux surfaces are determined by the helical flux function, $\phi = q_s \psi - \chi$, where q_s is the value of q at the separatrix. Flux surfaces which reconnect must have the same value of ϕ .

We describe the nonlinear evolution of the tearing mode using Kadomtsev's theory.¹⁰ Again, the plasma is assumed to obey ideal MHD except in a narrow region about the separatrix. Those flux surfaces which initially have the same value of ϕ (but different values of χ) all reconnect, so that at the end, ϕ uniquely labels the flux surfaces. When flux surfaces reconnect the enclosed toroidal fluxes must be subtracted,

$$\chi_f(\phi) = \chi_1(\phi) - \chi_2(\phi) \quad , \quad (5)$$

where the subscripts 1 and 2 distinguish the two values of the initially double-valued function $\chi(\phi)$, and $\chi_f(\phi)$ is the single valued function which results after the reconnection process has gone to completion.

Kadomtsev's theory has an infinite set of conserved quantities. It was recognized by Bhattacharjee et al.¹² that the invariants of Kadomtsev's theory could be expressed in the form $\int_V f(\phi) \mathbf{A} \cdot \mathbf{B} d^3x$, where V denotes an integral over the entire volume of the plasma and f is an arbitrary function. Depending on the helicity of the tearing mode we get different helical flux functions, ϕ , and thus different sets of conserved quantities. If we set $f(\phi) = 1$, we get the total magnetic helicity employed by Taylor. This is the only one of the invariants which is preserved by a mode of any helicity. Recall that this quantity is also conserved during the axisymmetric reconnection.

We look first at the case where ψ on the magnetic axis is the same for both coalescing spheromaks, and where the q 's on the axis sum to a value slightly above one. This will be of particular interest for the

refueling problem. Figure 4 shows the q profiles before and after the axisymmetric stage of coalescence. Because the unstable mode is localized near the magnetic axis, we can treat it by Taylor expanding the equilibrium quantities about the axis. (We also make use here of the fact that the Grad-Shafranov equation is nonsingular at the magnetic axis.) For the equilibrium after the first stage of the coalescence, we write

$$\phi \approx \phi_0 + a\chi + b\chi^2, \quad (6)$$

where ϕ_0 is the value of ϕ on the magnetic axis, and a and b are constants which determine the equilibrium in the neighborhood of the axis. Expressing ψ in terms of ϕ and χ , we get

$$\psi \approx \phi_0 + (a + 1)\chi + b\chi^2, \quad (7)$$

so that $\phi_0 = \psi_0 - \delta\psi_0$. Taking the conventions $\chi > 0$ and $q = d\chi/d\psi > 0$, we conclude from $\psi = 0$ at the separatrix that $\phi_0 < 0$. From Eq. (7), we calculate

$$q = 1/(1 + a + 2b\chi). \quad (8)$$

Comparing this result with the q profile of Fig. 4b we see that $a < 0$ and $b > 0$.

Figure 5 shows what $\phi(\chi)$ looks like before the tearing mode begins to grow (solid line). The minimum value of ϕ lies on the $n = 1, m = 1$

rational surface, which corresponds to $\chi = \chi_S = -a/(2b)$. The value of ϕ there is $\phi_S = \phi_0 - a^2/(4b)$. In terms of ϕ_S and χ_S we can rewrite Eq. (6) as $\phi - \phi_S = b(\chi - \chi_S)^2$, or

$$\chi = \chi_S \pm [(\phi - \phi_S)/b]^{1/2} . \quad (9)$$

The tearing mode leads to reconnection of flux surfaces having the same value of ϕ . From Eqs. (9) and (5) we see that after reconnection we have $\chi = 2[(\phi - \phi_S)/b]^{1/2}$, or

$$\phi = b \chi^2/4 + \phi_S , \quad (10)$$

which is shown as the dashed line in Fig. 5. Only those flux surfaces having $\chi < 2 \chi_S$ are affected by this second stage of reconnection. The final q profile in this region is

$$q = 1/(1 + b \chi/2) . \quad (11)$$

Note that at the boundary of this region, Eq. (11) gives $q \rightarrow 1/(1 - a/2)$ while Eq. (8) gives $q \rightarrow 1/(1 - a)$. We anticipate that this discontinuity, which is characteristic of Kadomtsev's theory, is smoothed by further turbulence. On the magnetic axis q is equal to one.

Now we apply these results to the refueling problem. To do so, we require that the two stage reconnection process we have just described restore the total poloidal flux to ψ_0 , and that it approximately restore the q profile to that of the fusion spheromak at $t = 0$. Letting $q_0(\psi)$

be the q of the fusion spheromak at $t = 0$, which is assumed to have $q_0(\psi_0) \approx 1$ to maximize the allowable beta,²¹ our condition on q is

$$1 - 3a/4 = [q_0]_{x=2x_S}^{-1} . \quad (12)$$

This gives

$$b = - (4/3) q'_0(\psi_0) ,$$

where

$$q'_0(\psi_0) \equiv \frac{d}{d\psi} q_0(\psi) |_{\psi=\psi_0} .$$

The condition on the poloidal flux then gives

$$a^2 = (16/3) q'_0(\psi_0) \delta\psi_0 .$$

The $q(\psi)$ for the refueling spheromak is

$$q_2 = \delta q + 4 |q'_0(\psi_0) \delta\psi_0 / 3|^{1/2} + (5/3) q'_0(\psi_0) (\psi - \psi_0) \quad (13a)$$

for

$$\psi - \psi_0 < (4/5) |3\delta\psi_0 / q'_0(\psi_0)| \quad (13b)$$

and

$$q_2(\psi) \approx \delta q(\psi) \quad (14)$$

for larger values of $\psi - \psi_0$.

We can solve analytically for the refueling equilibrium specified by Eqs. (13) and (14) in the limit of small δq and $\delta \psi_0$ by the same methods used to solve the equilibrium of the previous section. This is discussed in the appendix. The equilibrium is now considerably more interesting, with no hollow current profile, a smaller flux hole, and a larger value of q on axis. The condition that $q_2(\psi_0) > .3$, required for stability,²⁴ is satisfied for $\delta \psi_0 / \psi_0 > .02$. In obtaining this estimate we have approximated $q'_0 \approx q_0(\psi_0) / \psi_0$.

We calculate the efficiency of our process for maintaining the spheromak currents in the limit that the fusion and refueling spheromaks both have a large flux hole. That is, we determine the total magnetic energy lost during coalescence as a fraction of the energy of the refueling spheromak. This lost magnetic energy is dissipated by ohmic heating near the x-point.

The magnetic energy of each of the (large aspect ratio) spheromaks can be written as

$$W \approx \frac{1}{2} LI^2 + MII_e, \quad (15)$$

where L is the self inductance of the spheromak currents, M is the mutual inductance between the spheromak currents and the external field coil, I is the total toroidal current of the spheromak, and I_e is the total current through the external field coil. The inductances are

related to the flux through the spheromak hole by the relations

$$MI_e = \psi_e / c, \quad (16)$$

and

$$LI = \psi_s / c \quad (17)$$

where ψ_e is the external flux and ψ_s is the self flux. The previously defined poloidal flux function is proportional to the total flux,

$$2\pi R\psi_o \approx \Psi_e + \Psi_s,$$

where R is the major radius. We define $\Psi_o \equiv 2\pi R\psi_o$. The self inductance corresponding to the refueling equilibrium is

$$L_2 \approx (4\pi R/c^2)[\ln(8/\delta q_m) - 2],$$

where δq_m is the value of δq on the magnetic axis. To estimate the self inductance of the fusion spheromak we use the standard expression for the inductance of a large aspect ratio circular torus,

$$L_1 \approx (4\pi R/c^2)[\ln(8R/a) - 2 + \ell_1/2],$$

where a is the minor radius, and $\ell_1 \gtrsim 1/2$ is the internal inductance per unit length.

Equilibrium force balance for a large aspect ratio spheromak gives

$$2\pi R B_v I/c \approx (1/2)I^2 dL/dR \approx I^2 L/(2R) \quad , \quad (18)$$

where B_v is the externally imposed vertical field. From Eq. (18) we get $\Psi_e = -(1/4)\Psi_s$ and $MII_e = -(1/4)LI^2$. It follows that the magnetic energy of each spheromak can be expressed as

$$W \approx (4/a) \Psi_o^2 / (c^2 L) \quad . \quad (19)$$

Using Eq. (19) in conjunction with Eq. (18), we find that when the total poloidal flux of the fusion spheromak decays by an amount $\delta\Psi_o$, the magnetic energy of this spheromak decays by an amount

$$\delta W_1 = (2/3) \Psi_o \delta\Psi_o / (L_1 c^2) \quad . \quad (20)$$

If our refueling process were perfectly efficient, the magnetic energy of the refueling spheromak would just be given by Eq. (20). For the refueling process we have described, the efficiency is given by

$$\delta W_1/W_2 = 3(\delta\Psi_o/\Psi_o)(L_2/L_1) \approx 3(\delta\Psi_o/\Psi_o)[\ln\delta q_m/\ln(a_1/R_1)] \quad . \quad (21)$$

For example, for $\delta\Psi_o/\Psi_o \approx \delta q_m \approx .1$,

$$\delta W_1/W_2 \approx .7/\ln(R_1/a_1) \quad .$$

A substantial fraction of the magnetic energy of the refueling spheromak gets transformed into thermal energy during coalescence. There are two other possible sources of thermal energy, the kinetic energy of the spheromaks due to their relative velocity on impact, and adiabatic heating. There could also, in principle, be adiabatic cooling. We will have more to say about these effects in the next section, where we will look more closely into the possibility of using coalescence to heat the spheromak plasma.

Thus far in this section we have assumed that the coalescing spheromaks have an equal amount of poloidal flux. Since the magnetic energy of the refueling spheromak is proportional to Ψ_0^2 , it clearly would be advantageous if the poloidal flux of this spheromak could be smaller. If the coalescing spheromaks have different amounts of poloidal flux, axisymmetric reconnection gives a q profile with a local maximum (as in Fig. 6). The corresponding equilibrium is unstable to a double-tearing mode. Assuming that the growth rate of the double-tearing mode is sufficiently small, our two step coalescence process now consists of an axisymmetric reconnection followed by a nonaxisymmetric double-tearing mode. We use Kadomtsev's theory of the nonlinear evolution of the double-tearing mode.

The $n = 1$, $m = 1$ helical flux function corresponding to the q profile of Fig. 6 is shown schematically in Fig. 7 (dashed line). There are two $q = 1$ rational surfaces, corresponding to the two local extrema in ϕ (a local minimum and a local maximum). We again Taylor expand ϕ as a function of χ , expanding about $\chi = 0$ for $\chi < \chi_{S1}$, and about $\chi = \chi_{S1}$ for $\chi > \chi_{S1}$. We express the coefficient directly in terms of the q

profile at this intermediate stage. For $\chi < \chi_{S1}$, we get

$$\phi \approx \phi_0 + \frac{\delta q(\psi_0 - \delta\psi_0)}{q_1(\psi_0 - \delta\psi_0)} \chi - \frac{1}{2} \frac{q_1'(\psi_0 - \delta\psi_0)}{[q_1(\psi_0 - \delta\psi_0)]^3} \chi^2, \quad (22a)$$

with

$$\phi_0 = \psi_0 - \delta\psi_0 \quad (22b)$$

and

$$\chi_{S1} \approx (\psi_2 - \phi_0)q_1(\psi_0 - \delta\psi_0) + (1/2)(\psi_2 - \phi_0)^2 q_1'(\psi_0 - \delta\psi_0). \quad (23)$$

For $\chi > \chi_{S1}$ it is convenient to define

$$q_m \equiv q_1(\psi_0 - \delta\psi_0) + q_2(\psi_0 - \delta\psi_0) \quad (24a)$$

and

$$q_m' \equiv \left. \frac{dq_1}{d\psi} \right|_{\psi_0 - \delta\psi_0} + \left. \frac{dq_2}{d\psi} \right|_{\psi_0 - \delta\psi_0}. \quad (24b)$$

We then find, for $\chi > \chi_{S1}$,

$$\phi \approx \phi_1 + \frac{1 - q_m}{q_m} (\chi - \chi_{S1}) - \frac{q_m'}{2q_m^3} (\chi - \chi_{S1})^2, \quad (25)$$

with $\phi_1 = \psi_2 - \chi_{S1}$. The local minimum of Eq. (25) corresponds to the second $q = 1$ rational surface, at

$$\chi_{S2} = \chi_{S1} + q_m^2(1 - q_m)/q'_m, \quad (26)$$

$$\phi_2 = \phi_1 + (1/2)q_m(1 - q_m)^2/q'_m. \quad (27)$$

In Kadomtsev's theory, all flux surfaces having the same value of ϕ reconnect. The value of ψ on the magnetic axis is affected by this process only if $|\phi_2| > |\phi_0| = |\psi_0 - \delta\psi_0|$. When this condition is satisfied, the final value of ψ on the axis is just ϕ_2 . To restore the poloidal flux of the fusion spheromak, we must have $\phi_2 = \psi_0$, or

$$q_m(1 - q_m)^2/q'_m = (\psi_2 - \phi_0)^2 q'_1(\psi_0 - \delta\psi_0) + 2\delta\psi_0 - 2(\psi_2 - \phi_0) \delta q. \quad (28)$$

From Eqs. (22) - (27), we determine what $\chi(\phi)$ must be after all corresponding flux surfaces have reconnected,

$$\chi = \begin{cases} 2[2(\phi - \phi_2)q_m^3/q'_m]^{1/2} & \text{for } \phi_2 < \phi < \phi_0, \\ 2[2(\phi - \phi_2)q_m^3/q'_m]^{1/2} + (\phi - \phi_0)q_1(\psi_0 - \delta\psi_0)/\delta q & \text{for } \phi_1 > \phi > \phi_0. \end{cases} \quad (29)$$

The flux surfaces with $\phi > \phi_1$ are unaffected by the $n = 1, m = 1$ double-tearing mode. The final $n = 1, m = 1$ helical flux function is shown schematically in Fig. (7) as a dashed line. The corresponding $q(\psi)$ is not monotonic in the neighborhood of $\phi = \phi_0$. We could expect that further turbulence will smooth out this q profile.

Equation (28) shows that as $\psi_2 - \phi_0$ increases, so does the required

value of $q_m - 1$. When $q_m - 1$ is not small, the growth rate of the $n = 1, m = 1$ nonaxisymmetric mode is not small, and we can no longer expect the coalescence to proceed in two separate stages. That is, we can no longer make any symmetry assumptions about the coalescence process. Similarly, our assumption of helical symmetry for the nonlinear tearing modes breaks down if reconnection occurs sufficiently far from the magnetic axis that toroidal effects are important, or if tearing modes of different helicity grow to large amplitude simultaneously and begin to overlap. In these cases, we introduce an increased amount of turbulence into the fusion spheromak, something that is clearly undesirable in any refueling scheme. Turbulent coalescence is, however, of interest for heating, and for building up larger spheromaks from smaller ones. We describe turbulent coalescence in the next section.

IV. TURBULENT COALESCENCE

Thus far we have been assuming that at each stage we have either axial symmetry or helical symmetry. Within the quasi-ideal model, there is an infinite set of constants of motion associated with each of these symmetries. It is these invariants, in essence, that we have used to determine the final equilibrium state resulting from the coalescence process. Recall, too, that the total magnetic helicity is the only quantity which is conserved by a tearing mode of arbitrary helical symmetry, and that it is also an invariant under axial symmetry.

Our symmetry assumptions can break down for a variety of reasons, as discussed at the end of the previous section. When the assumption of symmetry is not valid, we expect that there will be fewer invariants, and that the plasma will assume a state of minimum energy subject to whatever set of invariants is appropriate. As long as the quasi-ideal model remains valid, the total magnetic helicity will be conserved. In the region where ideal MHD is obeyed, away from the separatrix, the local magnetic helicity is conserved. Regardless of the shape of the separatrix, reconnection of two flux surfaces clearly does not affect the total magnetic helicity. In the limit that the reconnection process becomes extremely turbulent, we expect that only the total magnetic helicity is conserved. The final state is that having minimum energy subject to the conservation of magnetic helicity. This is the hypothesis that Taylor has used successfully to describe the initial formation of the reversed field pinch.¹³ We use Taylor's theory in this section to describe turbulent coalescence.

For our coalescing spheromaks, we specialize to the spherical minimum energy equilibria of Rosenbluth and Bussac,²⁶ which have a uniform external magnetic field far from the spheromak. In spherical coordinates, the poloidal flux function in the plasma is

$$\psi(r,\theta) = r \sin^2\theta j_1(\lambda r)/\lambda, \quad (29)$$

with

$$\nabla \times \underline{\tilde{B}} = \lambda \underline{\tilde{B}}, \quad (30)$$

where λ is a constant. Although this equilibrium is unstable to a variety of modes in the absence of a conducting wall,²⁶ the modes can be stabilized by a combination of modifications to the equilibrium and a relatively distant external conducting wall,^{21,27,28,29} neither of which should strongly affect our conclusions regarding turbulent coalescence.

The equilibrium solution, Eq. (29), has a free parameter, λ , which is a function of the total magnetic helicity, K . The radius of the spheromak is proportional to $K^{1/4}$, while the total poloidal flux is proportional to $K^{1/2}$, and the total magnetic energy $= (1/2c) \int \underline{j} \cdot \underline{A} d^3x = \lambda K/8\pi$ is proportional to $K^{3/4}$.

Now suppose that we coalesce two spheromaks of equal size, each having a total magnetic helicity of K_0 . The final equilibrium must have total magnetic helicity $2K_0$. If the turbulence level during coalescence is sufficiently large that Taylor's hypothesis is valid, the final equilibrium after coalescence will again satisfy Eq. (30) in the plasma,

with λ determined by $K = 2K_0$. With plasma out to the separatrix, the final equilibrium in the uniform external field again belongs to the class of spherical Rosenbluth-Bussac equilibria, with $\psi(r, \theta)$ in the plasma of the form of Eq. (29). The total poloidal flux in the final spheromak is 40% larger than in each of the initial spheromaks, while the total magnetic energy is 1.7 times what it was in each initial spheromak. Fifteen percent of the total magnetic energy has been ohmically dissipated.

The substantial increase in poloidal flux due to Taylor relaxation suggests that, in coalescence experiments, measurement of the total poloidal flux by itself would give a useful indication of what is happening. Recall that axisymmetric coalescence preserves the maximum value of ψ . (Due to the presence of finite resistivity, ψ actually decays slightly.) Nonaxisymmetric tearing modes produce poloidal flux, increasing somewhat the maximum value of ψ .

Next, let one of the coalescing spheromaks be very much smaller than the other. Let the total magnetic helicities be K_1 and K_2 , with $K_2 = \epsilon K_1$. The fraction of the magnetic energy of the second spheromak that gets dissipated by ohmic heating is then

$$\Delta W/W_2 = 1 - (3/4) \epsilon^{1/4} .$$

In terms of the initial radii of the two spheromaks, this can be rewritten as

$$\Delta W/W_2 = 1 - (3/4) R_2/R_1 .$$

If the second spheromak is much smaller than the first, almost all of its magnetic energy gets transformed into thermal energy during coalescence. It appears that this could be an effective means of heating the spheromak plasma. This effect also places a limit on the building up of larger spheromaks by repeated coalescence of smaller ones.

Suppose, by repeated coalescence, we merge n spheromaks of equal size. The total magnetic energy of the final spheromak is $n^{-1/4}$ times the sum of the energies of the initial spheromaks. The radius of the spheromak is increased by a factor $n^{1/4}$.

To determine the change in thermal energy of the spheromak plasma during coalescence we should include also the contribution due to the kinetic energy of the spheromaks on impact, and that due to adiabatic heating (or cooling). We examine the kinetic energy contribution first. Suppose first that we initially have an equilibrium state with two separated spheromaks. The external field is then modified in some way to allow the spheromaks to coalesce. We show, following Shafranov,¹⁶ that the kinetic energy in this case is small unless the initial separation between the spheromaks is quite large. Let Δt be the time it takes to make the necessary changes in the external field, and let L be the initial separation distance of the two spheromaks. The kinetic energy density on impact is of the order

$$\rho L^2 / \Delta t^2 = (t_A / \Delta t)^2 (L/2R)^2 (B^2/4\pi),$$

where ρ is the local mass density of the plasma, t_A is the Alfvén time, R is the major radius of the spheromak, and B is the local magnetic field. In general $t_A \ll \Delta t$, and the kinetic energy is a very small fraction of the magnetic field energy

unless $L \gg R$. The initial state need not be an equilibrium (in fact, in the PS-1 experiment, the initial state with separated islands is not an equilibrium); nevertheless, the contribution of the kinetic energy to the energy balance is not significant unless the velocity of the spheromaks is a substantial fraction of the Alfvén velocity.

We turn finally to the question of adiabatic heating and cooling. For the spherical Rosenbluth-Bussac equilibria, r^3 scales like $K^{3/4}$. After coalescence the plasma occupies a slightly smaller total volume than it does initially, so there is a slight heating effect. For coalescence of two equal spheromaks, for example, the total volume occupied goes down by about 15%. The effect of adiabatic heating is clearly small compared to that of ohmic heating near the x-point.

V. CONCLUSIONS

Using the quasi-ideal model of magnetic reconnection, we have been able to describe spheromak coalescence in two analytically soluble limits: one in which Kadomtsev's theory applies, and one in which Taylor's theory applies. This has enabled us to develop a general picture of spheromak coalescence. We have found that a substantial fraction of the magnetic energy is transformed into thermal energy during coalescence due to ohmic heating near the x-point. Repeated coalescence with smaller spheromaks could provide a particularly effective means of heating a spheromak plasma. Repeated coalescence could also be used to build a large spheromak from smaller ones; but the relatively large amount of magnetic energy dissipated when a small spheromak coalesces with a much larger one places limits on the size of a spheromak that can be practically formed in this way. We have also found that spheromak currents can be maintained via repeated coalescence with "refueling spheromaks" having the appropriate q profile and total poloidal flux. Nonaxisymmetric effects play a key role in this process. Finally, our analysis suggests that coalescence experiments could lead to a better understanding of the Taylor and Kadomtsev models in nonlinear MHD.

ACKNOWLEDGMENTS

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APPENDIX

In this appendix we obtain analytic solutions for our refueling equilibria in the limit of small δq and $\delta\psi$. We look first at the equilibrium discussed at the end of section II, following the approach outlined there to obtain our solution.

In regions 1-3 the distance to the magnetic axis is much smaller than the distance to the symmetry axis, allowing us to use a large aspect ratio approximation there. Using Eq. (4) to specify the poloidal current distribution, the large aspect ratio limit of the Grad-Shafranov equation gives

$$\frac{d}{dV} \left(V \frac{d\psi}{dV} \right) + R^2 q \frac{d}{dV} \left(q \frac{d\psi}{dV} \right) = 0 \quad , \quad (\text{A1})$$

where $V \equiv r^2$, and r is the distance to the magnetic axis. In obtaining this equation we have specialized to circular flux surfaces near the magnetic axis.

In region 1 we expand

$$\psi = \psi_0 + \psi_1 V + \psi_2 V^2 + \dots \quad . \quad (\text{A2})$$

Taylor expanding the given $q(\psi) = q_0 + (\partial q / \partial \psi) (\psi - \psi_0) + \dots$, we substitute it along with Eq. (A2) into Eq. (A1) to determine ψ_2 , ψ_3 , etc. in terms of ψ_0 and ψ_1 . We get for ψ_2 ,

$$\psi_2 = -\psi_1 \left(1 + q_0 \frac{\partial q_0}{\partial \psi} \psi_1 \right) / (2q_0^2) \quad .$$

Its contribution to ψ is small everywhere in region 1 if $\psi_2 \delta a \ll \psi_1$, where $\delta a^2 \equiv V(\psi_0 - \delta\psi_0)$. This condition will be satisfied self-consistently by our solution. The constants ψ_0 and ψ_1 are to be determined by matching.

Region 2 is assumed to be sufficiently small that ψ is approximately constant across this region. In addition, we integrate Eq. (A1) across this region to get a jump condition on ψ' ,

$$\psi'_- [\delta a^2 + R^2 (q_0 - \delta\psi_0 \frac{\partial q_0}{\partial \psi})^2] = \psi'_+ (\delta a^2 + R^2 \delta q_0^2 / 2) , \quad (A3)$$

where δq_0 is the value of δq at $r=0$.

In region (3) we expand out Eq. (A1) and neglect the term containing dq/dV . Again, this approximation is justified self-consistently, taking $d\delta q/d\psi \sim \delta q/\psi$. The solution to the resulting equation is

$$\psi = C_1 \ln (r^2 + \delta q_0^2 R^2) + C_2 , \quad (A4)$$

where C_1 and C_2 are constants to be determined by matching.

In region 4 we can neglect δq entirely, getting the standard elliptic integral solution for the field of a ring with current I and major radius R_0 ,

$$\psi = (4IR_0 / ck^2) (R_0^2 + \tilde{R}^2 + 2\tilde{R}R_0 \sin\tilde{\theta})^{-1/2} [(2-k^2)\mathcal{K}(k^2) - 2E(k^2)] \quad (A5a)$$

$$k^2 = 4\tilde{R}R_0 (\tilde{R}^2 + R_0^2 + 2\tilde{R}R_0 \sin\tilde{\theta})^{-1} , \quad (A5b)$$

where $\tilde{R}, \tilde{\theta}$ are spherical coordinates relative to the center of the ring, and E and K are complete elliptic integrals (we follow here the notation of Abramowitz and Stegun³⁰).

Matching all our solutions together, we get the transcendental equation.

$$y = -\ln[\delta q_0^2(\alpha y + 1)] \quad , \quad (\text{A6a})$$

where

$$y \equiv c\psi_0/I \quad , \quad (\text{A6b})$$

and

$$\alpha \equiv (q_0^2/\delta q_0^2)(\delta\psi_0/\psi_0) \quad . \quad (\text{A6c})$$

This gives

$$I \approx c\psi_0/\ln(1/\delta q_0^2) \quad . \quad (\text{A7})$$

Expressing $4\pi j/c = \lambda B$, we have $\lambda \approx 2/(Rq_0)$ in region 1, and $\lambda \approx 2R\delta q_0(r^2 + \delta q_0^2 R^2)$ in region 3.

Now consider the equilibrium specified by Eqs. (13) and (14). We again use the same approach, but now we need to solve separately in only two different regions. Near the magnetic axis we use the same approximation used in region 3 of the previous equilibrium. Far from the magnetic axis we use the elliptic integral solution used in region 4 previously. Matching these solutions, we again get Eq. (A7).

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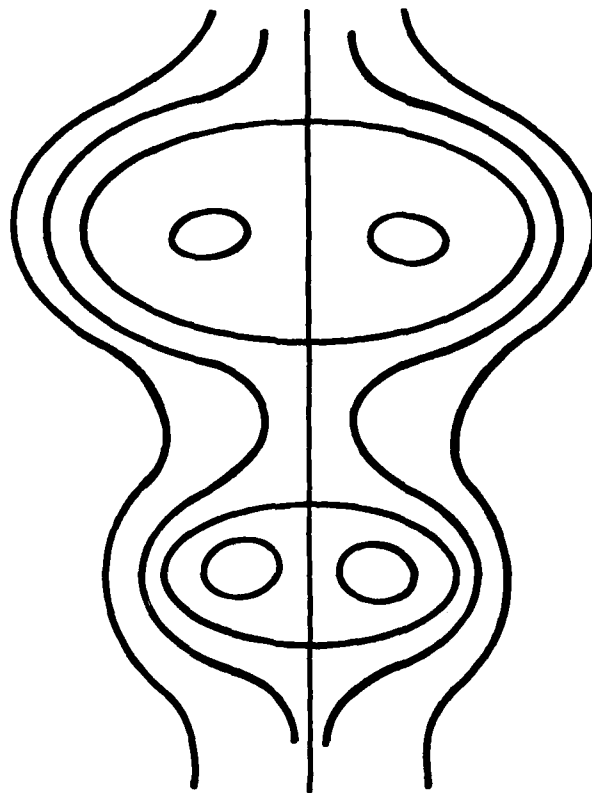
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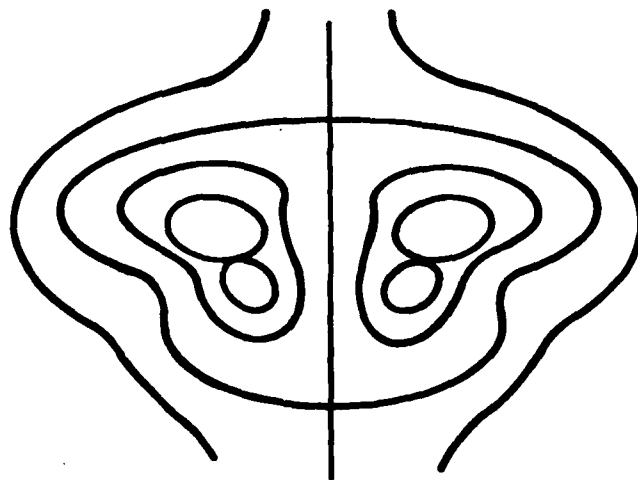
FIGURE CAPTIONS

- Fig. 1. Flux surfaces during axisymmetric coalescence
- a) Initial approach of the two spheromaks
 - b) Reconnection of the magnetic islands
 - c) Final equilibrium after coalescence.
- Fig. 2. Axisymmetric coalescence of spheromaks with different values of ψ on the magnetic axis, ψ_1 and ψ_2 . On the left we show the q profiles before coalescence, and on the right we have $q(\psi)$ after coalescence.
- Fig. 3. Restoring the q profile by purely axisymmetric coalescence.
- a) $q(\psi)$ of the fusion spheromak at $t=0$ and after a small time has elapsed
 - b) Refueling profiles
 - c) $q(\psi)$ after axisymmetric reconnection.
- Fig. 4. The q profiles before and after axisymmetric coalescence for the refueling scheme which makes use of an $n=1, m=1$ tearing mode.
- Fig. 5. The $n=1, m=1$ helical flux function before (solid line) and after (dashed line) the growth of the $n=1, m=1$ tearing mode.
- Fig. 6. The q profiles before and after axisymmetric coalescence for a refueling scheme which uses an $n=1, m=1$ double-tearing mode.
- Fig. 7. The $n=1, m=1$ helical flux function before (solid line) and after (dashed line) the growth of the $n=1, m=1$ double-tearing mode.

(a)



(b)



(c)

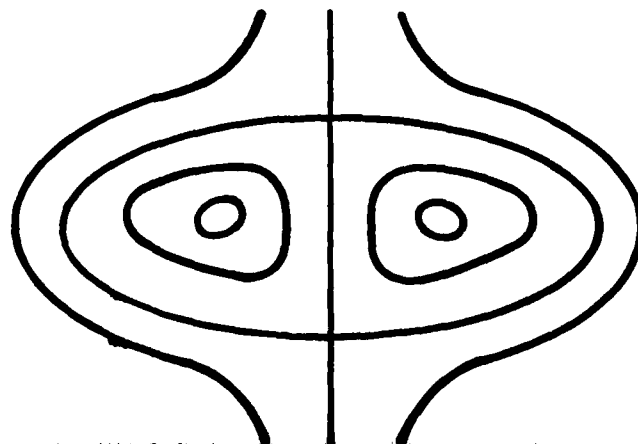


Figure 1

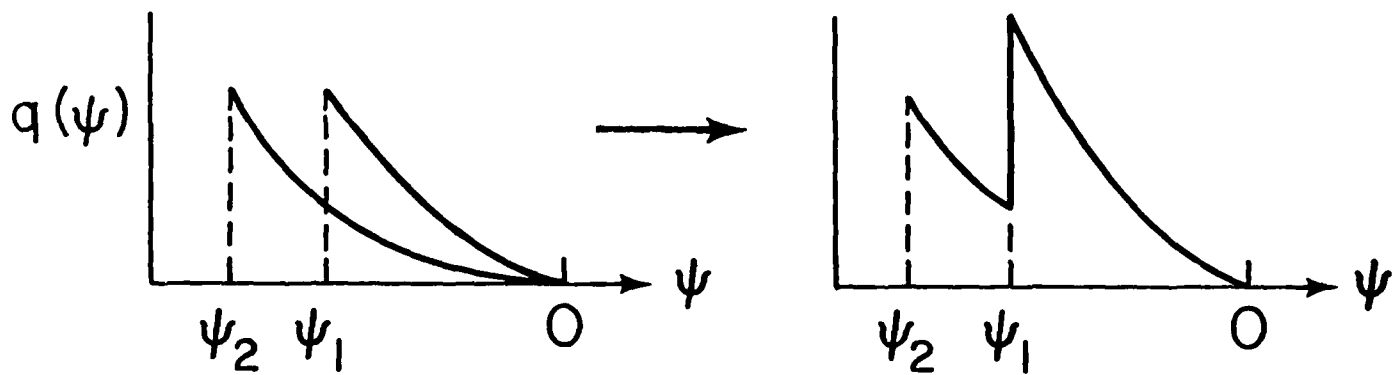


Figure 2

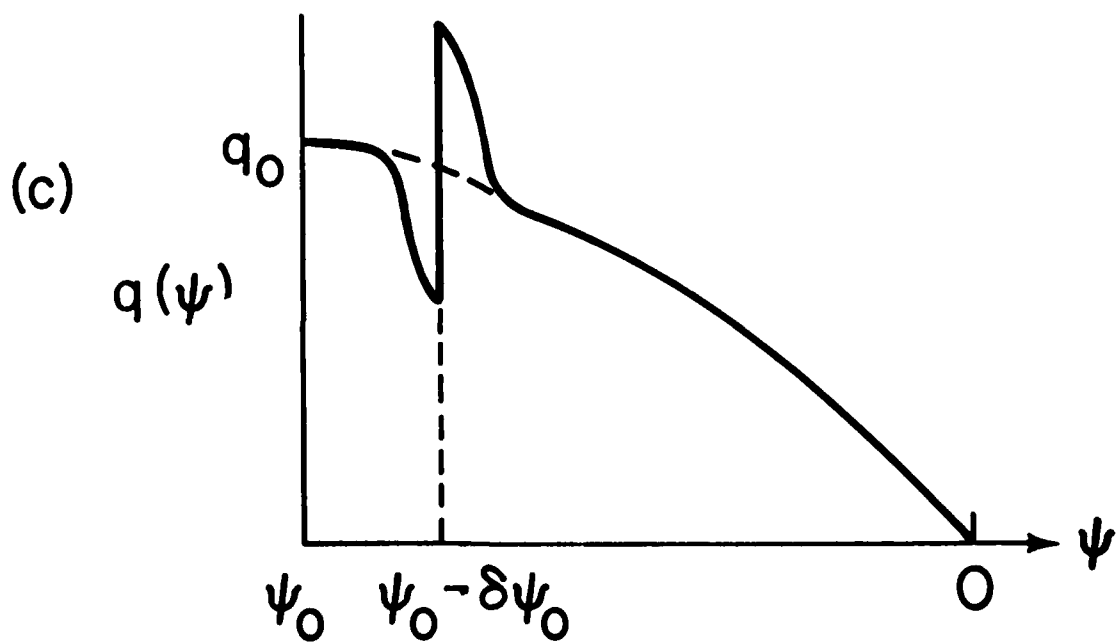
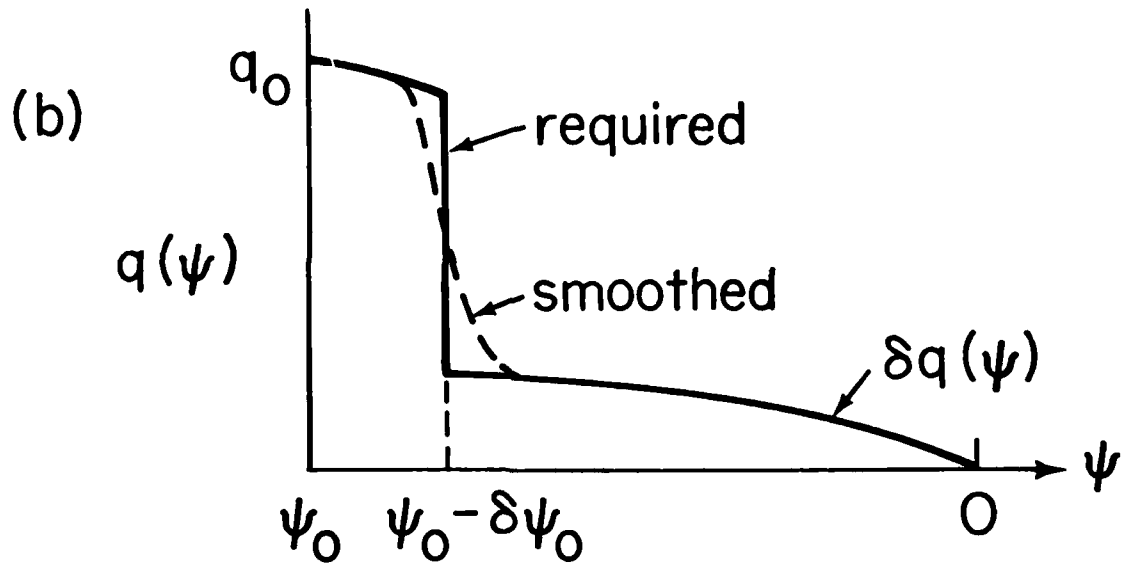
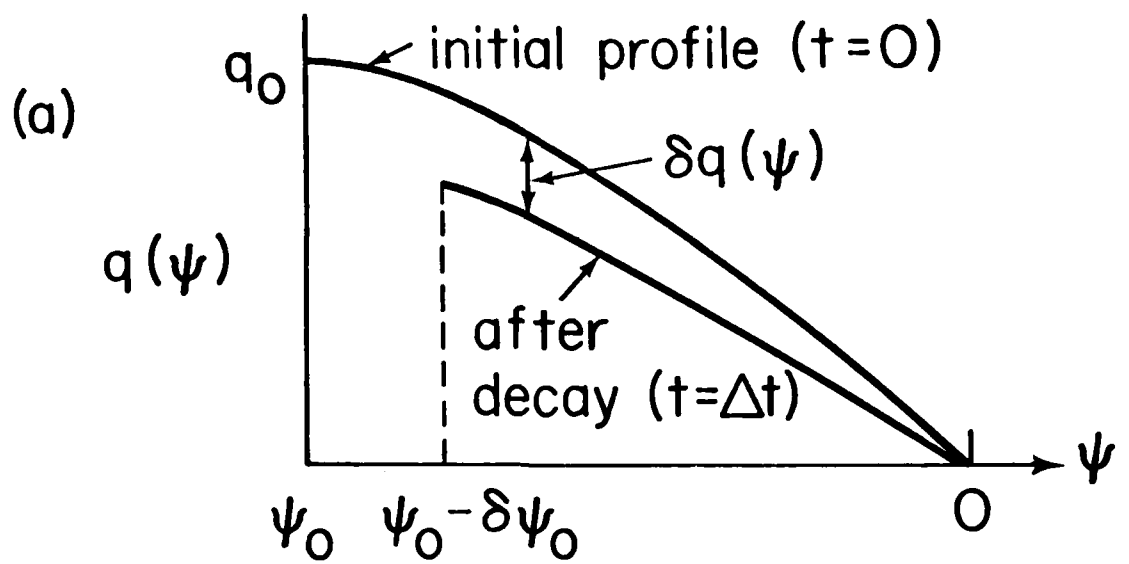


Figure 3

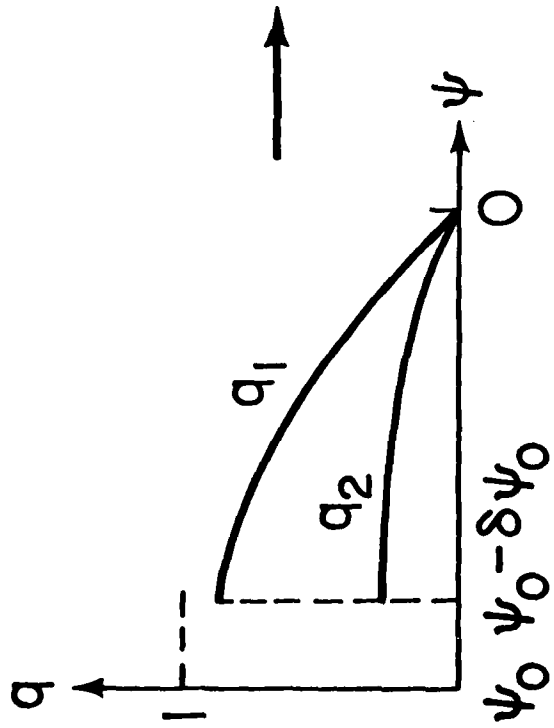
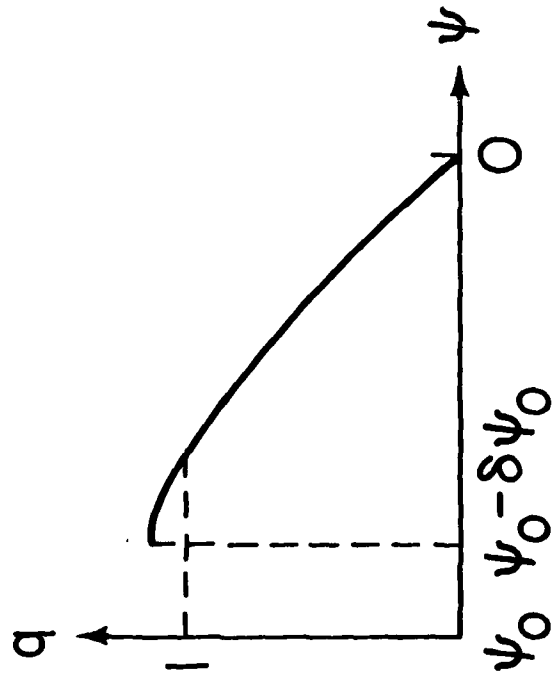


Figure 4

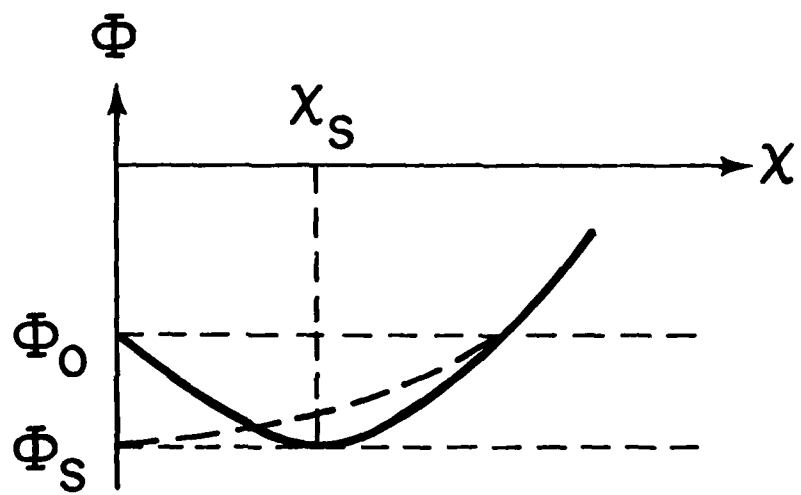


Figure 5

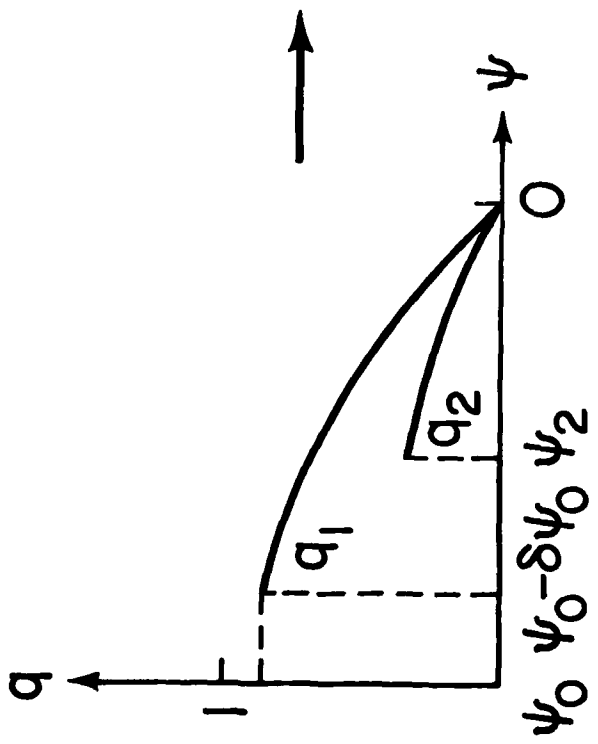
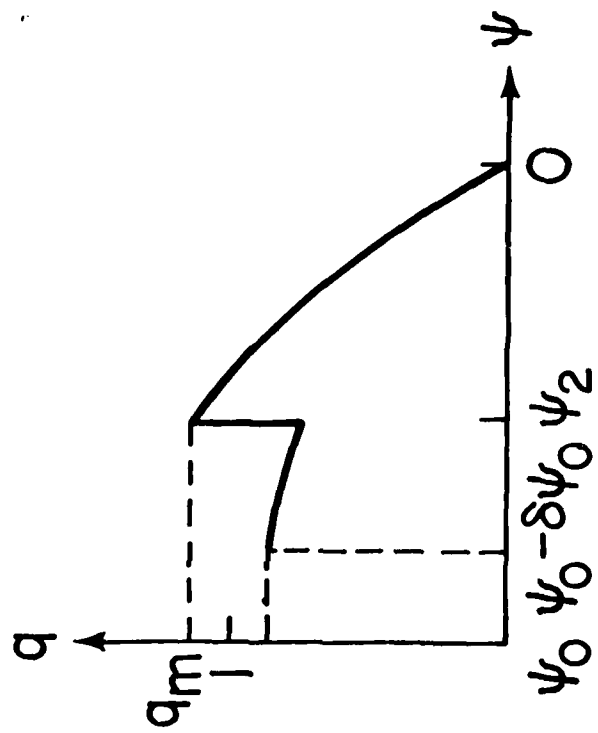


Figure 6

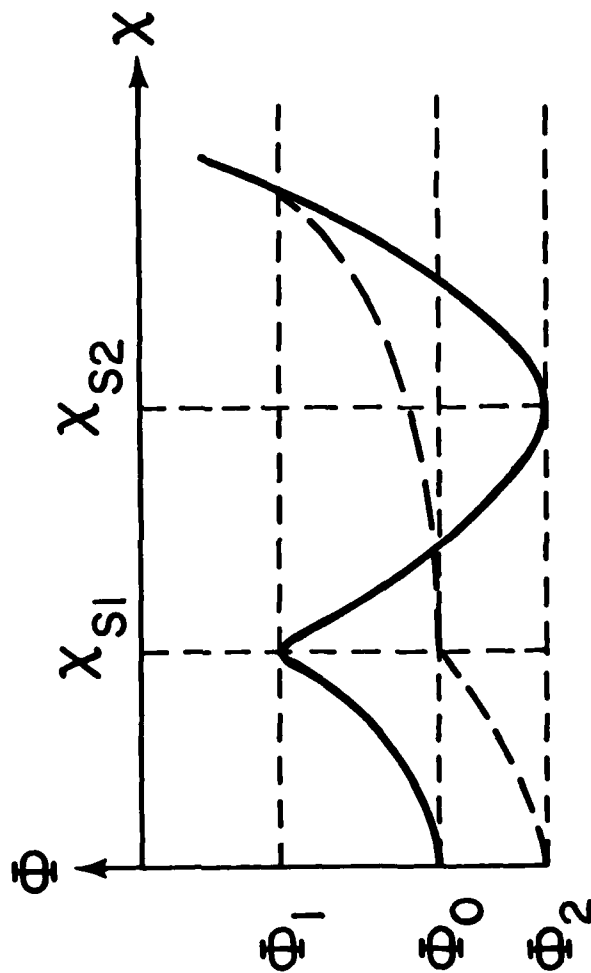


Figure 7

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