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AN APPROXIMATE FORMULA FOR THE FUNDAMENTAL  
FREQUENCY OF A UNIFORM ROTATING BEAM CLAMPED  
OFF THE AXIS OF ROTATION

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A semi-empirical method involving asymptotic expansions is used to obtain an approximate formula for the fundamental frequency of a uniform rotating beam clamped off the axis of rotation. Results from the formula are shown to be of the order of 0.1% different from the exact results for a wide range of rotor speeds and hub radii up to the order of blade length. Thus, the designer is provided with a rapid, very accurate estimate of the frequency, without having to interpolate results from a chart or run a digital computer program.

## 1. INTRODUCTION

The problem of approximating the fundamental frequency of a uniform rotating beam has application in the design of helicopter rotor blades, wind turbine blades, and flexible satellite booms. Although the Ritz method [1] can yield quite accurate estimates of the fundamental frequency with a moderate number of terms, it is sometimes desirable to have a fairly accurate formula that can be evaluated by using a pocket calculator rather than having to interpolate from charts or run a digital computer program. Such an expression is not trivial to come by, however, since the exact solution to the problem exists only as an infinite series [2]. Uniform approximations based on a small parameter, such as  $\eta = EI/m\Omega^2 L^4$ , are accurate only when the parameter is quite small and are thus not general enough for practical use [3]. A useful approach appears to be a semi-empirical method referred to as a "composite" expansion, as developed by Peters [4] for rotating beams clamped at the axis of rotation. In Peters' work [4] the expansion was accurate for all values of  $\eta$ , but did not include variation in off-clamping  $\alpha = R/L$  with  $R$  being the hub radius. It is the purpose of this paper to extend the results of reference [4] for the fundamental bending frequency to include variation in the off-clamping parameter.

To proceed with this expansion, one first notes that the ordinary differential equation for the mode shape of a rotating beam when the stiffness parameter  $\eta$  vanishes corresponds physically to that of a rotating string with its root offset from the axis of rotation. An approximate composite expansion solution for the fundamental frequency of the rotating string can then be used to extend the composite expansion of reference [4] to include the effect of off-clamping.

## 2. STATEMENT OF THE PROBLEM

The differential equation for free vibrations in bending of a uniform rotating beam in a plane at angle  $\theta$  with the rotation axis (see Figure 1) is

$$\eta u'''' - [\frac{1}{2}(1-x^2)u']' - \alpha[(1-x)u']' - \mu u = 0, \quad (1)$$

11

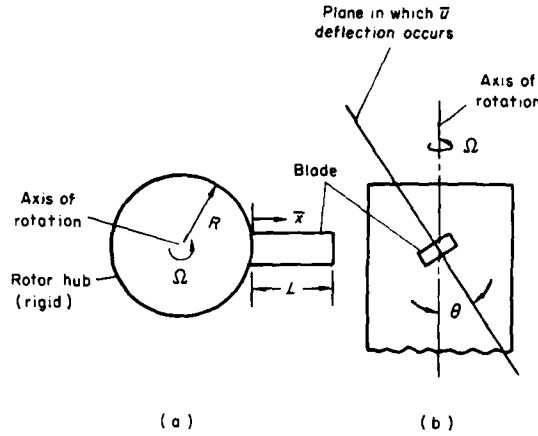


Figure 1. Schematic of rotating beam clamped off the axis of rotation.

where

$$\eta = EI/m\Omega^2 L^4, \quad \alpha = R/L, \quad u = \bar{u}/L, \quad x = \bar{x}/L, \quad 0 \leq x \leq 1, \\ \mu = \omega^2/\Omega^2 + \sin^2 \theta \begin{cases} \theta = 0 & \text{out-of-plane bending} \\ \theta = \pi/2 & \text{in-plane bending} \end{cases} \quad (2)$$

The differential equation is an eigenvalue problem in  $\mu$  for which approximations are sought as  $\eta$  and  $\alpha$  are varied. The boundary conditions for a cantilevered root end are

$$u(0) = 0, \quad u'(0) = 0, \quad u''(1) = 0, \quad u'''(1) = 0. \quad (3)$$

One can expand the quantities  $u$  and  $\mu$  in power series of the small quantity  $\eta^{1/2}$  as in reference [4] so that  $u = u_0 + u_1 \eta^{1/2} + \dots$  and  $\mu = \mu_0 + \mu_1 \eta^{1/2} + \dots$ . The equation corresponding to the zeroth power of  $\eta$  is

$$\mu_0 u_0 + \{[\alpha(1-x) + \frac{1}{2}(1-x^2)]u_0'\}' = 0. \quad (4)$$

For  $\alpha = 0$  it was shown in reference [4] that only the displacement boundary condition affects  $\mu$  to order  $\eta$ . Thus,

$$u_0(0) = 0. \quad (5)$$

It is assumed that equation (5) holds for the case when  $\alpha \neq 0$  as well. Equation (4) represents the differential equation for a rotating string with the root-end offset parameter  $\alpha$ . It is necessary to obtain a general approximation for  $\mu_0$  (i.e., a small  $\eta$  approximation for  $\mu$ ) from this equation which will serve as one ingredient in the process of approximating  $\mu$  for equation (1). Note that an approximation for  $\mu$  with  $\eta$  small and large compared to unity with  $\alpha = 0$  has already been obtained in reference [4]. Here the case for large  $\eta$  and  $\alpha \neq 0$  will be examined in a later section.

The exact solution of equation (4) can be expressed in terms of hypergeometric functions. A result more useful for present purposes can be obtained, however, by looking at limiting behavior for small and large  $\alpha$ , respectively. To this end, it is convenient to rewrite equation (4) temporarily with  $\lambda = \mu_0$  and  $v = u_0$ :

$$\lambda v + \{[\alpha(1-x) + \frac{1}{2}(1-x^2)]v'\}' = 0, \quad v(0) = 0. \quad (6)$$

## 3. ROTATING STRING WITH SMALL OFFSET

For small  $\alpha$  the variables  $\lambda$  and  $v$  can be expanded in powers of  $\alpha$ ,  $\lambda = \lambda_0 + \lambda_1\alpha + \dots$  and  $v = v_0 + v_1\alpha + \dots$ , which yields the following equations that correspond to powers of  $\alpha$  from equation (6):

$$\alpha^0: \lambda_0 v_0 + \frac{1}{2}[(1-x^2)v_0'] = 0, \quad (7a)$$

$$\alpha^1: \lambda_0 v_1 + \frac{1}{2}[(1-x^2)v_1'] = -\lambda_1 v_0 - [(1-x)v_0']. \quad (7b)$$

From reference [4] the solution is, for  $v_0(0) = 0$ , the odd Legendre polynomials. For the first mode ( $n = 1$ )

$$v_0 = P_{2n-1}(x) = x, \quad \lambda_0 = n(2n-1) = 1. \quad (8)$$

To obtain  $\lambda_1$ , it is observed that  $v_0$  must be orthogonal to the right-hand side of equation (7b). Thus,

$$\int_0^1 \{\lambda_1 v_0 + [(1-x)v_0']\} v_0 dx = 0, \quad (9)$$

or

$$\lambda_1 = \int_0^1 (1-x)v_0'^2 dx / \int_0^1 v_0^2 dx = \frac{3}{2}. \quad (10)$$

Since an estimate of the fundamental frequency is all that is of interest here, one can take

$$\lambda = 1 + \frac{3}{2}\alpha \quad (11)$$

as an indication of the behavior of  $\lambda$  for equation (6) with  $\alpha$  small. It should be noted that the linear term  $\lambda_1 = \frac{3}{2}$  does not agree with  $\lambda_1 = 6/\pi$  obtained in reference [3]. The reasons for this disagreement are not known; however, results from an exact numerical calculation (see below) seem to indicate that  $\lambda_1 = \frac{3}{2}$  is correct.

## 4. ROTATING STRING WITH LARGE OFFSET

The solution of equation (6) for large  $\alpha$  is similarly obtained with the definition  $\lambda/\alpha = \nu$  and  $\varepsilon = 1/\alpha$  so that equation (6) becomes

$$\nu v + \{[(1-x) + (\varepsilon/2)(1-x^2)]v'\}' = 0, \quad (12)$$

and one assumes

$$\nu = \nu_0 + \varepsilon \nu_1, \quad v = v_0 + \varepsilon v_1, \quad v_0(0) = v_1(0) = 0. \quad (13)$$

The equations for zeroth and first powers of  $\varepsilon$  are

$$\varepsilon^0: \nu_0 v_0 + [(1-x)v_0'] = 0, \quad (14a)$$

$$\varepsilon^1: \nu_0 v_1 + [(1-x)v_1'] = -\nu_1 v_0 - \frac{1}{2}[(1-x^2)v_0']. \quad (14b)$$

The solution for  $v_0$  involves a zeroth order Bessel function

$$v_0 = J_0[2\sqrt{\nu_0(1-x)}] = \sum_{n=0}^{\infty} \frac{(-1)^n \nu_0^n (1-x)^n}{(n!)^2}. \quad (15)$$

The boundary condition  $v_0(0) = 0$  yields

$$\nu_0 = 1.445796491. \quad (16)$$

To avoid secular terms,  $v_0$  must be orthogonal to the right-hand side of equation (14b) so that

$$\nu_1 = \frac{1}{2} \int_0^1 (1-x^2) v_0'^2 dx / \int_0^1 v_0^2 dx \approx 1.038163742. \quad (17)$$

Thus for large  $\alpha$

$$\lambda = \alpha \nu = \alpha(\nu_0 + \nu_1 \epsilon) = \alpha \nu_0 + \nu_1, \quad (18)$$

with  $\nu_0$  and  $\nu_1$  given in equations (16) and (17), respectively. It is interesting, upon comparison of equations (11) and (18), to note that the behavior for large and small  $\alpha$  is quite similar. Therefore, a simple exponential patching to produce an accurate approximation of  $\lambda$  for all  $\alpha$  will now be constructed.

#### 5. COMPOSITE EXPANSION FOR THE FUNDAMENTAL FREQUENCY OF A ROTATING STRING

If it is assumed that the behavior of  $\lambda$  for all  $\alpha$  can be expressed in the form

$$\lambda = \alpha \nu_0 + \nu_1 + e^{-\alpha}(a + b\alpha), \quad (19)$$

then, for small  $\alpha$ ,

$$\lambda = \alpha \nu_0 + \nu_1 + (1 - \alpha + \dots)(a + b\alpha). \quad (20)$$

For small  $\alpha$ ,  $\lambda = \lambda_0 + \lambda_1 \alpha$  so that

$$a = \lambda_0 - \nu_1, \quad b = \lambda_1 - \nu_0 - \nu_1 + \lambda_0. \quad (21)$$

Thus, for small  $\eta$ , the smallest eigenvalue of equation (4) is approximately

$$\mu_0 = \alpha \nu_0 + \nu_1 + g(\alpha), \quad (22)$$

where

$$g(\alpha) = e^{-\alpha}[\lambda_0 - \nu_1 + (\lambda_0 + \lambda_1 - \nu_0 - \nu_1)\alpha], \quad (23)$$

TABLE 1  
 $\mu_0$  versus  $\alpha$

$\alpha$	$\mu_0$ , expansion from equation (22)	$\mu_0$ , exact
0	1	1
1	2.47582	2.46802
2	3.92893	3.91965
3	5.37605	5.36815
4	6.82183	6.81551
5	8.26743	8.26232
6	9.71309	9.70883
7	11.1588	11.1552
8	12.6046	12.6014
9	14.0503	14.0475
10	15.4961	15.4935
11	16.9419	16.9396
12	18.3877	18.3855

and the various constants are defined above. The exact value of the frequency parameter for the first mode is easily obtained (for the purpose of comparison) by the method of reference [5] in which  $v$  is expanded in a simple power series or equivalently, a series of *all* Legendre polynomials. The exact values, to six places, are obtained with only a few terms in the series (ten terms are used to report the results here). It is evident from Table 1 that the simple expansion, equation (22), yields results that are quite accurate (within 0.3% for *all*  $\alpha$ ). Thus, equation (22) is suitable for extending the expansion for the fundamental frequency in reference [4], which applies only to rotating beams clamped at the axis of rotation ( $\alpha = 0$ ), to the more general case involving off-clamping ( $\alpha \neq 0$ ).

## 6. EXTENSION OF PETERS' FORMULA

In reference [4], Peters obtained a formula for  $\mu$  when  $\alpha = 0$  that was approximately valid for all  $\eta$ . In this section a generalization is obtained that is approximately valid for all  $\eta$  and  $\alpha$ .

For small  $\eta$  the formula will have the form

$$\mu = \mu_0(\alpha) + \mu_1(\alpha)\eta^{1/2}, \quad (24)$$

where the expression for  $\mu_0(\alpha)$  is given by equation (22). Results in reference [4] for  $\alpha = 0$  and in reference [6] for  $\alpha = O(\eta^{1/2})$  both suggest  $\mu_1(\alpha) = 3/\sqrt{2}$ . One can therefore assume that for all  $\alpha$

$$\mu = \nu_0\alpha + \nu_1 + g(\alpha) + (3/\sqrt{2})\eta^{1/2}. \quad (25)$$

The ultimate justification for such an assumption is that it works, although it is definitely a possible source of error for large  $\alpha$  and small  $\eta^{1/2}$ .

Before proceeding further one must obtain a large  $\eta$  approximation for  $\mu$ . For large  $\eta$  and  $\alpha = O(1)$ , let  $\varepsilon = 1/\eta$  and  $\gamma = \mu/\eta$ . Thus, equation (1) becomes

$$u'''' - (\varepsilon/2)[(1-x^2)u']' - \alpha\varepsilon[(1-x)u']' - \gamma u = 0. \quad (26)$$

For  $u = u_0 + u_1\varepsilon + \dots$  and  $\gamma = \gamma_0 + \gamma_1\varepsilon + \dots$  one obtains

$$\varepsilon^0: u_0'''' - \gamma_0 u_0 = 0, \quad (27a)$$

$$\varepsilon^1: u_1'''' - \gamma_0 u_1 = [\tfrac{1}{2}(1-x^2)u_0']' + \alpha[(1-x)u_0']' + \gamma_1 u_0. \quad (27b)$$

The boundary conditions in equation (3) yield the first mode shape for  $u$ :

$$u_0 = \cosh \beta_1 x - \cos \beta_1 x - \xi_1 (\sinh \beta_1 x - \sin \beta_1 x), \\ \gamma_0 = \beta_1^4, \quad \beta_1 = 1.875104069, \quad \xi_1 = 0.7340955138. \quad (28)$$

To suppress secular terms,  $u_0$  must be orthogonal to the right-hand side of equation (27b) which yields

$$\gamma_1 = \left\{ \frac{1}{2} \int_0^1 (1-x^2)u_0'^2 dx + \alpha \int_0^1 (1-x)u_0' dx \right\} / \int_0^1 u_0^2 dx = A + B\alpha, \quad (29)$$

where

$$A = 1.193336374 \quad \text{and} \quad B = 1.570878190. \quad (30)$$

The integrals in equation (29) are calculated as outlined in the appendix of reference [7]. Therefore, for large  $\eta$ ,

$$\mu = \beta_1^4 \eta + A + B\alpha. \quad (31)$$

Although  $\alpha = O(1)$  was assumed, it is observed from reference [6] that when the frequency parameter vanishes for large  $\eta$  and  $\alpha$  (negative),  $\eta$  is linear with  $\alpha$ . Hence, equation (31) may be approximately valid for large  $\eta$  regardless of the magnitude of  $\alpha$ . One can now assume a composite expansion for all  $\eta$  similar to that in reference [4] of the form

$$\mu = \beta_1^4 \eta + \nu_0 \alpha + \nu_1 + g(\alpha) + (3\sqrt{2}/\pi) \eta^{1/2} \tan^{-1} [\pi(a + b\alpha)/(3\sqrt{2}\eta^{1/2})]. \quad (32)$$

For small  $\eta^{1/2}$ , equation (32) reduces to

$$\mu \cong \nu_0 \alpha + \nu_1 + g(\alpha) + (3/\sqrt{2}) \eta^{1/2} + O(\eta), \quad (33)$$

which agrees with equation (25). For large  $\eta$ , equation (32) becomes

$$\mu \cong \beta_1^4 \eta + \nu_0 \alpha + \nu_1 + g(\alpha) + a + b\alpha + O(1/\eta), \quad (34)$$

so that, from equation (31)

$$A + B\alpha = \nu_0 \alpha + \nu_1 + g(\alpha) + a + b\alpha, \quad (35)$$

or

$$a = A - \nu_1 - g(\alpha), \quad b = B - \nu_0. \quad (36)$$

The final composite expansion is then

$$\mu = \beta_1^4 \eta + \nu_0 \alpha + \nu_1 + g(\alpha) + (3\sqrt{2}/\pi) \eta^{1/2} \tan^{-1} \{ \pi / (3\sqrt{2}\eta^{1/2}) [A - \nu_1 - g(\alpha) + (B - \nu_0)\alpha] \}, \quad (37)$$

or

$$\gamma = \beta_1^4 + [\nu_0 \alpha + \nu_1 + g(\alpha)] \bar{\Omega}^2 + (3\sqrt{2}/\pi) \bar{\Omega} \tan^{-1} \{ \pi \bar{\Omega} / (3\sqrt{2}) [A - \nu_1 - g(\alpha) + (B - \nu_0)\alpha] \}, \quad (38)$$

with

$$\begin{aligned} \beta_1^4 &= 12.36236337, & \nu_0 &= 1.445796491, & \nu_1 &= 1.038163742, \\ g(\alpha) &= e^{-\alpha} [\lambda_0 - \nu_1 + (\lambda_0 - \nu_1 + \lambda_1 - \nu_0)\alpha], \\ \lambda_0 &= 1, & \lambda_1 &= 3/2, & A &= 1.193336374, & B &= 1.570878190, \\ \gamma &= \mu/\eta = (mL^4/EI)(\omega^2 + \Omega^2 \sin^2 \theta), & \bar{\Omega} &= \eta^{-1/2} = \Omega \sqrt{mL^4/EI}. \end{aligned} \quad (39)$$

TABLE 2  
Frequency versus angular speed for  $\alpha = 0$

$\bar{\Omega}$	$\gamma^{1/2}$ equation (38)	$\gamma^{1/2}$ exact from [1, 2]
0	3.51602	3.51602
1	3.68163	3.68165
2	4.13710	4.13732
3	4.79644	4.79728
4	5.58316	5.58500
5	6.44650	6.44954
6	7.35614	7.36037
7	8.29440	8.29964
8	9.25085	9.25684
9	10.2192	10.2257
10	11.1956	11.2023
11	12.1775	12.1843
12	13.1634	13.1702

The exact solution of equation (1) has been calculated in reference [2] by using the method of Frobenius. It is far more convenient for present purposes, however, to obtain the exact solution from the analysis of reference [1] with enough terms in the analysis to ensure convergence. Note that in the finite element analysis of reference [1] the element displacement is expanded in a simple power series which is, of course, equivalent to a series of Legendre polynomials. Convergence to the exact solution will only rapidly occur if, when  $N + 1$  terms are taken, *all* terms of degree  $N$  or less are included. This explains the relatively slow convergence in reference [8] where only odd Legendre polynomials were taken. In reference [1], only a few terms are required to obtain the exact solution to six places for the first mode with one element. The exact solution for the dimensionless frequency parameter  $\gamma^{1/2}$  is given in Tables 2-5 for dimensionless rotor speed values  $\bar{\Omega} = 0, 1, \dots, 12$  and dimensionless off-clamping parameter values  $\alpha = 0, 0.1, 1$ , and 10.

Results from equation (38) are presented in the tables for comparison with the exact

TABLE 3  
Frequency versus angular speed for  $\alpha = 0.1$

$\bar{\Omega}$	$\gamma^{1/2}$ , equation (38)	$\gamma^{1/2}$ , exact from [1]
0	3.51602	3.51602
1	3.70288	3.70290
2	4.21208	4.21225
3	4.94054	4.94115
4	5.80129	5.80256
5	6.73944	6.74142
6	7.72347	7.72603
7	8.73548	8.73839
8	9.76513	9.76815
9	10.8063	10.8092
10	11.8552	11.8578
11	12.9094	12.9116
12	13.9675	13.9692

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TABLE 4  
Frequency versus angular speed for  $\alpha = 1$

$\bar{\Omega}$	$\gamma^{1/2}$ , equation (38)	$\gamma^{1/2}$ , exact from [1, 2]
0	3.51602	3.51602
1	3.88874	3.88882
2	4.83279	4.83369
3	6.07891	6.08175
4	7.46955	7.47505
5	8.93210	8.94036
6	10.4331	10.4439
7	11.9563	11.9691
8	13.4931	13.5074
9	15.0388	15.0541
10	16.5905	16.6064
11	18.1464	18.1625
12	19.7055	19.7215

A 20

TABLE 5

*Frequency versus angular speed for  $\alpha = 10$* 

$\bar{\Omega}$	$\gamma^{1/2}$ , equation (38)	$\gamma^{1/2}$ , exact from [1]
0	3.51602	3.51602
1	5.38018	5.40186
2	8.79660	8.88430
3	12.5274	12.6722
4	16.3558	16.5410
5	20.2260	20.4393
6	24.1177	24.3514
7	28.0220	28.2709
8	31.9342	32.1948
9	35.8518	36.1215
10	39.7731	40.0503
11	43.6971	43.9804
12	47.6232	47.9116

solution. The correlation is excellent for  $\alpha = 0, 0.1$  and  $1$ , the maximum error being of the order of  $0.1\%$ . For  $\alpha = 10$  the correlation is not as good; the errors reach  $1.1\%$ . Note, however, that the error decreases as  $\bar{\Omega}$  becomes large or tends toward zero. Evidently the formula is limited to  $\alpha = O(1)$  as assumed in the large  $\eta$  expansion, but gives a reasonably good estimate even for  $\alpha = 10$ . For helicopter and wind turbine blades,  $\alpha$  is of the order of  $0.1$ ; for gas turbine blades and satellite booms,  $\alpha$  may be  $1$  or larger.

It is evident that equation (38) provides an accurate and rapid estimate of the frequency of a rotating cantilever without the need for either digital computer programs or interpolation of charts. It would be desirable to include tip mass and taper parameters in the development, but the feasibility of such an extension remains to be seen. The work could be extended to higher modes, but only if one were willing to work through a great deal of algebra.

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