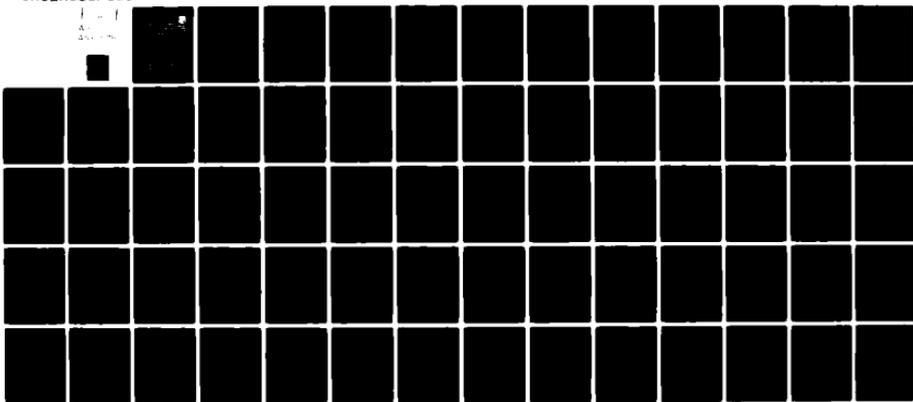


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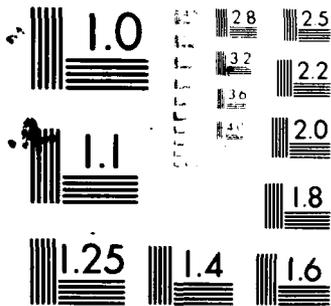
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October 1981



**INTEGRAL EQUATION APPROACH
TO THE PROPAGATION OF LOW-FREQUENCY
GROUNDWAVES OVER IRREGULAR TERRAINS:
II. TWO-DIMENSIONAL TERRAIN FEATURES
AND ELEVATED RECEIVERS**

Pacific-Sierra Research Corp.

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other frequencies. It is well known that the one-dimensional equation is invalid unless the terrain is nearly uniform across a Fresnel zone. It has been found that for obstacles narrower than about 10 km and typical path-lengths the one-dimensional equation erroneously predicts propagation anomalies that (1) are independent of width, and therefore too large, and (2) diminish too slowly at long distances. Considerable error can be incurred by applying the one-dimension equation to moderately sized terrain features. For example, for a path-length of 500 km, that equation overstates by a factor of four the effect of an obstacle 6 km in diameter. It cannot give accurate results unless the diameter approaches a Fresnel zone width-which exceeds 10 km for long propagation paths. Even for wide obstacles, the one-dimensional formulation neglects reflection and interference phenomena.

The two-dimensional integral equation has different forms for elevated and ground-based receivers. The two forms become equivalent in the limit of zero elevation. The stationary-phase approximation can sometimes be used to derive one-dimensional equations, which are widely used for most applications. This reduction in dimensionality causes elevation-angle and end-point errors, both of which are associated with elevated receivers. Elevation errors are large unless the elevation is much smaller than the path-length. End-point errors occur because the stationary-phase approximation improperly accounts for the region near the receiver, and are large for elevated receivers at heights below about one-sixth of a wavelength. The one-dimensional approximations for grounded and elevated receivers are therefore discontinuous near the ground. Neither type of error is large for most ranges and altitudes pertaining to airborne receivers. However, end-point errors are significant below 1 to 20 km, and elevation-angle errors can be important for path-lengths of several tens of kilometers or less. In practice, for receivers that are above the ground but no higher than 1 or 2 km, much of the end-point error can be avoided-without the expense of the two-dimensional solution-simply by using the one-dimensional solution for grounded receivers.

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SUMMARY

This report presents solutions of the one- and two-dimensional integral equations that describe groundwave propagation. We consider the effects of (1) terrain irregularities that are narrower than a Fresnel zone and (2) receiver elevation. Our results define the conditions under which the simpler one-dimensional equation can be used, as well as those that demand the more complex two-dimensional form. We assume a frequency of 100 kHz throughout, although certain results are easily scaled to other frequencies.

It is well known that the one-dimensional equation is invalid unless the terrain is nearly uniform across a Fresnel zone. We quantify the errors and find that, for obstacles narrower than about 10 km and typical pathlengths, the one-dimensional equation erroneously predicts propagation anomalies that (1) are independent of width, and therefore too large, (2) diminish too slowly at long distances, and (3) do not exhibit a diffraction pattern. Considerable error can be incurred by applying the one-dimensional equation to moderately sized terrain features. For example, for a pathlength of 500 km, that equation overstates by a factor of 4 the effect of an obstacle 6 km in diameter. It cannot give accurate results unless the diameter approaches a Fresnel zone width--which exceeds 10 km for long propagation paths. Even for wide obstacles, the one-dimensional formulation neglects reflection and interference phenomena.

The two-dimensional integral equation has different forms for elevated and ground-based receivers. The two forms become equivalent in the limit of zero elevation. The stationary-phase approximation can sometimes be used to derive one-dimensional equations, which are widely used for most applications. This reduction in dimensionality causes elevation-angle and end-point errors, both of which are associated with elevated receivers.

Elevation-angle errors are large unless the elevation is much smaller than the pathlength. End-point errors occur because the stationary-phase approximation improperly accounts for the region

near the receiver, and are large for elevated receivers at heights below about one-sixth of a wavelength. The one-dimensional approximations for grounded and elevated receivers are therefore discontinuous near the ground. Neither type of error is large for most ranges and altitudes pertaining to airborne receivers. However, end-point errors are significant below 1 to 2 km, and elevation-angle errors can be important for pathlengths of several tens of kilometers or less. In practice, for receivers that are above the ground but no higher than 1 or 2 km, much of the end-point error can be avoided--without the expense of the two-dimensional solution--simply by using the one-dimensional solution for grounded receivers.

PREFACE

This report continues Pacific-Sierra Research Corporation's (PSR) analysis of low-frequency groundwave propagation over irregular terrain. In earlier reports, Field and Allen derived validity criteria for application of the one-dimensional integral equation when the terminals are on the ground;^{*} and Gayer, Field, and D'Ambrosio gave extensive numerical results based on this equation for ground-based terminals and generic terrain models.[†]

The present report extends that earlier work by presenting and comparing numerical solutions of the one- and two-dimensional integral equations for elevated receivers and terrain features narrower than a Fresnel zone. In addition to identifying important two-dimensional phenomena, the results define the conditions under which the more convenient one-dimensional equation can be used, as well as those that demand the more complex two-dimensional form.

^{*}E. Field and R. Allen, *Propagation of the Low-Frequency Groundwave over Nonuniform Terrain*, Rome Air Development Center, RADC-TR-78-68, March 1978.

[†]S. Gayer, E. Field, and B. D'Ambrosio, *An Integral Equation Approach to the Propagation of Low-Frequency Ground Waves over Irregular Terrain: I. Ground-Based Terminals*, Rome Air Development Center, RADC-TR-80-334, November 1980.

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I. INTRODUCTION

Historically, two theoretical approaches to the analysis of ground-wave propagation have evolved: the differential equation method of Van der Pol, Norton, Bremmer, and Fock [Bremmer, 1958]; and the integral equation method [Hufford, 1952; Feinberg, 1959]. The differential equation yields virtually exact solutions for plane or spherical surfaces but is awkward for analyzing propagation over terrain whose conductivity is variable or whose topography is uneven. Under conditions of topographical irregularity, numerical solution of the integral equation is the more practical approach.

The general, two-dimensional form of the integral equation is valid provided (1) the conductivity of the earth is great enough to justify applying impedance boundary conditions, and (2) terrain irregularities are not too severe. However, because numerical solution of the two-dimensional integral equation is costly, its use has been limited to highly idealized irregularities [de Jong, 1975], with a much simpler, one-dimensional approximation more commonly used [Johler and Berry, 1967; Johler, 1977; Gressang and Horowitz, 1978].

The one-dimensional equation is derived by means of a stationary-phase integration that reduces the dimensionality of the general version; being an approximation, it is not valid for all antenna elevations or terrain types. It is therefore necessary to define the conditions under which the more convenient one-dimensional equation can be used, as well as those that demand the more complex two-dimensional form.

Field and Allen [1978] derived validity criteria for the one-dimensional equation as applied to ground-based terminals; Gayer, Field, and D'Ambrosio [1980] gave one-dimensional numerical results for ground-based terminals and generic terrain models. The present report extends the earlier work by comparing numerical solutions of the one- and two-dimensional integral equations. For elevated receivers, we show that the one-dimensional results are invalid at heights (1) lower than a reduced wavelength $\lambda/2\pi$, or (2) so large

as to constitute a significant fraction of the pathlength. For ground-based receivers, we quantify the differences between the one- and two-dimensional solutions, finding serious disparities for terrain whose transverse irregularities vary significantly across the first Fresnel zone. For long paths, there are hence severe restrictions on the terrains to which the one-dimensional equation can apply.

Section II presents the one- and two-dimensional integral equations and summarizes the assumptions on which they are based. Section III demonstrates analytically that the one-dimensional equation incorrectly predicts discontinuous fields near the ground, but that the two-dimensional equation correctly predicts continuous fields; it also compares one- and two-dimensional numerical solutions as functions of receiver elevation above a plane earth. Section IV presents numerical solutions for propagation over terrain that contains isolated inhomogeneities, and identifies two-dimensional phenomena not accounted for by the one-dimensional formulation.

The combined results of Secs. III and IV define the regimes of receiver altitude and severity of terrain irregularity for which the one-dimensional treatment may be used. Section V summarizes the conclusions; the appendixes give certain mathematical details.

II. INTEGRAL EQUATIONS FOR GROUND-BASED AND ELEVATED RECEIVERS

This section summarizes the one- and two-dimensional integral equations for ground-wave propagation used throughout this report. Each has two forms, according to whether the receiver is assumed on the ground or elevated. Detailed derivations, having been given elsewhere [Hufford, 1952; Field and Allen, 1978], are not repeated.

We work with the ground-wave attenuation function W , defined by

$$\psi = 2W\psi_0, \quad (1)$$

where ψ is the vertical Hertz potential and ψ_0 is the Hertz potential in free space. The function W accounts for the earth's imperfect conductivity and topographic features. For a flat, perfectly conducting earth, $W = 1$ and $\psi = 2\psi_0$.

TWO-DIMENSIONAL FORMS

If surface impedance is small, terrain variations not too severe, and the terminals on the ground, W satisfies the two-dimensional integral equation.

$$W(p) = 1 + \frac{ik}{2\pi} \int da \frac{r_0}{r_1 r_2} W(Q) e^{ik(r_1+r_2-r_0)} \left[\frac{1}{n_g} + \left(1 + \frac{i}{kr_2} \right) \frac{\partial r_2}{\partial n} \right]. \quad (2)$$

If the receiver is elevated, the following equation for W applies:

$$2W(P) = 1 + \frac{ik}{2\pi} \int da \frac{R_0}{r_1 R_2} W(Q) e^{ik(r_1+R_2-R_0)} \left[\frac{1}{n_g} + \left(1 + \frac{i}{kR_2} \right) \frac{\partial R_2}{\partial n} \right]. \quad (3)$$

In Eqs. (2) and (3), the transmitter is at the origin, and

- k = free-space wave number ω/c ,
- n = unit vector normal to ground,
- n_g = complex refractive index of ground,
- p, P = coordinates of ground-based and elevated receivers, respectively,
- r_0, R_0 = distance from transmitter to receiver,
- da = element of area on earth's surface,
- Q = integration point on earth's surface,
- r_1 = distance between transmitter and integration point Q ,
- r_2, R_2 = distance between Q and receiver points p, P .

Formally, there is no difference between the definitions of p, r_2, r_0 and P, R_2, R_0 . However, following Hufford [1952], we use capital letters in Eq. (3) to emphasize that the receiver is elevated above the ground a distance z_0 .

Although Eqs. (2) and (3) appear similar, they differ fundamentally. Equation (2) is an integral *equation* in that the unknown function $W(p)$ on the left side also appears in the integrand on the right. Equation (3) is an integral *formula*, rather than equation, because the unknown function $W(P)$ on the left refers to the attenuation function above the surface, whereas the integrand on the right contains the attenuation function as defined on the surface. Thus, to obtain W from Eq. (3) for an elevated receiver, we must first solve Eq. (2) to obtain W for points on the surface, then insert those values into the integrand of Eq. (3).

Equations (2) and (3) are the most general forms of the integral equation for the ground-wave attenuation function W . The quantity $1/n_g$ (square brackets) represents the earth's surface impedance and accounts for its imperfect conductivity. The function $\partial r_2/\partial n$ in Eq. (2) accounts for terrain shape. For terminals on a perfectly conducting flat earth, both $1/n_g$ and $\partial r_2/\partial n$ vanish, and $W = 1$, as before.

The function $\partial R_2 / \partial n$ in Eq. (3) accounts for both terrain shape and receiver elevation, and does not vanish for a flat earth. A careful limiting procedure--given in Sec. III--is needed to recapture the well-known result that for a flat earth, $W = 1$ if $n_g \rightarrow \infty$ and $z_0 \rightarrow 0$.

ONE-DIMENSIONAL APPROXIMATION

The two-dimensional Eqs. (2) and (3) are expensive to solve numerically. A major simplification results if the stationary-phase method is used for one of the integrations, thereby reducing Eqs. (2) and (3) to one dimension. Details of the reduction for ground-based receivers [Eq. (2)] are presented in Hufford [1952] and Field and Allen [1978]; the one-dimensional approximation given in those sources is

$$W(x_0) \approx 1 - e^{-\pi i/4} \left(\frac{k}{2\pi}\right)^{1/2} \int_0^{x_0} dx \left[\frac{x_0}{x(x_0 - x)} \right]^{1/2} \\ \times W(x) \left[\frac{1}{n_g} + \left(1 + \frac{1}{kr_2}\right) \frac{\partial r_2}{\partial n} \right] e^{ik(r_1 + r_2 - r_0)}, \quad (4)$$

where r_1 , r_2 , and r_0 are defined as above, the transmitter and receiver both lie on the $y = 0$ plane, and x and x_0 are projections of r_1 and r_0 on the $z = 0$ plane. Hufford also uses the stationary-phase approximation to reduce Eq. (3) (for elevated receivers) to the one-dimensional form

$$2W(x_0, z_0) = 1 - e^{-\pi i/4} \left(\frac{k}{2\pi}\right)^{1/2} \int_0^{x_0} dx \left[\frac{x_0}{x(x_0 - x)} \right]^{1/2} \\ \times W(x) \left[\frac{1}{n_g} + \left(1 + \frac{1}{kR_2}\right) \frac{\partial R_2}{\partial n} \right] e^{ik(r_1 + R_2 - R_0)}. \quad (5)$$

Both Eqs. (4) and (5) are expressed in the classic form derived by Hufford, which assumes that $r_0(R_0)$, r_1 , and $r_2(R_2)$ nearly equal x_0 , x , and $x_0 - x$, respectively; that assumption requires terrain or receiver elevation to be small compared with pathlength. Field and Allen [1978] show that somewhat greater elevation can be accommodated by substituting--as we occasionally do below--

$$\left[\frac{r_0^2}{r_1 r_2 (r_1 + r_2)} \right]^{1/2} \quad \text{for} \quad \left[\frac{x_0}{x(x_0 - x)} \right]^{1/2} \quad (6)$$

in Eq. (4) [and in Eq. (5), if we use R_0 , R_2 instead of r_0 , r_2].

APPLICATION

Although some solutions have been obtained for two-dimensional equations equivalent to Eqs. (2) and (3) (see de Jong [1975] and Secs. III and IV, below), most calculations for irregular terrain have been based on the far simpler Eqs. (4) and (5) [Johler and Berry, 1967; Johler, 1977; Gressang and Horowitz, 1978]. It is therefore important to define the constraints on applicability imposed by the stationary-phase reduction of the two-dimensional equation.

In analyzing the one-dimensional equation for ground-based receivers, Field and Allen [1978] show that the reduction in dimensionality is valid provided (1) the propagation path is much longer than a wavelength, (2) terrain features have small lateral gradients transverse to the direct propagation path (that is, gentle hills or valleys), and (3) terrain features vary only slightly across the first Fresnel zone. We show in Sec. IV that the third condition severely limits the applicability of the one-dimensional equation.

The validity criteria for reducing the two-dimensional equation for elevated receivers to one-dimensional form have not received much attention in the literature. We show in the following section that Eq. (5) is valid provided (1) the receiver is not too close to

the ground, and (2) the receiver height is much less than the path-length. Those restrictions are additional to the ones already given for ground-based receivers.

III. VALIDITY REGIME OF ONE-DIMENSIONAL EQUATION
FOR ELEVATED RECEIVERS

Here, we quantify the errors that result from transforming the two-dimensional equations to one-dimensional form. We begin by demonstrating that the two-dimensional formulation correctly represents the continuity of the Hertz potential at low receiver elevations, whereas the approximation does not. Next, we show that the stationary-phase reduction from two dimensions to one is inaccurate unless the elevation angle of the receiver is quite low. Finally, we compare numerical results for both formulations and determine the accuracy of the one-dimensional equations as a function of receiver height and elevation angle.

Because the discussion pertains solely to the effects of the receiver's elevation above the ground, derivations and examples are for a smooth-plane earth. The question of the one-dimensional form's ability to accurately represent propagation over irregular terrain is addressed in Sec. IV.

CONTINUITY OF TWO-DIMENSIONAL EQUATIONS NEAR THE GROUND

If the receiver is on a plane earth, $\partial r_2 / \partial n$ is identically zero, and Eq. (2) becomes

$$W(p) = 1 + \frac{ik}{2\pi} \int da \frac{r_0}{r_1 r_2} W(Q) \left(\frac{1}{n_g} \right) e^{ik(r_1 + r_2 - r_0)}. \quad (7)$$

however, if the receiver is a height z_0 above the ground,

$$\frac{\partial R_2}{\partial n} = \frac{-z_0}{(r_2^2 + z_0^2)^{1/2}} = \frac{-z_0}{R_2}, \quad (8)$$

and Eq. (3) becomes

$$2W(P) = 1 + \frac{ik}{2\pi} \int da \frac{R_0}{r_1 R_2} W(Q) \left[\frac{1}{n_g} - \left(1 + \frac{i}{kR_2} \right) \frac{z_0}{R_2} \right] e^{ik(r_1 + R_2 - R_0)} \quad (9)$$

Continuity of the fields requires that Eqs. (7) and (9) become equal in the limit $z_0 \rightarrow 0$. However, the equality is by no means immediately apparent, given the 2 on the left of Eq. (9) and the terms proportional to z_0/R_2 on the right.

To demonstrate the low-elevation equivalence of Eqs. (7) and (9), we must recognize that even though

$$\lim_{z_0 \rightarrow 0} R_0 \rightarrow r_0 \quad (10a)$$

and

$$\lim_{z_0 \rightarrow 0} R_2 \rightarrow r_2, \quad (10b)$$

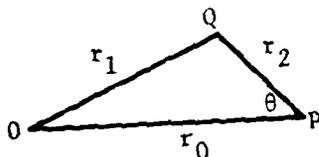
there is a small region centered about the subreceiver point (near $r_2 = 0$) in which

$$\lim_{z_0 \rightarrow 0} \frac{\partial R_2}{\partial n}$$

is undefined [refer to Eq. (8)], and that a careful limiting procedure is therefore required. Subtracting Eq. (7) from Eq. (9) and applying Eqs. (8) and (10), it follows that continuity of the fields near the ground requires

$$\begin{aligned} 2W(P) - W(P) = & \left(\frac{-ikr_0}{2\pi} \right) \int_0^{2\pi} d\theta \int_0^\infty dr_2 \frac{W(r_2, \theta)}{r_1} \left(\frac{r_2 z_0}{r_2^2 + z_0^2} \right) e^{ik(r_1 + r_2 - r_0)} \\ & + \frac{r_0}{2\pi} \int_0^{2\pi} d\theta \int_0^\infty dr_2 \frac{W(r_2, \theta)}{r_1} \left[\frac{r_2 z_0}{(r_2^2 + z_0^2)^{3/2}} \right] e^{ik(r_1 + r_2 - r_0)} \end{aligned} \quad (11)$$

in the limit $z_0 \rightarrow 0$. The coordinate system for Eq. (11) is



where $da = r_2 dr_2 d\theta$ and the point P (not shown) is a height z_0 directly above p.

The first term on the right of Eq. (11) vanishes because of the well-known relations

$$\lim_{z_0 \rightarrow 0} \frac{z_0}{r_2^2 + z_0^2} = \pi \delta(r_2) \quad (12)$$

and

$$r_2 \delta(r_2) = 0, \quad (13)$$

where $\delta(r_2)$ denotes the Dirac delta function. The integrand in the second term of Eq. (11) contains the term

$$\lim_{z_0 \rightarrow 0} \frac{r_2 z_0}{(r_2^2 + z_0^2)^{3/2}},$$

which is as strongly peaked as the delta function at $r_2 = 0$ but is zero elsewhere. We may therefore evaluate all other terms in the integrand at $r_2 = 0$ and factor them outside the integration sign, giving

$$\lim_{z_0 \rightarrow 0} [2W(P) - W(p)] = \frac{W(p)}{2\pi} \int_0^{2\pi} d\theta \int_0^{\infty} dr_2 \frac{r_2 z_0}{(r_2^2 + z_0^2)^{3/2}}. \quad (14)$$

The r_2 integration in Eq. (14)--found in standard tables--gives unity, and the θ integration obviously equals 2π . Thus, we have the correct continuity relation

$$\lim_{z_0 \rightarrow 0} [2W(P) - W(p)] = W(p) . \quad (15)$$

Two points should be noted regarding this demonstration of continuity. First, the essential term [the second on the right in Eq. (11)] stems from $(i/kR_2) (\partial R_2/\partial n)$ in Eq. (3). That quantity is often ignored (e.g., Hufford [1952]) on the grounds that it is a near-field correction unimportant for long paths--although ignoring it is clearly an error for low-elevation receivers. Second, a two-dimensional integration is needed to achieve continuity of the fields.

DISCONTINUITY OF ONE-DIMENSIONAL EQUATIONS NEAR THE GROUND

The discontinuity of the one-dimensional equations is easily demonstrated by attempting to apply the limiting procedure to Eqs. (4) and (5) rather than to Eqs. (2) and (3). We find that continuity of the one-dimensional equations requires

$$\begin{aligned} 2W(x_0, z_0) - W(x_0, 0) = & e^{-\pi i/4} \left(\frac{k}{2\pi}\right)^{1/2} \int_0^{x_0} dx W(x, 0) \left(\frac{x_0}{xR_2}\right)^{1/2} \frac{z_0}{R_2} \\ & + e^{\pi i/4} \left(\frac{1}{2\pi k}\right)^{1/2} \int_0^{x_0} dx W(x, 0) \left(\frac{x_0}{xR_2}\right)^{1/2} \frac{z_0}{R_2} \end{aligned} \quad (16)$$

in the limit $z_0 \rightarrow 0$ [just as Eq. (11) is required for the two-dimensional case]. Equation (16) incorporates Eq. (6)--the substitution form--and uses the cartesian notation $P = (x_0, z_0)$ and $p = (x_0, 0)$, all points lying in the plane $y = 0$.

It is evident that Eq. (16) cannot satisfy the continuity requirement expressed by Eq. (15), since the dependence on the wave

number k cannot be eliminated from either right-side term. For continuity to apply at all wavelengths, the k dependence must disappear in the limit $z_0 \rightarrow 0$ --as was accomplished in proceeding from Eq. (11) to Eq. (15) for the two-dimensional form.

The integrations in Eq. (16) produce numerical inconsistencies in addition to the conceptual problem just mentioned. To illustrate, we proceed as before [Eqs. (11) to (15)], arguing that $1/R_2$ peaks sharply about $x = x_0$ and factoring all terms outside the integrals. After some rearrangement and a change of variable, we have the following one-dimensional relation to replace Eq. (14):

$$\begin{aligned} \lim_{z_0 \rightarrow 0} [2W(x_0, z_0) - W(x_0, 0)] &= e^{-\pi i/4} \left(\frac{k}{2\pi}\right)^{1/2} \\ &\times W(x_0, 0) \int_0^{x_0} dx \frac{z_0}{(x^2 + z_0^2)^{3/4}} \\ &+ e^{\pi i/4} \left(\frac{1}{2\pi k}\right)^{1/2} W(x_0, 0) \int_0^{x_0} dx \\ &\times \frac{z_0}{(x^2 + z_0^2)^{5/4}}. \end{aligned} \quad (17)$$

The integrals can be evaluated in terms of the incomplete beta function [Gradshteyn and Ryzhik, 1965]. Here, we simply note that the first integral is proportional to $z_0^{1/2}$ and vanishes as $z_0 \rightarrow 0$, and that the second is proportional to $1/z_0^{1/2}$ and diverges as $z_0 \rightarrow 0$. We also recall that the second integral results from the (often ignored) near-field term in Eq. (5)-- $(i/kR_2)(\partial R_2/\partial n)$.

If near-fields are retained, the one-dimensional equation for elevated receivers thus gives an undefined result as altitude approaches zero. If near-fields are ignored, however, the right side of Eq. (17) vanishes and continuity, as expressed by Eq. (15), is

violated by a factor of 2. In either case, the one-dimensional equation for elevated receivers is clearly in error if the altitude is less than what we define below.

END-POINT AND ELEVATION-ANGLE ERRORS IN ONE-DIMENSIONAL EQUATIONS

The stationary-phase reduction of the two-dimensional equations is summarized in Appendix A. For receivers on a plane earth, the reduction in dimensionality is accurate provided $kx_0 \gg 1$. For elevated receivers, the reduction is subject to two additional constraints, which we call "end-point" and "elevation-angle" errors.*

The end-point error arises because (as discussed above) if receiver elevation is low, the term $(i/kR_2) (\partial R_2/\partial n)$ in Eq. (3) peaks very sharply near the end of the propagation path. The end-point term, which is not accounted for in the standard stationary-phase treatment, must undergo a full two-dimensional integration over a small region centered about the subreceiver point. The correction--details of which are given in Appendix A--that must be added to the right side of the one-dimensional Eq. (5) is as follows:†

$$\text{end-point correction} \approx \begin{cases} W(x_0, 0), & \text{if } 2\pi z_0/\lambda \ll 1; & (18) \\ W(x_0, 0)/kz_0, & \text{if } 2\pi z_0/\lambda \gg 1. & (19) \end{cases}$$

Equation (18) shows that the end-point correction will cancel half the left side of Eq. (5) if $2\pi z_0/\lambda \ll 1$, thereby removing the factor of 2 and causing Eqs. (4) and (5) to yield the same value for W near the ground. Omitting the correction under the same conditions will cause significant errors. However, Eq. (19) shows that as

* Although corrections to these errors are derived for a plane earth, they pertain solely to receiver elevation and should apply even for irregular terrain.

† To avoid counting the end-point region twice, and to avoid the low-elevation divergence discussed in connection with Eq. (17), we must also exclude the region surrounding $x \approx x_0$ when integrating the terms in Eq. (5) that contain $\partial R_2/\partial n$.

receiver elevation increases, the end-point error in the one-dimensional equation becomes smaller--being in fact negligible if $2\pi z_0/\lambda \gg 1$.

Certain of the approximations used to transform Eq. (3) into Eq. (5) break down (see Appendix A) if the receiver elevation angle z_0/x_0 becomes too large. That constraint on elevation angle can be combined with Eqs. (18) and (19) to give the following criteria for the domain of validity of the one-dimensional equation:

$$\frac{\lambda}{2\pi} \ll z_0 \ll x_0 . \quad (20)$$

The lower limit on receiver height is the end-point error; the upper limit is the elevation-angle error.

NUMERICAL COMPARISON OF ONE- AND TWO-DIMENSIONAL SOLUTIONS FOR RECEIVERS ABOVE A PLANE EARTH

The one-dimensional equations, much less complicated and costly to solve numerically than the two-dimensional forms, should be used whenever possible. Below, we quantify the altitude regime over which the simpler equations may be used by comparing one-dimensional solutions with those of the more accurate two-dimensional version. To isolate inaccuracies due solely to elevation, we make the comparisons for a plane earth. Section IV presents a complementary analysis that compares one- and two-dimensional solutions for terrain irregularities of various sizes.

Figure 1 shows the phase of W calculated from the one- and two-dimensional equations for a frequency of 100 kHz and a plane earth having a conductivity of 10^{-2} mhos/m. The curves trace phase as a function of distance for several receiver altitudes between 1 m and 10 km.

For comparison, we also show the phase of W as computed from Norton's [1937] well-known solution of the differential wave equation--which is inherently two-dimensional. Although Norton's solution is a

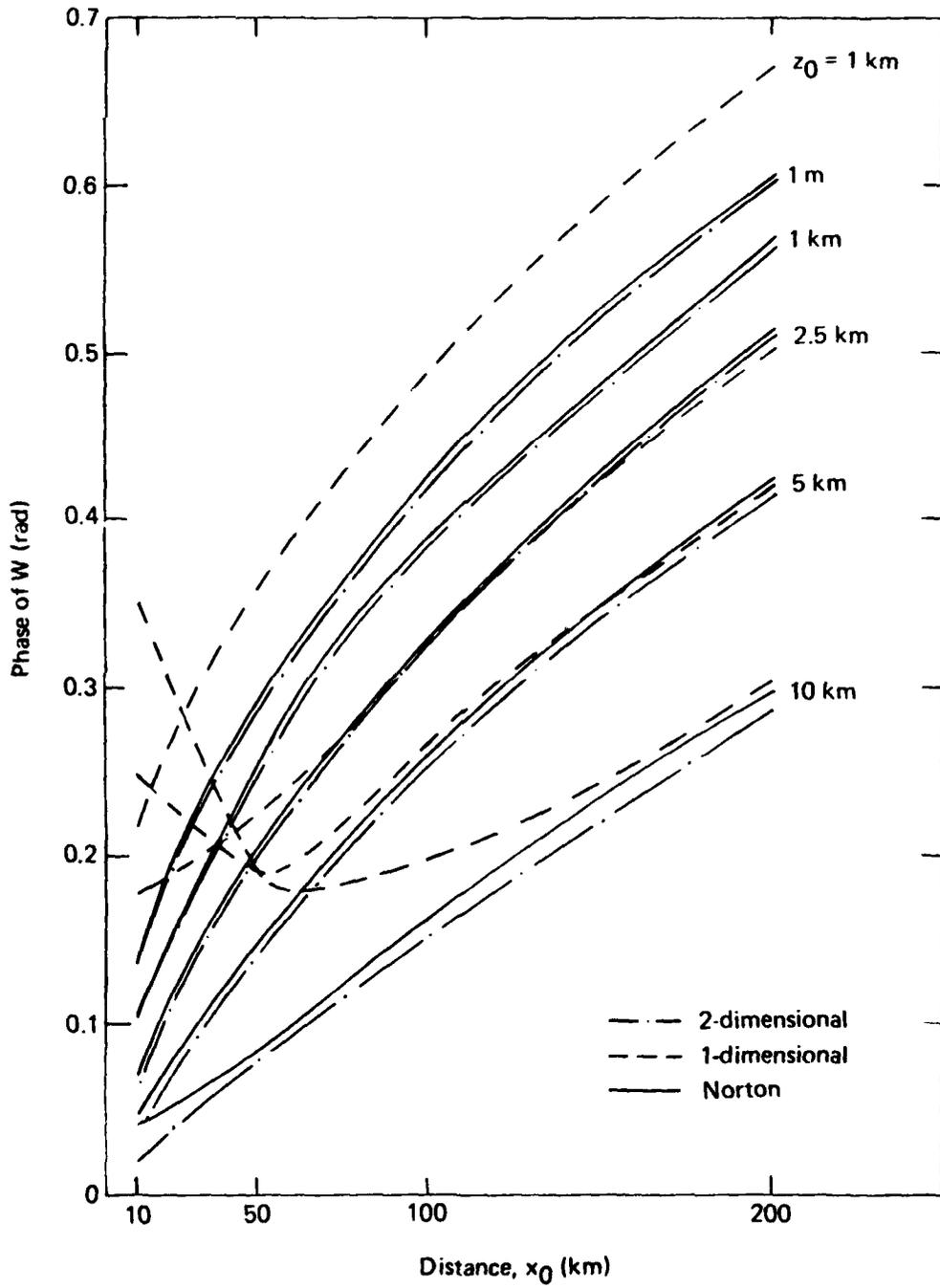


Fig. 1--Phase of W versus distance x_0 for several receiver heights z_0 :
plane earth, $\sigma = 10^{-2}$ mhos/m, frequency = 100 kHz

far simpler way of obtaining W for a plane earth than is either of the integral equations, it is valid only for a plane earth, whereas the integral equations are also valid--subject to the constraints given in Sec. IV--for propagation over irregular terrain. The comparison in Fig. 1 is useful for defining the elevations at which each integral solution agrees with the classic results for the idealized model. The integral equations are presumably also valid at the same elevations for more realistic terrain, where Norton's solution cannot be used.

As expected, our two-dimensional solutions agree well with Norton's for all distances and elevations shown. The small differences are due to rounding in our numerical solution and, at very short distances, to Norton's ignoring terms of order $1/(kx_0)^2$ and $(z_0/x_0)^2$. Also as expected, our one-dimensional results are accurate provided the constraints on z_0 expressed by Eq. (20) are satisfied. However, they exhibit large errors if the elevation is 1 km or less (end-point error) or if z_0/x_0 is too large (elevation-angle error). The error in the one-dimensional results for $z_0 = 1$ m is so great as to be off scale--a consequence of the $z_0^{-1/2}$ divergence discussed in connection with Eq. (17).

The reliability of phase predictions made using the one-dimensional equation is illustrated in Fig. 2, which shows contours of constant one-dimensional errors in the elevation-distance plane. We define phase errors as the differences (in meters) between the one- and two-dimensional results shown in Fig. 1. Figure 2 also shows several contours of constant elevation angle z_0/x_0 , and on the right axis, the elevation z_0 expressed in units of reduced wavelength $\lambda/2\pi$. The upper branches of the contours correspond to elevation-angle errors; the lower branches correspond to end-point errors.

The contours in Fig. 2 conform closely to the locus of points in the $x_0 - z_0$ domain corresponding to the validity conditions of Eq. (20). For example, to achieve a phase error smaller than, say, 50 m, the receiver must be inside the 50 m contour. From Fig. 2, we see that the receiver must then be below the line $z_0/x_0 \approx 0.25$ (elevation-angle error) and above the line $2\pi\lambda_0/z_0 \approx 2$ (end-point

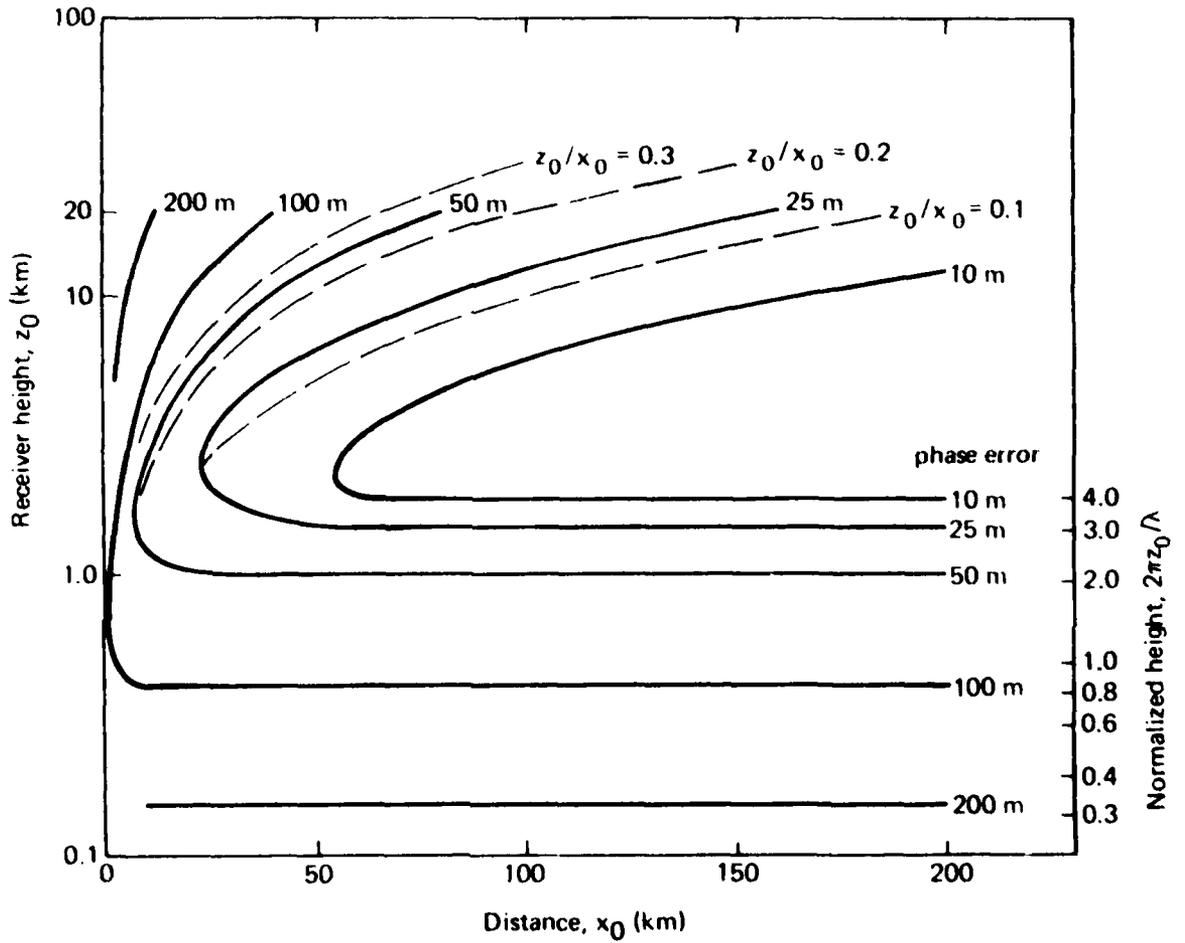


Fig. 2--Contours of constant phase error for the one-dimensional equation and contours of constant elevation angle: plane earth, $\sigma = 10^{-2}$ mhos/m, frequency = 100 kHz. Note the conversion 1 radian \approx 477 m.

error). An accuracy of 25 m requires the receiver to be below the line $z_0/x_0 \approx 0.13$ and above the line $2\pi z_0/\lambda \approx 3$. The usable domain of the one-dimensional equation thus shrinks as error tolerance tightens.

Fortunately, the end-point and elevation-angle errors caused by the one-dimensional approximation are small at most ranges and altitudes relevant to airborne receivers. Johler and Berry [1967], for example, present one-dimensional calculations at 100 kHz for airborne receivers at distances of 150 to 400 km and altitudes of 2.5 to 10 km; Fig. 2 shows that range of parameters to be almost entirely within the 10 m error contour. However, the one-dimensional equation can yield significant phase errors at 100 kHz for receivers positioned below 1 km or--for higher altitudes--at ranges of less than several tens of kilometers.

In practice, much of the error for receivers that are above the ground, but no higher than 1 or 2 km, can be avoided--without incurring the expense of the two-dimensional solution--by using the one-dimensional formulation for grounded receivers even though the receivers are actually slightly elevated. The situation is illustrated in Fig. 3, which again pertains to a frequency of 100 kHz and a plane earth of conductivity 10^{-2} mhos/m. The curves shown apply to all ranges, provided $z_0/x_0 < 0.1$ (to make elevation-angle errors unimportant). For an airborne receiver at an altitude of 10 km, for example, the curves apply to all ranges beyond 100 km. The end-point error decreases with elevation, whereas the difference between the phase on the ground [computed from Eq. (4)] and that at z_0 [computed from Eq. (3)] increases. The two curves cross at 1.3 km, where the phase error is about 30 m.

Thus, at least for the conductivity and frequency in question, one-dimensional end-point errors can be kept below 30 m simply by computing the phase as if it were on the ground [Eq. (4)] for altitudes under 1.3 km, and by computing the phase from Eq. (5) for greater heights. The accuracy that results is contained within the shaded region in Fig. 3; greater accuracy requires numerical solution of the two-dimensional equation.

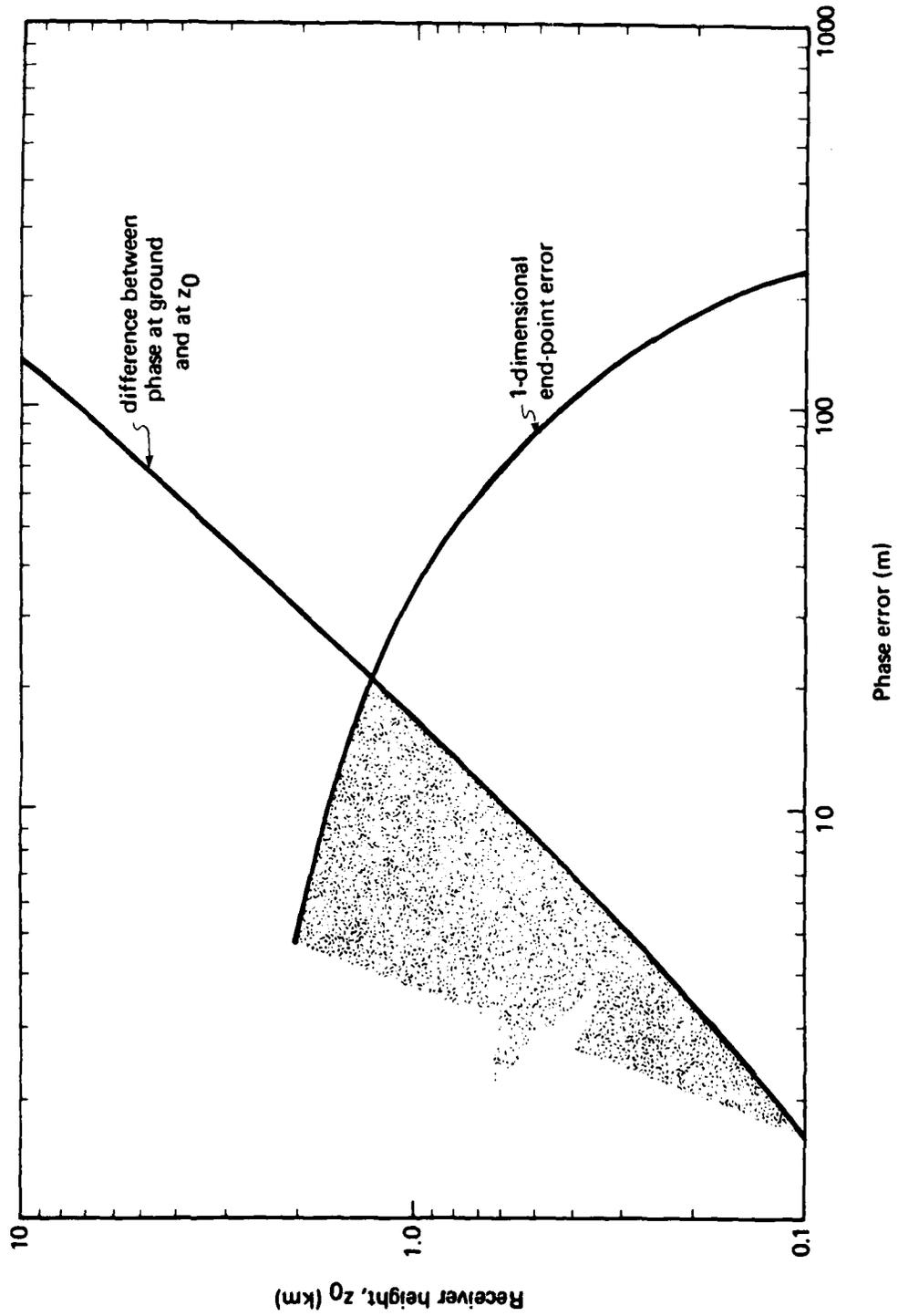


Fig. 3--Envelope of minimum phase error (shaded region) achievable with one-dimensional solutions: plane earth, $\sigma = 10^{-2}$ mhos/m, frequency = 100 kHz. Note the conversion 1 radian \approx 477 m.

IV. TWO-DIMENSIONAL SOLUTIONS FOR ISOLATED IRREGULARITIES

We showed in Sec. III and Appendix A that, even if the terminals are on the ground, the stationary-phase derivation of the one-dimensional equation is valid only for terrain that is relatively uniform across the width of the first Fresnel zone, and that more severe irregularities require the two-dimensional equation. For typical pathlengths of a few hundred kilometers, the Fresnel zone width $(\lambda x_0)^{1/2}$ is on the order of tens of kilometers at a frequency of 100 kHz. Since irregularities in actual terrain are often smaller than tens of kilometers, the regime of the one-dimensional equation's applicability is quite restricted.

This section compares numerical solutions of the one- and two-dimensional equations for propagation over surfaces having irregularities smaller than a Fresnel zone. First, we modify the two-dimensional equation to a form more efficient for computing the effects of isolated irregularities than the standard form, Eq. (2). Then we give an approximate analytic solution to this equation which--although valid only under limited conditions--lends insight to the dependence of W on the location and transverse dimension of the irregularity. Finally, we present solutions to the full two-dimensional equation.

INTEGRAL EQUATIONS FOR ISOLATED IRREGULARITIES ON A HOMOGENEOUS PLANE

The standard form of the two-dimensional integral equation--given by Eq. (2)--requires integration over the entire surface. Such an integration is necessary for continuously varying or undulating terrain, but not for terrain that is uniform except for a few isolated irregularities. In the case of mostly regular terrain, we can take advantage of the fact that the attenuation function is already known on a plane or sphere having the properties of the homogeneous part of the surface. Specifically, we write

$$W = W_1 W_H, \quad (21)$$

where W_H is the known solution for a uniform surface, and W_1 is the new unknown function. For propagation on a spherical earth, W_H would be the well-known residue-series solution [Bremmer, 1958]. Here, we consider irregularities on an otherwise uniform plane for which W_H is the Norton-Sommerfeld function [Norton, 1937] given by

$$W_H(r) = 1 + i \sqrt{\pi\eta} e^{-\eta} \operatorname{erfc}(-i \sqrt{\eta}) , \quad (22)$$

where

$$\eta = ikr/2n_g^2 \quad (23)$$

and erfc is the complement of the error function [Abramowitz and Stegun, 1972].

Consider first a plane where the refractive index is n_g everywhere except for a certain region where it is \tilde{n}_g --a lake on flat land or a flat island in the ocean, for example. In such a case, Appendix B shows that the integral equation governing W_1 is

$$W_1(r_0) = 1 + \frac{ik}{2\pi} \iint dx dy \Delta(x, y) \frac{r_0}{r_1 r_2} e^{ik(r_1+r_2-r_0)} \times \left[\frac{W_H(r_1)W_H(r_2)}{W_H(r_0)} \right] W_1(x, y) , \quad (24)$$

where

$$\Delta(x, y) = \frac{1}{\tilde{n}_g(x, y)} - \frac{1}{n_g} \quad (25)$$

is the impedance contrast between the lake or island and the surrounding region.

At first, Eq. (24) looks more formidable than Eq. (2) because W_H appears in the integrand. It is, however, much less costly to

solve numerically because the integration need be carried out only over regions where $\Delta \neq 0$, rather than over the entire $x - y$ plane. The savings are well worth the added complexity of including the Norton-Sommerfeld functions in the integrand. Equation (24) is related to a two-dimensional integral equation derived--but not solved--by Feinberg [1945].

As a second example, consider a uniform plane of refractive index n_g on which sits an isolated hill. Unlike the first example, we have no impedance contrast here, but have instead a region where $\partial r_2 / \partial n \neq 0$. For such a case, we show in Appendix B that W_1 satisfies

$$W_1(r_0) = 1 + \frac{ik}{2\pi} \iint dx dy \left[\left(1 + \frac{i}{kr_2} \right) \frac{\partial r_2}{\partial n} \right] \frac{r_0}{r_1 r_2} e^{ik(r_1 + r_2 - r_0)} \\ \times \left[\frac{W_H(r_1) W_H(r_2)}{W_H(r_0)} \right] W_1(x, y), \quad (26)$$

which is similar to Eq. (24) except that terms involving $\partial r_2 / \partial n$ appear in place of terms involving Δ . Equation (26) is numerically more convenient than Eq. (2) because integration is required only where $\partial r_2 / \partial n$ is significant. However, the integration region can extend well beyond the hill itself; Eq. (26) is thus more costly to solve than Eq. (24), which requires integration only over the region actually occupied by the irregularity. That is, the "irregularity" caused by an isolated hill is characterized by $\partial r_2 / \partial n$, which affects a region much larger than the hill itself.

BORN-APPROXIMATION SOLUTION FOR WEAK, ISOLATED IRREGULARITY

We can solve Eq. (24) approximately for a weak impedance irregularity embedded in a perfectly conducting plane, where W_H is unity and $W_1 = W$. We set $W_1 \approx 1$ on the right side to obtain

$$W_1 \approx 1 + \frac{ik}{2\pi} \iint dx dy \Lambda(x, y) \frac{x_0}{r_1 r_2} e^{ik(r_1 + r_2 - x_0)}, \quad (27)$$

for terminals on the x axis. Equation (27) is valid if the second term on the right is small, giving a posteriori justification of our assumption that $W_1 \approx 1$. If the irregularity were a small hill not too near the terminals, $\partial r_2 / \partial n$ would replace Δ in Eq. (27). This perturbation solution is equivalent to the well-known Born approximation.

Equation (27) requires a double integration over the impedance contrast $\Delta(x, y)$. We assume the simple form

$$\Delta(x, y) = \Delta_0 \exp \left[- \frac{(x - \bar{x})^2}{(\Delta x)^2} - \frac{(y - \bar{y})^2}{(\Delta y)^2} \right], \quad (28)$$

which allows us to illustrate the dependence of W_1 on the lateral gradients and position of the perturbation by adjusting (Δx) , (Δy) , \bar{x} , or \bar{y} , respectively. We can perform the integrations using the saddle-point approximation, provided $kx_0 \gg 1$, $\Delta x \ll x_0$, and $2\Delta x < \bar{x} < 2(x_0 - \Delta x)$. The procedure is described by Field and Joiner [1979], and the result is:

$$W_1 \approx 1 + i\Delta_0(\Delta x) \left[\frac{ikx_0}{2\bar{x}(x_0 - \bar{x})} \right]^{1/2} \Lambda^{1/2} \exp \left[- \Lambda \frac{\bar{y}^2}{(\Delta y)^2} \right], \quad (29)$$

where

$$\Lambda(x_0, \bar{x}, \Delta y) = \frac{1}{1 + \frac{i}{\pi} \left(\frac{d}{\Delta y} \right)^2 \left[\frac{4\bar{x}(x_0 - \bar{x})}{x_0^2} \right]} \quad (30)$$

and

$$d = \frac{\sqrt{\lambda x_0}}{2}. \quad (31)$$

In Eqs. (29) through (31), d is the maximum half-width of the first Fresnel zone. Λ accounts for the width and longitudinal position of

the perturbation, and the exponent in Eq. (29) for its transverse location. The factor $4\bar{x}(x_0 - \bar{x})/x_0^2$ in Eq. (30) accounts for the narrowing of the Fresnel zone away from midpath, and has a maximum value of unity when $\bar{x} = x_0/2$. Although valid only under very restricted conditions, Eqs. (29) through (31) reveal several important characteristics of the attenuation function.

Weak One-Dimensional Irregularity

By taking the limits $\Delta y \gg d$ and \bar{y} , we find that Eq. (29) assumes the following form for a wide (i.e., one-dimensional) on-path perturbation:

$$W_1 \xrightarrow{\Delta y \rightarrow \infty} 1 + \Delta_0 \frac{(\Delta x)(ikx_0)^{1/2}}{\left[2\bar{x}(x_0 - \bar{x})\right]^{1/2}} \quad (32)$$

The denominator in Eq. (32) is largest at midpath, $\bar{x} = x_0/2$, and small near the endpoints, $\bar{x} \approx 0$ or x_0 . The effects of the perturbation are therefore greatest if it is near either terminal, and smallest near midpath. This enhanced importance of regions near the terminals is well known [Feinberg, 1959].

Narrow On-Path Irregularity

For perturbations much narrower than a Fresnel zone $\Delta y \ll d$, and Eq. (30) becomes

$$\Lambda \xrightarrow{\Delta y \ll d} - \frac{i\pi(\Delta y/d)^2}{4\bar{x}(x_0 - \bar{x})/x_0^2} \quad (33)$$

The second term on the right of Eq. (32) must therefore be reduced by a factor of magnitude

* Recall that the derivation of Eq. (29) prohibits moving the perturbation to within $\sim 2\Delta x$ of either terminal.

$$|\Lambda^{1/2}| \approx \frac{\sqrt{\pi} \left(\frac{\Delta y}{d} \right)}{2 \sqrt{\bar{x}(x_0 - \bar{x})/x_0^2}} \quad (34)$$

to account for the limited width of the perturbation. Equation (34) gives the intuitively reasonable result that the two-dimensional correction factor is proportional to the fraction of the Fresnel zone effectively occupied by the impedance contrast.

For narrow perturbations the reduction indicated by Eq. (34)--but omitted by one-dimensional equations--can be considerable. For example, for $\Delta y = 3$ km (a fairly sizable terrain feature), $\lambda = 3$ km, $x_0 = 500$ km, and $\bar{x} = x_0/2$, this two-dimensional correction factor is about 0.27. For these parameters, the one-dimensional formulation therefore overstates the impact of the perturbation by nearly a factor of four.

Although strictly valid only for the weak Gaussian impedance contrast assumed above, the above discussion shows that the one-dimensional integral equation is in serious error unless the perturbation is nearly as wide as a Fresnel zone. Details will depend on the structure of a given irregularity, but the conclusion is general; *viz. two-dimensional solutions are needed unless terrain features are nearly uniform over transverse distances on the order of $\sqrt{\lambda x_0}$.* The notion that the stationary-phase reduction of the two-dimensional equations requires slight variation across a Fresnel zone is certainly not new [Bremmer, 1958; Feinberg, 1945]. Nonetheless, the inaccuracy of the one-dimensional equation at low frequencies for terrain features narrower than, say, ten kilometers is often overlooked in applications.

Next, we examine the dependence of W_1 on pathlength by rewriting Eq. (34) in the form

$$|\Lambda^{1/2}| = \frac{\sqrt{\pi} \Delta y}{\sqrt{\bar{x}\lambda}} \left[\frac{1}{1 - \bar{x}/x_0} \right]^{1/2}, \quad (35)$$

which shows that, for fixed \bar{x} , the effect of the perturbation diminishes with increasing pathlength according to $|1 - \bar{x}/x_0|^{-1/2}$. This "recovery" of W_1 occurs because the width of the Fresnel zone increases as the path is made longer; and the fraction occupied by the perturbation therefore decreases. This two-dimensional phenomena causes W_1 to recover beyond an obstacle more quickly than predicted by the one-dimensional approximation.

Isolated Off-Path Irregularity

We showed above that the one-dimensional result, Eq. (32), must be reduced by the factor in Eq. (34) to account for an on-path perturbation being narrower than a Fresnel zone. If a narrow terrain feature is centered off-path, the result in Eq. (32) must be reduced by still another factor. By combining Eq. (30) with the exponential term in Eq. (29), we find this "off-path" factor F to be

$$F \approx \exp \left[-\frac{\bar{y}^2}{(\Delta y)^2 + \frac{id^2}{\pi}} \right], \quad (36)$$

where, for simplicity, we have assumed $\bar{x} = x_0/2$. Equation (36) shows that the "off-path" factor is governed by the Fresnel zone width for narrow perturbations ($\Delta y \ll d$), and depends only weakly on off-path distance \bar{y} , for wide ones. F is related to a more complicated off-path factor used by King and Wait [1976].

NUMERICAL SOLUTIONS OF THE TWO-DIMENSIONAL EQUATION

The algorithm used to solve Eqs. (24) and (26) is given in Appendix C. The first two examples given below compare the numerical solutions against analytic and experimental results, thereby validating our approach. The third example compares solutions of the one- and two-dimensional equations for terrain features of various widths. Because computational expense is a consideration, all examples pertain to impedance--rather than topographic--irregularities.

Weak Gaussian Impedance Contrast

Equations (29) and (32) give two- and one-dimensional analytic solutions, respectively, for a weak Gaussian impedance contrast [Eq. (26)] in a conducting plane. Their main value is to lend insight into the behavior of W , since their range of validity is too restricted for application to actual terrain. Nonetheless, if we confine ourselves to weak perturbations where $|W_1 - 1| \ll 1$, we can use Eqs. (29) and (32) to validate numerical solutions of Eqs. (24) and (4), respectively.

The following table gives the phase of W_1 as computed by the analytic approximations and numerical solutions. The assumed maximum impedance contrast Δ_0 corresponds to a conductivity σ of 10^{-2} mhos/m, and the frequency is 100 kHz. The other parameters (see table) were chosen to satisfy the validity restrictions on Eqs. (29) and (32). If we had selected parameters to give a more substantial W_1 --e.g., greater Δx or lower conductivity--we would have violated these restrictions. Moreover, since $\sqrt{\lambda x_0}/2 \approx 8.7$ for the case shown, our use of $\Delta y = 10$ km corresponds to a relatively wide perturbation that does not cause major two-dimensional effects. Use of a much narrower perturbation would reduce $W_1 - 1$ to where it could not be resolved within our numerical accuracy.

COMPARISON BETWEEN ANALYTICAL AND NUMERICAL SOLUTIONS
FOR A WEAK GAUSSIAN IMPEDANCE CONTRAST

	Phase of W_1 (rad)	
	One-Dimensional	Two-Dimensional
Numerical solution		
Eq. (4)	1.97×10^{-2}	
Eq. (24)		1.90×10^{-2}
Analytical solution		
Eq. (32)	1.97×10^{-2}	
Eq. (29)		1.89×10^{-2}

NOTE: $\Delta y = 10$ km, $\bar{x} = 50$ km, $\bar{y} = 0$, $\Delta x = 50$ km,
 $x_0 = 200$ km.

For the above reasons, the phase anomalies shown in the table are very small and the one- and two-dimensional results do not differ substantially. Nonetheless, these results confirm two important facts. First, our numerical algorithms are accurate. Second, the one-dimensional equation overstates the effect of an on-path irregularity that is narrower than a Fresnel zone.

Conducting Region on Dielectric Plane

King and Tsukamoto [1966] made laboratory measurements of surface waves propagating over a plane dielectric surface containing isolated conducting regions. De Jong [1975] obtained good agreement with these data by numerically solving a two-dimensional integral equation similar to our Eq. (24), albeit of somewhat different form. To our knowledge, his results are the only full two-dimensional solutions available in the literature. Before proceeding to models representing groundwave propagation, we compare our calculations with de Jong's.

Figures 4 and 5 show the propagation model and our numerical solutions for the amplitude and phase of W_1 . A rectangular "island" of infinite conductivity is embedded in a dielectric plane having an impedance $(n_g)^{-1}$ of $0.11 - 9.15 \times 10^{-3} i$. The width and length of the island are 3 and 6.75 km, respectively, and the frequency is 100 Hz. The width of the Fresnel zone is $\sqrt{3 \times 30} \approx 9.5$ km for a path-length of 30 km. The island is therefore much narrower than a Fresnel zone, and we expect large differences between the one- and two-dimensional solutions.

Our two-dimensional results are in excellent agreement with de Jong's, if allowance is made for his assumed transmitter and receiver being a small fraction of a wavelength above the surface. Moreover, they agree well with the data (not repeated here) everywhere except in a narrow region beginning at the rear of the island and extending for one or two wavelengths. Even our calculated standing-wave structure agrees closely with King and Tsukamoto's data and de Jong's calculations.

We concur with de Jong's explanation that the region of poor agreement with experiment is caused by the well-known failure of

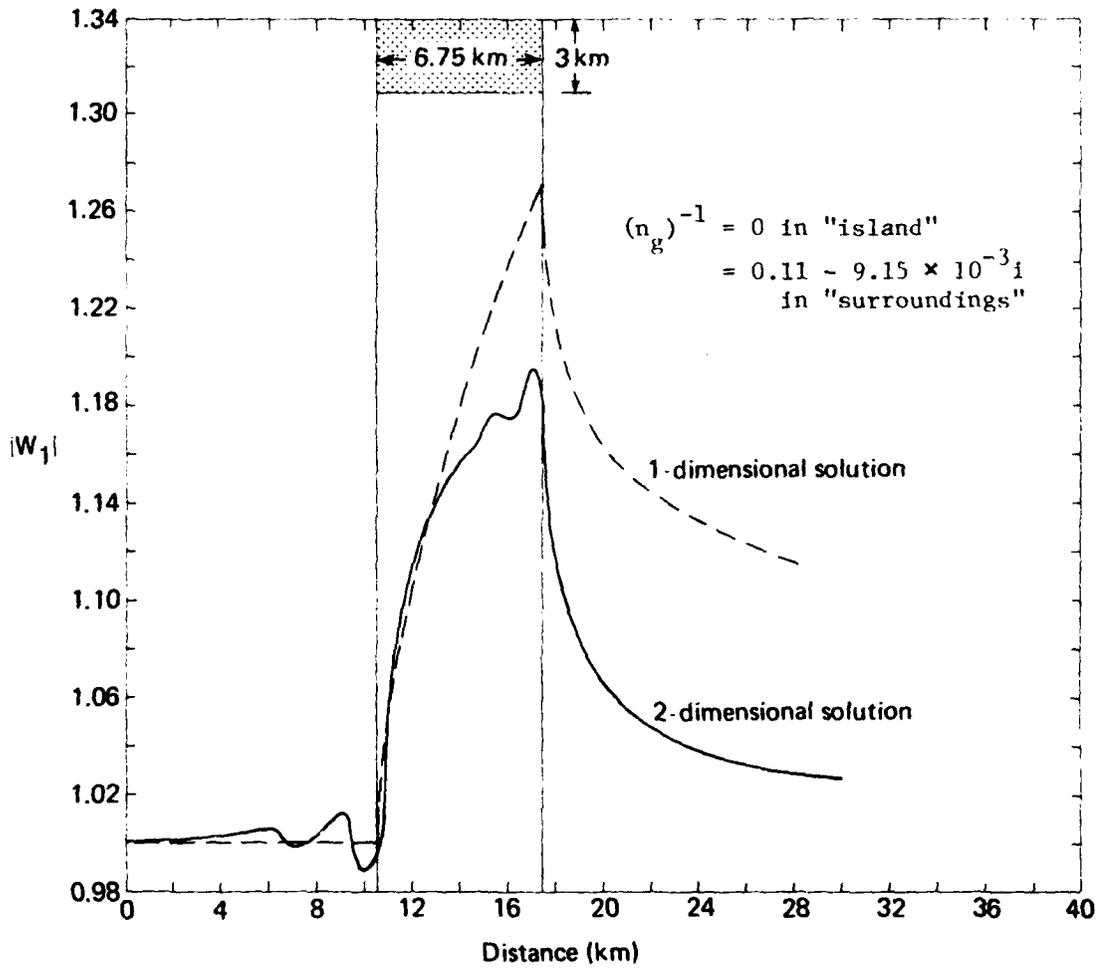


Fig. 4--Calculated magnitude of W_1 for a conducting "island" (shaded region) on a dielectric plane: frequency = 100 kHz, nonelevated terminals

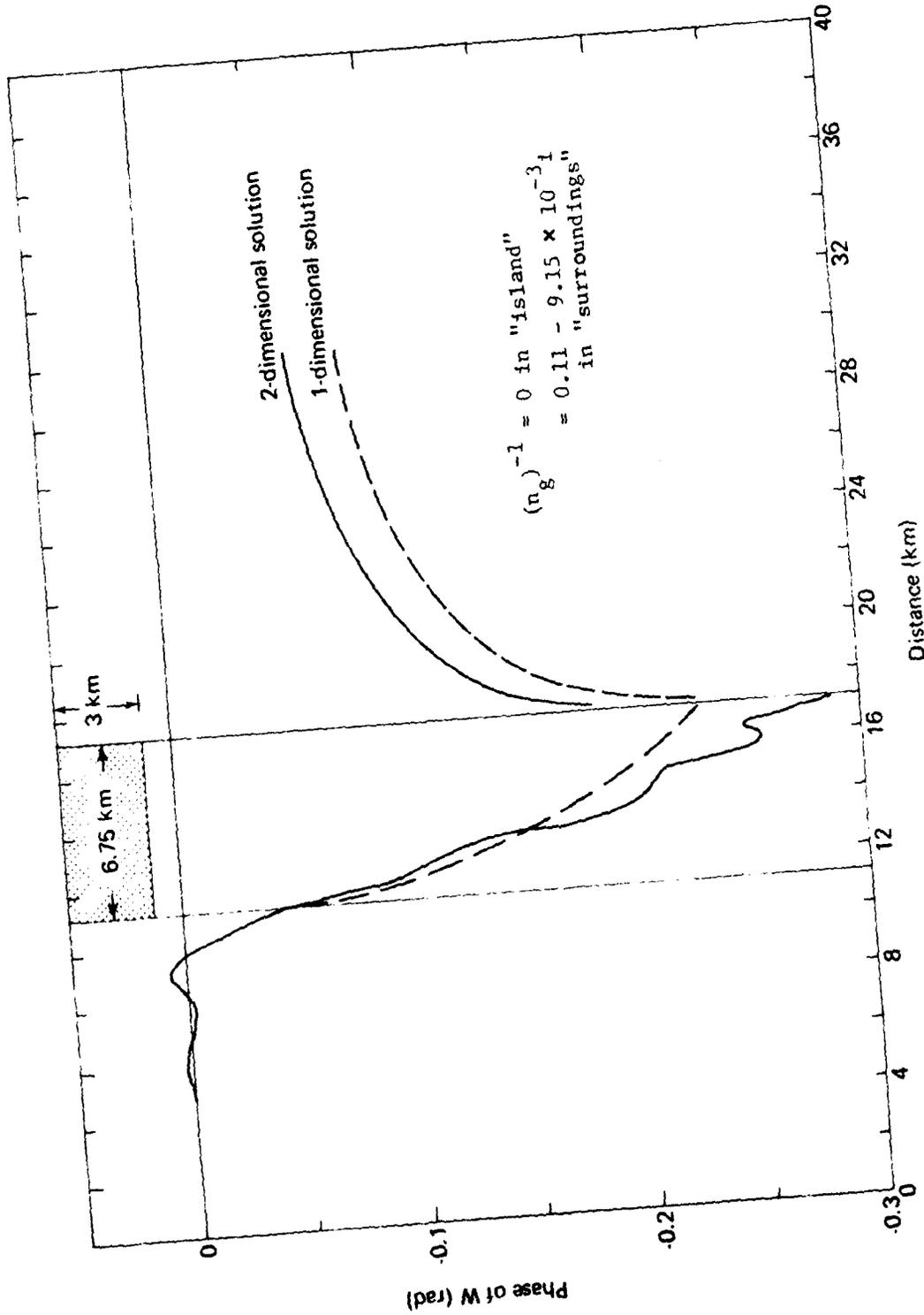


Fig. 5--Calculated phase of W_1 for a conducting "island" (shaded region) on a dielectric plane: frequency = 100 kHz, nonelevated terminals

impedance boundary conditions near abrupt conductivity changes. The discontinuous behavior of the phase at the rear of the island indicates incorrect boundary conditions. It is encouraging that this poor accuracy persists for only one or two wavelengths beyond the boundary, after which our results (and de Jong's) recover to the correct values.

Figures 4 and 5 quantify several defects of the one-dimensional solution which are expected on intuitive grounds. First, the one-dimensional solution overstates the amplitude anomaly caused by the island--a consequence of its treating the island as infinitely wide. Second, it recovers too slowly beyond the island--a consequence of not accounting for Fresnel zone widening as the pathlength increases. Third, it omits the standing-wave patterns evident from the two-dimensional solutions--a consequence of the stationary-phase approximation neglecting terrain features beyond the receiver.

Recall that Figs. 4 and 5 pertain to W_1 , which accounts solely for the effect of the island. These results must be multiplied by W_H [see Eq. (21)], given in Fig. 6, to obtain the full attenuation function W .

Saltwater "Lake" on Poor Ground

Having verified our approach for two idealized cases, we next consider parameters more representative of long-wave propagation on the earth. Even here we use a simple model to avoid unnecessary computer expense. Our method can readily treat more complicated situations, albeit at increased cost.

Our illustrative model assumes a conductivity $\sigma(x, y)$ given by

$$\begin{aligned} \sigma(x, y) &= 10^{-3} + 4 \cos^2 \left| \frac{\pi}{2} \left(\frac{x - 50}{10} \right) \right| \cos^2 \left| \frac{\pi}{2} \frac{y}{\Delta y} \right| \text{ mhos/m ,} \\ & \quad |x - 50| \leq 10 \text{ km} \\ & \quad \text{if } |y| \leq \Delta y , \\ & = 10^{-3} \text{ mhos/m ,} \quad \text{otherwise .} \end{aligned} \tag{37}$$

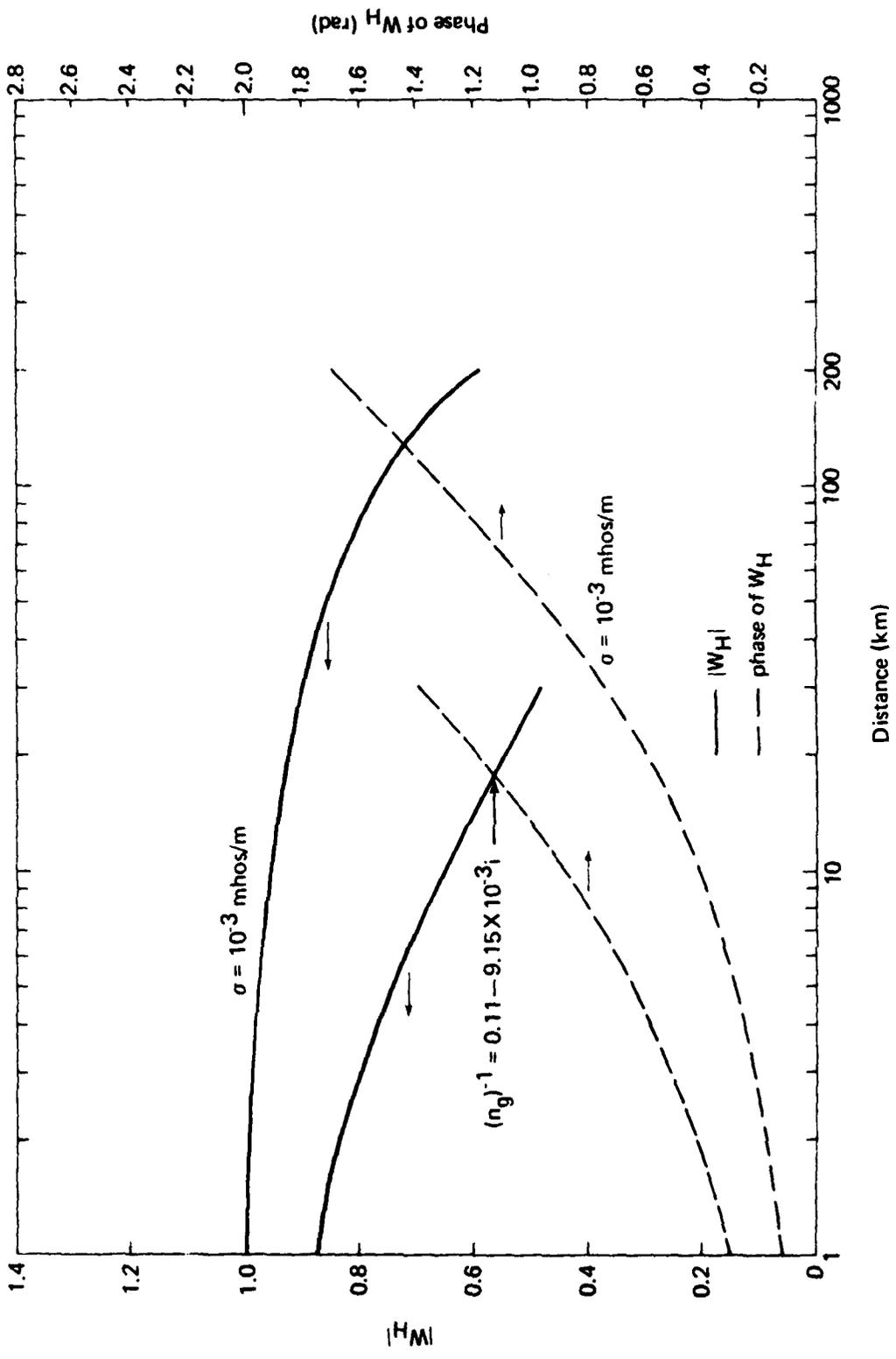


Fig. 6--Attenuation function for plane earth: frequency = 100 kHz

This conductivity might correspond to a saltwater lake or bay of variable depth. We assume this "lake" to be embedded in ground of conductivity 10^{-3} mhos/m and dielectric constant 10, and to extend from $x = 40$ km to $x = 60$ km. Its half-width is specified by Δy , which we will vary. The model given by Eq. (37) has the desirable feature of continuous lateral derivatives. However, our assumed large conductivity contrast causes significant impedance changes within a fraction of a kilometer of the boundaries.

Figure 7 shows two-dimensional solutions [Eq. (24)] for W_1 versus distance for half-widths Δy of 3, 5, and 7 km. We also show for reference the one-dimensional solution [Eq. (4)]. All show little or no effect in front of the lake, a pronounced enhancement in it, and various degrees of recovery beyond it. The levels of enhancement and recovery shown should *not* be generalized to other parameters. Wait [1964] gives comprehensive one-dimensional results for land-sea-land propagation, and shows that the enhancement in the sea portion, and subsequent recovery, depend strongly on virtually all parameters of the problem. Even simple-looking one-dimensional results shown in Fig. 7 could change drastically if we altered the lake's location or length, or the ground's conductivity. The results in Fig. 7 must be multiplied by W_H , given in Fig. 6, to recover the full attenuation function W .

The accuracy of the solutions is poor near the rear boundary, because the impedance boundary conditions fail near strong lateral gradients. However, the preceding example showed that the accuracy recovers at some distance beyond the irregularity. Thus, we should restrict attention to regions at least a few wavelengths from the boundaries of our model lake.

The maximum half-width of the first Fresnel zone is about 12 and 17 km for pathlengths of 200 and 400 km, respectively. All models shown in Fig. 7 are therefore fairly narrow, although the one having $\Delta y = 7$ km occupies a sizable fraction of the first zone. As expected, the two-dimensional solutions more closely approach the one-dimensional solution as Δy increases. Also, as predicted by Eq. (35), the one-dimensional solution recovers too slowly beyond the lake. Finally, and most important, the one-dimensional solution erroneously gives

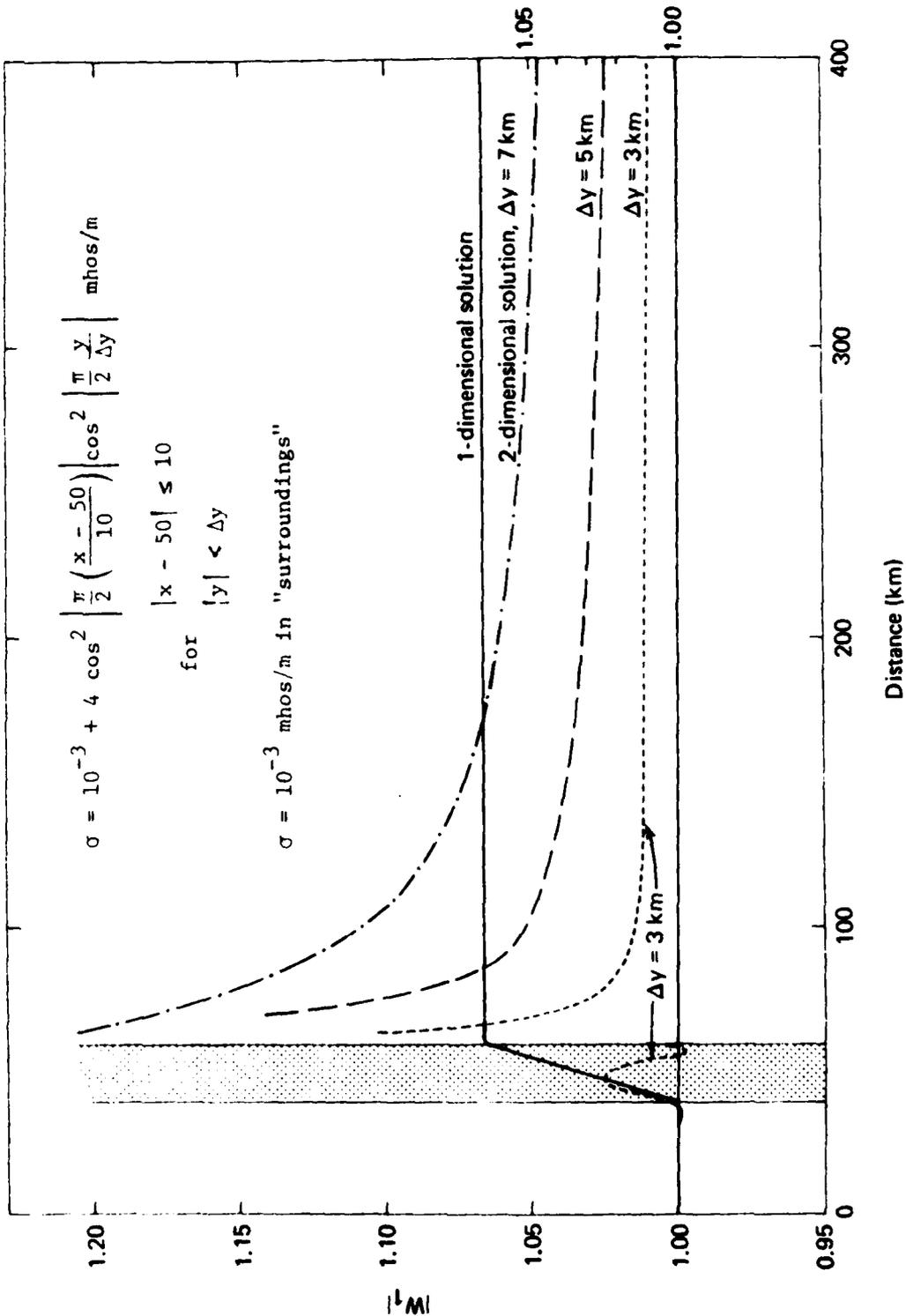


Fig. 7--Magnitude of W_1 for seawater regions (shaded area) of various widths on a poorly conducting plane earth: frequency = 100 kHz

the same result for all values of Δy . Even for $\Delta y = 7$ km--wider than many natural impedance or topographic irregularities--the one-dimensional solution strongly overstates the propagation anomaly for long pathlengths.

Figure 7 shows that, although the one- and two-dimensional solutions approach one another for wide disturbances and long pathlengths, they differ for several tens of kilometers beyond the lake. Thus, we have the apparent paradox that--near a terrain feature--the solution of the two-dimensional equation *does not* approach that of the one-dimensional equation as $\Delta y \rightarrow \infty$.^{*} We believe that this disagreement occurs because the two-dimensional equation is more accurate, even for a one-dimensional terrain feature--e.g., one that has no y-dependence. One-dimensional features cause reflections, resonances, and standing waves--all of which are lost in the stationary-phase approximation, but retained in the two-dimensional formulation. Such phenomena diminish in importance as the distance from boundaries becomes large, and the one-dimensional result should be accurate for large values of $x_0 - (\bar{x} + \Delta x)$. This behavior is evident in Fig. 7. Of course, the accuracy of even the two-dimensional equation is adversely affected by the failure of impedance boundary conditions near abrupt conductivity changes.

Figures 8 and 9 show W_1 as a function of transverse receiver position y for several values of x and lakes having, respectively, half-widths of 3 and 7 km. The function W_1 exhibits a classic diffraction pattern of maxima and minima in the transverse direction. That pattern is to be expected because the obstacle is smaller than a Fresnel zone. Although the details of the signal structure shown in Figs. 7 through 9 pertain to the model and geometric factors assumed, a qualitatively similar structure will occur for any terrain feature having lateral dimensions smaller than a Fresnel zone. Diffraction patterns such as those shown in Figs. 8 and 9 are, of course, omitted by the widely used one-dimensional equation.

This conclusion does not contradict Eq. (32) and the results in the table (see p. 27), which are valid only for perturbations too weak to cause significant reflection.

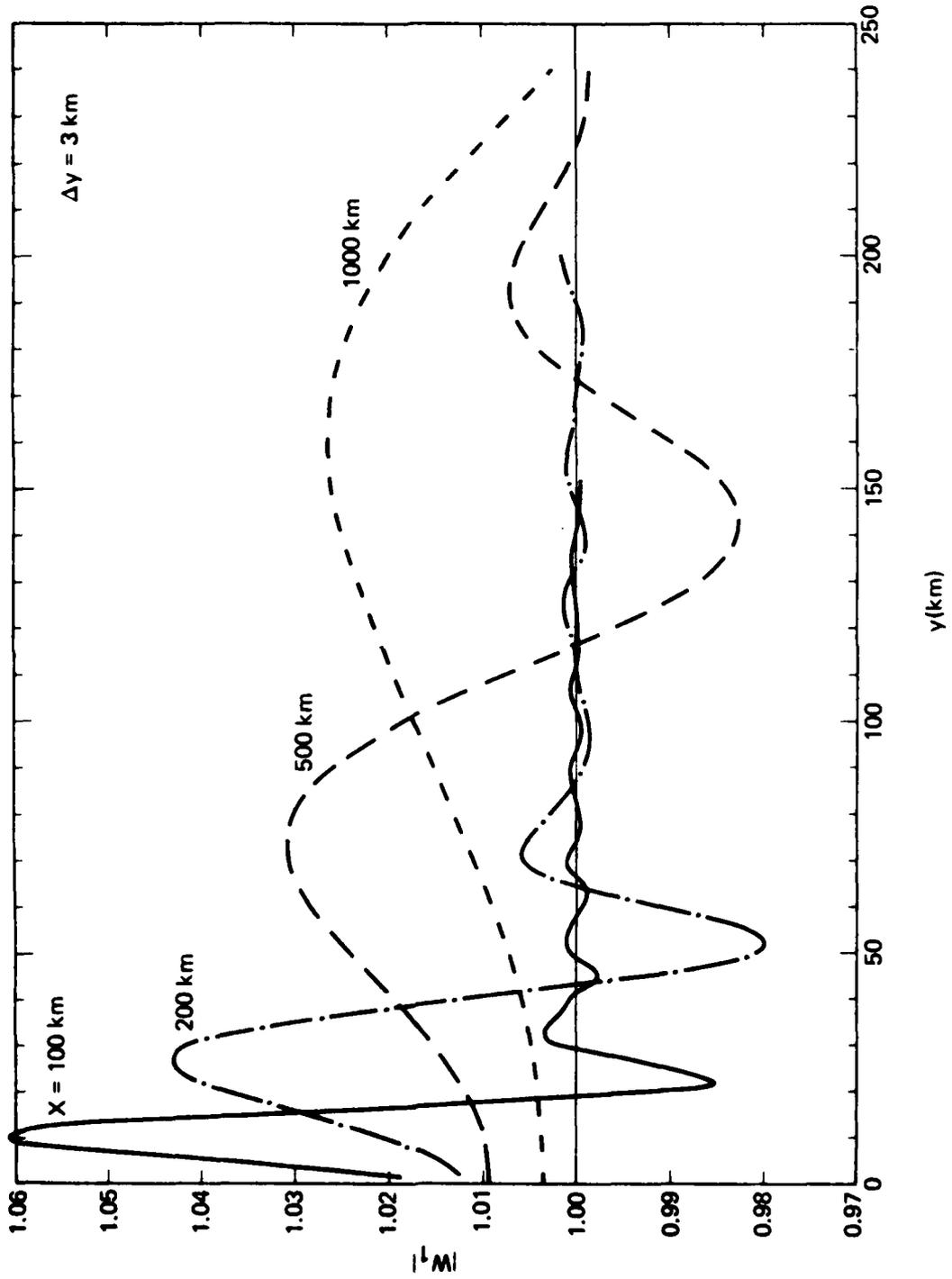


Fig. 8--Magnitude of W_1 versus transverse receiver position for a seawater region of half-width 3 km on a poorly conducting plane earth: frequency = 100 kHz

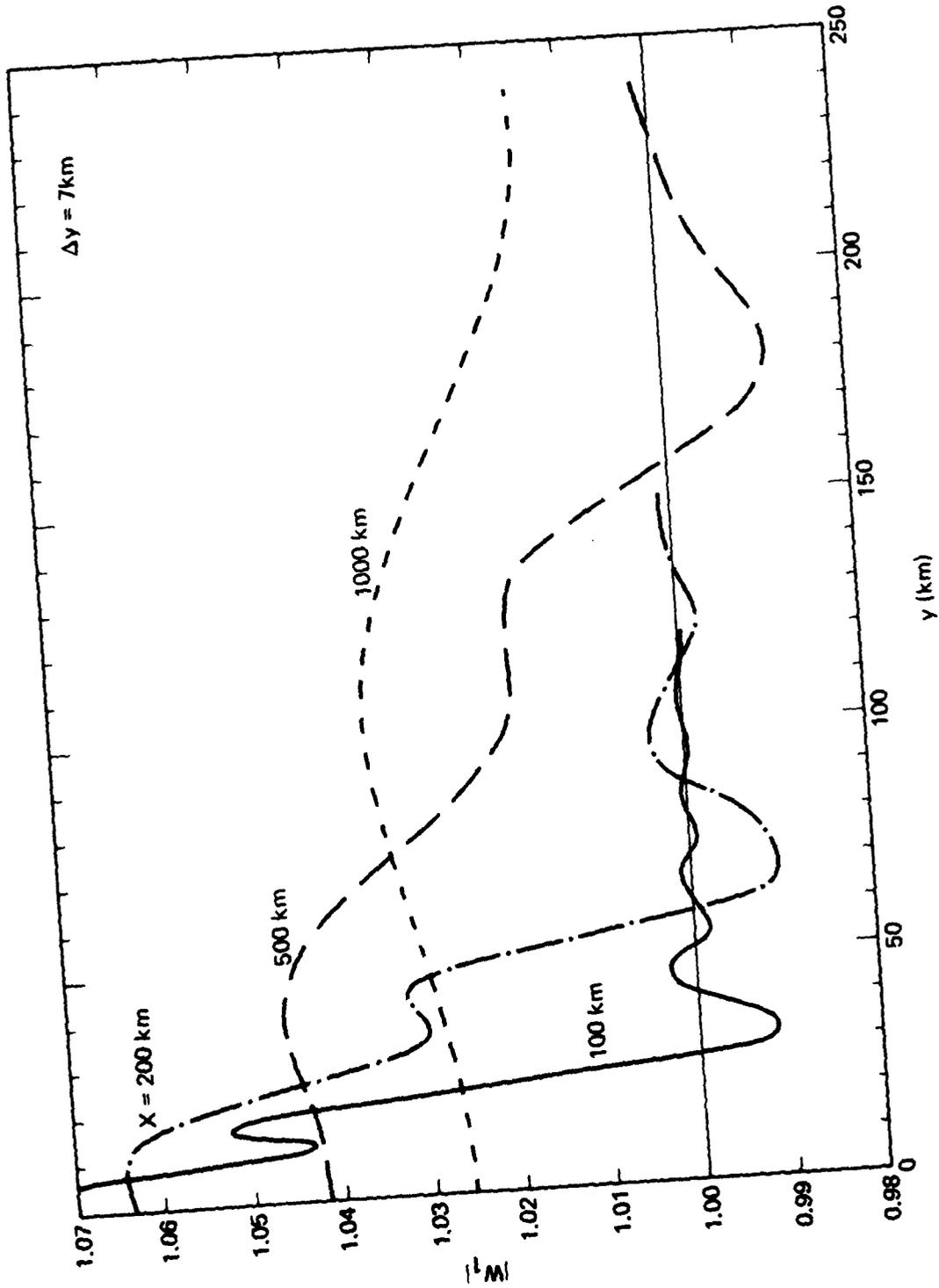


Fig. 9--Magnitude of W_1 versus transverse receiver position for a seawater region of half-width 7 km on a poorly conducting plane earth: frequency = 100 kHz

V. CONCLUSIONS

The two-dimensional integral equation for groundwave propagation has different forms for elevated and ground-based receivers. The two forms become equivalent in the limit of zero elevation. The stationary-phase approximation can sometimes be used to derive much simpler one-dimensional equations, which are widely used for most applications. This stationary-phase reduction in dimensionality causes so-called elevation-angle and end-point errors, both of which are associated with elevation. We have quantified these errors by comparing our one- and two-dimensional numerical solutions with Norton's attenuation function for a uniform flat earth. The two-dimensional solutions agree closely, whereas the one-dimensional solutions fail at certain elevations.

Elevation-angle errors are large unless the elevation is much smaller than the pathlength. End-point errors occur because the stationary-phase approximation improperly accounts for the region near the receiver, and are large for elevated receivers at heights below about one-sixth of a wavelength. The one-dimensional approximations for grounded and elevated receivers are therefore discontinuous near the ground. Fortunately, neither type of error is too serious at a frequency of 100 kHz for most ranges and altitudes pertaining to airborne receivers. However, end-point errors are significant at altitudes below 1 to 2 km, and elevation-angle errors can be important for pathlengths of tens of kilometers or less. In practice, for receivers that are above the ground but no higher than 1 or 2 km, much of the end-point error can be avoided--without the expense of the two-dimensional solution--simply by using the one-dimensional solution for grounded receivers.

Our numerical solutions of the two-dimensional equation for isolated terrain features agree well with (1) approximate solutions from first-order perturbation theory, (2) numerical solutions of de Jong [1975] for an idealized model, and (3) laboratory measurements by King and Tsukamoto [1966].

It is well known that the one-dimensional equation is invalid unless the terrain is nearly uniform across a Fresnel zone. Comparisons between our one- and two-dimensional solutions for on-path terrain quantify the errors. For a frequency of 100 kHz and features narrower than about 10 km, we find that the one-dimensional equation erroneously predicts propagation anomalies that (1) are independent of width, and therefore too large, (2) diminish too slowly at long distances, and (3) do not exhibit a diffraction pattern in the transverse direction.

The one- and two-dimensional solutions approach one another at large distances beyond very wide terrain features. However, they disagree near boundaries because the stationary-phase approximation neglects reflections and interference effects. Therefore, even for terrain that exhibits no transverse gradients, only the two-dimensional equation accounts for detailed signal structure.

Considerable error can be incurred at low frequencies by applying the one-dimensional equation to moderately sized terrain features. For example, for a pathlength of 500 km, that equation overstates by a factor of 4 the effect of an obstacle 6 km in diameter. It cannot give accurate results unless the diameter approaches a Fresnel zone width, which for this example is several tens of kilometers. Nonetheless, the one-dimensional equation is often applied to terrain that does not satisfy the validity requirements.

For the above reasons, we believe the one-dimensional equation to be incapable of treating many terrains, even if the input data were perfect. Unfortunately, routine application of the more accurate two-dimensional equation to irregular terrain is probably impractical. A more fruitful approach is to devise means of averaging terrain over the Fresnel zone to obtain equivalent one-dimensional models. The statistical approach of Feinberg [1944] can be used to derive a one-dimensional equation for the average field if the terrain has many small, randomly located irregularities. His approach is invalid for terrain exhibiting, say, a few hills or lakes, however.

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Appendix A

STATIONARY-PHASE APPROXIMATION FOR ELEVATED RECEIVERS

For terminals on a plane earth, the phase of the integrand in Eq. (2) is dominated by the term $\exp [ik(r_1 + r_2 - r_0)]$, whose phase is constant on a family of ellipses defined by the expression $r_1 + r_2 =$ constant, but varies rapidly along the hyperbolas normal to the ellipses (see Fig. A.1). That behavior led Hufford to use elliptical coordinates in his solution; the hyperbolas shown in Fig. A.1 define his transverse coordinate. The main contributions to the integral occur within the ellipse defined by $r_1 + r_2 = \lambda/2$ --that is, within the first Fresnel zone (illustrated in Fig. A.2).

Ellipses and hyperbolas are natural coordinates for a receiver on a plane earth, but not for an elevated receiver. For the elevated case, we must begin with Eq. (3), which contains the term $\exp [ik(r_1 + R_2 - R_0)]$ and for which the relation $r_1 + R_2 =$ constant does not trace ellipses in the plane $x = 0$. Another objection to elliptical coordinates is that for either ground-based or elevated receivers, they are awkward for computing higher order corrections to the stationary-phase integration.

Field and Allen [1978], in rederiving Eq. (2) for nonelevated receivers, used cartesian coordinates and applied the stationary-phase approximation along the y coordinate. Despite those differences from Hufford's approach, the two results agree. The reason is that for either method to be valid in the case of ground-based receivers, $kr_0 \gg 1$ must hold, which in turn implies $r_0 \gg \sqrt{\lambda r_0}$. The last condition stretches the Fresnel zone into a highly elongated ellipse. The hyperbolic trajectories, except near the ends, then conform nearly to the paths defined by $x =$ constant. Hence, integrating along the hyperbolas is nearly equivalent to integrating along the y coordinate over most of the path.

STATIONARY-PHASE RESULT FOR ELEVATED RECEIVERS

To assess the end-point and elevation-angle errors in the one-dimensional equation, we extend the stationary-phase integration in

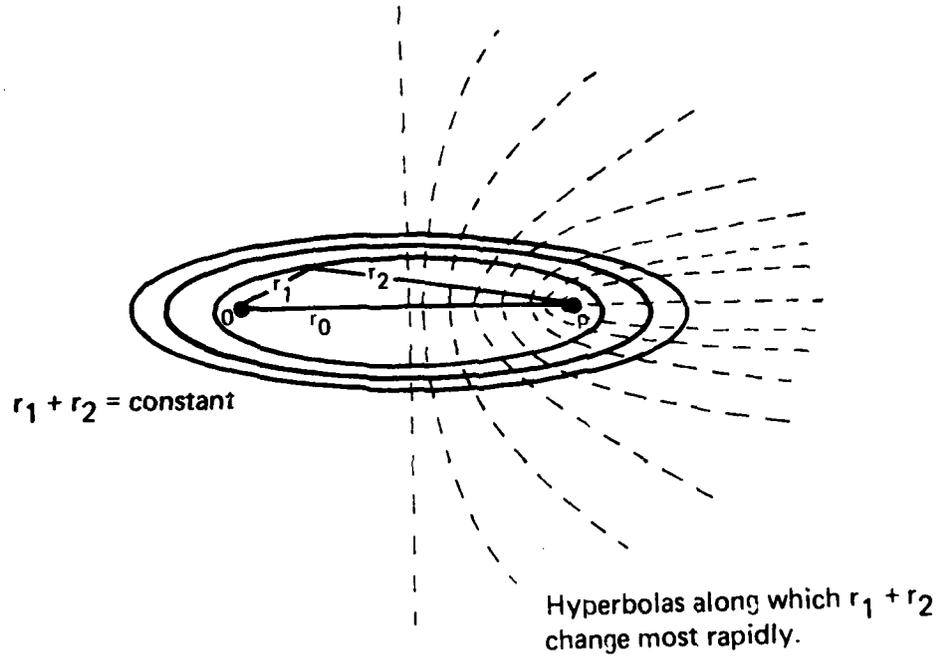


Fig. A.1--Elliptical coordinate system

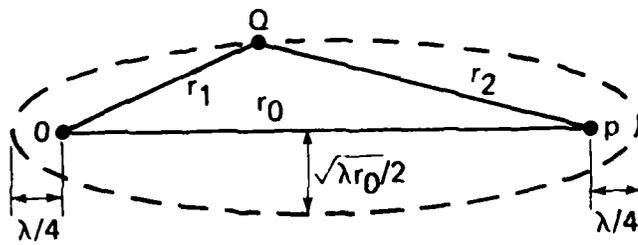


Fig. A.2--Geometry of first Fresnel zone

cartesian coordinates so that it applies to elevated receivers, then evaluate correction terms to the usual result. By performing the derivation for a plane earth, we concentrate solely on effects related to receiver elevation. We seek to determine when the two-dimensional equation for elevated receivers,

$$2W(P) = 1 + \frac{ik}{2\pi} \int da \frac{R_0}{r_1 R_2} W(Q) \left[\frac{1}{n_g} - \left(1 + \frac{i}{kR_2} \right) \frac{z_0}{R_2} \right] e^{ik(r_1 + R_2 - R_0)}, \quad (9)$$

can be approximated by the usual one-dimensional formula

$$2W(x_0, z_0) = 1 - e^{-\pi i/4} \left(\frac{k}{2\pi} \right)^{1/2} \int_0^{x_0} dx \left[\frac{R_0^2}{xR_2(x + R_2)} \right]^{1/2} \\ \times W(x, 0) \left[\frac{1}{n_g} - \left(1 + \frac{i}{kR_2} \right) \frac{z_0}{R_2} \right] e^{ik(x + R_2 - R_0)}. \quad (A.1)$$

Equation (9) is simply repeated from the text; Eq. (A.1) results from inserting Eqs. (6) and (8) into Eq. (5) and using a cartesian coordinate system with the ground plane at $z = 0$, the transmitter at the origin, and the receiver at x_0, z_0 . For Eq. (9), the integration point Q has coordinates $x, y, 0$, and the following relations hold:

$$R_0^2 = x_0^2 + z_0^2; \quad (A.2)$$

$$r_1^2 = x^2 + y^2; \quad (A.3)$$

and

$$r_2^2 = (x_0 - x)^2 + y^2; \quad (A.4)$$

$$R_2^2 = r_2^2 + z_0^2. \quad (A.5)$$

For the one-dimensional equations, the same relations apply, with y set equal to zero.

Equation (9) can be written

$$2W(x_0, z_0) = 1 + \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \Gamma(x, y) \exp \left[ikR_0 \left(\frac{r_1 + R_2}{R_0} - 1 \right) \right] . \quad (A.6)$$

where

$$\Gamma(x, y) = \frac{ik}{2\pi} \left(\frac{R_0}{r_1 R_2} \right) W(x, y, 0) \left[\frac{1}{n_g} - \left(1 + \frac{i}{kR_2} \right) \frac{z_0}{R_2} \right] . \quad (A.7)$$

The inner integral in Eq. (A.6) is given by

$$I(x) = \int_{-\infty}^{\infty} dx \Gamma(x, y) e^{ikR_0 h(x,y)} , \quad (A.8)$$

where

$$h = \frac{r_1 + R_2}{R_0} - 1 . \quad (A.9)$$

For long transmission paths, $kr_0 \gg 1$, and the integral [Eq. (A.8)] is of the classic form amenable to approximate evaluation by the stationary-phase method [Erdelyi, 1956], provided $\Gamma(x, y)$ varies more slowly than $\exp(ikR_0 h)$. If it does, the stationary-phase formula can be used to write

$$I(x) \approx e^{\pi i/4} e^{ikR_0 h} \left(\frac{2\pi}{kR_0 h''} \right)^{1/2} \Gamma \Big|_{y=0} . \quad (A.10)$$

By inserting Eq. (A.10) and the relations

$$R_0 h \Big|_{y=0} = x + R_2 - R_0 \quad (\text{A.11})$$

and

$$kR_0 h'' \Big|_{y=0} = k \left(\frac{x + R_2}{xR_2} \right) \quad (\text{A.12})$$

into Eq. (A.6), we recover the standard form of the one-dimensional equation [Eq. (A.1)]. The x integration is terminated at 0 and x_0 , rather than at $\pm\infty$, because--as discussed below--the exponent varies slowly for $0 \leq x \leq x_0$ but oscillates rapidly for $x < 0$ or $x > x_0$.

END-POINT ERROR IN ONE-DIMENSIONAL APPROXIMATION

The above treatment is valid provided $\Gamma(x, y)$ varies only slightly over distances the order of

$$\delta y \sim (kR_0 h'')^{-1/2} \Big|_{y=0} = \left(\frac{xR_2}{k(x + R_2)} \right)^{1/2} \quad (\text{A.13})$$

That condition is satisfied for most values of x . However, near the receiver, where $x \approx x_0$ and $R_2 \approx z_0$, Eq. (A.13) gives

$$\delta y \sim \left(\frac{z_0 \lambda}{2\pi} \right)^{1/2} \quad (\text{A.14})$$

as the distance over which Γ must be relatively constant; whereas the term

$$\frac{i}{kR_2} \frac{z_0}{R_2} = \frac{iz_0}{k(r_2^2 + z_0^2)}, \quad (\text{A.15})$$

which appears in Γ [see Eq. (A.7)], has a peak at $x = x_0$ and a half-width of z_0 . Therefore, unless

$$z_0 \gg \left(\frac{z_0 \lambda}{2\pi} \right)^{1/2},$$

or, equivalently,

$$z_0 > \lambda/2\pi, \quad (\text{A.16})$$

the stationary-phase approximation fails near the end point. Stated more practically, the canonical form of the one-dimensional equation for elevated receivers is inaccurate unless the receiver height exceeds about one-sixth of a wavelength. Note that this problem does not occur for ground-based receivers, where $\partial r_2/\partial n \equiv 0$.

To better quantify the end-point error caused by the stationary-phase approximation, we again examine Eqs. (A.6) and (A.7). The integration just outlined is valid except for a small region of dimension $\sim z_0$ near the end point. Even within that region, only the term proportional to z_0/R_2^2 in Eq. (A.7) varies so rapidly as to violate the stationary-phase approximation. Thus, we can apply the approximation over the entire path save the end-point region, where the z_0/R_2^2 term--which stems from the $(1/kR_2)(\partial R_2/\partial n)$ term in Eq. (3)--requires special attention. The equation that results is identical to Eq. (A.1), except that the "end-point correction" must be added to the right side:

$$\begin{aligned} \text{end-point correction} &= \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^a dr_2 \\ &\times W(r_2, \theta) \left(\frac{r_0}{r_1} \right) \frac{z_0 r_2}{R_2^3} e^{ik(r_1 + R_2 - R_0)}, \quad (\text{A.17}) \end{aligned}$$

where the upper limit, a , should be taken somewhat larger than z_0 , but much smaller than x_0 ; and r_2 , θ , and z are cylindrical coordinates centered with origin on the ground beneath the receiver. For long paths, the following approximations are valid within the end-point region, provided $z_0 \ll (\lambda x_0)^{1/2}$:

$$r_1 \approx R_0 \approx x_0 ; \quad (\text{A.18})$$

$$W(r_2, \theta) \approx W(x_0, 0, 0) . \quad (\text{A.19})$$

With these approximations, we have

$$\text{end-point correction} \approx W(x_0, 0, 0) \int_0^a dr_2 \frac{z_0 r_2}{R_2^3} e^{ikR_2} . \quad (\text{A.20})$$

The integral in Eq. (A.20) cannot be evaluated in closed form because of its finite upper integration limit. The rapid falloff of the integrand for $r_2 > z_0$ permits us to set $a = \infty$ with little loss of accuracy. Then, by making the change of variable

$$\zeta = \sqrt{1 + r_2^2/z_0^2} ,$$

we find

$$\text{end-point correction} \approx W(x_0, 0) \int_1^\infty \frac{e^{-ikz_0\zeta}}{\zeta^2} d\zeta = W(x_0, 0) E_2(-ikz_0) , \quad (\text{A.21})$$

where E_2 is the exponential integral whose properties are given by Abramowitz and Stegun [1972]. By using the power and asymptotic series for E_2 , we find

$$\text{end-point correction} \approx \begin{cases} W(x_0, 0) , & \text{if } kz_0 \ll 1 , \quad (\text{A.22}) \\ \frac{W(x_0, 0) e^{-ikz_0}}{ikz_0} , & \text{if } kz_0 \gg 1 , \quad (\text{A.23}) \end{cases}$$

the magnitudes of which correspond to Eqs. (18) and (19) of the main text. Note that adding Eq. (A.22) to the right side of the standard

one-dimensional equation for elevated receivers [Eq. (A.1) or Eq. (5)] recaptures the continuity of the fields as $z_0 \rightarrow 0$. Equations (22) and (23) explain why, for long paths, the one- and two-dimensional numerical solutions given in Sec. III agree well at elevations above a few kilometers but differ substantially near the ground.

ELEVATION-ANGLE ERROR IN ONE-DIMENSIONAL APPROXIMATION

In addition to the end-point errors, which depend on the ratio of receiver altitude to wavelength, the one-dimensional equation suffers from an elevation-angle error that depends on the ratio z_0/x_0 of receiver altitude to pathlength. It is intuitively evident that several aspects of the stationary-phase integration over the transverse coordinate, which reduces Eq. (9) to Eq. (A.1), breaks down at large elevation angles. Hufford's [1952] transformation to elliptical coordinates fails because the intersection of the Fresnel zone ellipsoids with the ground strongly deviates from an ellipse if z_0/x_0 is too large.* Similarly, Field and Allen's [1978] treatment encounters errors because as z_0/x_0 increases, the stationary-phase integration paths--the intersection of hyperboloids with the earth's surface--are less accurately approximated by the lines $x = \text{constant}$ (refer to Fig. A.1).

One way to quantify the elevation-angle error is by noting that the approximation

$$\int_{-\infty}^{\infty} dx \approx \int_0^{x_0} dx, \quad (\text{A.24})$$

used to reduce Eq. (A.6) to Eq. (A.1), fails unless z_0/x_0 is small. When the receiver is on the ground, the following relations apply along the x axis--which closely approximates the line of stationary points for long propagation paths:

* Godzinski [1961] shows that if $z_0 > \sqrt{\lambda x_0}$, higher order Fresnel zones intersect the ground in a manner that deemphasizes the contribution of points near the receiver.

$$ik(r_1 + r_2 - r_0) = \begin{cases} 0, & \text{if } 0 \leq x \leq x_0, \\ 2ik(x - x_0), & \text{if } x > x_0, \\ -2ikx, & \text{if } x < 0. \end{cases} \quad (\text{A.25})$$

The exponents in Eqs. (A.6) and (A.1) are thus invariant between $x = 0$ and $x = x_0$, but oscillate rapidly outside that range. However, if z_0 is not zero, we have the following relations:

$$\left. \begin{aligned} ik(r_1 + R_2 - R_0) &\approx \frac{ikz_0^2}{(x_0 - x)}, & \text{if } x_0 - x > z_0; \\ ik(r_1 + R_2 - R_0) &\approx ikz_0, & \text{if } x_0 - x \ll z_0. \end{aligned} \right\} \quad (\text{A.26})$$

If $kz_0 \ll 1$, no substantial oscillation arises in the region $0 \leq x \leq x_0$, and the upper integration limit may reasonably be taken as x_0 . Recall however that the end-point error becomes substantial if $kz_0 \ll 1$.

If $kz_0 \gtrsim 1$, Eq. (A.26) shows that $\exp [ik(r_1 + R_2 - R_0)]$ does oscillate in the region defined by $x_0 - z_0 \lesssim x \lesssim x_0$. The implication is that the integrand does not change abruptly from nonoscillatory to oscillatory at $x = x_0$ as it does for a nonelevated receiver; instead, change occurs over a transition region of width about equal to z_0 around the subreceiver point. Therefore, unless

$$z_0/x_0 \ll 1, \quad (\text{A.27})$$

the assumption that the integrand slowly oscillates between 0 and x_0 will be invalid over a significant fraction of the path.

Although the criterion given as Eq. (A.27) derives from Field and Allen's [1978] reduction to a one-dimensional equation, it applies to any stationary-phase integration over the transverse coordinates [Hufford, 1952; Johler and Berry, 1977]. That point is borne out in Sec. III, where the disagreement between the one- and two-dimensional numerical results is shown to be nearly directly proportional to the elevation angle z_0/x_0 .

Appendix B

INTEGRAL EQUATIONS FOR ISOLATED IRREGULARITIES
ON A UNIFORM PLANE

This appendix derives Eqs. (24) and (26), which are more convenient numerically than Eq. (2) for small terrain irregularities. We use the Green's function

$$G(r_2) = W_H(r_2)\psi_0(r_2) \quad (B.1)$$

and an auxiliary attenuation function W_1 , defined by

$$\psi(p) = 2W_1(p)W_H(p)\psi_0(p) , \quad (B.2)$$

instead of $G = \psi_0$ and $\psi = 2W\psi_0$, as was done [Hufford, 1952] to derive Eq. (2). As before, ψ and ψ_0 are the vertical Hertz potential and the Hertz potential in free space, respectively. W_H is the attenuation function for a uniform plane given by Eq. (22). W_1 therefore accounts solely for impedance or topographic irregularities. As shown below, use of the auxiliary function W_1 instead of W itself (recall $W \equiv W_1W_H$) offers computational advantages.

ISOLATED LAKE OR ISLAND

We consider a plane containing no topographic features so that $\partial r_2/\partial n$ is everywhere equal to zero. The refractive index is $\tilde{n}_g(x, y)$, which equals a constant, n_g , except over a limited region. The impedance boundary conditions become

$$\frac{\partial \psi}{\partial n} = - \frac{ik\psi}{\tilde{n}_g} , \quad (B.3)$$

$$\begin{aligned} \frac{\partial G}{\partial n} &= \psi_0 \frac{\partial W_H}{\partial n} \\ &= -\frac{ikG}{n_g}, \end{aligned} \quad (B.4)$$

where we have used the fact that $\partial W_H / \partial n = -ikW_H / n_g$.

We apply Green's theorem to obtain

$$\frac{\psi(p)}{2} = G(p) - \frac{1}{4\pi} \iint dx dy \left[G(r_2) \frac{\partial \psi(Q)}{\partial n} - \psi(Q) \frac{\partial G(r_2)}{\partial n} \right] \quad (B.5)$$

and use Eqs. (B.3) and (B.4) to cast Eq. (B.5) into the following form

$$\psi(p) = 2G(p) + \frac{ik}{2\pi} \iint dx dy G(r_2) \psi(r_1) \Delta(x, y), \quad (B.6)$$

where

$$\Delta(x, y) = \frac{1}{\tilde{n}_g} - \frac{1}{n_g} \quad (B.7)$$

is the impedance contrast. Then, using the fact that

$$\psi_0(r) = \frac{e^{ikr}}{r} \quad (B.8)$$

and inserting Eqs. (B.1) and (B.2) into Eq. (B.6), we find

$$\begin{aligned} 2W_1(p)W_H(r_0) \frac{e^{ikr_0}}{r_0} &= 2W_H(r_0) \frac{e^{ikr_0}}{r_0} + \frac{ik}{\pi} \iint dx dy \\ &\times W_1(Q)W_H(r_1)W_H(r_2) \Delta e^{ik(r_1+r_2)}. \end{aligned} \quad (B.9)$$

By dividing both sides of Eq. (B.9) by $2W_H(r_0) \exp(ikr_0)/r_0$, we obtain Eq. (24) in the text. Equation (24) is unchanged if more than one hill or lake exists, but the integration on the right must be performed over each irregularity.

ISOLATED HILL

The derivation of the integral equation for W_1 in the presence of an isolated hill (or hills) proceeds exactly as outlined above for an isolated lake or island. We note that here the refractive index is equal to n_g over the whole plane, but $\partial r_2/\partial n$ does not vanish everywhere. The impedance boundary conditions are:

$$\frac{\partial \psi}{\partial n} = -\frac{ik\psi}{n_g} \quad (\text{B.10})$$

$$\begin{aligned} \frac{\partial G}{\partial n} &= W_H \frac{\partial \psi_0}{\partial n} + \psi_0 \frac{\partial W_H}{\partial n} \\ &= ikG \left[-\frac{1}{n_g} + \left(1 + \frac{i}{kr_2} \right) \right] \quad (\text{B.11}) \end{aligned}$$

By inserting Eqs. (B.10) and (B.11) into Eq. (B.5), and repeating the steps leading from Eq. (B.5) to Eq. (B.9), we arrive at Eq. (26) in the text.

Appendix C

ALGORITHM FOR NUMERICAL SOLUTION OF TWO-DIMENSIONAL
INTEGRAL EQUATION

This appendix gives the algorithm used to obtain numerical solutions of Eqs. (24) and (26), which can be written in the form

$$W(x, y) = 1 + \iint dx' dy' K(x, y, x', y') W(x', y') . \quad (C.1)$$

If terrain features are localized, the kernel of Eq. (C.1) can be assumed to vanish outside some bounded region. The solution consists of first solving Eq. (C.1) to find W within this irregular region, and then integrating over the region to obtain W on the uniform portion of the plane.

SOLUTION WITHIN REGION OF IRREGULARITY

We divide the isolated region into a square grid as shown in Fig. C.1, where n_x and n_y are the number of squares in the x and y directions. Equation (C.1) then becomes

$$W(x, y) = 1 + \sum_{i=0}^{n_x-1} \sum_{j=0}^{n_y-1} \iint_{s_{ij}} dx' dy' K(x, y, x', y') W(x', y') , \quad (C.2)$$

where on the square s_{ij} , centered at x_i, y_j ,

$$x_i - \frac{\Delta s}{2} \leq x' \leq x_i + \frac{\Delta s}{2} ,$$
$$y_j - \frac{\Delta s}{2} \leq y' \leq y_j + \frac{\Delta s}{2} . \quad (C.3)$$

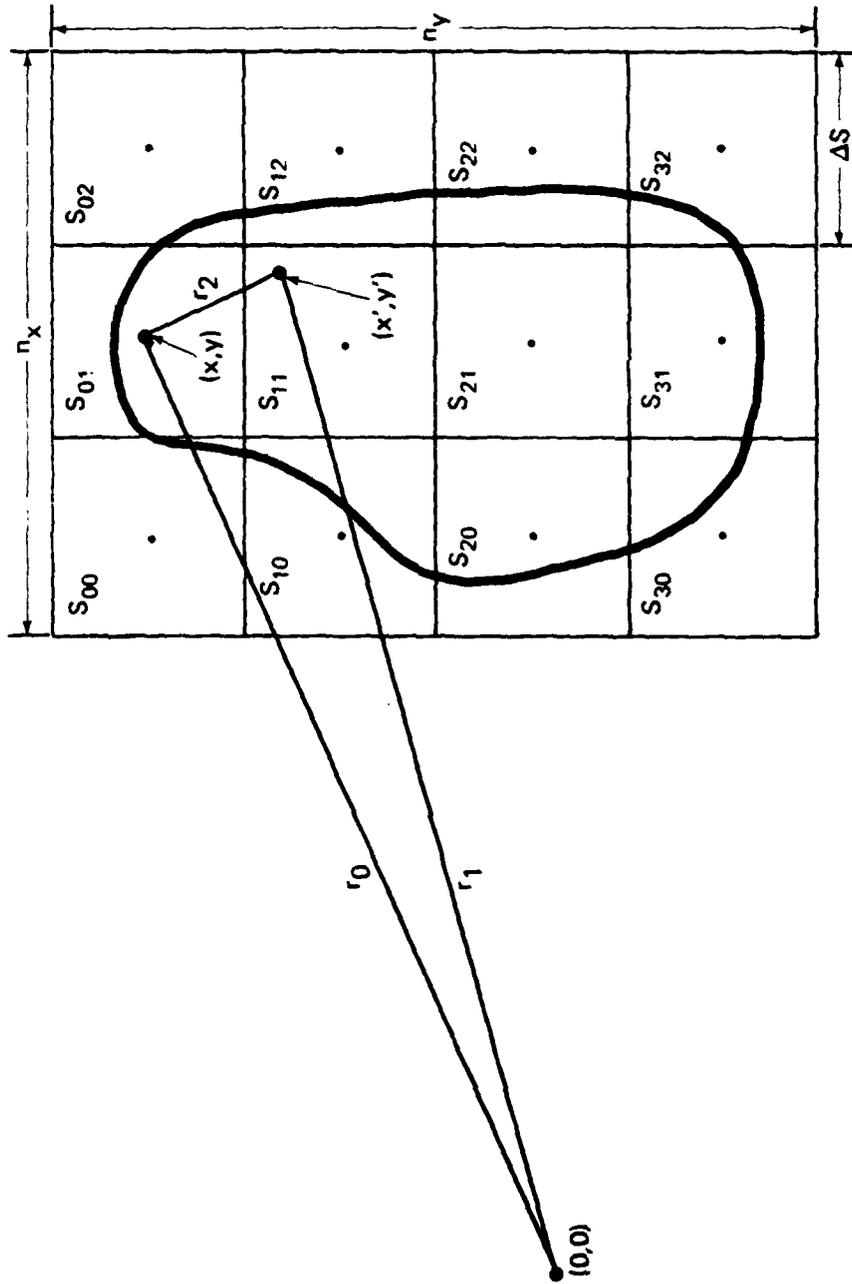


Fig. C.1--Partition for numerical solution of Eqs. (24) and (26)

Here Δs is the length of the side of a square taken to be small enough that

$$W(x, y) \approx W(x_i, y_j) \quad (C.4)$$

on each square. We can then rewrite Eq. (C.2) as

$$W(x_k, y_\ell) \approx 1 + \sum_{i=0}^{n_x-1} \sum_{j=0}^{n_y-1} W(x_i, y_j) \iint_{s_{ij}} dx' dy' K(x_k, y_\ell, x', y') \quad (C.5)$$

By defining

$$M_{k\ell ij} \equiv \iint_{s_{ij}} dx' dy' K(x_k, y_\ell, x', y') \quad (C.6)$$

the solution to Eq. (C.1) can be expressed as

$$W(x_k, y_\ell) = \frac{1 + \sum_{\substack{i=0 \\ i \neq k}}^{n_x-1} \sum_{\substack{j=0 \\ j \neq \ell}}^{n_y-1} W(x_i, y_j) M_{k\ell ij}}{1 - M_{k\ell k\ell}} \quad (C.7)$$

The integration in Eq. (C.6) is performed with standard quadrature methods. We use a higher density of quadrature points when $(i, j) = (k, \ell)$ to accommodate the proximity of the singularity in K at $r_2 = 0$.

Equation (C.7) is a system of $n_x \times n_y$ linear equations, which we solve iteratively using the Gauss-Siedel method with each W initially set equal to one. To evenly distribute errors throughout the grid, we begin each iteration at a different point. Arbitrary convergence criteria can be used. Computation time varies as $(n_x \times n_y)^2$, so care must be exercised in selecting the parameter Δs . In most problems,

the optimum value for Δs depends on wavelength, rather than on terrain features. Computation time then varies as the square of the area of the perturbed region, and limits the size of irregularities that can be analyzed at reasonable cost. For a frequency of 100 kHz, we found the practical limit on size imposed by computational cost to be about 10 km \times 10 km.

SOLUTION OUTSIDE REGION OF IRREGULARITY

Once W has been found inside the irregular region, its value outside is calculated from Eq. (C.5), using well-known quadrature methods. Here W is given by a double integral over this region, rather than by a solution to a two-dimensional integral equation. Also, the singularity at $r_2 = 0$ causes no difficulty.

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