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MONOTONE OPTIMAL PREVENTIVE MAINTENANCE POLICIES FOR STOCHASTICALLY FAILING EQUIPMENT

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ABSTRACT

This paper examines various models for maintenance of a machine operating subject to stochastic deterioration. Three alternative models are presented for the deterioration process. For each model, in addition to the replacement decision, the option exists of performing preventive maintenance. The effect of this maintenance is to "slow" the deterioration process. With an appropriate reward structure imposed on the processes, the models are formulated as continuous time Markov decision processes, the optimality criterion being the maximization of expected discounted reward earned over an infinite time horizon. For each model conditions are presented under which the optimal maintenance policy exhibits the following monotonic structure. First, there exists a control limit rule for replacement. That is, there exists a number i^* such that if the state of machine deterioration exceeds i^* the optimal policy replaces the machine by a new machine. Secondly, prior to replacement the optimal level of preventive maintenance is a nonincreasing function of the state of machine deterioration. The conditions which guarantee this result have a cost/benefit interpretation.

INTRODUCTION

There is a substantially large collection of papers in the machine repair/maintenance/replacement literature. Two fairly comprehensive surveys are McCall [22] and Pierskalla and Voelker [23]. Specific articles of interest are Barlow and Proschan [4], Derman [8], [9], [10], Klein [17], Kolesar [18], Kalymon [15], Thompson [29], Kamien and Schwartz [16], Ross [25], and Rosenfield [24]. An interesting semi-Markovian treatment of shock models is given by Feldman [11]. The focus of these models has been on the replacement decision. In such models, gradual operating deterioration (or "wearing out") is not subject to control. The model in [16] differs somewhat in that preventive maintenance can be applied to decrease the failure probability over time.

In this paper we formulate three machine maintenance models that in addition to the replacement decision, incorporate the option of performing preventive maintenance. The effect of the maintenance is to "slow" the rate of machine deterioration. Each model presents a different version of the nature by which maintenance action affects the deterioration process.

Results

For each model conditions are given under which an optimal maintenance policy exhibits the following monotonic structure:

1. *Control limit rule for replacement.* There exists $0 \leqslant i^* \leqslant +\infty$ such that when the state of the machine $i \geqslant i^*$, the machine is to be replaced.
2. *Decreasing preventive maintenance.* For $0 \leqslant i < i^*$ denote an optimal maintenance level by $a^*(i)$. Then a^* is a nonincreasing function of i. (Compare to [16].)

The technique employed to obtain the above results first involves transforming the continuous time Markov decision process (CTMDP) to an equivalent discrete time Markov process (DTMDP). The infinite horizon problem will then be approached via the associated finite horizon discrete time process, thus permitting induction arguments on the "n-period" return functions. We then use the fact that the n-period return converges to the infinite horizon return function (see Bertsekas [5], Denardo [7], Ross [26]). This technique has been successfully employed by others, notably Lippman [19], [20] in queueing optimization.

Papers which focus on other aspects of optimal control of the machine maintenance/repair problem are Albright [1], Crabill [6], Goheen [12], and Winston [32], [33].

MODEL DESCRIPTION—MODEL 1

We consider a machine operating over an infinite time horizon. The state of the machine is represented by $i \in S = S_0 \cup \{f\}$, where $S_0 = \{0, 1, 2, \ldots\}$. State zero represents a new machine, state f a failed machine, and increasing integer state values represent increasing stages of machine deterioration.

The action space is denoted by A and consists of triples $(a_1, a_2, a_3) \in R^3$ and is defined as follows: $a_3 = 0$ or 1 according to whether the machine is not or is replaced by a new machine. If a machine in state i is replaced by a new machine, the transition $i \to 0$ takes place instantaneously. When $a_3 = 0$, the transition $i \to i + 1$ occurs at (exponential) rate $\lambda_1(i, a_1)$ and the transition to failure $i \to f$ occurs at rate $\lambda_f(i, a_2)$. The term "rate" is used here in the usual sense, see Ross [26]. For example, let $p_{i,i+1}(a_1)$ be the transition probability for the transition $i \to i + 1$ given action a_1 and suppose the transition time probability out of state i is exponential with mean $1/\lambda(i, a_1)$. Then $\lambda_1(i, a_1) = \lambda(i, a_1)p_{i,i+1}(a_1)$. Finally, if the machine is in the failed state, it must be immediately replaced ($a_3 = 1$ when $i = f$). a_1 and a_2 are viewed as maintenance actions in a sense to be made precise below.

The reward structure is specified as follows. A lump sum cost C is incurred when the machine is replaced by a new machine. If the machine fails while in service a lump sum cost $F \geqslant C$ is incurred for replacement. If the state is $i \in S_0$, the machine earns revenue at rate $r(i, a)$ per unit time, $a = (a_1, a_2, 0)$. The optimality criterion is the maximization of expected discounted revenue earned over an infinite time horizon. For the continuous discounting of cash flows denote the discount factor by $\alpha \in (0, 1)$.

Notation and Definitions. For a real-valued function g defined on the nonnegative integers, let $\Delta g(i) = g(i) - g(i-1)$, $i \geqslant 1$, and $\Delta^2 g(i) = g(i) - 2g(i-1) + g(i-2)$, $i \geqslant 2$. Thus, g is nonincreasing (nondecreasing) iff $\Delta g(i) \leqslant 0$ ($\geqslant 0$) and convex (concave) iff $\Delta^2 g(i) \geqslant 0$ ($\leqslant 0$).

Let f be a real-valued function defined on $C \times D \subseteq R^n$. f is *supermodular (submodular)* on D if for $d_1, d_2 \in D$, $f(c, d_1 \wedge d_2) + f(d, d_1 \vee d_2) \geqslant (\leqslant) f(c, d_1) + f(d, d_2)$, where $\wedge$ denotes component-wise minimum and $\vee$ denotes component-wise maximum. f has *antitone (isotone) differences* on $C \times D$ if for $c_1 \leqslant c_2$ and $d_1 < d_2$,

$$f(c_1, d_2) - f(c_1, d_1) \geqslant (\leqslant) f(c_2, d_2) - f(c_2, d_1).$$

Let $A' = A - \{a \in A \mid a_3 = 1\}$.

Define $\Lambda = \sup_{i,a} \{2\lambda_1(i, a_1) + \lambda_f(i, a_2)\}$.

We make the following assumptions for Model 1:

(A1) $\Lambda < +\infty$

(A2) A is a compact sublattice of R^3 (under the usual partial ordering and relative topology).

(A3) r is nonincreasing convex in i, nonincreasing in $(a_1, a_2, 0) \in A$ and has antitone differences on $S_0 \times A'$. Also, $r \geqslant 0$.

(A4) λ_1 and λ_f are each nondecreasing concave in i and nonincreasing on A'. λ_1 has isotone differences in (i, a_1) and λ_f isotone differences in (i, a_2).

(A5) λ_1, λ_f, and r are each continuous on A'.

REMARKS: Given state i and action a_2, the probability that the machine will fail in a small time dt is $\lambda_f(i, a_2)dt$. In this sense we think of λ_f as a failure rate function. Similarly, λ_1 is a measure of the tendency of the machine to "wear out." The fact that the rates λ_1 and λ_f are nondecreasing in i will play a major role in establishing a control limit rule for replacement (compare to Derman's IFR conditions [8]). The monotonicity of λ_1 and λ_f in $(a_1, a_2, 0)$ makes precise the notion of viewing the action as "preventive maintenance."

The concavity assumptions on λ_1 and λ_f can be given the interpretation that when the machine is new there is a strong tendency for wear-out and failure to increase sharply. As the machine breaks in, this rate of increase declines. This concavity assumption is used to prove the optimality of decreasing preventive maintenance.

From (A3), letting $b > a$ we have $r(i + 1, a) - r(i + 1, b) \geqslant r(i, a) - r(i, b)$ which says that increasing the level of maintenance from a to b incurs a greater cost increase in state $i + 1$ than in state i. Not only does the cost increase but the benefit decreases since from (A4) $\lambda_1(i + 1, a) - \lambda_1(i + 1, b) \leqslant \lambda_1(i, a) - \lambda_1(i, b)$ (and similarly for λ_f).

THEOREM 1: There exists an optimal stationary policy for Model 1.

PROOF: Note that from (A2), (A3), and (A5) the reward function is bounded. It is then trivial to verify that the conditions specified in [7] are satisfied. (The functional operator H_g defined in that paper turns out to be a two stage contraction due to the possibility of instantaneous transitions $i \to 0$. For unbounded return functions see Lippman [21].)

Functional Equations

Using the method described in Lippman [19] (see also Kakumanu [14], Serfozo [28], Winston [32], Howard [13], Anderson [2]) we can transform the continuous time process to an

equivalent discrete time Markov decision process (DTMDP). Denote the optimal return function for the n-period horizon in the DTMDP by V_n, $n \leqslant +\infty$.

Then

$$V_0(i) \equiv 0 \tag{1}$$

and for $n \geqslant 1$,

$$V_n(i) = \begin{cases} \text{Max } \{T_n(C, 0), T_n(i)\} & i \neq f \\ T_n(F, 0) & i = f \end{cases} \tag{2}$$

For $x \in \{C, F\}$,

$$T_0(x, 0) \equiv 0$$

and for $n \geqslant 1$,

$$\begin{aligned} T_n(x, 0) = \max_{a \in A'} \{ & r(0, a) - x(\alpha + \lambda_1(0, a_1) + \lambda_f(0, a_2)) \\ & + \lambda_1(0, a_1) V_{n-1}(1) + \lambda_f(0, a_2)\, T_{n-1}(F, 0) \\ & + (\Lambda - \lambda_1(0, a_1) - \lambda_f(0, a_2)) V_{n-1}(0)\}/(\Lambda + \alpha) \end{aligned} \tag{3}$$

$$T_0(i) \equiv 0$$

$$T_n(i) = \max_{a \in A'} J(i, a, V_{n-1})/(\Lambda + \alpha)$$

$$\begin{aligned} J(i, a, V_{n-1}) = r(i, a) + \lambda_1(i, a_1) V_{n-1}(i+1) + \\ \lambda_f(i, a_2)\, T_{n-1}(F, 0) \quad + \\ (\Lambda - \lambda_1(i, a_1) - \lambda_f(i, a_2))\, V_{n-1}(i). \end{aligned} \tag{4}$$

(For $n = +\infty$ we write $V_\infty = V$, $T_\infty = T$). Equations (2) assume that a failed machine must be replaced. The presence of the lump sum rewards $-C$ and $-F$ requires special consideration in the discrete time recursions resulting in their somewhat complicated appearance involving the functions T_n. For a discussion on the treatment of lump sum rewards see Serfozo [28] or Anderson [2]. In (2), $T_n(C, 0)$ is the return if the replacement action is chosen, $T_n(F, 0)$ is the return if an in service failure occurs, and $T_n(i)$ is the return otherwise.

We denote the *optimal action* in state i by $a^*(i, n)$ and resolve ties in (3) by defining $a^*(i,n) = \inf \{a \in A' | J(i, a, V_{n-1}) = \max_{a \in A'} J(i, a, V_{n-1})\}$ which is well-defined since J is continuous on A' and A is compact.

Finally, if a tie occurs in (2) we define $V_n(i) = T_n(i)$.

LEMMA 1: For $0 \leqslant n \leqslant +\infty$ and $i \in S_0$, $T_n(F, 0) \leqslant V_n(i)$.

PROOF: Equation (2) implies $V_n(i) \geqslant T_n(C, 0) \geqslant T_n(F, 0)$ since $F \geqslant C$. □

LEMMA 2: For $0 \leqslant n \leqslant +\infty$, V_n is nonincreasing in $i \in S_0$.

PROOF: We prove the lemma for finite n (by induction). The infinite horizon case follows by taking limits ($V_n \to V$). For $n = 0$ the result is trivial. For $n > 0$ it suffices to show that $\Delta T_n(i) \leqslant 0$. Define $a = a^*(i) = (a_1, a_2, 0)$. From (3), (4), and the definition of a,

$$(\Lambda + \alpha)\,[T_n(i) - T_n(i-1)] \leqslant J(i, a, V_{n-1}) - J(i-1, a, V_{n-1}) \tag{5}$$

$$\leqslant \Delta V_{n-1}(i+1)\lambda_1(i, a_1) + \Delta V_{n-1}(i)\,[\Lambda - \lambda_1(i-a, a_1) - \lambda_f(i-1, a_2)]$$

$$\leqslant 0. \tag{6}$$

Inequality (5) follows from $\lambda_f(i, a_2) \geqslant \lambda_f(i-1, a_2)$ and $T_{n-1}(F, 0) - V_{n-1}(i) \leqslant 0$ (Lemma 1). Inequality (6) follows from the definition of Λ and the inductive hypothesis. □

THEOREM 2 (Control limit rule for replacement): There exists i^*, $0 \leqslant i^* \leqslant +\infty$, such that for $i \geqslant i^*$, $a_3^*(i) = 1$ (i.e., the optimal action is replacement).

PROOF: From (2) it suffices to show that T is nonincreasing on S_0. In the proof of Lemma 2 we showed that $\Delta T_n(i) \leqslant 0$ for finite n. Using $\Delta V(i) \leqslant 0$ (Lemma 2), the proof that $\Delta T_n(i) \leqslant 0$ holds in the same way for $n = +\infty$. Now set $i^* = \inf\{i: T(C, 0) \geqslant T(i)\}$. □

In order to establish the optimality of nonincreasing preventive maintenance we first need the following lemma.

LEMMA 3: For $0 \leqslant n \leqslant +\infty$, V_n is convex in i.

PROOF: The convexity of V will follow from that of $V_n (n < +\infty)$ by taking limits on n. For $n < +\infty$ we employ induction. Since the maximum of convex functions is convex, from (2) it suffices to show that T_n is convex. Since $V_0 \equiv 0$ the convexity of V_0 is trivial. Assume then that V_{n-1} is convex. To show that T_n (and, therefore, V_n) is convex we need $\Delta^2 T_n(i) = (T_n(i) - T_n(i-1)) - (T_n(i-1) - T_n(i-2)) \geqslant 0$, $i \geqslant 2$. Define $a = a^*(i-1, n) = (a_1, a_2, 0)$. From (3) we have

$$(\Lambda + \alpha)\,[T_n(i) - T_n(i-1)] \geqslant J(i, a, V_{n-1}) - J(i-1, a, V_{n-1}) \tag{7}$$

$$(\Lambda + \alpha)\,[T_n(i-1) - T_n(i-2)] \leqslant J(i-1, a, V_{n-1}) - J(i-2, a, V_{n-1}). \tag{8}$$

From (7) and (8) in order to establish $\Delta^2 T_n(i) \geqslant 0$ it suffices to prove that $\Delta^2 J(i, a) \equiv J(i, a, V_{n-1}) - 2J(i-1, a, V_{n-1}) + J(i-2, a, V_{n-1}) \geqslant 0$.

From (4),

$$\Delta^2 J(i, a) = r(i,a) - 2r(i-1, a) + r(i-2, a) \tag{9}$$

$$+ \lambda_1(i, a_1)\, V_{n-1}(i+1) \tag{10}$$

$$+ \Delta V_{n-1}(i)[\Lambda - 2\lambda_1(i-1, a_1)] \tag{11}$$

$$+ \Delta V_{n-1}(i-1)[-\Lambda + \lambda_1(i-2, a_1)] \tag{12}$$

$$+ T_{n-1}(F, 0)\,[\lambda_f(i, a_2) - 2\lambda_f(i-1, a_2) + \lambda_f(i-2, a_2)] \tag{13}$$

$$+ V_{n-1}(i)\,[-\lambda_f(i, a_2)] \tag{14}$$

$$+ V_{n-1}(i-1)\,[2\lambda_f(i-1,\,a_2)] \tag{15}$$

$$+ V_{n-1}(i-2)\,[-\lambda_f(i-2,\,a_2)] \tag{16}$$

Consider the last four terms of the above sum ((13)-(16)). From Lemma 1, $T_{n-1}(F,\,0) \leqslant V_{n-1}(i-1)$ and using the concavity of λ_f we see that the sum of (13)-(16) is greater than or equal to

$$\Delta V_{n-1}(i)[-\lambda_f(i,\,a_2)] + V_{n-1}(i-1)\,[\lambda_f(i-2,\,a_2)]. \tag{17}$$

Replacing (13)-(16) by (17), using the convexity of r, the definition of Λ and the induction hypothesis that V_{n-1} is convex ($\Delta V_{n-1}(i) \geqslant \Delta V_{n-1}(i-1)$), we see that

$$\begin{aligned}\Delta^2 J(i,\,a) &\geqslant \Delta V_{n-1}\,(i-1)\,[\lambda_1(i,\,a_1) - 2\lambda_1(i-1,\,a_1) + \lambda_1(i-2,\,a_1)\\ &\qquad + \lambda_f(i-2,\,a_2) - \lambda_f(i,\,a_2)]\\ &\geqslant 0\end{aligned}$$

where the last inequality follows from the concavity of λ_1 and the monotonicity of λ_f. □

THEOREM 3: Prior to replacement, the optimal preventive maintenance level is a nonincreasing function of i, i.e., for $0 \leqslant i < i^*$, $a^*(i) \geqslant a^*(i+1)$.

PROOF: From Theorem 6.2 of [30] and assumption (A2) it suffices to show that (1) for each i, J is supermodular in $a \in A'$ and (2) J has antitone differences on $A' \times S_0$. (1) follows from the supermodularity of r and the fact that $J - r$ is a separable function of a_1 and a_2.
Let $a = (a_1,\,a_2,\,0) < b = (b_1,\,b_2,\,0) \in A'$. Define $\Delta_{i,a,b} \equiv J(i+1,\,b,\,V) + J(i,\,a,\,V) - J(i+1,\,a,\,V) - J(i,\,b,\,V)$.
Then,

$$\Delta_{i,a,b} = r(i+1,\,b) + r(i,\,a) - r(i+1,\,a) - r(i,\,b) \tag{18}$$

$$+ \Delta V(i+2)\,[\lambda_1(i+1,\,b_1) - \lambda_1(i+1,\,a_1)] \tag{19}$$

$$+ \Delta V(i+1)\,[\lambda_1(i,\,a) - \lambda_1(i,\,b)] \tag{20}$$

$$+ (T(F,\,0) - V(i+1))\,[\lambda_f(i+1,\,b_2) - \lambda_f(i+1,\,a_2)] \tag{21}$$

$$+ (T(F,\,0) - V(i))\,[\lambda_f(i,\,a_2) - \lambda_f(i,\,b_2)] \tag{22}$$

$$\leqslant \Delta V(i+1)\,[\lambda_1(i+1,\,b_1) - \lambda_1(i+1,\,a_1) + \lambda_1(i,\,a_1) - \lambda_1(i,\,b)] \tag{23a}$$

$$+ (T(F,\,0) - V(i+1))\,[\lambda_f(i+1,\,b_2) - \lambda_f(i+1,\,a_2) + \lambda_f(i,\,a_2) - \lambda_f(i,\,b_2)] \tag{23b}$$

$$\leqslant 0. \tag{24}$$

The inequality yielding (23a) and (23b) results from the following: From (A3), summand (18) is nonpositive; use the convexity of V and the fact that λ_1 is nonincreasing on A' to obtain (23a) from (19) and (20). Now use Lemma 2 and the fact that λ_f is nonincreasing on A' to obtain (23b) from (21) and (22).

Inequality (24) follows using Lemma 1, and assumption (A4) that λ_1 and λ_2 have isotone differences.

□

This completes the specification of the structure of the optimal maintenance/replacement policy for Model 1. Note that the results hold for the special cases $\lambda_f \equiv 0$, or $\lambda_1 \equiv 0$. Also observe that from (4), $J(i, a, V) = r(i, a) + \lambda_1(i, a_1)\Delta V(i+1) + \lambda_f(i, a_2)\ (T(F, 0) - V(i)) + \Lambda V(i)$. Thus, if r is convex in a, and λ_1 and λ_f are concave in a, J will be convex in a. (Recall Lemmas 1 and 2.) In this case, a^* is a bang-bang policy for $i \in S_0$: $a^* = \sup\{a \in A'\}$ or $a^* = \inf\{a \in A'\}$. (A' is compact by assumption (A2).) λ_1 concave nonincreasing in a_1 means that small increments in a_1 above inf A' have little effect in decreasing λ_1 (and similarly for a_2 and λ_f). However, r convex nonincreasing in a implies that the increase in cost for such an increment is great. Whether $a^*(0) = \sup A'$ or $a^*(0) = \inf A'$ will depend on how effective preventive maintenance is against protecting the new machine relative to the cost of the machine.

MACHINE MAINTENANCE — MODEL 2

The specification of Model 2 is identical to Model 1 regarding the state space, reward structure, and optimality criterion. However, Model 2 differs in the nature of machine deterioration. Preventive maintenance is represented by a pair $a = (a_1, a_2, 0) \in A \subseteq R^3$. The transition $i \rightarrow f$ (failure) occurs at rate $\lambda_f(i, a_2)$. Also, the machine may deteriorate to state $i + j$ from state i at rate $\lambda(j, a_1)$, $j \in \{1, 2, \ldots\}$.

Suppose the previous assumptions in Model 1 on A, r and λ_f hold here. In addition, we assume that $\lambda(j, \cdot)$ is nonincreasing and continuous on A' for $j \in S_0$. Also, assume that $\Lambda \equiv \sup\{\Sigma\lambda(j, a_1) + \lambda_f(i, a_f)\} < +\infty$. Then routine application of the techniques used for Model 1 establishes that the optimal maintenance/replacement policy for Model 2 exhibits the same monotonic structure. Since the proofs are completely straightforward we omit the details.

MACHINE MAINTENANCE—MODEL 3

Again Model 3 differs from Model 1 only in the nature of the deterioration process. Specifically, denote an action by $a = (a_1, a_2, a_3) \in A \subseteq R^3$. $a_3 = 1$ indicates replacement as before. Suppose $a_3 = 0$. Then failure $i \rightarrow f$ occurs at rate $\lambda_f(i, a_2)$ as before. In addition the machine may also deteriorate from state i to a state $i + m(i, a_1, p)$ at rate $\lambda dF(p)$. Here p is a random variable taking values in a set P and having distribution F. Let Z denote the set of integers. Then m is a map m: $S_0 \times A \times P \rightarrow Z$. We think of $p \in P$ as a measure of the magnitude of a randomly occuring shock. The time between shocks is exponential with mean $1/\lambda$. The three variables i, a_1, and p together determine the subsequent "damage" $m(i, a_1 p)$ to the machine.

We make the following assumptions:

(B1) $\Lambda \equiv \sup_{i, a_2}\{\lambda_f(i, a_2)\} + \lambda < +\infty$

(B2), (B3) same as (A2) and (A3), respectively.

(B4) The conditions on λ_f are the same as stated in (A4).

(B5) The function m is nonincreasing in a_1 ("preventive maintenance"), is nondecreasing in $i \in S_0$, and has isotone differences on $S_0 \times A$.

(B6) m, λ_f, and r are each continuous on A.

These assumptions are to be interpreted in a manner similar to the discusssion following (A1)-(A5) of Model 1.

It is easy to establish Theorem 1 for Model 3. We now present the recursions for the present model.

Recursions for Model 3

The DTMDP recursions equivalent to the CTMDP are the following:

$$V_0(i) \equiv 0 \tag{25}$$

and for $1 \leqslant n \leqslant +\infty$,

$$V_n(i) = \begin{cases} \text{Max}\{T_n(C,0),\ T_n(i)\} & i \neq f \\ T_n(F,\ 0) & i = f \end{cases} \tag{26}$$

For $x \in \{C,\ F\}$,

$$T_0(x,\ 0) \equiv 0$$

and for $n \geqslant 1$, with $A' = A - \{a \in A \mid a_3 = 0\}$,

$$\begin{aligned} T_n(x,\ 0) = \max_{a \in A'} \{ & r(0,\ a) - x(\alpha + \lambda + \lambda_f(0,\ a_2)) \\ & + \lambda \int_P V_{n-1}(m(0,\ a_1,\ p)dF(p) + \lambda_f(0,\ a_2) T_{n-1}(F,\ 0) \\ & + (\Lambda - \lambda - \lambda_f(0,\ a_2))\ V_{n-1}(0)\}/(\Lambda + \alpha) \end{aligned}$$

$$T_0(i) \equiv 0$$

for $n \geqslant 1$,

$$T_n(i) = \max_{a \in A'} J(i,\ a,\ V_{n-1})/(\Lambda + \alpha) \tag{27}$$

$$\begin{aligned} J(i,\ a,\ V_{n-1}) = {} & r(i,\ a) + \lambda \int_P V_{n-1}\ (i + m(i,\ a_1,\ p))\ dF(p) \\ & + \lambda_f(i,\ a_2)\ T_{n-1}(F,\ 0) \\ & + (\Lambda - \lambda - \lambda_f(i,\ a_2))\ V_{n-1}(i) \end{aligned} \tag{28}$$

As before for $n = +\infty$ we write $V_\infty = V$ and $T_\infty = T$, and $a^*(i,\ n)$ denotes the optimal maintenance action in state i.

LEMMA 4: For $0 \leqslant n \leqslant +\infty$ and $i \in S_0$,

$$T_n(F,\ 0) \leqslant V_n(i).$$

PROOF: The proof is similar to the proof of Lemma 1. □

LEMMA 5: For $0 \leqslant n \leqslant +\infty$, V_n is nonincreasing in $i \in S_0$.

PROOF: As before, the infinite horizon case will follow from the finite horizon result. The result is trivial for $n = 0$ since $V_0 \equiv 0$. Assume $\Delta V_{n-1}(i) \leqslant 0$ for $i \in S_0$. From (26) it suffices to show $\Delta T_n(i) \leqslant 0$. Let $a = a^*(i, n) = (a_1, a_2, 0) \in A'$.

From (27),

$$\begin{aligned}(29)\qquad \Delta T_n(i) &\leqslant J(i, a, V_{n-1}) - J(i-1, a, V_{n-1}) \\ &= \Delta r(i, a) \\ &\quad + \lambda \int_P V_{n-1}(i + m(i, a_1, p)) - V_{n-1}(i - 1 + m(i-1, a_1, p))\, dF(p) \\ &\quad + \lambda_f(i, a_2)\,[T_{n-1}(F, 0) - V_{n-1}(i-1)] \\ &\quad - \lambda_f(i-1, a_2)\,[T_{n-1}(F, 0) - V_{n-1}(i-1)] \\ &\quad + (\Lambda - \lambda)\Delta V_{n-1}(i) \\ &\leqslant \Delta r(i, a) + \\ &\quad \lambda_P \int [V_{n-1}(i + m(i, a_1, p)) - V_{n-1}(i - 1 + m(i-1, a_1, p))]\, dF(p) \\ &\quad + (\Lambda - \lambda)\Delta V_{n-1}(i)\end{aligned}$$

$$(30)\qquad \leqslant 0.$$

Inequality (29) follows from Lemma 4 and the monotonicity of λ_f in i. To obtain the last inequality, (30), use (A3) on r, (B5) on $\underline{M}$, and the induction hypothesis that V_{n-1} is nonincreasing, along with the definition of Λ. □

THEOREM 4 (Control limit rule for replacement): There exists i^*, $0 \leqslant i^* \leqslant +\infty$ such that for $i \geqslant i^*$, $a_3^*(i) = 1$ (replacement).

PROOF: The proof is similar to the proof of Theorem 2. □

LEMMA 6: For $0 \leqslant n \leqslant +\infty$, V_n is convex in i.

PROOF: As in Lemma 3, it suffices to show that T_n is convex. $V_0 \equiv 0$ so assume that V_{n-1} is convex. Let $a = a^*(i-1, n) = (a_1, a_2, 0)$ and

$$\Delta^2 J(i, a) \equiv J(i, a, V_{n-1}) - 2J(i-1, a, V_{n-1}) + J(i-2, a, V_{n-1}).$$

From (27),

$$\begin{aligned}\Delta^2 T_n(i) &\geqslant \Delta^2 J(i, a) \\ (31)\qquad &= \Delta^2 r(i, a) \\ &\quad + \lambda \int_P [V_{n-1}(i + m(i, a_1, p)) - 2V_{n-1}(i - 1 + m(i-1, a_1, p)) \\ (32)\qquad &\quad + V_{n-1}(i - 2 + m(i-2, a_1, p))]\, dF(p) \\ (33)\qquad &\quad + \Delta^2\lambda_f(i, a_2)\, T_{n-1}(F, 0) \\ (34)\qquad &\quad + V_{n-1}(i)[\Lambda - \lambda - \lambda_f(i, a_2)] \\ (35)\qquad &\quad + V_{n-1}(i-1)\,[-2\Lambda + 2\lambda + 2\lambda_f(i-1, a_2)]\end{aligned}$$

(36) $$+ V_{n-1}(i-2)[\Lambda - \lambda - \lambda_f(i-2, a_2)].$$

From (B3), $\Delta^2 r(i, a) \geqslant 0$. Define $m_0 = i + m(i, a_1, p)$, $m_1 = i - 1 + m(i-1, a_1, p)$, and $m_2 = i - 2 + m(i-2, a_1, p)$. From (B5) $m_0 \geqslant m_1 \geqslant m_2$ and m concave in $i \Rightarrow m_0 - m_1 \leqslant m_1 - m_2$. Since V_{n-1} is convex (the inductive hypothesis) and nonincreasing (Lemma 5), $V_{n-1}(m_0) - V_{n-1}(m_1) \geqslant V_{n-1}(m_1) - V_{n-1}(m_2)$ which establishes that the integrand in (32) is nonnegative.

Finally, the proof that the sum of the last four terms (33)-(36) is nonnegative is the same as in Lemma 3. Thus, $\Delta^2 J(i, a) \geqslant 0$. □

THEOREM 5: Prior to replacement the optimal preventive maintenance level is a nonincreasing function of i.

PROOF: As in the proof of Theorem 3 it suffices to show that (1) for each i, J is supermodular in $a \in A'$ and (2) J has antitone differences on $A' \times S_0$. (1) follows as before. To establish antitone differences we first invoke (B3) on r. Also, the proof that J has antitone differences in i and a_2 is the same as in Theorem 3. It thus suffices to prove that the integrand in (28) exhibits antitone differences in i and a_1. Let $a_1 < b_1$. Define $m_1 = i + 1 + m(i+1, b_1, p)$, $m_2 = i + 1 + m(i+1, a_1, p)$, $m_3 = i + m(i, b_1, p)$, and $m_4 = i + m(i, a_1, p)$. From (B5) $m_1 \leqslant m_2$, $m_3 \leqslant m_1$, $m_3 \leqslant m_4$, $m_4 \leqslant m_2$. Define $g(i, a_1) = V(i + m(i, a_1, p))$. Then

$$[g(i+1, b_1) - g(i+1, a_1)] - [g(i, b_1) - g(i, a_1)]$$
$$= [V(m_1) - V(m_2)] - [V(m_3) - V(m_4)] \leqslant 0.$$

The inequality follows from the fact that V is nonincreasing convex and the fact that $m_2 - m_1 \leqslant m_4 - m_3$ (namely (B5) which assumes that m has isotone differences in (i, a_1)). This completes the proof. □

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A NEW BIVARIATE NEGATIVE BINOMIAL DISTRIBUTION

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ABSTRACT

A new bivariate negative binomial distribution is derived by convoluting an existing bivariate geometric distribution; the probability function has six parameters and admits of positive or negative correlations and linear or nonlinear regressions. Given are the moments to order two and, for special cases, the regression function and a recursive formula for the probabilities. Purely numerical procedures are utilized in obtaining maximum likelihood estimates of the parameters. A data set with a nonlinear empirical regression function and another with negative sample correlation coefficient are discussed.

1. INTRODUCTION

In this paper we develop a new bivariate negative binomial (bnb) distribution by convolving a certain bivariate geometric distribution. The univariate negative binomial with parameters $\nu > 0$ and $\theta > 0$ is defined (Johnson and Kotz [6]) as the distribution of a random variable (r.v.) X for which

(1)
$$Pr[X = x] = \frac{\Gamma(x+\nu)}{x!\Gamma(\nu)}\left(\frac{1}{1+\theta}\right)^{\nu}\left(\frac{\theta}{1+\theta}\right)^{x} = \binom{x+\nu-1}{x}\left(\frac{1}{1+\theta}\right)^{\nu}\left(\frac{\theta}{1+\theta}\right)^{x}, \quad x = 0, 1, 2, \ldots$$

where $\binom{x+\nu-1}{x}$ shall be taken to be defined as the ratio of gamma functions; the characteristic function is

$$E[e^{it'X}] = [1 + \theta(1 - e^{it})]^{-\nu}.$$

The mean and variance of X are $\nu\theta$ and $\nu\theta(1+\theta)$, respectively. For $\nu = 1$ we have the geometric distribution. We shall find it more convenient to use probability generating functions than characteristic functions because of the discrete nature of the random variables we consider. The correspondence between characteristic functions and probability generating functions is effected by letting $u = e^{it}$ which gives

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$$\phi(u) = E[u^X] = [1 + \theta(1 - u)]^{-\nu}. \tag{2}$$

Another common representation is to let $\theta = p/(1-p)$, or equivalently, $p = \theta/(1+\theta)$, in (1) and so

$$Pr[X = x] = \frac{\Gamma(x+\nu)}{x!\Gamma(\nu)}(1-p)^{\nu} p^x, \quad x = 0, 1, 2, \ldots \tag{3}$$

$$\phi(u) = \left[\frac{1-pu}{1-p}\right]^{-\nu} = \left[1 + \frac{p}{1-p}(1-u)\right]^{-\nu} \tag{4}$$

This latter representation is referred to as a negative binomial distribution with parameters ν and p. We use both representations throughout.

The probability function of a bnb distribution (Mardia [9]) (or negative multinomial (Johnson and Kotz, [6]) is

$$Pr[X = x, Y = y] = \frac{\Gamma(x+y+\nu)}{x!y!\Gamma(\nu)}(1-p-q)^{\nu} p^x q^y, \; x, y = 0, 1, 2, \ldots \tag{5}$$

where $\nu > 0$, $p = \theta/[1 + (\alpha+1)\theta]$, $q = \alpha p$, $0 < p < 1$, $0 < q < 1$, and $0 < p + q < 1$. The probability generating function is

$$\phi(u, v) = E[u^X v^Y] = [1 + \theta(1-u) + \alpha\theta(1-v)]^{-\nu}. \tag{6}$$

Guldberg [5] introduced this distribution and Bates and Neyman [3] fitted it to several data sets. We designate this bnb distribution as $G - B - N(\alpha, \theta, \nu)$. The distribution admits only of positive correlation and linear regressions.

Certain data sets do not exhibit empirical regressions which are linear nor do some data sets show positive correlation and so it is natural, for these cases, to work with a bivariate probability function which allows for nonlinear regressions or negative correlations or both. The classical Bates and Neyman paper exhibited empirical data which would seemingly be best fit by regression curves which were obviously nonlinear, and, consequently, their results were not entirely satisfactory. Furthermore, in Table 3 we provide some new bivariate data related to *aircraft flight aborts which has a negative sample correlation* coefficient. We would thus be reluctant to use a bivariate distribution which did not admit of the possibility of both negative correlation and nonlinear regressions to describe the observed abort phenomenon. These few instances (and several others we do not provide) indicate there is a need for a study of the properties and potentialities of new multivariate negative binomial distributions which provide for more flexibility than those which have heretofore been discussed in the literature. We also isolate some situations which call for even more general distributions than we consider herein. Before we perform the data analysis we discuss an *inventory model* which generates the new distributions and examine the resulting distributions in some detail.

2. A STORAGE MODEL

Consider the storage system consisting of two facilities, one of which is used to store commodity A alone, the other to store commodity B alone. At each epoch of time, $n = 1, 2, \ldots$, facility 1 receives a random amount of commodity A, say a_n, and facility B receives a random amount of commodity B, say b_n. Take the a_n and b_n to be mutually independent, identically distributed random variables. Let the total of A and B at epoch n be

A_n and B_n. The total A_n and B_n are related to the totals A_{n-1} and B_{n-1} in the following fashion. Between epoch $n-1$ and n the totals A_{n-1} and B_{n-1} may randomly be reduced to zero or remain intact. The manner in which A_{n-1} and B_{n-1} are carried over to the nth epoch determines storage characteristics. The system equations are

$$A_n = a_n + \xi_n A_{n-1}$$
$$B_n = b_n + \zeta_n B_{n-1},$$

and without loss of generality we take $A_0 = B_0 = 0$. There are four cases of interest:

(I) $\Pr(\xi_n = \zeta_n = 0) = a$
$\Pr(\xi_n = \zeta_n = 1) = d = 1 - a.$

(II) $\Pr(\xi_n = 0, \zeta_n = 0) = a$
$\Pr(\xi_n = 1, \zeta_n = 0) = b$
$\Pr(\xi_n = 1, \zeta_n = 1) = d = 1 - a - b.$

(III) $\Pr(\xi_n = 0, \zeta_n = 0) = a$
$\Pr(\xi_n = 0, \zeta_n = 1) = c$
$\Pr(\xi_n = 1, \zeta_n = 1) = d = 1 - a - c.$

(IV) $\Pr(\xi_n = 0, \zeta_n = 0) = a$
$\Pr(\xi_n = 1, \zeta_n = 0) = b$
$\Pr(\xi_n = 0, \zeta_n = 1) = c$
$\Pr(\xi_n = 1, \zeta_n = 1) = d = 1 - a - b - c.$

For example, (I) implies that the contents of facilities A and B are either both discharged or both kept in the time between epoch $n-1$ and n. (II) implies that the contents of both are simultaneously discharged with probability a; the contents of B is discharged while those of A are kept with probability b; and the contents of both A and B are kept with probability d. Finally, take both a_n and b_n to be geometric. Provided $b + d < 1$, $c + d < 1$, the system defined by the above system equations soon reaches steady state and is governed by a bivariate geometric probability distribution (Paulson and Uppuluri, [11]). If there are m identical systems of this type and the quantities A and B are summed across the m systems, a bivariate negative binomial distribution results. Curiously, cases (I), (II), and (III) give rise to infinitely divisible bivariate distributions (as we show later) whereas (IV) does not in general The analytical details, starting from the steady state system whose behavior is defined by a functional equation in the probability generating function, are given in the next section.

3. THE NEW BIVARIATE NEGATIVE BINOMIAL DISTRIBUTION AND PROPERTIES

Paulson and Uppuluri [11] showed that the bivariate r.v. (X, Y) (identify X and Y with the steady state A_n and B_n, respectively), where each element in the pair is defined on the nonnegative integers, has a bivariate geometric distribution if its probability generating function, $\phi(u, v)$, satisfies the functional equation

$$\phi(u, v) = \psi_1(u, 1)\psi_2(1, v)(a + b\phi(u, 1) + c\phi(1, v) + d\phi(u, v)) \tag{7}$$

where

$$\psi_1(u, 1) = \left[1 + \frac{p}{1-p}(1-u)\right]^{-1},$$

$$\psi_2(1, v) = \left[1 + \frac{q}{1-q}(1-v)\right]^{-1},$$

and a, b, c, d are all nonnegative quantities with $a + b + c + d = 1$, $b + d < 1$, $c + d < 1$, and $0 < p < 1$, $0 < q < 1$. It is easy to show from (7) that

$$\phi(u, 1) = (1 + \theta_1(1-u))^{-1}, \tag{8a}$$

$$\phi(1, v) = (1 + \theta_2(1-v))^{-1}, \tag{8b}$$

with

$$\theta_1 = p/[(1-p)(a+c)],\ \theta_2 = q/[(1-q)(a+b)]. \tag{8c}$$

We thus see that the marginals are geometric. Paulson and Uppuluri obtained the moments of (X, Y) in (7) to order two and showed that the correlation coefficient ρ varies over $-.25 \leqslant \rho < 1$; they also presented recursive formulae for determining the probability function.

Consider the case $b = c = 0$ in (7). We have (suppressing the arguments of $\psi_1(u, 1)$ to give $\psi_1 = \psi_1(u, 1)$ and similarly for ψ_2)

$$\phi(u, v) = \psi_1\psi_2((1-d) + d\phi(u, v)) \tag{9}$$

$$= \frac{(1-d)\psi_1\psi_2}{1 - d\psi_1\psi_2}$$

$$= (1-d)\psi_1\psi_2(1 + d\psi_1\psi_2 + d^2\psi_1^2\psi_2^2 + \ldots). \tag{10}$$

The inverse transform of $\phi(u, v)$, the probability function $g_1(x, y)$, say, may be obtained termwise from (10) since the resultant series converges uniformly and absolutely for all $x, y = 0, 1, 2, \ldots$ (Titchmarsh, [12]). We obtain

$$g_1(x, y) = (1-d)(1-p)p^x(1-q)q^y \sum_{j=0}^{\infty} \binom{x+j}{x}\binom{y+j}{y}[d(1-p)(1-q)]^j, \tag{11}$$

where $x, y = 0, 1, 2, \ldots$. Expansion of the combinatoric terms in (11) gives

$$g_1(x, y) = (1-d)(1-p)p^x(1-q)q^y F(x+1, y+1; 1; d(1-p)(1-q)), \tag{12}$$

where $F(a, b; c; z)$ is the Gaussian hypergeometric series given by

$$F(a, b; c; z) = 1 + \sum_{j=1}^{\infty} \frac{(a)_j(b)_j}{(c)_j}\frac{z^j}{j!}, \tag{13}$$

and the term $(n)_j$ is defined by

$$(n)_j = \frac{\Gamma(n+j)}{\Gamma(n)} = n(n+1)\ldots(n+j-1).$$

It can be shown that Theorem 12.2.3 of Lukacs [8] can be appropriately extended to yield the result that $\phi(u, v)$ is infinitely divisible, or equivalently, $g_1(x, y)$ is infinitely divisible (see also Block, Paulson, and Kohberger [4].

Thus, the ν-fold convolution of $g_1(x, y)$ with itself yields a bnb distribution which we designate $g_\nu(x, y)$ where ν need not be restricted to integer values. The probability generating function $\phi_\nu(u, v)$ of $g_\nu(x, y)$ is

$$\phi_\nu(u, v) = ((1 - d)\psi_1\psi_2)^\nu (1 - d\psi_1\psi_2)^{-\nu} \tag{14}$$

$$= ((1 - d)\psi_1\psi_2)^\nu \left(1 + \nu d\psi_1\psi_2 + \frac{\nu(\nu + 1)}{2!} (d\psi_1\psi_2)^2 + \dots\right)$$

which gives in the same manner as above

$$g_\nu(x, y) = (1 - d)^\nu h_1(x) h_2(y) F(x + \nu, y + \nu; \nu; z), \tag{15}$$

where

$$h_1(x) = \binom{\nu + x - 1}{x} (1 - p)^\nu p^x$$

$$h_2(y) = \binom{\nu + y - 1}{y} (1 - q)^\nu q^y$$

$$z = d(1 - p)(1 - q)$$

and $x, y = 0, 1, 2, \dots$. It is clear by construction that the marginals of $g_\nu(x, y)$, say $g_{x\nu}(x)$ and $g_{y\nu}(y)$, are negative binomial with $\theta_1 = p/(a(1 - p))$ and $\theta_2 = q/(a(1 - q))$ as may be determined from (8) with $b = c = 0$.

The infinite divisibility of $g_\nu(x, y)$ is a result of considerable appeal in applications since ν need not be restricted to integer values.

We remark in passing that the distribution (15) is the discrete analogue to the Wicksell-Kibble bivariate gamma distribution. The k-variate extension may be readily derived as above and has probability distribution (in obvious notation)

$$g_\nu(x_1, x_2, \dots, x_k) = C(k) F_{k-1}(x_1 + \nu, \dots, x_k + \nu; \nu, \dots, \nu; d \prod_{i=1}^{k} (1 - p_i)),$$

with

$$C(k) = \prod_{j=1}^{k} \binom{\nu + x_j - 1}{x_j} (1 - p_j)^\nu (p_j)^{x_j}.$$

Next we consider the case with b and c different from zero in (7). For notational convenience, set $\phi_1(u, 1) = \phi_1$ and $\phi_2(1, v) = \phi_2$. Denote the probability function corresponding to (7) by $f_1(x, y)$. Observe that when $b = c = 0$, $f_1(x, y) = g_1(x, y)$. The (integral) m-fold convolution of $f_1(x, y)$ with itself has probability generating function $\phi_m(u, v)$, probability function $f_m(x, y)$, and marginal probability functions $f_{xm}(x)$ and $f_{ym}(y)$, respectively. With some minor calculation we find

$$\phi_m(u, v) = \left[\frac{(1-d)\psi_1\psi_2}{1-d\psi_1\psi_2}\right]^m \left[\frac{a}{1-d} + \frac{b}{1-d}\phi_1 + \frac{c}{1-d}\phi_2\right]^m. \tag{16}$$

From (8) or (16) it is easily shown that

$$\phi_m(u, 1) = (1+\theta_1(1-u))^{-m}, \tag{17a}$$

$$\phi_m(1, v) = (1+\theta_2(1-v))^{-m}, \tag{17b}$$

and accordingly,

$$f_{Xm}(x) = \frac{\Gamma(x+m)}{\Gamma(x+1)\Gamma(m)}\left[\frac{1}{1+\theta_1}\right]^m\left[\frac{\theta_1}{1+\theta_1}\right]^x, \tag{18a}$$

$$f_{Ym}(y) = \frac{\Gamma(y+m)}{\Gamma(y+1)\Gamma(m)}\left[\frac{1}{1+\theta_2}\right]^m\left[\frac{\theta_2}{1+\theta_2}\right]^y, \tag{18b}$$

for $x = 0, 1, \ldots, y = 0, 1, \ldots,$ and θ_1 and θ_2 as given in (8c) and $m = 1, 2, \ldots$ Define $\lambda_1 = a/(1-d)$, $\lambda_2 = b/(1-d)$, $\lambda_3 = c/(1-d)$ in (16). Since products of probability generating functions correspond to convolutions of probability functions, an application of the trinomial expansion to (16) coupled with a termwise inversion of products of the form (17) with (15) gives

THEOREM 1: The probability function of (16) with $a, b, c, d \geq 0$, $a+b+c+d = 1$, $b+d < 1$, $c+d < 1$, is

$$f_m(x, y) = \sum_{\alpha,\beta,\gamma} \frac{m!}{\alpha!\beta!\gamma!}\lambda_1^\alpha\lambda_2^\beta\lambda_3^\gamma f_{X\beta}(x) \overset{x}{*} g_m(x, y) \overset{y}{*} f_{Y\gamma}(y), \tag{19}$$

$$= \sum_{\alpha,\beta,\gamma} \frac{m!}{\alpha!\beta!\gamma!}\lambda_1^\alpha\lambda_2^\beta\lambda_3^\gamma \sum_{\xi=0}^{x}\sum_{\eta=0}^{y} f_{X\beta}(x-\xi) f_{Y\gamma}(y-\eta) g_m(\xi, \eta),$$

where Σ runs over all $\alpha, \beta, \gamma \geq 0$ such that $\alpha+\beta+\gamma = m$. The operator $\overset{x}{*}$ for convolution over x is defined for two functions $h_1(x, y)$ and $h_2(x, y)$ by

$$h_1 \overset{x}{*} h_2 = \sum_{\xi=0}^{x} h_1(\xi, y) h_2(x-\xi, y).$$

The operator $\overset{y}{*}$ is defined similarly. When $\beta = 0$ or $\gamma = 0$ in (19), $f_{X0}(x)$ and $f_{Y0}(y)$ are taken to be unity.

The distribution $f_m(x, y)$ will be henceforth referred to as the $BNB(a, b, c, p, q, m)$ distribution. The distribution $g_\nu(x, y)$, only when $\nu = m$ an integer, is a special case of $f_m(x, y)$; the distribution $f_m(x, y)$ is not in general infinitely divisible. The $BNB(a, b, c, p, q, m)$ distribution is the discrete analogue to Paulson's [10] bivariate gamma distribution.

As a direct consequence of Theorem 1 we obtain a closed form representation of Paulson and Uppuluri's [11] bivariate geometric distribution as

COROLLARY 1: For $m = 1$, $d < 1$, the $BNB(a, b, c, p, q, 1)$ distribution is

$$f_1(x, y) = \frac{a}{1-d} g_1(x, y) + \frac{b}{1-d}\left[\frac{1}{1+\theta_1}\right]\left[\frac{\theta_1}{1+\theta_1}\right]^x {}^x_* g_1(x, y) + \frac{c}{1-d}\left[\frac{1}{1+\theta_2}\right]\left[\frac{\theta_2}{1+\theta_2}\right]^y {}^y_* g_1(x, y). \tag{20}$$

The requirement that the shape parameter m of $f_m(x, y)$ in (19) be integer valued limits the practical utility of the probability function. We are thus led to seek a representation for which the shape parameter may take on arbitrary real values. This is equivalent to first determining whether $f_m(x, y)$ is infinitely divisible, and if so, to secondly determining an expression which is not tied to integer values m. In general, the $BNB(a, b, c, p, q, m)$ distribution is not infinitely divisible (Block, Paulson, Kohberger [4]) and no such representation will exist. The $BNB(a, b, 0, p, q, \nu)$ and $BNB(a, 0, c, p, q, \nu)$ distributions are, however, infinitely divisible as are the $BNB(a, 0, 0, p, q, \nu)$ distributions.

THEOREM 2: The $BNB(a, b, 0, p, q, \nu)$ and $BNB(a, 0, c, p, q, \nu)$ distributions, $b + d < 1$, $c + d < 1$, $d < 1$, $\nu > 0$, are infinitely divisible.

PROOF: In (16) replace m by ν and write the resulting probability generating function when $c = 0$ as $\phi_\nu(c = 0)$. Then it follows that

$$\phi_\nu(c = 0) = \left[\frac{(1-d)\psi_1\psi_2}{1-d\psi_1\psi_2}\right]^\nu \left\{\frac{a}{1-d}\left[\frac{1-d\psi_1}{1-(b+d)\psi_1}\right]\right\}^\nu. \tag{21}$$

The term in square brackets is infinitely divisible. The term in braces is infinitely divisible if it is a probability generating function for arbitrary $\nu > 0$. But

$$\begin{aligned}\left\{\frac{(1-d\psi_1)}{1-(b+d)\psi_1}\right\}^\nu &= \exp\left[\nu\left(\log\left[\frac{1-d\psi_1}{1-(b+d)\psi_1}\right]\right)\right] \\ &= \exp[\nu(\log(1-d\psi_1) - \log(1-(b+d)\psi_1))] \\ &= \exp\left[\nu\sum_{j=1}^{\infty}((b+d)^j - d^j)\frac{\psi_1^j}{j}\right] \\ &= 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n}\binom{\nu}{k}\binom{n-1}{n-k}b^k(b+d)^{n-k}\psi_1^n \\ &= 1 + \sum_{n=1}^{\infty} h(b, d, n, \nu)\psi_1^n,\end{aligned} \tag{22}$$

say, on successive expansion of $\log(1 - d\psi_1)$, $\log(1 - (b + d)\psi_1)$ and the exponential function. For all $\nu > 0$, $0 < b < 1$, $0 < b + d < 1$, $h(b, d, n, \nu) > 0$. An application of Theorem 12.2.3 of Lukacs [8] yields the infinite divisibility of the probability generating function $\{a(1 - d\psi_1)/((1 - d)(1 - (b + d)\psi_1))\}^\nu$. Since $\phi_\nu(c = 0)$ is a product of infinitely divisible probability generating functions, it, too, is infinitely divisible.

The constructive nature of Theorem 2 provides the representation we seek. The term $[\cdot]^\nu$ in (21) has probability function $g_\nu(x, y)$. The term $\{\cdot\}^\nu$ in (21) has probability function given by the termwise inverse of

$$\left[\frac{a}{1-d}\right]^\nu \left[1 + \sum_{n=1}^{\infty} h(b, d, n, \nu)\psi^n\right].$$

Hence, the inverse of $\phi_\nu(c = 0)$ of (21) is

$$f_\nu(x, y|c = 0) = \left[\frac{a}{1-d}\right]^\nu \left\{ g_\nu(x, y) + g_\nu(x, y) * \sum_{n=1}^{\infty} h(b, d, n, \nu)(1-p)^n p^x \binom{x+n-1}{x} \right\} \tag{23}$$

We thus have proven

COROLLARY 2: The $BNB(a, b, 0, p, q, \nu)$ distribution, $b < 1$, $b + d < 1$, has probability function $f_\nu(x, y|c = 0)$ given by (23).

Elementary computations provide the moments of the $BNB(a, b, c, p, q, \nu)$ family. In particular, the means and covariance matrix are provided in

THEOREM 3: The mean vector and covariance matrix for the $BNB(a, b, c, p, q, \nu)$ distribution are

$$\underline{\mu} = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix} = \begin{bmatrix} \nu\theta_1 \\ \nu\theta_2 \end{bmatrix} \tag{24a}$$

and

$$\Sigma = \begin{bmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{bmatrix} = \begin{bmatrix} \nu\theta_1(1+\theta_1) & \dfrac{\nu(ad-bc)}{1-d}\theta_1\theta_2 \\ \dfrac{\nu(ad-bc)}{1-d}\theta_1\theta_2 & \nu\theta_2(1+\theta_2) \end{bmatrix} \tag{24b}$$

where θ_1 and θ_2 are given in (8c).

The conditional mean of Y given x is plotted against x in Figure 1. The data, taken from Bates and Neyman [3], as depicted in this figure, strongly suggests that a nonlinear regression function would be most suitable for describing the empirical relationship between X and Y. Unfortunately, the Bates-Neyman distribution does not admit of nonlinear regressions. The capability of allowing for nonlinear regressions seems to be fundamental in data analysis situations. The regression function for $BNB(a, b, c, p, q, 1)$ (and, hence, also $BNB(a, b, c, p, q, m)$) is nonlinear.

THEOREM 4: The regression function $E(Y|x)$ of the $BNB(a, b, c, p, q, 1)$ distribution, $b \neq 0$, is

$$E(Y|x) = \frac{q}{1-q}\left[\frac{b+d}{b} + \left[\frac{A}{m} - \frac{d}{b}\right]k^{x+1}\right], \quad b \neq 0, \tag{25a}$$

$$E(Y|x) = \frac{q}{1-q}\left[\frac{\nu + d(1-p)x}{1-d(1-p)}\right], \quad b = c = 0, \tag{25b}$$

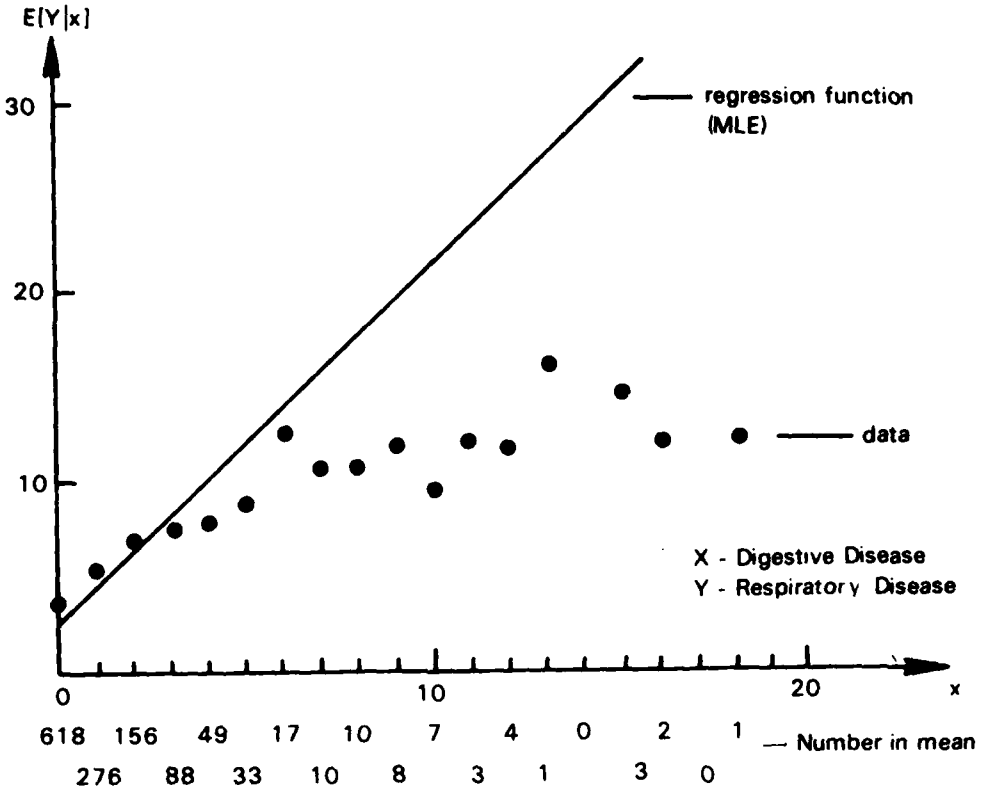


FIGURE 1. Theoretical and observed regression functions for Bates-Neyman Data

where

(26) $$m = p + (1-p)(a+c),\quad A = cm/[(a+c)(a+b)]$$
$$k = m/[m + b(1-p)],\quad d = 1 - a - b - c.$$

PROOF: The z-transform of $E(Y|x)$ is defined by

(27) $$z(E[Y|x]) = g(z) = \sum_{x=0}^{\infty} z^{-x} E[Y|x] = \sum_{x=0}^{\infty} z^{-x} y f_1(y|x),$$

(28) $$= (1+\theta_1) \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} \left[\frac{(1+\theta_1)}{z\theta_1}\right]^x y f_1(x, y),$$

since the marginal density of X is $\Pr[X = x] = \left(\frac{1}{1+\theta_1}\right)\left(\frac{\theta_1}{1+\theta_1}\right)^x$.

But (28) may be expressed in terms of the probability generating function $\phi_1(u, v)$ as

$$g(z) = (1+\theta_1)\frac{\partial}{\partial\lambda}\phi_1\left[\frac{1+\theta_1}{z\theta_1}, \lambda\right]\Bigg|_{\lambda=1} \tag{29}$$

$$= \left[\frac{q}{1-q}\right]\left[\frac{1}{m+b(1-p)}\right]\left[\frac{Az}{z-k} + \frac{z(z-m)}{(z-k)(z-1)}\right].$$

after some computation and simplification. The inversion of $g(z)$ (Jury, [7]) gives (25a), an exponential regression function. Letting $b \to 0$, $c \to 0$ in (25a) gives (25b).

The nonlinearity of regression is embodied in the term k^{x+1} in (25a) with k defined in (26). The parameter b must be sizeable for small m in order that the regressions be nonlinear over a range of x not far removed from zero. Thus, bivariate negative binomial data for which this form of regression would be most applicable would possess appreciably more frequency counts along the x-axis than along the y-axis.

The distribution $g_\nu(x, y)$ is central in the discussions of all the probability distributions discussed to this point. The availability of an easily effected computational procedure for $g_\nu(x, y)$ would facilitate the utilization of the negative binomial distributions. Since $g_\nu(x, y)$ is expressible in terms of the hypergeometric functions, the Gaussian contiguous functions play a dominant role in any computational scheme. First, observe from the definition of $F(a, b, c, z)$ that

$$F(\nu + x, \nu; \nu; z) = (1-z)^{-\nu-x},$$

$$F(\nu, \nu + y, \nu; \nu; z) = (1-z)^{-\nu-y}.$$

Thus, it follows immediately that

$$g_\nu(x, 0) = \left[\frac{(1-d)(1-p)(1-q)}{1-z}\right]^\nu \left[\frac{p}{1-z}\right]^x \binom{\nu+x-1}{x}. \tag{30a}$$

$$g_\nu(0, y) = \left[\frac{(1-d)(1-p)(1-q)}{1-z}\right]^\nu \left[\frac{q}{1-z}\right]^y \binom{\nu+y-1}{y}. \tag{30b}$$

The use of relationship 15.2.18 of Abramowitz and Stegun [1] allows (15) to be written as

$$g_\nu(x, y) = (1-d)^\nu h_1(x) h_2(y) \left\{\frac{1}{x+y+\nu}[xF(\nu+x-1, \nu+y; \nu; z)\right.$$

$$\left. + (\nu+y)(1-z)F(\nu+x, \nu+y+1; \nu; z)]\right\}.$$

A few elementary manipulations give

$$g_\nu(x, y+1) = \frac{q}{(y+1)(1-z)}[(x+y+\nu)g_\nu(x, y) \tag{31}$$

$$- p(\nu+x-1)g_\nu(x-1, y)],$$

which allows the entire probability distribution to be computed recursively from $g_\nu(x, 0)$ and $g_\nu(0, y)$, $x \geq 0$, $y \geq 0$.

Estimation of the parameters of these probability distributions generally require nonlinear optimization routines. Some of the computational burden can be reduced through judicious use of

THEOREM 5: For the $BNB(a, 0, 0, p, q, \nu)$ distribution and if ν is known the likelihood equations for a random sample of size n are

$$\text{(32a)} \qquad \frac{\partial \log L}{\partial a}: \quad \frac{\nu}{a} + \bar{y} - \bar{R} = 0$$

$$\text{(32b)} \qquad \frac{\partial \log L}{\partial p}: \quad \frac{(1-p)}{p}\bar{x} + \bar{y} - \bar{R} = 0$$

$$\text{(32c)} \qquad \frac{\partial \log L}{\partial q}: \quad \frac{(1-q)\bar{y}}{q} + \bar{y} - \bar{R} = 0$$

where L is the likelihood function, $\bar{x}$ and $\bar{y}$ are the sample means for the marginal distributions, $\bar{R} = \frac{1}{n}\sum_{x,y} n_{xy}\left[\frac{y+1}{q}\right]\frac{g_\nu(x, y+1)}{g_\nu(x, y)}$, n_{xy} is the number of observations for which $X = x$, $Y = y$, and $g_\nu(x, y)$ is the probability function in (15).

PROOF: If the probability function in (15) is differentiated with respect to the parameters a, p, and q the following differential-difference equations

$$\text{(33a)} \qquad \frac{\partial g_\nu(x, y)}{\partial a} = \left[\frac{\nu}{ad} + \frac{y}{d}\right] g_\nu(x, y) - \frac{1}{d}\left[\frac{y+1}{q}\right] g_\nu(x, y+1),$$

$$\text{(33b)} \qquad \frac{\partial g_\nu(x, y)}{\partial p} = \left[\frac{x}{p} + \frac{y}{1-p}\right] g_\nu(x, y) - \frac{1}{1-p}\left[\frac{y+1}{q}\right] g_\nu(x, y+1),$$

$$\text{(33c)} \qquad \frac{\partial g_\nu(x, y)}{\partial q} = \left[\frac{y}{q} + \frac{y}{1-q}\right] g_\nu(x, y) - \frac{1}{1-q}\left[\frac{y+1}{q}\right] g_\nu(x, y+1),$$

result; these equations follow by using (15.2.1, Abramowitz and Stegun [1])

$$\frac{\partial F(a, b; c; z)}{\partial z} = \frac{ab}{c} F(a+1, b+1; c+1; z)$$

and (exercise 1, page 296, Whittaker and Watson [13]), namely

$$F(a, b+1; c; z) - F(a, b; c; z) = \frac{az}{c} F(a+1, b+1; c+1; z).$$

The log likelihood function, log L, for a random sample of size n is $\sum_{x,y} n_{xy} \log g_\nu(x, y)$ and so

$$\frac{\partial \log L}{\partial a} = \sum_{x,y} n_{xy} \frac{1}{g_\nu(x, y)} \frac{\partial g_\nu(x, y)}{\partial a}.$$

Using (33a) and a few simple operations leads to (32a). Similarly, (32b) and (32c) are obtained.

From (32) it is clear that

$$\frac{1-p)}{p}\bar{x} = \frac{(1-q)}{q}\bar{y} = \frac{\nu}{a} \tag{34}$$

and these relationships are very useful in estimating the parameters via the method of maximum likelihood. For ν known or not, the conditions in (32) are necessary for a maximum likelihood solution to the likelihood function for $b = c = 0$. Therefore, (34) can be used to reduce the dimensionality of the unknown parameter space from four if ν is unknown, to two by taking, say, $p = q\bar{x}/[q\bar{x} + (1-q)\bar{y}]$ and $a = \nu q/[(1-q)\bar{y}]$. We have used a nonlinear optimization computer program to solve for the parameter estimates. The dimensionality reduction permits extremely shorter running times.

4. APPLICATIONS OF THE NEW BNB DISTRIBUTIONS

This paper was originally motivated in part by the visual disagreeableness of the fit of the regression function

$$E(Y \mid x) = \frac{q(\nu + x)}{1-q}$$

of the Guldberg-Bates-Neyman distribution of (5) to the observed data as depicted in Figure 1. The maximum likelihood fit of the Guldberg-Bates-Neyman model to this data is dismal as measured by a χ^2 goodness-of-fit, indeed, a linear regression is clearly not appropriate for this data. We fitted the $BNB(a, b, c, p, q, 1)$ and the $BNB(a, b, 0, p, q, \nu)$ to these data (Bates-Neyman, [3]) but obtained $\hat{b} = \hat{c} = 0$ in the first case and $\hat{b} = 0$ in the latter. Even though the degree of fit as measured by χ^2 increased substantially, the overall fit was still very poor. The conclusion that the Bates-Neyman data possesses characteristics which preclude the possibility of a good representation by a bivariate negative binomial seems inescapable. There is thus no reason to present any of our results concerning this data.

TABLE 1 — *Bivariate Data Sets*

Description of Data	Sample Size and Correlation[1]	Marginals: Mean and Variance	Parameters[2] of Univariate Negative Binomial (EQ. (1)) (ν, θ)	Fig of Univariate Negative Binomial to Marginals (χ^2, df, P)[3]
Arbous-Sichel Absenteeism	248, 0.73	(X) 1947 :4.70,18.66 (Y) 1948 :4.48,18.66	1.58,2.99 1.56,2.88	8.9, 13,0.78 11.0,12,0.53
Bates-Neyman Diseases	1286, 0.42	(X) Dig. D. :1.40, 5.06 (Y) Res. D. :5.32,22.13	0.53,2.62 1.69,3.16	12.3,10,0.26 25.4,20,0.20
Aircraft Flight Aborts	109,−0.16	(X)1st 6 mos :0.62, 1.03 (Y)2nd 6 mos :0.72, 1.08	0.95,0.66 1.51,0.48	0.24,1,0.62 6.3, 1,0.01

[1]Pearson product-moment [2]Via method of moments [3]Probability of exceeding computed χ^2 value with indicated degrees of freedom

Table 1 shows selected summary results for all the data examined in detail. The first column identifies the data, column 2 provides the sample size and correlation, column 3 specifies the marginal random variables and shows the associated sample means and variances,

columns 4 and 5 show the parameters of the univariate negative binomial fitted to the marginals and the associated χ^2 values, degrees of freedom and probability levels, respectively. Results from this table will be presented along with a discussion of the individual data sets. The absenteeism data of Arbous-Sichel [2] is examined first.

From Table 1 we see that the marginal distributions are fitted rather nicely by the univariate negative binomial. Coupled with the fairly large sample correlation coefficient, it seems reasonable to expect a bnb distribution to adequately describe the data. Arbous and Sichel fit the $G - B - N(1, \theta, \nu)$ model to their data and report a χ^2 of 17.0 on 13 degrees of freedom ($P = 0.20$), indicating reasonable agreement of the data with the model. We give in Table 2 the expected cell frequencies obtained from the $BNB(a, 0, 0, p, q, \nu)$ distribution along with the observed cell frequencies for all but 12 of the nonzero observed cells. The expected cell frequencies for the two models are about the same and we thus expect a similar probability P to obtain for the fit of the $BNB(a, 0, 0, p, q, \nu)$. Since Arbous and Sichel do not show their grouping, we have not computed the χ^2 statistic. The fits are comparable as judged by cell residuals.

Although the fit of the $G - B - N(1, \theta, \nu)$ model to the observed data is reasonably good, the authors point out that 12 of the 18 observed means lie below the theoretical regression function $E(Y|x) = q(\nu + x)/(1 - q)$. The $BNB(a, 0, 0, p, q, \nu)$ model, via maximum likelihood estimates, gives rise to a regression function for which only 10 of the 18 observed means are less than the predicted values.

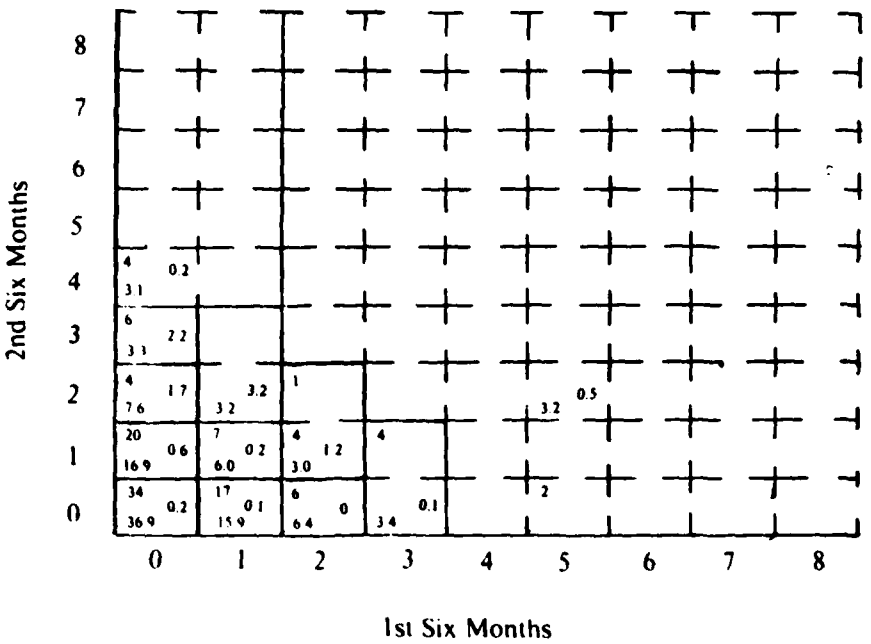
An attempt to fit the $BNB(a, b, c, p, q, 1)$ and $BNB(a, b, 0, p, q, \nu)$ models to these data was made but we obtained maximum likelihood estimates $\hat{b} = \hat{c} = 0$ in the first case and $\hat{b} = 0$ in the second. The lack of influence of the parameters indicates the apparent adequacy of linear regressions for the data. Although the fit of $BNB(a, 0, 0, p, q, \nu)$ is adequate, a bivariate beta binomial is a more appropriate distribution for this data.

Table 3 provides observed and expected cell frequencies for flight aborts for 109 aircraft. The fundamental data consist of 109 pairs of observations (x_i, y_i), where x_i represents the number of aborts by aircraft i in the first 6 months and y_i the number of aborts by aircraft i in the 2nd 6 months of a one year period. Most of the data fall in the $(x, 0)$ and $(0, y)$ cells and this implies that the correlation should be negative. Note that the probability contours are unlike those traditionally associated with negative correlation and that the distribution has its probability concentrated along the two axes. The maximum likelihood estimates for the $BNB(a, b, c, p, q, 1)$ model are $\hat{a} = 0$, $\hat{b} = .68$, $\hat{c} = .32$, $\hat{p} = .17$, $\hat{q} = .33$. These values produce an estimate of correlation of $-.13$. The solid borders in Table 3 indicate the grouping we have used for χ^2. We find the χ^2 statistic to be 10.2 on 6 degrees of freedom or $P = 0.12$, a marginally acceptable fit. The observed $x + y$ has frequencies 34, 37, 17, 10, 9, 2 for $x + y = 0, 1, \ldots, 5$, respectively, and 0 for all $x + y > 5$. The mode is at $x + y = 1$ which makes more plausible a $BNB(a, b, c, p, q, \nu)$ model with $\nu > 1$ since with $\nu = 1$, the mode is at $x + y = 0$. Unfortunately, the $BNB(a, b, c, p, q, 1)$ is not infinitely divisible (Block, H.W., A. S. Paulson and R.C. Kohberger [4]) and hence we will not be able to produce a better fit to this data in the class of models introduced herein. One might reasonably be hesitant to call a distribution which is not infinitely divisible a "true" negative binomial.

TABLE 2 — *Observed and Expected Cell Frequencies for Arbous-Sichel Data (248 Workers) under a BNB* (0.037,0,0.101,0.097,1.548) *Model*

X \ Y	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	7 11.51	7 7.92	8 4.49	1 2.36	1 1.19	0.59	0.29	0.14	0.06	0.03	0.01	0.01			
1	9 8.25	6 8.54	8 6.46	7 4.25	1 2.58	3 1.48	0.82	0.44	0.23	0.12	0.06	0.03	0.02	0.01	
2	9 4.86	9 6.73	6 6.31	5 4.94	1 3.47	3 2.27	1.41	2 0.84	0.49	1 0.27	0.15	0.08	0.04	0.02	0.01
3	1 2.66	7 4.61	5 5.14	4 4.65	3 3.71	2.71	2 1.85	1 1.21	1 0.76	0.46	1 0.27	0.16	0.09	1 0.05	0.03
4	1 1.40	7 2.91	5 3.76	4 3.86	3 3.43	2.76	2 2.06	1 1.46	1 0.99	0.64	1 0.41	0.25	0.15	1 0.09	0.05
5	2 0.72	1 1.74	1 2.56	2 2.93	5 2.87	2 2.52	1 2.04	1 1.55	1.12	1 0.78	3 0.52	0.34	0.22	0.13	0.08
6	0.36	1 1.01	1 1.66	2 2.09	2 2.24	3 2.13	1 1.85	2 1.51	1.16	2 0.86	0.61	1 0.42	0.28	1 0.18	0.11
7	0.18	0.57	1.03	1 1.42	1 1.65	2 1.69	1 1.57	1.36	3 1.11	0.87	0.65	0.47	0.33	0.22	0.15
8	0.09	0.31	1 0.62	0.93	2 1.16	2 1.27	1.26	1.16	2 1.00	0.82	1 0.64	1 0.48	0.35	0.25	0.17
9	0.04	0.17	0.36	0.59	1 0.79	0.92	3 0.97	0.94	2 0.85	0.73	0.60	0.47	2 0.36	0.27	0.19
10	0.02	0.09	0.21	1 0.36	0.52	0.64	1 0.71	0.73	1 0.70	0.63	0.54	1 0.44	2 0.35	0.27	0.20
11	0.01	0.05	0.12	0.22	1 0.33	0.43	0.51	0.55	0.55	0.51	0.46	0.39	1 0.32	0.26	0.20
12	0.01	0.02	0.07	0.13	0.21	1 0.29	0.35	0.40	1 0.42	1 0.41	1 0.38	0.34	0.29	0.24	1 0.19
13		0.01	0.04	0.08	0.13	1 0.19	0.24	0.28	0.31	1 0.31	1 0.30	1 0.28	1 0.25	1 0.21	0.17
14		0.01	0.02	0.04	1 0.08	0.12	0.16	1 0.20	0.22	0.24	0.24	0.23	0.21	1 0.18	0.16
15			0.01	0.02	0.05	0.07	0.10	1 0.13	0.16	0.17	0.18	0.18	0.17	0.15	1 0.13
16			0.01	0.01	0.03	0.04	0.07	0.09	0.11	0.12	1 0.13	0.13	0.13	0.12	0.11
17				0.01	0.02	0.03	0.04	0.06	0.07	0.09	1 0.10	0.10	0.10	0.10	1 0.09

TABLE 3 — *Observed and Expected*[1] Cell Frequencies of Flight Aborts for 109 Aircraft



[1]Bivariate negative binomial distribution of (20). Maximum likelihood parameter estimates are: $\hat{a} = 0$, $\hat{b} = 0.6820$, $\hat{c} = 0.3179$, $\hat{p} = 0.1655$, $\hat{q} = 0.3299$. There results $\chi^2 = 10.2$ and $P = 0.12$ for $df = 6$.

5. DISCUSSION

We have produced and examined properties of a class of negative binomial distributions which admit of nonlinear regressions and negative correlations. The most useful subset of this class is infinitely divisible and therefore allows for a flexible fit to data. A nonlinear optimization routine was used to produce the maximum likelihood parameter estimates.

Even though the development of models which admitted of nonlinear regressions and negative correlations provided some of the impetus to this work, it soon became clear that bivariate models with a different shape parameter for each margin would be eminently useful in the modeling of bivariate data. A large number of attempts at developing such appropriate negative binomial distributions of this type met with only limited success. It appears that a model of a character much different from those heretofore discussed in the literature is required to produce a multivariate negative binomial distribution which is simultaneously infinitely divisible, allows for nonlinear regressions, and different marginal shape parameters. The comparison of adequacy of agreement between data and model was based on the usual χ^2 goodness-of-fit statistic, a procedure with serious shortcomings. An alternative procedure for multivariate goodness-of-fit would be a welcome addition to the multivariate literature.

The most appealing properties of our $BNB(a, b, 0, p, q, \nu)$ and $BNB(a, 0, c, p, q, \nu)$ distributions are their infinite divisibility and nonlinear regressions. However, unless nonlinear regressions are called for, it appears that any of the bivariate negative binomials would produce, via maximum likelihood, an acceptable match to bivariate negative binomial data. It should be emphasized that the maximum likelihood estimates need not produce the best fit of a model to the data as measured, for example, by χ^2. Further, under maximum likelihood and ordinary least squares, each observation is weighted equally; if the number of "tail" observations relative to the complete frequency count is small, then these procedures need not produce a visibly nonlinear regression even when one is clearly appropriate. In this case an "anti-robust" procedure may be preferable to maximum likelihood and least squares in order that legitimate outlying observations be weighted more heavily. A case in point is provided in Figure 1 where the bivariate negative binomial model is not appropriate but the data are correct and, unfortunately, the model has been given preference.

Extension of our results to dimensionality in excess of two would require a little care in ensuring the simultaneous infinite divisibility and nonlinear regressions. The computational problems would increase with the complexity of the model which implies a necessity for parsimony vis-a-vis realism tradeoff.

Finally, we wish to emphasize that all of our results and discussion has a dual counterpart in the bivariate gamma distributions of Paulson [10]. Accordingly we would define and work with bivariate gamma distributions $BVG(a, b, 0, p, q, \nu)$ and $BVG(a, 0, c, p, q, \nu)$ in complete analogy with the preceeding results.

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MINIMAX INSPECTION STRATEGIES FOR SINGLE UNIT SYSTEMS

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ABSTRACT

A model for proper scheduling of inspections is considered, if system failures can be detected only by checking. Two cases are analyzed: replacement and no replacement of a failed system. On condition that no or only partial information on the lifetime distribution of the system is available, minimax inspection strategies are obtained with respect to cost criterions.

1. INTRODUCTION

The following situation is considered: At time $t = 0$ a system starts working. The time to its failure (lifetime) is a random variable X with the probability distribution function $F(t)$, $F(+0) = 0$. A system failure is assumed to be known only by inspecting. Each inspection entails a fixed cost c_1 and takes only negligible time. On the other hand, a downtime t of the system (= time between system failure and its detection) gives rise to cost $v(t)$, where $v(t)$ is a continuous, strictly increasing function of t, $t \geqslant 0$; $v(0) = 0$.

Let be $S = \{t_k\}$ an inspection strategy (at time t_k the kth inspection takes place, when no failure has been detected before, $0 = t_0 < t_1 < t_2 < \ldots$). The aim of this paper consists in deriving inspection strategies, which are optimal with respect to some cost criterions. Results are obtained in the cases where the lifetime distribution is completely or partially unknown. In the latter case, the expected system lifetime is assumed to be known.

To the knowledge of the author, the model described has been first analyzed by Derman [10] (unknown lifetime distribution) and Barlow, Hunter and Proschan [2] (known lifetime distribution). Modifications of the basic model described have been treated e.g., in [1, 3, 12, 14, 15]. This paper summarizes and extends results obtained by the author [4, 5, 6, 8].

2. INSPECTION WITHOUT REPLACEMENT

If an inspection strategy (short: strategy) $S = \{t_k\}$ is in effect, then the expected (total) loss cost $K(S, F)$ up to detection of a system failure amounts to

$$K(S, F) = \sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} g_k(t, t_{k+1})\, dF(t), \tag{1}$$

where

$$g_k(x, y) = (k+1)c_1 + v(y-x), \quad 0 \leqslant x \leqslant y.$$

If a downtime cost can arise only in the finite interval (O, T), then inspections after T are not necessary. Applying in this case a strategy $S_n = \{t_k\}$ with $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = T$ the expected loss cost is

$$K(S_n, F) = \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} g_k(t, t_{k+1}) dF(t) + (n+1)c_1(1 - F(T)). \tag{2}$$

Next $F(t)$ is assumed to be entirely unknown. This assumption leads to the cost criterion

$$K(S) = \sup_{F \in \tilde{F}} K(S, F), \tag{3}$$

where $\tilde{F}$ is the set of all probability distribution functions $F(t)$ with $F(+0) = 0$ and $K(S, F)$ is given by (2). A straightforward estimation of $K(S_n, F)$ yields

$$K(S_n) = \max_{k=0, 1, \ldots, n} g_k(t_k, t_{k+1}).$$

A strategy, $S_{n^*}^*$, satisfying

$$K(S_{n^*}^*) = \min_{S} K(S)$$

is called "minimax strategy."

THEOREM 1: Let n^* be the largest integer n satisfying

$$\sum_{k=0}^{n} v^{-1}(kc_1) < T, \tag{4}$$

where $v^{-1}(x)$ is the inverse function of $v(t)$. Then there exists a unique solution $S_{n^*}^* = \{t_k^*\}$, $0 < t_1^* < t_2^* < \ldots < t_{n^*}^* < T$, of

$$g_0(0, t_1) = g_1(t_1, t_2) = \ldots = g_{n^*}(t_{n^*}, T) \tag{5}$$

and $S_{n^*}^*$ is minimax strategy.

PROOF: With $n = n^*$ Equation (5) is equivalent to

$$v(\delta_k) = v(\delta_0) - kc_1, \quad k = 0, 1, \ldots, n; \tag{6}$$

$$\sum_{k=0}^{n} \delta_k = T, \ \delta_k = t_{k+1} - t_k, \ k = 0, 1, \ldots, n.$$

Thus, in view of the definition of n^*, $v(t_1^*) \leqslant (n^*+1)c_1$ holds, and for any strategy S_r with $r = n^* + 1$

$$K(S_{n^*}^*) - K(S_r) < v(t_1^*) + c_1 - (r+1)c_1 \leqslant v(t_1^*) - (n^*+1)c_1 \leqslant 0.$$

Therefore, a strategy S_r with $r > n^*$ cannot be minimax strategy.

The functions $g_k(x, y)$, $k = 0, 1, \ldots$, have two obvious properties:

(i) $g_k(x, y)$ is strictly increasing in y, if x is fixed, $0 \leqslant x \leqslant y$.

(ii) $g_k(x, y)$ is strictly decreasing in x, if y is fixed, $0 \leqslant x \leqslant y$.

Let $S_n' = \{t_k'\}$ the solution of (6) with $0 \leqslant n \leqslant n^*$. Assume there exists a strategy $\bar{S}_n = \{\bar{t}_k\}$ satisfying

(iii) $K(\bar{S}_n) < K(S_n')$.

Let $\bar{t}_i$ be the smallest of the inspection times $\bar{t}_k$ with the property $\bar{t}_i \neq t_i'$. Because of (iii), $g(\bar{t}_{i-1}, \bar{t}_i) < K(S_n')$. Taking into account (i) this implies $\bar{t}_i < t_i'$. Analogously, (i) and (ii) yield $\bar{t}_{i+1} < t_{i+1}'$ (for otherwise (iii) could not be satisfied). Continuing in this way, one finally obtains $\bar{t}_n < t_n'$ and, therefore, $g(\bar{t}_n, T) > K(S_n')$ (see [16]). Hence, there cannot exist a strategy $\bar{S}_n$ satisfying (iii). If S_{n_1}' and S_{n_2}', $0 \leqslant n_1 < n_2 \leqslant n^*$, are solutions of (6) with $n = n_1$ and $n = n_2$, respectively, then it can be proved analogously that $K(S_{n_2}') < K(S_{n_1}')$. This completes the proof of the theorem.

COROLLARY: n^* is the largest integer n for which there exists a strategy $S_n' = \{t_k'\}$ satisfying (6), $0 < t_1' < t_2' < \ldots < t_n' < t_{n+1}' = T$. Similarly, t_1^* is the smallest time point for the first inspection, which can belong to a solution of (6).

An analogous property of the first optimum inspection moment with respect to $K(S, F)$, if $f(t) = F'(t)$ is known and of type PD_2 (polýa density function of order 2), has been shown in [5].

Derman [10] has already obtained the minimax strategy in case of $v(t) = c_2 t$, $c_2 > 0$. It is given by

$$t_k^* = k\left[\frac{T}{n^*+1} + \frac{c_1}{2c_2}(n^* - k + 1)\right], \quad k = 1, 2, \ldots, n^*,$$

where n^* is the largest integer satisfying

$$n(n+1) < \frac{2c_2 T}{c_1}.$$

Now suppose the lifetime distribution of the system to be unknown with the exception of the expected lifetime $\mu = E(X)$, $0 < \mu < \infty$. The condition that downtime cost occurs only in $(0, T)$ is now dropped. Let

$$K_\mu(S) = \sup_{F \in F_\mu} K(S, F),$$

where $K(S, F)$ is given by (1) and

$$F_\mu = \left\{F;\ F \in \tilde{F},\ \mu = \int_0^\infty t\, dF(t)\right\}.$$

A strategy, S_μ^*, satisfying

$$K_\mu(S_\mu^*) = \min_S K_\mu(S)$$

is called "partial minimax strategy."

Within a more general model Hoeffding [11] has shown that for any $S = \{t_k\}$

$$\sup_{F \in F_\mu} K(S, F) = \sup_{F \in F_\mu^{(2)}} K(S, F),$$

where $F_\mu^{(2)}$ is the subset of those elements of F_μ having exactly two points of increase. Routine calculations yield

$$K_\mu(S) = \sup_{\substack{0 \leqslant i \leqslant m \\ m < j}} G_{ij}(S), \tag{7}$$

where

$$G_{ij}(S) = \frac{t_j - \mu}{t_j - t_i}\, g_i(t_i,\ t_{i+1}) + \frac{\mu - t_i}{t_j - t_i}\, g_j(t_j,\ t_{j+1})$$

and $m = m(S)$ is defined by

$$t_m \leqslant \mu < t_{m+1} \text{ (see [8] for details).} \tag{8}$$

A strategy, $S = \{t_k\}$, is called strictly periodic with the inspection interval δ, if $\delta = \delta_k = t_{k+1} - t_k$ for all $k = 0, 1, \ldots$; notation: $S = S^{(\delta)}$. $K_\mu(S)$ has a simple structure for strictly periodic strategies:

$$K_\mu(S^{(\delta)}) = \frac{c_1}{\delta}\,\mu + \mathrm{v}(\delta) + c_1. \tag{9}$$

This results also from a straightforward estimation of $K(S^{(\delta)},\ F)$ if $F \in F_\mu$:

$$\begin{aligned} K(S^{(\delta)},\ F) &= \sum_{k=0}^{\infty} \int_{k\delta}^{(k+1)\delta} [(k+1)c_1 + \mathrm{v}((k+1)\delta - t)]\, dF(t) \\ &\leqslant \sum_{k=0}^{\infty} \int_{k\delta}^{(k+1)\delta} [(k+1)c_1 + \mathrm{v}(\delta)]\, dF(t) \\ &= \frac{c_1}{\delta} \sum_{k=0}^{\infty} (k\delta)\, [F((k+1)\delta) - F(k\delta)] + \mathrm{v}(\delta) + c_1 \\ &\leqslant \frac{c_1}{\delta}\mu + \mathrm{v}(\delta) + c_1. \end{aligned}$$

Thus, if $\mathrm{v}(t)$ is differentiable the optimum inspection interval δ_μ^* with respect to $K_\mu(S^{(\delta)})$ satisfies

$$\delta^2\, \mathrm{v}'(\delta) = \mu c_1. \tag{10}$$

There exists always a positive solution of this equation. Especially, for $\mathrm{v}(t) = c_2 t$ the unique solution is

$$\delta_\mu^* = \sqrt{\mu c_1 / c_2}. \tag{11}$$

THEOREM 2: There exists a strictly periodic partial minimax strategy.

PROOF: Assuming there exists a strategy $S = \{t_k\}$ with the property

$$K_\mu(S) < K_\mu(S^{(\delta_\mu^*)}). \tag{12}$$

$G_{ij}(S)$ can be written in the form

$$G_{ij}(S) = (i+1)c_1 + \mathrm{v}(\delta_i) + (\mu - t_i)c_1\, a_{ij}$$

with $$a_{ij} = \frac{(j-i)c_1 + \mathrm{v}(\delta_j) - \mathrm{v}(\delta_i)}{(t_j - t_i)c_1},\quad 0 \leqslant i \leqslant m < j.$$

Further, let $a_i = \sup_{j>m} a_{ij}$ and $a = \min_{0 \leqslant i \leqslant m} a_i$. Without loss of generality it can be assumed $0 < a < \infty$. Putting $\delta = 1/a$ there holds in view of (9) and (12) for all $0 \leqslant i \leqslant m$

$$(i+1)c_1 + v(\delta_i) + (\mu - t_i)c_1 a < \mu c_1 a + v\left(\frac{1}{a}\right) + c_1.$$

Starting with $i = 0$ induction yields

(13) $$\delta_i < 1/a, \quad 0 \leqslant i \leqslant m.$$

Let be i_0 defined by $a_{i_0} = a$. Then it must be

$$\frac{(j - i_0)c_1 + v(\delta_j) - (\delta_{i_0})}{(t_j - t_{i_0})c_1} \leqslant a, \; j > m.$$

Using (13) it can be inductively seen that $\delta_j < \delta_{i_0}$, $j > m$. But this implies $a = \sup_{j>m} a_{i_0 j} \geqslant 1/\delta_{i_0}$, contradictory to (13). Hence, there cannot exist a strategy S satisfying (12) and the theorem is proved.

The proof given shows, moreover, that every partial minimax strategy must be strictly periodic. Its inspection interval is a solution of Equation (10).

Analogous problems, where, instead of the expected lifetime, a percentile of the lifetime distribution is assumed to be known, have been treated in [5, 8, 13, 15].

3. INSPECTION AND REPLACEMENT

In this section an obvious extension of the basic model with μ known is considered. Immediately on discovery of a failure the system is replaced by a statistically equivalent one. Each replacement requires a fixed cost c_3 and a fixed time d. The inspection-replacement process is continued unlimitedly.

In what follows it is assumed $v(t) = c_2 t$, where c_2 is restricted by

(14) $$(c_1 + c_3)/\mu < c_2.$$

Otherwise, the expected loss cost per unit downtime of the system would be smaller or equal than the mean loss per unit time which arises by "ideal inspection and replacement" (i.e., failure of the system is detected immediately and replacement takes negligible time). But then inspection and replacement are uneconomical from the first.

The expected length of a cycle (time between two neighboring replacements) by application of the strategy $S = \{t_k\}$ is

$$L(S, F) = \sum_{k=0}^{\infty} t_{k+1} [F(t_{k+1}) - F(t_k)] + d.$$

Thus, the expected loss cost per unit time $C(S, F)$ amounts to

$$C(S, F) = \frac{K(S, F) + c_3}{L(S, F)},$$

where $K(S, F)$ is given by (1).

Brender [9] has dealt with the problem of calculating an optimum inspection strategy with respect to $C(S, F)$ if F is known. If F is completely unknown the corresponding mimimax inspection strategy is easily seen to be trivial—no inspection and replacement at all (a nontrivial minimax inspection strategy exists if $v(t)$ is strictly convex [7]).

Let

$$C_\mu(S) = \sup_{F \in \mathcal{F}_\mu} C(S, F).$$

To get the corresponding partial minimax strategy, the following lemma is needed. For being only a slight generalization of an approach used by Brender [9] its straightforward proof can be omitted (see [6] for details).

LEMMA: Let be

$$Q(S, x) = \sup_{F \in \mathcal{F}_\mu} [K(S, F) + c_3 - xL(S, F)]$$

and assume for all x, $0 \leqslant x < c_2$, the existence of a unique strategy $S(x)$ and of a number x_0, $0 < x_0 < c_2$, so that

$$Q(S(x), x) = \min_S Q(S, x) \quad \text{and} \quad Q(S(x_0), x_0) = 0.$$

Then $S(x_0)$ is the unique partial minimax strategy and $x_0 = C_\mu(S(x_0))$.

$K(S, F) - xL(S, F)$ has for every x, $0 \leqslant x < c_2$, the same functional structure as $K(S, F)$. Hence, Theorem 2 secures the existence of a strictly periodic strategy $S(x)$. According to (11) its inspection interval $\delta(x)$ is given by

$$\delta(x) = \sqrt{\mu c_1/(c_2 - x)}. \tag{15}$$

It follows from (9) and (15)

$$Q(S(x), x) = 2\sqrt{\mu c_1(c_2 - x)} - x(\mu + d) + c_1 + c_2 d + c_3.$$

Of course, $Q(S(0), 0) > 0$, and in virtue of (14) $Q(S(c_2), c_2) < 0$. Hence, a unique solution x_0 of $Q(S(x), x) = 0$ exists:

$$x_0 = \frac{1}{\mu + d}\left[c_1 + c_2 d + c_3 - \frac{2\mu c_1}{\mu + d} + 2\sqrt{\frac{\mu c_1}{\mu + d}\left[c_2 \mu - c_1 - c_3 + \frac{\mu c_1}{\mu + d}\right]}\right].$$

Minimax inspection problems for replaceable systems, where, instead of the expected lifetime, a percentile of the lifetime distribution is known, have been treated in [6, 16].

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THE ESTIMATION OF $P(X < Y)$ FOR DISTRIBUTIONS USEFUL IN LIFE TESTING*

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ABSTRACT

In this paper the reliability function $R = P(X < Y)$ has been estimated when X and Y follow gamma, exponential or bivariate exponential distributions. The paper is partly expository.

1. INTRODUCTION

Let X and Y be two random variables with cumulative distribution functions $F(x)$ and $G(y)$ respectively. Suppose Y is the strength of a component subject to a stress X. Then the component fails if at any moment the applied stress (or load) is greater than its strength. The stress is a function of the environment to which the component is subjected. Strength depends on material properties, manufacturing procedures, and so on. The reliability of a component is the probability that its strength exceeds the stress. From practical considerations it is desirable to draw inference about the reliability function.

The above model was first considered by Birnbaum [2] and has since found an increasing number of applications in many different areas, especially in the structural and aircraft industries. For a bibliography of available results see Basu [1].

In many situations, the distribution of X (or of both X and Y) will be completely known except possibly for a few unknown parameters and it is desired to obtain parametric solutions. Thus, in case of missile flights, the stress may be expensive to sample, but the physical characteristics of the missile system, such as the propulsive force, angle of elevation, changes in atmospheric condition, and so on, may all have known distributions; consequently, the distribution of stresses can be calculated. Church and Harris [4], Owen, Craswell and Hanson [8], and Govindarajulu [6] have considered the above problem under the assumption that X and Y have normal distributions. Since in many physical situations, especially in reliability, exponential and other distributions provide more realistic models, it is desirable to obtain estimators of R for distributions useful in life testing. In section 2 we consider gamma and exponential distributions under the assumption that X and Y are independently distributed. The case of a bivariate exponential distribution is studied in section 3. Two distribution free procedures are mentioned in section 4. The effect of misspecifying the model is considered in section 5 and a numerical example is given in section 6.

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2. GAMMA AND EXPONENTIAL DISTRIBUTION

Let X and Y be independently distributed with density functions

$$f(x) = \frac{\alpha^p}{\Gamma(p)} e^{-\alpha x} x^{p-1}, \quad x > 0, \ p > 0 \tag{2.1}$$

$$g(y) = \frac{\beta^q}{\Gamma(q)} e^{-\beta y} y^{q-1}, \quad y > 0, \ q > 0 \tag{2.2}$$

respectively. Then

$$R_{pq} = P(X < Y) = \int_0^\infty \left[\int_x^\infty \frac{\beta^q}{\Gamma(q)} e^{-\beta y} y^{q-1} dy \right] \frac{\alpha^p}{\Gamma(p)} e^{-\alpha x} x^{p-1} dx \tag{2.3}$$

$$= \sum_{k=0}^{q-1} \frac{\Gamma(p+k)}{\Gamma(p)\Gamma(k+1)} \frac{\alpha^p \beta^k}{(\alpha+\beta)^{p+k}}.$$

Here p and q are assumed to be known integers.

Note: $R_{11} = \alpha/(\alpha+\beta)$, and $R_{p1} = R_{11}^p$ for all p. Also,

$$R_{1q} = \sum_{k=0}^{q-1} (\beta/\alpha)^k R_{11}^{k+1} = \frac{\alpha}{\beta} \sum_{k=0}^{q-1} (1 - R_{11})^{k+1}.$$

In particular,

$$R_{21} = R_{11}^2, \ R_{31} = R_{11}^3,$$
$$R_{12} = R_{11} + (\beta/\alpha) R_{11}^2$$
$$R_{13} = R_{11} + (\beta/\alpha) R_{11}^2 + (\beta/\alpha) R_{11}^3,$$
$$R_{22} = R_{11}^2 (3 - 2R_{11}).$$

If R_{11} is close enough to one, as is expected for items with high reliability,

$$R_{1q} \cong \frac{\alpha}{\beta} (1 - R_{11}).$$

Expressions for R_{p1} and R_{1q} indicate that in this case the expression for the reliability is not strongly dependent on the choice of the parameter p (especially if p is small) and the distribution of X can be approximated by the exponential distribution without much distortion in the value of R_{p1}. However, so far as the parameter q is concerned, the situation is quite reversed. The value of reliability is heavily dependent on the choice of the underlying distribution of Y and one has to choose the value of q more carefully. Later in section 5, we shall further study the effect of misspecifying the parameters p and q and confirm our conclusions by numerical studies.

If two independent random samples, $(X_1, X_2, \ldots, X_m)$ and $(Y_1, Y_2, \ldots, Y_n)$, from the two gamma populations are available, maximum likelihood estimators (m.l.e.) of α and β are given by $\hat{\alpha} = \frac{p}{\bar{X}}$ and $\hat{\beta} = \frac{q}{\bar{Y}}$. Hence, m.l.e. of R_{pq} is

$$\hat{R}_{pq} = \sum_{k=0}^{q-1} \frac{\Gamma(p+k)}{\Gamma(p)\Gamma(k+1)} \frac{\hat{\alpha}^p \hat{\beta}^k}{(\hat{\alpha}+\hat{\beta})^{p+k}}. \tag{2.4}$$

As special cases, if $q = 1$, that is, if X follows the gamma distribution and Y follows the exponential distribution,

$$\hat{R}_{p1} = \left(\frac{\hat{\alpha}}{\hat{\alpha} + \hat{\beta}}\right)^p. \tag{2.5}$$

Finally, if both p and q are equal to 1, we have the case of two independent exponential distributions, and we have

$$\hat{R}_{11} = \frac{\hat{\alpha}}{\hat{\alpha} + \hat{\beta}} = \frac{\bar{Y}}{\bar{X} + \bar{Y}}. \tag{2.6}$$

Since $\hat{R}$ in 2.6 is a function of $\frac{\bar{X}}{\bar{Y}}$, the exact distribution $\hat{R}$ can be obtained in the exponential case. It is well known that $\frac{\bar{X}}{\bar{Y}}\frac{\alpha}{\beta}$ follows the F distribution with $(2m, 2n)$ degrees of freedom. Thus, the distribution of $\hat{R}_{11}$ follows. The result will be used later in section 5 to compare the performance of independent exponentials with those of dependent exponential models. Using a theorem in Rao [10], Theorem 6a.2, page 321, the distribution of $\hat{R}_{pq}$ in each of the above cases, for large m and n, can be shown to be normal and hence an estimate of the asymptotic confidence interval for R can be obtained. Thus, in this case with $m = n$ we have for large n, $\sqrt{n}\,(\hat{R}_{pq} - R_{pq}) \sim N(0, \sigma^2_{pq})$. Expressions for R_{pq}, $\hat{R}_{pq}$ and σ^2_{pq}, for a few selected values of p and q are given below.

$$R_{11} = \frac{\alpha}{\alpha + \beta},\quad \hat{R}_{11} = \frac{\bar{Y}}{\bar{X} + \bar{Y}},\quad \sigma^2_{11} = \frac{2\alpha^2\beta^2}{(\alpha + \beta)^4}$$

$$R_{21} = \left(\frac{\alpha}{\alpha + \beta}\right)^2,\quad \hat{R}_{21} = \frac{4\bar{Y}^2}{(\bar{X} + 2\bar{Y})^2},\quad \sigma^2_{21} = \frac{6\alpha^4\beta^2}{(\alpha + \beta)^6}$$

$$R_{12} = \frac{\alpha}{\alpha + \beta} + \frac{\alpha\beta}{(\alpha + \beta)^2},\quad \hat{R}_{12} = \frac{\bar{Y}}{(\bar{Y} + 2\bar{X})} + \frac{2\bar{X}\bar{Y}}{(\bar{Y} + 2\bar{X})^2},\quad \sigma^2_{12} = \frac{6\alpha^2\beta^4}{(\alpha + \beta)^6}.$$

$$R_{22} = \frac{\alpha^2}{(\alpha + \beta)^2} + \frac{2\alpha^2\beta}{(\alpha + \beta)^3},\quad \hat{R}_{22} = \frac{\bar{Y}^2}{(\bar{X} + \bar{Y})^2} + \frac{2\bar{X}\bar{Y}^2}{(\bar{X} + \bar{Y})^3},$$

$$\sigma^2_{22} = 36(\alpha\beta)^4/(\alpha + \beta)^4.$$

3. BIVARIATE EXPONENTIAL DISTRIBUTION

Since the exponential distribution is considered a useful model in life testing problems, it is desirable to consider bivariate analogues of univariate exponential distributions which will have properties similar to those of the univariate exponential distribution. Marshall and Olkin [7] have proposed a very important bivariate exponential distribution (BVE), which is given by

$$\bar{F}(x, y) = P(X > x,\ Y > y) = e^{-\lambda_1 x - \lambda_2 y - \lambda_{12}\max(x,y)}, \tag{3.1}$$

$$0 \leqslant \lambda_1, \lambda_2, \lambda_{12} < \infty,\ \lambda_1 + \lambda_{12} > 0,\ \lambda_2 + \lambda_{12} > 0 (x > 0,\ y > 0).$$

The BVE arises in several natural ways and is considered a useful model in reliability with appealing properties.

Let (X_i, Y_i), $i = 1, 2, \ldots, n$ be a random sample from (3.1). We shall estimate $P(X < Y)$ when (X, Y) follows the BVE. It can be readily obtained from (3.1) that

$$R = P(X < Y) = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_{12}}. \tag{3.2}$$

Hence, R is estimated by

$$\hat{R} = \frac{\hat{\lambda}_1}{\hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_{12}} \tag{3.3}$$

where $\hat{\lambda}_1$, $\hat{\lambda}_2$, and $\hat{\lambda}_{12}$ are the maximum likelihood estimators of λ_1, λ_2, and λ_{12}, respectively. Various authors have considered maximum likelihood estimation of the parameters λ_1, λ_2, λ_{12}. However, no explicit solutions of these parameters are available. In order to obtain an explicit form for $\hat{R}$, we replace the m.l.e. by some special ad hoc estimators called the "INT" estimators of Proschan and Sullo [9] which have very high asymptotic relative efficiency compared with the m.l.e. estimators. Let n_1 = number of pairs such that $X_i < Y_i$, n_2 = number of pairs with $X_i > Y_i$ and n_0 = numbers of pairs with $X_i = Y_i$.

The "INT" estimators are given by

$$\mathring{\lambda}_1 = \left[\frac{n_1}{n_1 + n_0}\right] \frac{n}{\sum_{i=1}^{n} X_i}, \quad \mathring{\lambda}_2 = \left[\frac{n_2}{n_2 + n_0}\right] \frac{n}{\sum_{i=1}^{n} Y_i} \tag{3.4}$$

and

$$\mathring{\lambda}_{12} = n_0 \left[1 + \frac{n_2}{n_1 + n_0} + \frac{n_1}{n_2 + n_0}\right] \Big/ \sum_{i=1}^{n} \max(X_i, Y_i).$$

Proschan and Sullo also prove the following theorem.

THEOREM (Proschan and Sullo): $n^{1/2}(\underline{\mathring{\lambda}}_n - \underline{\lambda})$ is asympotically trivariate normal with mean $\underline{0}$ and dispersion matrix $\Sigma = (\sigma_{ij})$ where, using the suffix n to denote dependence on the sample size, $\underline{\mathring{\lambda}}_n = (\mathring{\lambda}_1, \mathring{\lambda}_2, \mathring{\lambda}_{12})'$, $\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_{12})'$, $\underline{0} = (0,0,0)'$

$$\begin{aligned}
\sigma_{11} &= (\lambda_1(\lambda - \lambda_1\lambda_2\gamma_1^{-1}),\\
\sigma_{12} &= \lambda_1^2\lambda_2^2\lambda_{12}(\lambda\gamma_1\gamma_2)^{-1},\\
\sigma_{13} &= -\lambda_1\lambda_2\lambda_{12}(\lambda_1\lambda_2\gamma_2^{-2} + \lambda\gamma_1^{-1})/(\theta_0\gamma_1),\\
\sigma_{22} &= \lambda_2(\lambda - \lambda_1\lambda_2\gamma_2^{-1}),\\
\sigma_{23} &= -\lambda_1\lambda_2\lambda_{12}(\lambda_1\lambda_2\gamma_1^{-2} + \lambda\gamma_2^{-1})/(\theta_0\gamma_2),\\
\sigma_{33} &= \lambda_{12}[\lambda - \lambda_0\theta_0^{-2}(\lambda_1\gamma_2^{-3} + \lambda_2\gamma_1^{-3} + 2\lambda_1\lambda_2\gamma_1^{-2}\gamma_2^{-2})],
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
\lambda &= \lambda_1 + \lambda_2 + \lambda_{12}, \ \gamma_1 = \lambda_1 + \lambda_{12},\\
\gamma_2 &= \lambda_2 + \lambda_{12}\\
\theta_0 &= E[\max(X, Y)] = \gamma_1^{-1} + \gamma_2^{-1} - \lambda^{-1}.
\end{aligned}$$

R is then estimated by $\mathring{R} = \mathring{\lambda}_1/(\mathring{\lambda}_1 + \mathring{\lambda}_2 + \mathring{\lambda}_{12})$ rather than by $\hat{R}$. Since $\mathring{R}$ is a totally differential function of $\mathring{\lambda}_1$, $\mathring{\lambda}_2$, and $\mathring{\lambda}_{12}$ by Theorem 6a.2(ii), page 321 of Rao [10], $\sqrt{n}\,(\mathring{R} - R)$ is asympototically normally distributed with mean zero and variance σ^2, where

$$\sigma^2 = \frac{\gamma_2^2}{\lambda^4}\sigma_{11} + \frac{\lambda_1^2}{\lambda^4}(\sigma_{22} + \sigma_{33}) - 2\frac{\lambda_1\gamma_2}{\lambda^4}(\sigma_{12} + \sigma_{13}) + 2\frac{\lambda_1^2}{\lambda^4}\sigma_{23}. \tag{3.6}$$

Thus, from (3.3), (3.4) and (3.6), one-sided and two-sided confidence limits for R can readily be obtained.

To check the adequacy of the large sample approximation when the underlying distribution is BVE, computer simulations were made. For various values of $(\lambda_1, \lambda_2, \lambda_{12})$ 500 sets of random samples of size n ($n = 10, 15, 20$) from bivariate exponential distributions were obtained and the empirical distribution of $\hat{R}$ was obtained. Even for sample size as low as 10 the exact distribution is found to be well approximated by the normal approximation. The situation improves as the sample size increases.

4. DISTRIBUTION FREE PROCEDURES

In all the cases considered in the previous section to check adequacy of normal approximation, the value of R is rather small ($R \leqslant .75$), whereas applications of interest would be for systems with high reliability ($R > .90$). Unfortunately, in all the cases with $R > .90$ considered, the sample estimate of the variance of R came out to the negative or very close to zero. (The situation is similar to the problem of having "negative" estimator of variances in analysis of variance problems.)

To study the cases for which $R > .90$, we therefore consider the following estimators of R:

For bivariate data, a rather natural way to estimate $R = P(X < Y)$ would be based on the binomial distribution. Let T be the number of cases for which $X < Y$. Then T is a binomial random variable with mean R and variance $nR(1 - R)$. We therefore can obtain an exact binomial confidence interval or an approximation two-sided 100 $(1 - \gamma)$% confidence interval given by

$$\left(\hat{R} - Z_\gamma \sqrt{\frac{\hat{R}(1-\hat{R})}{n}},\ \hat{R} + Z_\gamma \sqrt{\frac{\hat{R}(1-\hat{R})}{n}}\right), \tag{4.1}$$

where $\hat{R} = T/n$ and Z_γ is such that $\Phi(Z_\gamma) = (1 - \gamma/2)$, where $\Phi(\cdot)$ is the standard normal distribution.

If X and Y are assumed independent, a second estimator is the following nonparametric confidence interval proposed by Govindarajulu [6] and is based on the Wilcoxon-Mann-Whitney statistic.

Let $(X_1, X_2, \ldots, X_m)$ and $(Y_1, Y_2, \ldots, Y_n)$ be two independent samples of measurements from populations with distribution functions $F(x)$ and $G(y)$ respectively. Let

$$\psi(X_i, Y_j) = \begin{cases} 1 & \text{if } X_i < Y_j \\ 0 & \text{otherwise} \end{cases}$$

then $U = \sum_{i=1}^{m} \sum_{j=1}^{n} \psi(X_i, Y_j)$ is the well-known two sample Mann-Whitney statistic. That is, $U =$ number of pairs (X_i, Y_j) such that $X_i < Y_j$. Govindarajulu [6] has explicitly derived one-sided and two-sided distribution free confidence bounds for R (actually Govindarajulu derived confidence bounds for $1 - R$) based on the asymptotic normality of $\hat{R} = U/mn$. In particular, for the two-sided case, Govindarajulu showed that for all F and G and large m or n, the solution ϵ_γ of the inequality

(4.2) $$P(|\hat{R} - R| \leqslant \epsilon_\gamma) \geqslant 1 - \gamma, \quad 0 < \gamma < 1$$

is given by

$$\epsilon_\gamma \geqslant (4v)^{-1/2}\Phi^{-1}(1 - \gamma/2).$$

Here $v = \min(m, n)$ and $\Phi(\cdot)$ is the cumulative distribution function of standard normal distribution. In particular, if $m = n$, a $100(1 - \gamma)\%$ confidence interval is given by

(4.3) $$(\hat{R} - \epsilon_\gamma, \hat{R} + \epsilon_\gamma),$$

where $\hat{R}$ and Z_γ are as defined before, and

$$\epsilon_\gamma = \frac{Z_\gamma}{2\sqrt{n}}.$$

5. EFFECT OF MISSPECIFYING THE MODEL

In section 2 it was pointed out that for value of $P(X < Y)$ close to unity, R_{pq} is less sensitive to the variation of p and varies considerably for varying values of q. To study the effect of misspecifying the model we carry out the following Monte Carlo experiment.

Let $G(\alpha, p)$ and $G(\beta, q)$ be two given gamma populations with known parameters (α, p) and (β, q) respectively, where $\alpha = 19$ and $\beta = 1$. In this case, if $p = q = 1$, $R_{11} = .95$. By choosing different pairs, (p, q), we get different pairs of gamma distributions.

Let $(X_1, X_2, \ldots, X_n)$ and $(Y_1, Y_2, \ldots, Y_n)$ be two random samples from $G(\alpha, p_1)$ and $G(\beta, q_1)$. Since p_1 and q_1, the true values are not known, there is a possibility that we will choose a different pair of distributions as the true model. Let us assume that the above samples have come from populations with distributions $G(\alpha, p_2)$ and $G(\beta, q_2)$. Thus, we would estimate $R_{p_2q_2}$ instead of estimating the true value $R_{p_1q_1}$, and compute a confidence interval based on $\hat{R}_{p_2q_2}$. We would not commit much specification error if this confidence interval contains the true unknown value $R_{p_1q_1}$. For a given n and two pairs of values (p_1, q_1) and (p_2, q_2) the above procedure is repeated 1000 times and a count is made of how many times the true value $R_{p_1q_1}$ is contained in the confidence interval based on $\hat{R}_{p_2q_2}$. We repeat the procedure for different values of n and different combinations of (p_1, q_1) and (p_2, q_2). The results are given in Table 1. Here n is chosen to be 5, 10 and 25. All combinations of the following pairs of values are chosen for (p_1, q_1) and (p_2, q_2): (1, 1), (1, 2), (2, 1) and (2, 2).

From Table 1 we can make a number of conclusions. First, note that no parametric method performs well in all situations. For all the gamma models considered, the procedure, as anticipated in section 2, is robust for small variation in p. However, it is sensitive to variation in q. The nonparametric confidence intervals based on the Wilcoxon-Mann-Whitney statistic performs well in all cases. However, in each case the width of the confidence interval for an assumed model is too large compared with the corresponding width based on the parametric models. In each case the parametric method is to be preferred, especially if the parameter q is reasonably well specified.

TABLE 1 — *a. Number of Counts for N = 5*

ASSUMED MODEL		TRUE MODEL			
		$G(1, 1) \equiv$ Exponential	$G(2, 1)$	$G(1, 2)$	$G(2, 2)$
	$R_{p_1 q_1}$	.95	.9025	.9975	.99275
	$G(1, 1)$ exact	933	967	46	77
	$G(1, 1)$ normal approx.	855	934	1000	1000
	$G(2, 1)$	763	819	66	94
	$G(1, 2)$	287	487	828	938
	$G(2, 2)$	209	371	725	871
	*NP pr	998	991	1000	1000

*Nonparametric Procedure

TABLE 1 — *b. Number of Counts for N = 10*

ASSUMED MODEL		TRUE MODEL			
		$G(1, 1) \equiv$ Exponential	$G(2, 1)$	$G(1, 2)$	$G(2, 2)$
	$R_{p_1 q_1}$	.95	.9025	.9975	.99275
	$G(1, 1)$ exact	947	976	0	2
	$G(1, 1)$ normal approx.	913	952	188	348
	$G(2, 1)$	775	838	0	1
	$G(1, 2)$	133	293	869	968
	$G(2, 2)$	58	192	750	906
	*NP pr	999	996	1000	1000

*Nonparametric Procedure

TABLE 1 — *c. Number of Counts for N = 25*

ASSUMED MODEL		TRUE MODEL			
		$G(1, 1) \equiv$ Exponential	$G(2, 1)$	$G(1, 2)$	$G(2, 2)$
	$R_{p_1 q_1}$	.95	.9025	.9975	.99275
	$G(1, 1)$ exact	945	964	0	0
	$G(1, 1)$ normal approx.	934	964	0	0
	$G(2, 1)$	768	812	0	0
	$G(1, 2)$	6	82	925	982
	$G(2, 2)$	1	27	754	928
	*NP pr	1000	998	1000	999

*Nonparametric Procedure

6. AN EXAMPLE

To illustrate the computation of confidence intervals let us consider the following example. Fifteen items of random strengths $Y_1, \ldots, Y_{15}$ are subject to random stresses $X_1, X_2, \ldots, X_{15}$. To estimate the reliability function $P(X < Y)$ random samples of 15 pairs of (X, Y) values are drawn and given below.

Pair No.		
1	.0352	1.7700
2	.0397	.9457
3	.0677	1.8985
4	.0233	2.6121
5	.0873	1.0929
6	.1156	.0362
7	.0286	1.0615
8	.0200	2.3895
9	.0793	.0982
10	.0072	.7971
11	.0245	.8316
12	.0251	3.2304
13	.0469	.4373
14	.0838	2.5648
15	.0796	.6377

From past record it is known that $R = .95$. Estimates of $R_{p_1q_1}$ and corresponding confidence interval will depend on the model chosen and the method used to compute the confidence interval. Thus, if X and Y are assumed independent, we have two independent samples of size 15 each. Table 2 lists the values of $\hat{R}_{p_2q_2}$ and confidence interval for the cases considered in section 5. We also use the notation of section 5 for convenience.

TABLE 2 — *Comparison of Various Confidence Intervals and Estimates of R*

Model Used	$\hat{R}$	Confidence Interval
$G(1, 1)$ exact	.9639	(.9280, .9823)
$G(1, 1)$ normal approximation	.9639	(.9300, .9978)
$G(2, 1)$	.9639	(.9416, .9856)
$G(1, 2)$	.9952	(.9896, 1.000*)
$G(2, 2)$	.9962	(.9925, .9999)
Binomial Procedure	.9333	(.8071, 1.000*)
NP Procedure	.9600	(.7070, 1.000*)

*Computed as being larger than 1

Let us illustrate the calculation for a couple of cases. From the above data we have $\bar{X} = .0509$ and $\bar{Y} = 1.3602$. If we assume $p_2 = q_2 = 1$ we have, using results in section 2,

$$R_{11} = \frac{\bar{Y}}{\bar{X} + \bar{Y}} = .9639$$

and corresponding 95% confidence interval for R_{11}, using normal approximation, is (.9300, .9978).

On the other hand, if we used the exact distribution of $\hat{R}_{11}$, we can obtain the required confidence interval from an F-table since $U = \frac{\bar{Y}}{\bar{X}} \frac{\beta}{\alpha}$ follows the F-distribution with $(2n, 2n)$ degrees of freedom we obtain, after some simplification (.9280, .9823) as the required 95% confidence interval.

From Table 2 it seems any of the first three procedures based on two independent exponential distributions or $G(2, 1)$ (X gamma with shape parameter $p = 2$ and Y independent exponential) will be quite satisfactory.

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OPTIMAL SELECTION FROM A RANDOM SEQUENCE WITH OBSERVATION ERRORS

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ABSTRACT

A form of sequential decision problem is introduced in which options are presented in sequence, with no recall of rejected options (as in the secretary problem), but in which the value of each option may only be inferred from experiments. Decisions have thus to be made concerning both the acceptance and rejection of each option and the degree of experimentation. General properties of the optimal policy are derived, and an algorithm is obtained for the solution in a special case. This special case suggests a heuristic rule for more general situations, the performance of which rule has been investigated by a Monte Carlo study.

1. INTRODUCTION

We consider a sequential decision problem in which a known finite number of options, having values drawn from a known probability distribution, are presented in random order to the decision maker who cannot observe the values directly. The decision maker has to perform a number of tests on each option (for instance by some form of sampling) in order to make inferences about its unknown value, and must finally decide on the acceptance or rejection of this current option before the next becomes available. We assume that the total number of tests that can be performed is strictly bounded above (owing, for instance, to budgetary constraints). According to a criterion to be discussed in the next section, only one option may be chosen.

The problem sketched above is a simplified representation of a number of practical situations. A mining development company may, for example, obtain options on various mineral deposits from time to time, but have to make a decision on each (either giving up the option to another or committing its resources to developing a mine at that site) before prospecting on the next site can even start. The value of each mineral deposit is, of course, unknown to the company and must be estimated by bore-hole samples. These are, however, expensive, and a limited number only can be afforded before a final decision has to be made.

A similar situation may be faced in the procuring of major capital equipment, where expensive operational tests may be necessary to select between alternatives, and where, for practical or political reasons, negotiations with only one potential supplier at a time is possible.

These practical situations are generally more complex than the decision model discussed here. Nevertheless, a model of this nature can be used to gain valuable understanding of the real problem, as discussed in some detail in Stewart [9].

It may be recognized that the problem described in the opening paragraph is essentially a generalization of the well-known 'secretary' or 'dowry' problem in the case of values drawn from a known distribution (for instance, the 'full information game' of Gilbert and Mosteller [3]), extended to allow for imperfect information (i.e., observation errors) on the value of each candidate. Note that we use 'information' in a different sense from that of Gilbert and Mosteller, where 'full information' refers to complete knowledge of the distribution from which the values are drawn, but where observation of each value is implied to be error-free. The antithetical case of those writers is such that only relative ranking of candidates is possible, but in our model even ranking may be in error owing to observation errors (even though 'full information' concerning the underlying distribution is assumed to be available).

MacQueen [4] has treated a somewhat similar problem, involving an unbounded number of options in a random sequence, but with a cost incurred in obtaining each new option. In his problem some imperfect information is available on receipt of each option, and further information can be obtained, at a cost, by executing a single test on the option. The present problem differs from that of MacQueen in three main aspects:

(1) the number of options is bounded;

(2) the degree of testing is variable (under control of the decision maker), and

(3) expenditure on testing is strictly limited.

The problem is also related to recent work on partially observable Markov processes (see, for instance, Smallwood and Sondik [7], Sondik [8] and White [11]). In our case we have sufficient special structure to warrant special attention, but there are also some differences, especially in respect to the variable degree of testing and the budgetary constraints.

In Section 2 the problem is formulated more precisely and the decision criterion discussed.

General properties of the solution under certain mild assumptions are developed in Section 3. In particular it is demonstrated that given a realization of a sufficient statistic $\bar{y}$ for the value of the option based on the tests conducted, there exist two critical values u and w such that the option is rejected for $\bar{y} \leqslant u$, accepted for $\bar{y} > w$, and further tested otherwise. These critical values depend on the numbers of options and permitted tests remaining, and on the number of tests already conducted on the option.

In Section 4 a special case is introduced, for which a specific algorithm for computing these critical values can be obtained. Not only is this algorithm of use in its own right, but the special problem suggests an adaptation of its use as a heuristic for more general problems.

Section 5 consists of a discussion of a Monte Carlo investigation of the behaviour of this heuristic.

2. PROBLEM FORMULATION AND DECISION CRITERIA

We shall throughout the paper denote random variables by capital letters, and their specific realizations by the corresponding lower case letters. Conditional probabilities of the form $\Pr[Z|Y = y]$ will for convenience be written $\Pr[Z|y]$, and a similar convention will apply to expectations, probability distributions and density functions. In order to avoid technical difficulties we shall always assume that the probability distributions have densities. The random values Z_n and observations Y_{nj} (defined below) will be assumed to be real-valued scalars.

Now assume that a *known number* N of options are to be presented, and that the value of option $n (n = 1, \ldots, N)$ is the unobserved random variable Z_n, drawn from a *known probability distribution* $F(\cdot)$:

$$F(z) = \Pr[Z_n \leqslant z].$$

Inferences concerning the actual realization of Z_n are to be made on the basis of observations on r independent random variables $Y_{n1}, Y_{n2}, \ldots, Y_{nr}$. (The choice of r is discussed later). The probability distribution of Y_{nj} conditional on $Z_n = z_n$ is given by the known function $G(y_{nj}|z_n)$, with corresponding density $g(y_{nj}|z_n)$. Adopting a Bayesian approach, the prior distribution of Z_n is $F(z_n)$, while the posterior distribution is given by $F_{nr}(z_n|y_{n1}, \ldots, y_{nr})$ for $r > 1$:

$$F_{nr}(z_n|y_{n1}, \ldots, y_{nr}) = K \int_{z \leqslant z_n} g(y_{nr}|z)\, dF_{n,r-1}(z|y_{n1}, \ldots, y_{n,r-1}) \tag{1}$$

where K is an appropriate integration constant and where for convenience $F_{n0}(z)$ is defined as $F(z)$.

In setting up decision criteria two factors have to be considered, viz. the cost of observations and the objective of selecting a satisfactory option. These factors could, in principle, be handled in a variety of ways, but for practical reasons it is necessary to adopt an approach that shows promise of yielding workable solutions. We shall thus assume the following structure for the problem:

(1) a fixed total number of observations R will be allowed, which is equivalent to assuming a fixed budget;

(2) the objective of the decision-maker is to maximize the expected value of the option chosen.

Now let:

$\gamma_{nr\rho}(y_{n1}, \ldots, y_{nr}) \triangleq$ expected optimal value of the option chosen, given that option n is under consideration (the first $n - 1$ having been rejected), that r observations $Y_{n1} = y_{n1}, \ldots, Y_{nr} = y_{nr}$ have been made on n, and that altogether ρ observations have been made on the first n options;

and then let

$$V_{nr\rho}(y_{n1}, \ldots, y_{n,r-1}) \triangleq E[\gamma_{nr\rho}(y_{n1}, \ldots, y_{nr})|y_{n1}, \ldots, y_{n,r-1}].$$

In evaluating the expectation in the latter definition we use the posterior distribution of Y_{ni}, i.e., we use the probability density for Y_{ni} defined by:

$$g_{n,r-1}(y|y_{n1}, \ldots, y_{n,r-1}) \triangleq \int_{\mathbf{R}^1} g(y|z)\,dF_{n,r-1}(z|y_{n1}, \ldots, y_{n,r-1}). \tag{2}$$

For $r > 1$, $\rho < R$ and $n < N$, three options are available at this stage, viz.

(i) to accept option n;

(ii) to make another observation on n;

(iii) to reject option n, implying that option $n + 1$ and the first observation on it become available.

We thus have the following expression for $\gamma_{nr\rho}(y_{n1}, \ldots, y_{nr})$:

$$\gamma_{nr\rho}(y_{n1}, \ldots, y_{nr}) = \text{Max}\,[E\{Z_n|y_{n1}, \ldots, y_{nr}\};\ V_{n,r+1,\rho+1}(y_{n1}, \ldots, y_{nr});\ V_{n+1,1,\rho+1}] \tag{3}$$

where we note that $V_{n+1,1,\rho+1}$ does not depend on any observations. Then by invoking a dynamic programming argument (see, for example, Chow, Robbins and Sigmund [2], Section 3.2), we have the following recursive relationship for $r > 1$, $\rho < R$ and $n < N$:

$$V_{nr\rho}(y_{n1}, \ldots, y_{n,r-1}) = \int_{\mathbf{R}^1} [\text{Max}\,\{E(Z_n|y_{n1}, \ldots, y_{n,r-1}, y);\ V_{n,r+1,\rho+1}(y_{n1}, \ldots, y_{n,r-1}, y);\ V_{n+1,1,\rho+1}\}]\,g_{n,r-1}(y|y_{n1}, \ldots y_{n,r-1})\,dy. \tag{4}$$

For $r = 1$, $\rho < R$ and $n < N$ the same form as the above holds, with a null argument list for $V_{n1\rho}$ and using $g_{n0}(\cdot)$ implied by (2) (from $F_{n0}(\cdot) \equiv F(\cdot)$) in place of $g_{n,r-1}(\cdot)$. When $n = N$, this option must be accepted, and the expectation of its expected value after further sampling is the current expected value. When $\rho = R$, either the current option or the $(n + 1)$-th (without observation) has to be accepted. All these cases can be accounted for by the relationship (4) if the following definitions are used as boundary conditions:

$$V_{nr,R+1} = -\infty \text{ for } r > 1 \text{ and all } n \leqslant N$$

$$V_{n1,R+1} = \int_{\mathbf{R}^1} z\,dF(z) \triangleq \mu \text{ for all } n \leqslant N$$

$$V_{1,1,\rho} = \mu \qquad \text{for all } \rho \leqslant R + 1$$

$$V_{1+1,1,\rho} = -\infty \qquad \text{for all } \rho.$$

The recursive relationship (4) now holds for $1 \leqslant n \leqslant N$, $1 \leqslant \rho \leqslant R$, $1 \leqslant r \leqslant \rho - n$, and also defines the optimal policy which is to

(i) select option n if

$$E(Z_n|y_{n1}, \ldots, y_{nr}) > V_{n,r+1,\rho+1}(y_{n1}, \ldots, y_{nr}) \text{ and } V_{n+1,1,\rho+1};$$

(ii) take another observation on n if

$$V_{n,r+1,\rho+1}(y_{n1}, \ldots, y_{nr}) > E(Z_n|y_{n1}, \ldots, y_{nr}) \text{ and } V_{n+1,1,\rho+1};$$

(iii) reject option n otherwise.

Quite clearly any general solution to (4) would be very difficult, if not impossible, to obtain in practice. Nevertheless, we obtain in the next section some general solution properties which may be useful in specific cases. An actually soluble problem is presented in Section 4, and the use of this result as a general rule is discussed.

3. GENERAL SOLUTION PROPERTIES

Two assumptions regarding the distributions F and G will be invoked in this section at various times. These assumptions are stated first.

ASSUMPTION (A1): The distributions are such that

(i) given observations $y_{n1}, \ldots, y_{nr}$, there exists a scalar-valued sufficient statistic $\bar{y}_{nr}$ for z_n;

(ii) $\bar{y}_{nr}$ can be recursively generated (given a suitable definition of $\bar{y}_{n0}$) as a function of $\bar{y}_{n,r-1}$ and y_{nr} which is nondecreasing in each of its arguments;

(iii) $E[Z_n|\bar{y}_{nr}]$ is a nondecreasing function of $\bar{y}_{nr}$;

(iv) the posterior probability, $\Pr(Y_{nr} \geq y|\bar{y}_{n,r-1})$ is nondecreasing in $\bar{y}_{n,r-1}$ for all y.

Assumptions (i) and (ii) together imply that the dynamic programming recursion formulae (4) requires only $\bar{y}_{nr}$ as state variable, rather than the full observation vector. Assumptions (iii) and (iv) are introduced in similar spirit to assumption 2^0 of MacQueen [4]. The requirement is that the probability mass of the observations should follow the true underlying value. More specifically (iii) requires that posterior inferences on z_n should in turn follow the observations, while conversely (iv) requires that posterior probability mass on future observations should follow the inferred underlying value through the sufficient statistic $\bar{y}_{n,r-1}$. Assumption (iv) is, in a sense, analogous to the 'increasing failure rate' assumption of reliability theory. In our case, however, it simply implies that high observation values should not decrease the posterior probability of future high observation values (i.e., a form of consistency assumption).

In fact, (A1) may be expected to hold quite widely. It holds, for example, when $Z_n \sim N(0, 1)$ and $Y_{nr} \sim N(z_n, \sigma^2)$, in which case:

$$\bar{y}_{nr} = \frac{1}{r} \sum_{i=1}^{r} y_{ni}$$

or

$$\bar{y}_{nr} = \{(r-1)\bar{y}_{n,r-1} + y_{nr}\}/r$$

$$\bar{y}_{n0} = 0, \text{ say (arbitrarily)}$$

and

$$EZ_n|\bar{y}_{nr} = \left\{\frac{r}{\sigma^2 + r}\right\} \bar{y}_{nr} \qquad Y_{nr} \sim N\left[\frac{(r-1)\bar{y}_{n,r-1}}{\sigma^2 + r - 1}; \frac{\sigma^2(\sigma^2 + r)}{\sigma^2 + r - 1}\right].$$

The normal case also satisfies the second assumption below, which is really an extension of (iii) and (iv) of (A1), now requiring a certain consistency between the probability mass of the observations and the true value in the tails of the distributions as well.

ASSUMPTION (A2): (A1) holds, and for any $\epsilon > 0$ there exist y_1 and y_2 such that $\{\bar{Y}_{nr} < y_1\}$ and $\{\bar{Y}_{nr} > y_2\}$ have nonzero probability and for any $m > n$

(i) $\bar{y}_{nr} < y_1$ implies

$$\Pr[Z_m > z_n | \bar{y}_{nr}] = \int_{\mathbb{R}^1} F_{mr}(z|\bar{y}_{nr}) dF(z) > 1 - \epsilon$$

(ii) $\bar{y}_{nr} > y_2$ implies

$$\Pr[Z_m > z_n | \bar{y}_{nr}] < \epsilon.$$

We now consider certain general properties of the solution. First, however, we prove the following lemma.

LEMMA: If (A1) holds, then $V_{nrp}(\bar{y}_{n,r-1})$ $(= V_{nrp}(y_{n1}, \ldots, y_{n,r-1})$, see comment after assumption (A1)) is a nondecreasing function of $\bar{y}_{n,r-1}$, which is, furthermore, such that:

$$D_{nrp}(\bar{y}_{n,r-1}) \triangleq [E Z_n | \bar{y}_{n,r-1}] - V_{nrp}(\bar{y}_{n,r-1})$$

is a nonpositive, nondecreasing function of $\bar{y}_{n,r-1}$. (Note that by virtue of (A1) (i), conditional expectations depend only on $\bar{y}_{n,r-1}$.)

PROOF: By assumptions (A1) (ii) and (A1) (iii) it follows that for given $y_{nr} = y$:

$$E[Z_n | \bar{y}_{n,r-1}, y_{nr}] = E[Z_n | \bar{y}_{nr}(\bar{y}_{n,r-1}, y)]$$

is a nondecreasing function of $\bar{y}_{n,r-1}$ and y.

The boundary conditions on V_{nrp} imply that for any $r (1 \leqslant r \leqslant R)$ and given $y_{nr} = y$:

$$\begin{aligned} \gamma_{1R}(y_{11}, \ldots, y_{1r}) &= \gamma_{1R}(\bar{y}_{1,r-1}, y) \\ &= E[Z_1 | \bar{y}_{1,r-1}, y] \end{aligned}$$

while for $n < N$

$$\gamma_{nR}(\bar{y}_{n,r-1}, y) = \text{Max}[E\{Z_n | \bar{y}_{nr}(\bar{y}_{n,r-1}, y)\}; \mu].$$

(Here and elsewhere in the proof $\gamma_{nR}(\bar{y}_{n,r-1}, y)$ is a compressed form of $\gamma_{nR}(\bar{y}_{nr}(\bar{y}_{n,r-1}, y))$ and similarly for the other functions.)

Thus, for all $n \leqslant N$ and $r \leqslant R$ and a given value for y, $\gamma_{nrR}(\bar{y}_{n,r-1}, y)$ is non-decreasing in $\bar{y}_{n,r-1}$ and y. Now define y, $\bar{\gamma}_{n,r-1}$ and y. Now define

$$\begin{aligned} \delta_{nrR}(\bar{y}_{n,r-1}, y) &\triangleq E\{Z_n | \bar{y}_{n,r-1}, y\} - \gamma_{nR}(\bar{y}_{n,r-1}, y) \\ &= \text{Min}[0; E\{Z_n | \bar{y}_{nr}(\bar{y}_{n,r-1}, y)\} - \mu]. \end{aligned}$$

Clearly, $\delta_{nrR}(\bar{y}_{n,r-1}, y)$ is also nondecreasing in $\bar{y}_{n,r-1}$ and y, and is nonpositive.

Now define the following probability density function on y:

$$\bar{g}_{n,r-1}(y) \triangleq g_{n,r-1}(y | \bar{y}_{n,r-1})$$

with corresponding distribution function $\tilde{G}_{n,r-1}(y)$. This distribution is defined to be independent of any perturbation of $\bar{y}_{n,r-1}$ around the nominal value.

Since $\gamma(\cdot)$ and $\delta(\cdot)$ are nondecreasing in $\bar{y}_{n,r-1}$ for every y, it follows that their expectations with respect to the distribution defined by $\tilde{g}_{n,r-1}(\cdot)$ are also nondecreasing in $\bar{y}_{n,r-1}$. Thus, defining for any $\bar{y}'_{n,r-1}$ (not necessarily the nominal value $\bar{y}_{n,r-1}$):

$$\tilde{V}_{nrR}(\bar{y}'_{n,r-1}) \triangleq \int_{\mathbb{R}^1} \gamma_{nrR}(\bar{y}'_{n,r-1}, y)\, \tilde{g}_{n,r-1}(y)\,dy \quad \text{and}$$

$$\tilde{D}_{nrR}(\bar{y}'_{n,r-1}) \triangleq \int_{\mathbb{R}^1} \delta_{nrR}(\bar{y}'_{n,r-1}, y)\, \tilde{g}_{n,r-1}(y)\,dy$$

we have for any $h > 0$:

$$\tilde{V}_{nrR}(\bar{y}_{n,r-1}+h) \geq \tilde{V}_{nrR}(\bar{y}_{n,r-1}) = V_{nrR}(\bar{y}_{n,r-1}) \quad \text{and}$$

$$\begin{aligned}\tilde{D}_{nrR}(\bar{y}_{n,r-1}+h) \geq \tilde{D}_{nrR}(\bar{y}_{n,r-1}) &= \int_{\mathbb{R}^1} \delta_{nrR}(\bar{y}_{n,r-1}, y)\, g_{n,r-1}(y|y_{n,r-1})\,dy \\ &= E\{Z_n|\bar{y}_{n,r-1}\} - V_{nrR}(\bar{y}_{n,r-1}) \\ &= D_{nrR}(\bar{y}_{n,r-1}).\end{aligned}$$

Now consider the expectations of $\gamma_{nrR}(\cdot,\cdot)$ and $\delta_{nrR}(\cdot,\cdot)$ when y has distribution function $G_{n,r-1}(y|\bar{y}_{n,r-1}+h)$. By assumption (A1) (iv), if this distribution differs from $\tilde{G}_{n,r-1}(y)$, the difference is a shift in probability weight to higher values of y, and thus, since $\gamma_{nrR}(\cdot,\cdot)$ and $\delta_{nrR}(\cdot,\cdot)$ are nondecreasing in y, we have for $h > 0$:

$$V_{nrR}(\bar{y}_{n,r-1}+h) \geq \tilde{V}_{nrR}(\bar{y}_{n,r-1}+h) \geq V_{nrR}(\bar{y}_{n,r-1})$$

$$D_{nrR}(\bar{y}_{n,r-1}+h) \geq \tilde{D}_{nrR}(\bar{y}_{n,r-1}+h) \geq D_{nrR}(\bar{y}_{n,r-1}).$$

Furthermore, taking expectations, it is also clear that $D_{nrR}(\bar{y}_{n,r-1})$ inherits the nonpositivity of $\delta_{nrR}(\bar{y}_{n,r-1}, y)$. The lemma thus holds for $\rho = R$. For other values of ρ, the $V_{nr\rho}(\cdot)$ values can be obtained by backward induction from the $V_{nrR}(\cdot)$ computed above and from the boundary condition:

$$V_{n1\rho} = \mu, \qquad \rho \leq R+1.$$

We thus assume that the lemma holds for $n, r+1, \rho+1$, i.e., that $V_{n,r+1,\rho+1}(\bar{y}_m)$ and $D_{n,r+1,\rho+1}(\bar{y}_m)$ are nondecreasing functions of $\bar{y}_m$ and that:

$$D_{n,r+1,\rho+1}(\bar{y}_m) \leq 0.$$

Subtracting the expression for $\gamma_{nr\rho}(\bar{y}_{n,r-1}, y)$ given by Equation (3) (where, as above, $y_{n1}, \ldots, y_{n,r-1}$ is replaced by $\bar{y}_{n,r-1}$ and $y_{nr} = y$) from $E\{Z_n|\bar{y}_m(\bar{y}_{n,r-1}, y)\}$, we have

$$\begin{aligned}\delta_{nr\rho}(\bar{y}_{n,r-1}, y) = \text{Min}\,[0;\, &E\{Z_n|\bar{y}_m(\bar{y}_{n,r-1}, y)\} - V_{n,r+1,\rho+1}\{\bar{y}_m(\bar{y}_{n,r-1}, y)\}; \\ &E\{Z_n|\bar{y}_m(\bar{y}_{n,r-1}, y)\} - V_{n+1,1,\rho+1}]. \\ = \text{Min}\,[0;\, &D_{n,r+1}\{\bar{y}_m(\bar{y}_{n,r-1}, y)\}; \\ &E\{Z_n|\bar{y}_{nr}(\bar{y}_{n,r-1}, y)\} - V_{n+1,1,1}].\end{aligned}$$

Since $V_{n+1,1,\rho+1}$ is a constant not dependent on $\bar{y}_{n,r-1}$ or y we have from (3) and the above, and using (A1) (ii) and (A1) (iii), that $\gamma_{nr\rho}(\bar{y}_{n,r-1}, y)$ and $\delta_{nr\rho}(\bar{y}_{n,r-1}, y)$ are nondecreasing functions of $\bar{y}_{n,r-1}$ and y, and that $\delta_{nr\rho}(\bar{y}_{n,r-1}, y)$ is nonpositive. In the same way as above it can be shown that $V_{nr\rho}(\bar{y}_{n,r-1})$ and $D_{nr\rho}(\bar{y}_{n,r-1})$ are nondecreasing in $\bar{y}_{n,r-1}$ and that

$$D_{nr\rho}(\bar{y}_{n,r-1}) \leqslant 0. \qquad \|$$

The lemma follows by induction.

Lemma 1 may be seen to be a special case of a wider class of results derived by Topkis [10], where conditions under which optimization problems yield solutions which are isotonic in a parameter are discussed. In the present case, however, not only is the direct proof relatively simple, but it directly yields the structure of the optimal policy. In fact, by the optimality principle of the last section, there is implicit in the proof also the demonstration of the following theorem.

THEOREM 1: If (A1) holds, then the optimal policy is defined by numbers $u_{nr\rho}$ and $w_{nr\rho}$ for all n, r, ρ such that

$$u_{nr\rho} \leqslant w_{nr\rho}$$

(they may be equal), and

(i) option n is rejected if $\bar{y}_{nr} \leqslant u_{nr\rho}$;

(ii) option n is accepted if $\bar{y}_{nr} > w_{nr\rho}$;

(iii) a further sample is taken otherwise.

This is a result which is intuitively fairly obvious although difficult to prove, and it is similar in spirit to related results of Rosenfield [6] and MacQueen [4]. General conditions for the optimality of structured classes of policies, such as that of Theorem 1, have been derived by Porteus [5]. Thus, as for Lemma 1, Theorem 1, is a special case of a more general result. Our derivation has, however, allowed the specific form of structuring to be developed constructively, and thus Theorem 1 is in a form suitable for direct practical implementation.

This structure of the optimal policy is important for a number of reasons. Certainly it is easier to store $u_{nr\rho}$ and $w_{nr\rho}$ values than to store $V_{nr\rho}(\cdot)$ for all values of its argument. In some cases it may even be possible to compute $u_{nr\rho}$ and $w_{nr\rho}$ directly, although we have not found such an example. Furthermore, once this structure of the optimal policy is known, various heuristics can be proposed for the problem, based on assumed or heuristically derived values for the $u_{nr\rho}$ and $w_{nr\rho}$.

We now state and prove a further property of the optimal policy under the stronger assumption (A2). This result is given here not only because it is an aid to intuitive understanding of the problem, but also because it may suggest heuristic derivations for the $u_{nr\rho}$ and $w_{nr\rho}$ above, based on approximations for $V_{nr\rho}(\bar{y}_{n,r-1})$.

THEOREM 2: If (A2) holds and $\bar{y}_{nr} \in [y_l, y_u]$ for all n and r (we may have $y_l = -\infty$ and/or $y_u = \infty$), then

$$\lim_{\bar{y}_{n,r-1} \downarrow y_l} V_{nr\rho}(\bar{y}_{n,r-1}) = V_{n+1,1,\rho+1}$$

$$\lim_{\bar{y}_{n,r-1} \uparrow y_u} V_{nr\rho}(\bar{y}_{n,r-1}) = E\,[E\{Z_n | \bar{y}_{nr}(\bar{y}_{n,r-1}, y_{nr})\} | \bar{y}_{n,r-1}].$$

(Note that, although the second expression above is in a rather cumbersome form, it is in most practical applications quite easily evaluated).

PROOF: By assumption, for small enough $\bar{y}_{n,r-1}$ it will be optimal to reject option n immediately, giving:

$$\gamma_{nr\rho}(\bar{y}_{n,r-1}, y) = V_{n+1,1,\rho+1}$$

for all $\bar{y}_{n,r-1}$ and y, and the first limit holds.

Similarly, for large enough $\bar{y}_{n,r-1}$, option n is immediately accepted, giving

$$\gamma_{nr\rho}(\bar{y}_{n,r-1}, y) = E\{Z_n | \bar{y}_{n,r-1}, y\},$$

and the second limit follows. ||

In this paper we have not attempted to develop any heuristics on the basis suggested above. Rather, we introduce in the next section a particular case in which the exact solution can be computed. Not only is this case of some interest in its own right, but it suggests another heuristic approach for the general case. The results of a Monte Carlo study of the performance of this heuristic are presented in Section 5.

4. BETA-DISTRIBUTED VALUES WITH BINOMIAL TRIALS

In this section we consider the case in which the Z_n are independently distributed according to a Beta distribution, with parameters α and β, i.e., $Z_n \in [0, 1]$ and $F(\cdot)$ has density:

$$f(z) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} z^{\alpha-1}(1-z)^{\beta-1}.$$

The observations Y_{ni} are of a simple binomial success/failure form, with success probability z_n. This is a somewhat simplified model of many real situations in which z_n represents, for each option, a certain degree of satisfaction, and the y_{nk} represent either a simple sampling (e.g., defectives in a batch) or the results of expert opinions (each declaring the option 'satisfactory' where z_n is now the fraction of all experts who would find the option satisfactory).

Since $y_{ni} \in \{0, 1\}$, we can define

$$\bar{y}_{n0} = \alpha$$

$$\bar{y}_{nr} = \alpha + \sum_{i=1}^{r} y_{ni} = \bar{y}_{n,r-1} + y_{nr}.$$

Then, a posteriori, the uncertainty in z_n can also be represented by a Beta distribution with parameters $\bar{y}_{nr}$ and $(\alpha + \beta + r - \bar{y}_{nr})$, i.e.,

$$E\, Z_n | \bar{y}_{nr} = \bar{y}_{nr}/(\alpha+\beta+r) = \Pr(Y_{n,r+1} = 1 | \bar{y}_{nr}).$$

Using the same boundary conditions as in Section 2, we thus have for $n < N$, $r < \rho$, $\rho \leqslant R$:

$$(5) \qquad V_{nr\rho}(\bar{y}_{n,r-1}) = \frac{\bar{y}_{n,r-1}}{\alpha+\beta+r-1}\left[\text{Max}\left\{\frac{\bar{y}_{n,r-1}+1}{\alpha+\beta+r};\ V_{n,r+1,\rho+1}(\bar{y}_{n,r-1}+1);\ V_{n+1,1,\rho+1}\right\}\right] + \frac{\alpha+\beta+r-1-\bar{y}_{n,r-1}}{\alpha+\beta+r-1}\left[\text{Max}\left\{\frac{\bar{y}_{n,r-1}}{\alpha+\beta+r};\ V_{n,r+1,\rho+1}(\bar{y}_{n,r-1});\ V_{n+1,1,\rho+1}\right\}\right].$$

Now because the state space of $\bar{y}_{nr}$ is finite, i.e., $\bar{y}_{nr} \in \{0, 1, \ldots, r\}$, it is in principle possible to compute $V_{nr\rho}(\bar{y}_{n,r-1})$ for all values of n, r, ρ, and $\bar{y}_{n,r-1}$. This would be achieved by starting for each $n < N$ with each value of s in $1 \leqslant s \leqslant R - n$, and then computing by a backward recursion for $j = 0, 1, \ldots, (s-1)$, $V_{nr\rho}(\bar{y}_{n,r-1})$ where $r = s - j$ and $\rho = R - j$ for all $\bar{y}_{n,r-1} \in \{0, 1, \ldots, r-1\}$. Note that in doing so we may discard all $V_{n,r+1,\rho+1}(\cdot)$ values after computing the $V_{nr\rho}(\cdot)$ values and storing the $u_{nr\rho}$ and $w_{nr\rho}$ values. Apart from this the only other intermediate results requiring storage are the $(R-n)$ values of $V_{n+1,1,\rho+1}$ ($\rho = n+2, n+3, \ldots, R+1$). Thus, at most, $3(R-1)$ intermediate storage positions are necessary. Finally, the policy variables $u_{nr\rho}$ and $w_{nr\rho}$ must be stored, but this need not be in-core; approximately $2NR^2$ values would be generated.

In order to achieve this computation, the following number of evaluations of (5) are required:

$$\sum_{n=1}^{N-1}\sum_{s=1}^{R-n}\sum_{j=0}^{s-1}(s-j) = \sum_{n=1}^{N-1}\sum_{s=1}^{R-n}\frac{s(s+1)}{2}$$

$$= \frac{1}{2}\sum_{n=1}^{N-1}\left[\frac{(R-n)(R-n+1)(2R-2n+1)}{6} + \frac{(R-n)(R-n+1)}{2}\right]$$

$$= \frac{1}{12}\sum_{n=1}^{N-1}[2R(R+1)(R+2) - 2(3R^2+6R+2)n + 6(R+1)n^2 - 2n^3].$$

$$= \frac{R(R+1)(R+2)(N-1)}{12} - \frac{(3R^2+6R+2)(N-1)N}{12}$$

$$+ \frac{(R+1)(N-1)N(2N-1)}{12} - \frac{N^2(N-1)^2}{24}.$$

For moderate values of N and R (e.g., 10 and 10^2, respectively,) this computation is indeed quite feasible, since evaluation of (5) is extremely simple (three additions/subtractions, six multiplications/divisions and four comparisons).

Although the solution is not very elegant, its feasibility is of interest. In particular, we consider the case in which $\alpha = \beta = 1$, i.e., the case of uniform distribution on $[0, 1]$. For a known distribution function $G(\cdot)$, Z_n can always be transformed into a variable Z_n' which is uniformly distributed. Maximizing the expected value of the Z_n' chosen is equivalent to maximizing the fractile value of Z_n' and thus of Z_n (which also minimizes the probability that one given later option will improve upon option n). This is in fact a quite satisfying objective in the general case (cf. the 'uniform game,' paragraph 5a of Gilbert and Mosteller [3]).

Now if the Y_{nr}' represent, as above, binomial trials on the uniformly distributed Z_n', then any event of the form $\{\bar{Y}_{nr}' \leqslant \xi\}$ is equivalent to the event {[estimated probability that $Z_m \leqslant z_n$, for any given $m > n$] $\leqslant \xi/(r+2)$}.

For general observations y_m on z_n this probability can also be computed: it is in fact given by $\bar{\eta}_m$ where

$$\bar{\eta}_m = \int_{\mathbb{R}^1}\{1 - F_m(z|\bar{y}_m)\}dF(z). \tag{6}$$

(It is easily confirmed that in the uniform/binomial case $\bar{\eta}_m' = \bar{y}_m/(r+2)$.) This suggests that a heuristic policy may be defined by the following: where $u_{nr\rho}'$ and $w_{nr\rho}'$ are the critical values computed in the binomial case and with $\alpha = \beta = 1$,

(i) option n is rejected if $\bar{\eta}_{nr} \leqslant u'_{nrp}/(r+2)$;

(ii) option n is accepted if $\bar{\eta}_{nr} > w'_{nrp}/(r+2)$;

(iii) a further sample is taken otherwise.

5. MONTE CARLO RESULTS

In order to test the heuristic suggested at the end of the last section, a series of Monte Carlo studies were performed. The example selected for these studies was chosen more to demonstrate performance under fairly extreme conditions than to represent any particular practical situation. We give first a description of this example.

The values Z_n are drawn from an inverse gamma distribution with shape parameter 3 and scale parameter 1, i.e., if we define $X_n = 1/Z_n$, then the distribution of X_n has density $\phi(x)$:

$$\phi(x) = \frac{1}{2} x^2 e^{-x}.$$

The observations Y_m are drawn from the exponential distribution with mean Z_n, i.e.,

$$g(y|z_n) = [\exp(-y/z_n)]/z_n.$$

A sufficient statistic for z_n is thus

$$\bar{y}_m = \left[\sum_{j=1}^{r} y_{nj}\right].$$

It is not difficult to show that (A1) (in fact (A2)) holds. The value of $\bar{\eta}_{nr}$ of the previous section is easily obtained, although in a form slightly different from that of Equation (6). We note first that the posterior distribution of x_n (note, not z_n) is also of the gamma form with density defined as $\phi_{nr}(x)$ given by:

$$\phi_{nr}(x) = \frac{(\bar{y}_{nr}+1)^{3+r} x^{2+r} \exp[-(\bar{y}_{nr}+1)x]}{\Gamma(3+r)}.$$

The probability that $Z_m \leqslant z_n$ for any given $m > n$, is equal to the corresponding probability that $X_m \geqslant x_n$. Conditional on x_n (i.e., on z_n) this probability is given by

$$\Pr[X_m \geqslant X_n | X_n = x_n] = \int_{x_n}^{\infty} \frac{1}{2} x^2 e^{-x} dx = \left[1 + x_n + \frac{1}{2} x_n^2\right] \exp(-x_n).$$

Taking the expectation of this conditional probability with respect to the posterior distribution for X_n conditional on $\bar{y}_{nr}$, we obtain the required expression for $\bar{\eta}_{nr}$ as follows:

$$\bar{\eta}_{nr} = \int_0^{\infty} \frac{(\bar{y}_{nr}+1)^{3+r} \exp[-(\bar{y}_{nr}+2)x]}{\Gamma(3+r)} \left[x^{2+r} + x^{3+r} + \frac{1}{2} x^{4+r}\right] dx$$

$$= \left[\frac{\bar{y}_{nr}+1}{\bar{y}_{nr}+2}\right]^{3+r} \left[1 + \frac{3+r}{\bar{y}_{nr}+2} + \frac{(3+r)(4+r)}{2(\bar{y}_{nr}+2)^2}\right].$$

With these results the proposed heuristic can be applied.

In the studies performed, $R = 10N$ throughout. Now it is important to note that simply to take 10 observations on each value, and to use the mean of these as the true value in one of standard models (as, for instance, in Gilbert and Mosteller [3]) can lead to substantial errors. For example, around the mean value of X_n (i.e., 3) the variance of the estimate of z_n is 0.03333, and the one standard deviation limits of this estimate is [0.1507; 0.5159]. In fact, this range within which the estimate is most likely to fall, covers approximately 65% of the probability mass of Z_n. Thus, even if all N options could be seen simultaneously, in ordering the options quite considerable errors could occur.

The Monte Carlo studies with the heuristic rule were carried out for four values of N, viz. 4, 6, 8 and 10. In each case 1,000 sets of N values were generated by a standard pseudo-random number generator. The cutoff points u'_{nrp} and w'_{nrp} had been computed previously and stored. For each set the rule was applied, and the rank and value of z_n for the option chosen was recorded. In this way the following comparisons could be made with optimal performance in perfect information models:

(a) Mean fractile of the option selected (i.e., the expected probability that the chosen option exceeds a random draw of Z_m; if Z_n is transformed to a random variable, U_n say, which is uniformly distributed on [0, 1] then this corresponds to EU_n for the option selected). The observed mean can be compared with the corresponding optimal expected value without observation error ([3], Table 11). It may be recalled that maximization of this value is the objective invoked as a basis for the heuristic.

(b) Frequency of selection of the best option. This can be compared with corresponding optimal probabilities in the standard cases with no observation error when the underlying distribution is either known ([3], *Equation (3c-1) and Tables 7 and 8) or* unknown (and ranking only is possible; [3] Table 2).

(c) Mean rank of the option chosen (where 1 = best). This can be compared with the corresponding optimal expected rank, if ranking only is allowed ([1]. Equations (6) and (7)).

These comparisons are shown in Table 1

Clearly the heuristic rule compares well with optimal rules when there are no observation errors. In particular, the rule performs substantially better than the optimal rules based on error-free ranking of the options only. Of more importance, however, is the fact that performance compares favourably with optimal rules based on full information on the underlying distribution and perfect information on the values in hand. This advantage is particularly evident for the mean fractile of the option chosen, the basic objective of the heuristic.

6. CONCLUDING COMMENTS

We have introduced a class of sequential decision problems which generalize the "secretary" or "dowry" problems. Apart from general solution properties, an exact solution has been found in principle for a particular case. What is more important, it has been shown that the solution in this case can form the basis of heuristic rule for more general situations, which rule has been validated by means of a Monte Carlo study. Admittedly, the rule is computationally convenient only for rather small numbers of options, but it must be noted that the motivating

TABLE 1 — *Comparison of Performance of the Heuristic Rule and Optimal Performance with Error-Free Information*

	Number of options (N)			
	4	6	8	10
Mean fractile of the option selected:				
estimate, case H	0,727	0,786	0,818	0,853
(standard deviation of estimate)	(0,007)	(0,006)	(0,006)	(0,005)
optimal expected value, case O_1	0,742	0,800	0,836	0,861
Probability of selecting the best option:				
observed frequency, case H	0,607	0,580	0,516	0,526
(estimated standard deviation of frequency)	(0,015)	(0,016)	(0,016)	(0,016)
optimal probability, case O_1	0,655	0,629	0,616	0,609
optimal probability, case O_2	0,458	0,428	0,410	0,399
Mean rank of option selected:				
estimate, case H	1,587	1,734	1,921	1,989
(standard deviation of estimate)	(0,027)	(0,035)	(0,043)	(0,049)
optimal expected rank, case O_2	1,875	2,217	2,400	2,558

Case H: heuristic rule of Section 4, estimates based on the Monte Carlo study
Case O_1: optimal rule given perfect information on values drawn from a known distribution
Case O_2: optimal rule given perfect information only on ranks of the observed options

examples in the introduction (in which costly examination of each option is required) will generally involve rather few options. In these cases the model and solution would appear to have some applicability.

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A DIFFERENTIAL EQUATION MODEL OF SEARCH FOR RANDOMLY ARRIVING AND DEPARTING TARGETS

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ABSTRACT

In this paper we analyze optimal search strategies in an environment in which multiple, independent targets arrive and depart at random. The analysis revolves around a continuous time differential equation model which captures the time dependent nature of the search process. We explore the impact on optimal strategies of nonzero travel times between regions as well as differing target arrival rates. We derive simple closed form expressions for determining if only one region should be searched.

INTRODUCTION

In this paper we present a differential equation model used to explore optimal search strategies in an environment which involves:

1. multiple independent targets
2. random arrival of targets
3. random departure of targets
4. different target types (mean duration)
5. time lost in travel between noncontiguous regions.

A wide range of target types exhibit characteristics similar to some or all of the above. This research evolved initially from research into crimes but has application to the following target types, a) smugglers, b) terrorists, c) transiting submarines, d) infiltrating enemy forces and e) groups of attacking aircraft.

After a brief literature review (see [13], [15], and [17] for more extensive reviews), we introduce the basic differential equation model and apply it to a series of increasingly more complex problems. The analysis considers only cyclical search strategies and focuses on the two region problem. Our goals are: (1) to determine the optimal cyclical strategy, and (2) to develop simple analytic expressions for when a region should be excluded from search.

The problems we address are as follows:

(1) Two regions with *equal* target arrival rates, *no time lost* in travel between regions.

We determine for this problem the optimal cycle length and the magnitude of the impact of using optimal search strategies.

(2) Two regions with *equal* target arrival rates, *time lost* in travel between regions.

For this problem we again determine the optimal cycle length and also develop an analytic expression for when to search only one region. This expression is a function of the travel time, and the departure and detection rates of the targets, but not the arrival rate.

(3) Two regions with *unequal* target arrival rates, *no time lost* in travel between regions.

We determine optimal cyclical policies and an analytic expression for when to search only the high target region which is a function of only the ratio of the departure to the detection rate. In addition, we explore the impact of constraining the minimum time to search a region.

In all of the above examples we consider only a single target type and only two regions. We close with a brief discussion of:

(4) Two regions with multiple target types.

(5) More than two regions.

Throughout the paper we evaluate the limiting properties of many of the equations and, for example, study the impact of infinitesimally short cycle times. Although these cycle lengths are not achievable in the real world, the results provide important bounds on the difference between optimal and nonoptimal strategies.

1. LITERATURE REVIEW

The pioneering work by Koopman [9], [10], and [11], along with the work in the discrete analog by Charnes and Cooper [3], has laid the foundation for the continuing expansion of search theory. The later work which is most directly applicable to our area are those papers which explore the sequencing of search effort. Blackman and Proschan [2] analyzed the optimal sequencing of search effort among a series of regions into which targets arrive randomly. Gilbert [6], and later, Kisi [8] analyzed a two-box search problem with the objective to minimize the expected time until detection. However, their work is not extendable to situations with departing targets [4]. Moore [14] analyzed the search for one randomly arriving and departing target in n regions solely from the perspective of how much search effort to allocate to each region, but not the sequencing problem. The paper closest in orientation to this work is that of Barnett [1]. In his problem, targets arrive in a Poisson process and survive for a random duration. Barnett, in considering a discretized search, (1) proves optimal strategies are cyclical, a fact we use in applying the differential equation model, (2) finds, analytically, the optimal

sequence of search, and (3) generates a sufficient condition for excluding regions from the optimal search strategy. We develop analogous results while introducing a more flexible continuous search problem which allows for travel time between regions and the possibility of searching a region and not discovering a target that is actually present.

2. DIFFERENTIAL EQUATION MODEL

The model consists of two differential equations for each region, one to describe the system when no search is in progress in that region, the other to describe the system when a search is in progress.

Let S = the state of the system (*expected value* of the number of targets present).

A = the rate at which targets arrive (assumed independent of the state of the system).

Targets depart from the system in either one of two ways. Either targets leave because they have finished their task or because they have been intercepted during a period of search.

F = the constant of proportionality for the rate at which targets finish their tasks. (This will depend upon the duration of their task.)

I = the constant of proportionality for the target interception rate. (This will depend on the size of the region and the *observability* of the target.)

Using the above parameters (S, A, F, I), we can write the following equations to describe the changing system. During a period of no search, the system is described by

$$\frac{dS}{dt} = S' = A - FS. \tag{1}$$

The equation states that the system is changing because targets are arriving at a rate A and are departing, as a result of finishing, at a rate FS, proportional to the state of the system. The solution of the differential equation is

$$S = C_1 \exp(-Ft) + A/F. \tag{2}$$

A/F represents the steady state number of targets if no search were carried out.

During a period of search, the equation becomes

$$\frac{dS}{dt} = A - (F + I)S \tag{3}$$

whose solution is

$$S = C_2 \exp(-(F + I)t) + A/(F + I). \tag{4}$$

$A/(F + I)$ is the steady state number of targets during an unending period of search.

These equations assume that the arrival process of targets is Poisson and that the duration of a target's presence is exponentially distributed. S is therefore the *expected* value of the number of targets. This can be shown by setting up queueing type equations for the probabilities P_n, where n is the number of targets present at any one time.

3. TWO REGIONS—EQUAL TARGET ARRIVAL RATES—NO TIME LOST IN TRANSFER

The first problem to which we apply the differential equation involves two identical regions. For now we will focus on the optimal cycle length and consider only equal allocations of search effort to the two regions; later, we will show that in fact the global optimal strategy is symmetric. In addition, we will consider only simple cyclical strategies of the form X minutes of search in $R1$, followed by X minutes in $R2$ and then back to R1 for X minutes, and so on. This second assumption is motivated by a number of earlier search theory papers [1, 6, 8] in which the optimal search strategies were cyclical.

Our objective will be to minimize the total average expected number of targets in the combined regions. However, because the regions generate independent target arrivals and departures, they will be treated separately as experiencing alternating periods of search and no search. In addressing this problem we will, in addition, answer simultaneously the following question:

Given that a searcher is unavailable 50% of the time, what is the distinction between short numerous intervals of search and a few long intervals of search?

The three steps common to this and the later examples are outlined as follows:

1. Two differential equations are defined for each region. One to describe the dynamics of the region while it is being searched, the other to describe the region while no search is going on.

2. For each region, two boundary conditions of the following form are established. The target level in each region at the end of a period of search must equal the level at the beginning of a period of no search and vice versa. These conditions are simply a continuity constraint on S, the number of targets present, not allowing discrete shifts in the values of S as a result of the searcher entering or departing a region. Once the boundary conditions are defined, each of the constants in the differential equations is determined.

3. The average target level, $\bar{S}$, in the total area is calculated by integrating each of the four equations over its respective period of search or no search, summing the four values and dividing by the cycle length. The resultant expression for $\bar{S}$ is a function of the parameters of interest in each of the particular examples and is subsequently analyzed to determine an optimal strategy.

In both regions, $R1$ and $R2$, the two equations describing the periods of search and no search are just the previously defined Equations (1) and (3) and whose solution are Equations (2) and (4).

The continuity constrainton S is used to generate the following equation:

$$C_2 \exp(-(F+I)X) + A/(F+I) = C_1 + A/F. \tag{5}$$

The left-hand side represents the end of a search period of duration X minutes and the right-hand side the beginning of a no-search period. A second analogous equation can be set up for the end of a period of no search and the beginning of a period of search. These equations are then solved to find C_1 and C_2 as functions of X. The expected target level in each of the regions at any instant in time can now be written as an integral function of only one parameter, X, the half-cycle length, which yields

$$\overline{S1} = -(C_1/(2FX)) \times (1 - \exp(-FX)) + A/2F - (C_2/2(F + I)X)) \times (1 - \exp(-(F + I)X) + A/2(F + I). \tag{6}$$

The average for the entire region, $\bar{S}$, is merely twice that of any individual region.

With some algebraic manipulation of the derivative, we have proved [4] that $\bar{S}$ increases monotonically as X increases. In other words, the average expected target level in each region decreases as the frequency of transfers between the two regions increases. This result is directly comparable to that of Gilbert [6]. This result also implies that if search were scheduled in long blocks of time (i.e., increase X), and no penalties were incurred for switching from search to no search the model predicts a decrease in the probability of interception.

Magnitude of Impact of Short Cycles

To determine the magnitude of improvement produced by the shorter cycles, we determined the average number of targets present as the cycle length, X, approaches infinity.

$$\overline{S1} = 1/2(A/F) + 1/2(A/(F + I)) = A(2F + I)/2F(F + I) \tag{7}$$

For short cycles as X approaches zero, Equation (6) becomes $2A/(2F + I)$ and thus the reduction of the average target level for short cycles as compared to long cycles is

$$1/(2F/I + 1)^2. \tag{8}$$

This expression, not surprisingly, does not depend on the arrival rate, A, and depends only on the *ratio* of the departure rate, F, to the interception rate, I. Thus, for example, if the departure and detection rates were equal, infinitesimally short cycles would reduce the target level by 1/9 as compared to long cycles. Table 1 summarizes the reduction for a range of ratios. As the departure rate grows (shorter mean duration) relative to the interception rate, the impact of shorter cycles decreases rapidly.

If, as an alternative, we measure the relative impact of long and short cycles against a no search policy then the reduction due to infinitely long searches is $I/2(F + I)$ and for short cycles it is $I/(2F + I)$. If, for example, $I = F$ then short cycles reduce the target level by 1/3 more than long cycles. The probability of interception under each policy is directly related to this proportional reduction in target level. For infinitely long cycles the fraction of targets intercepted while searching is just $I/(F + I)$ but we search only *half* of the time, $(I/2(F + I)$.

For infinitesimally short cycles during any target's lifetime, the searcher and target will be in the same region half of the time. Consequently, the probability of interception is

$$\int_0^\infty F \exp(-Ft) \times (1 - \exp(-It/2))\, dt. \tag{9}$$

This yields $[I/(2F + I)]$, which represents the reduction in target level for infinitesimally short cycles.

TABLE 1 — *A Comparison of Short and Long Cycles With Regard to Interception Probabilities and the Average Expected Number of Targets Present*

$R = F/I$	Reduction in Target Level Due to Short Cycles	Probability of Interception	
	$1 - S_s/S_l$	Short Cycles	Long Cycles
1	.111	.333	.250
2	.040	.200	.167
3	.020	.143	.125
4	.0123	.111	.100
5	.0083	.091	.083
10	.0023	.0476	.0455
20	.00059	.0244	.0238
30	.00027	.0164	.0161
40	.00015	.0123	.0122
50	.000098	.0099	.0098

4. TWO REGIONS - EQUAL - TARGET RATES - TIME LOST IN TRANSFER

In this section we modify the previous example by introducing a parameter L, which represents the time lost from search while traveling between the two regions. We will use the differential equation model to address two questions.

1. What is the optimal value of X, the time spent in $R1$ before switching to $R2$? The optimal value is obviously no longer infinitesimal.

2. Is there a simple analytic expression which specifies for which values of L it does not even pay to switch regions?

In tackling the problem with the differential equation model, we will again assume a cyclical strategy (both Kisi [8] and Gilbert [6] have cyclic strategies) and that the search effort will be divided equally between the two regions.

The key to our analysis is that optimal solutions which limit the search to only one region will appear in the analysis as solutions in which the optimal value of X is infinite.

In this second problem, which ascribes a penalty for switching regions, the basic equations which describe the periods of search and no search do not vary from those of the previous example (Equations 2 and 4). The introduction of the switching time, L, does affect the continuity boundary conditions and changes C_1 and C_2 since a no search period in each region has a duration of '$X + 2L$' minutes. The average $S1$ is calculated as in Equation (6) except that the no search time period extends over a time period of $X + 2L$ minutes and that the cycle length is now $2X + 2L$. The resultant expression is

$$\overline{S1} = A/F - [AIX/2(X + L)F(F + I)] + [A/2(X + L)] \times [I/F(F + I)]^2 \times \left[1 + \frac{2 - \exp(-(I + F)X) - \exp(-(X + 2L)F)}{\exp(-(2F + I)X - 2FL) - 1}\right]. \tag{10}$$

Once values have been assigned to F, I and L, this equation is a function of only one variable, X, whose optimum is easily determined. In Figure 1 the optimum value of X is shown for a range of L, F and R, where R is the ratio of F, the departure rate, to I, the interception rate. There are graphs for two values of F, one for F equal to 20—targets last an average of three minutes—and one for F equal to 10, with R as high as 20, and as low as .01, which would mean that targets are intercepted at a rate one hundred times as fast as they depart on their own. The lower values for R were included less for realism than to display how the curves behave as R approaches the limit of zero.

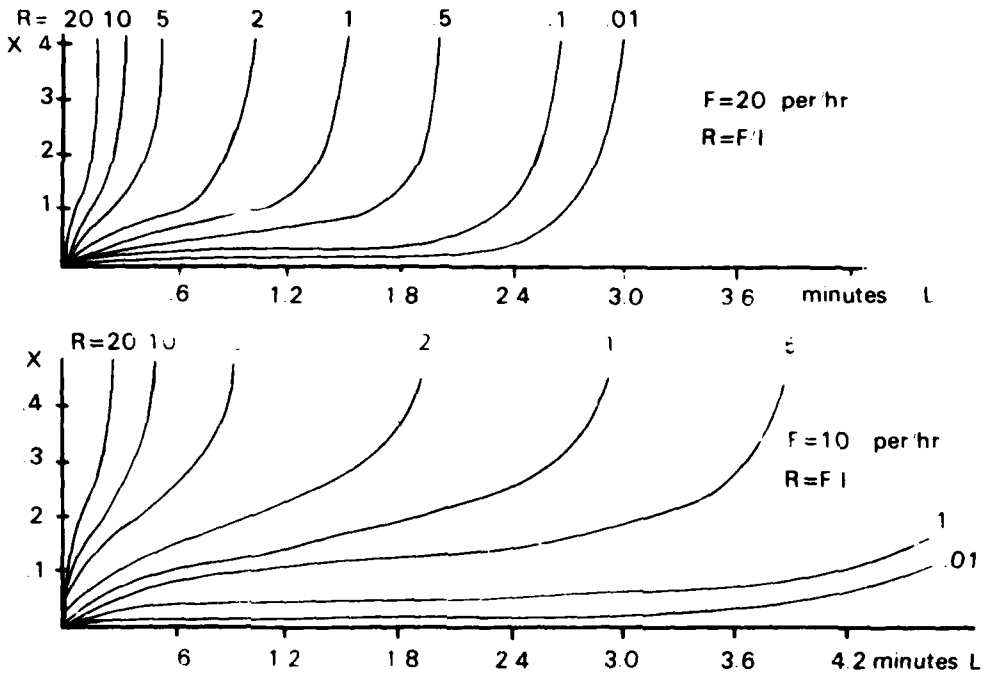

FIGURE 1. Optimal value of X as a function of L

The most striking characteristic of these curves is their asymptotic nature and the location of the asymptote.

The asymptote is always less than the mean duration of the target, $1/F$, and approaches the mean duration as R approaches 0. In addition, for a given value of L as the ratio, R, decreases, the optimum X also decreases; as F decreases (from 20 to 10) the value of the asymptote for a given R increases.

All of the above points will be addressed more formally in the succeeding section in which an analytic expression for the asymptote as a function of F and R is derived.

When to Search Only One Region

The development of an analytic expression for the asymptote as a function of F and R builds on the observation that $\overline{S1}$, the average target level will either be (1) a monotonically

decreasing function of X, or (2) a unimodal function. We have no formal proof of this but extensive computer analysis of $\overline{S1}$ for a wide range of values of F, R and X supports this claim. In both cases the limiting value as X tends to infinity is the same. Because of the complexity of the derivative, we found that the simplest way to determine monotonicity was to explore the limiting behavior of the derivative. If the derivative approached zero from the negative direction as X increased to infinity then the function was monotonic and the optimal strategy was to search only one region. It can be shown that after excluding all components of the derivative which approach zero faster than $1/(X + L)^2$ and consider only the direction of the derivative, we find the expression of interest to be

$$\lim_{X\to\infty} -L - \frac{I}{F(F+I)} \times \left\{1 - \frac{1}{D}\left[2 - \exp(-(I+F)X) - \exp(-F(2L+X))\right]\right\} \tag{11}$$

with $D = \exp(-2F(X + L) - IX) - 1$.

Its limit is

$$-L - I/F(I + F) + 2I/F(I + F) = -L + [1/F] \times [1/((F/I) + 1)]. \tag{12}$$

This will be negative and thus the optimal value of X infinite whenever

$$L \geqslant [1/F] \times [1/((F/I) + 1)]. \tag{13}$$

This confirms our earlier graphical analysis. The second component $[1/((F/I) + 1)]$ is always less than 1 and approaches 1 when R, the ratio of F to I, approaches 0. Thus, the asymptote for each curve in Figure 1 is less than the mean target duration, $1/F$, and approaches the mean target duration as R approaches 0. Of perhaps greater significance is that even if L is orders of magnitude shorter than the mean duration, the optimal strategy might be to search only one region. For example, assume targets depart at nine times the rate they are detected. If the travel time is more than one-tenth the average duration of a target (e.g., 12 seconds for two minute targets, 18 seconds for three minute ones), the optimal strategy is to search only one region.

5. DIFFERING TARGET ARRIVAL RATES

The final example involves two regions, RH and RL. In RL targets arrive at a rate A and in RH, the high target region, targets arrive at a rate $M \times A$, with M greater than 1. In both regions all targets have the same departure and interception rates, F and I, respectively. Once again, our analysis will revolve about cyclic policies, this time of the form, X minutes in RL followed by $K \times X$ minutes in RH.

Although one problem of interest is the finding of the (K, X) pair which minimizes the average expected number of targets in progress, the discussion will not be limited to that, since in all instances the optimum is approached as X tends towards zero, an unimplementable optimum. Therefore, the development will also address the issue of the optimum value of K for a given value of X. Setting X to be a specified value is interpretable as establishing a feasibility constraint on the search process in RL. We have not constrained K to be greater than 1. If K is less than one, less search time is spent in the high target region.

The discussion that will follow, then, can be categorized briefly as:

1. A single expression is developed to be used in obtaining the optimal value of K for a given X.

2. An analytic expression is found for the optimal K value as X approaches zero.

3. Evolving directly from 2, is an expression for identifying values of $\underline{M}$ for which the optimal K value is infinite for all values of X. (If the optimal K value is infinite then the optimal strategy limits search to the high target region.)

4. Lastly, an expression is found which specifies the values of $\underline{X}$, as a function of M, F, and I, for which the optimal K is again infinite.

In this example, the regions are not symmetric and separate differential equations are needed for each region. The equations for the expected number of targets in the low target region are the same as before, Equations (2) and (4). The differential equations for the high target region and their solutions are also basically the same as before except for replacing A, the arrival rate, by $M \times A$. We then set up the standard boundary conditions which constrain the target level to be continuous and solve for their constants, C_1, C_2, C_3, and C_4. In this case a cycle consists of KX minutes of search in RH followed by X minutes of search in RL.

The average expected number of targets present in the combined two-region area is then determined with the following equation

$$\begin{aligned}(14)\qquad \bar{S} = {} & \frac{1}{(K+1)X}\int_0^{KX}[C_1\exp(-Ft) + A/F \\ & + C_4\exp(-(F+I)t) + MA/(I+F)]dt \\ & + \frac{1}{(K+1)X}\int_0^{X}[C_2\exp(-(F+I)t) \\ & + A/(I+F) + C_3\exp(-Ft) + MA/F]dt.\end{aligned}$$

The first integral represents the average number of targets in both regions during the search (KX minutes) of RH and the second, the average during the search of RL. After integrating out, substituting where necessary, and combining terms wherever possible, the result is

$$\begin{aligned}(15)\qquad \bar{S} = {} & \frac{MA}{F+I} + \frac{A}{F} + \frac{AI(M-1)}{(K+1)F(F+I)} - \left[\frac{1}{(K+1)X}\right] \times \left[\frac{AI^2}{F^2(F+I)^2}\right] \\ & \times \left[\frac{M(1-\exp(-FX))\,(1-\exp(-(F+I)KX))}{1-\exp(-(F(K+1)+IK)X)}\right. \\ & \left. + \frac{(1-\exp(-FKX))\,(1-\exp(-(F+I)X))}{1-\exp(-(F(K+1)+I)X)}\right].\end{aligned}$$

Our analysis of this expression is delayed until later; for now we present just a graphical analysis. In Figures 2 and 3 we have graphed the optimal value of K as a function of X, the duration of a visit to RL. In each case targets are of a 5 minute duration, $F = 12$. Figure 3 graphs the relationship for a range of M (the ratio of high to low region target rates) with the ratio F to I set at 1. A common characteristic of the curves, except for the M equal to one curve, is that as X increases, the optimal K value grows asymptotically to infinity (i.e., search only one region). Not surprisingly, as M increases the asymptote decreases. As the difference in target arrival rates increases, the optimal strategy is more and more likely to focus only on the high target region.

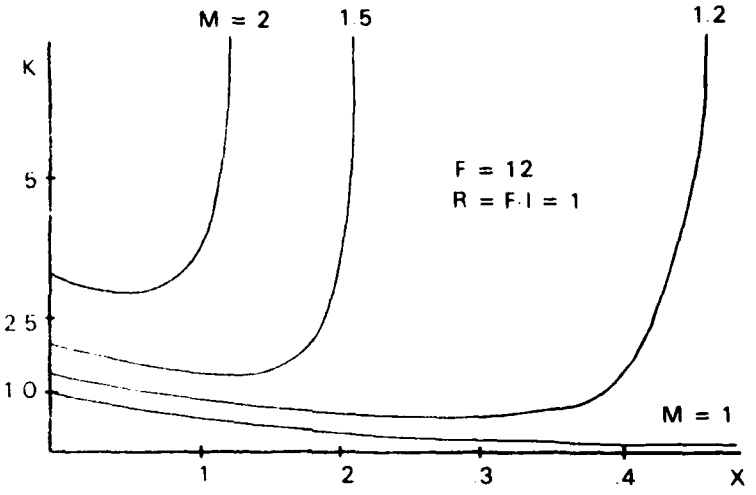

FIGURE 2. The optimal value of K as a function for a range of M values

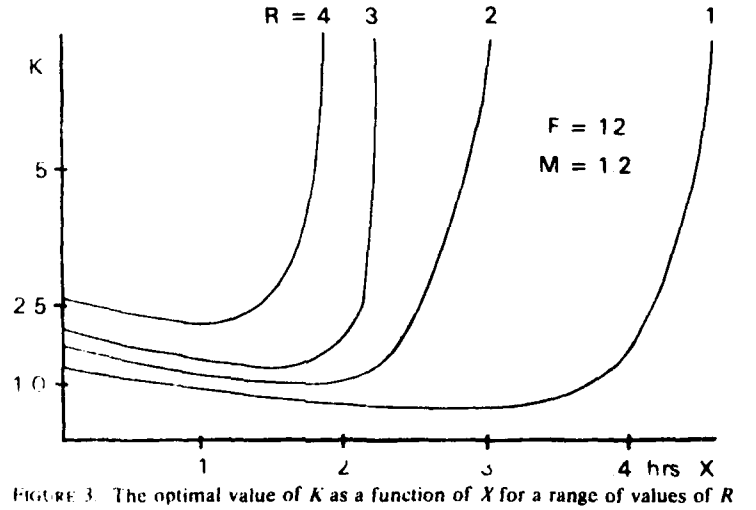

FIGURE 3. The optimal value of K as a function of X for a range of values of R

Figure 3 graphs K as a function of X for a range of R with M set at 1.2. As R increases, the cutoff value of X, above which only RH is searched, decreases. For $R = 1$ the cutoff value of X is approximately .43 hours and as R increases to 4 this becomes .18 hours.

One characteristic of the curves that seems counterintuitive is that as X increases initially, the optimal value of K decreases. A corollary of this is that the optimal strategy within the constraints we have imposed may allocate less than half the search effort to the high target region. For identical regions the optimal strategy divides the search equally only in the limit as X goes to zero.

There seem to be two conflicting forces at work. The more intuitive one is that as X increases the searcher is forced to incur increased diminishing return in any visit to the low target region and eventually the cost for visiting the region RL becomes so prohibitive that it no longer pays to search the region. This force dominates in the long run. However, there is an advantage, in general, to having short cycles and as X increases from 0 there is only one way of restraining the increase in cycle length and that is by decreasing K. This propensity for shorter cycles seems to dominate the behavior of the optimal value of K for small values of X turning K optimal initially into a decreasing function of X.

No Constraint on the Duration of a Search

In this section we focus on the limiting behavior of $\bar{S}$ as X approaches zero (i.e., no constraint on the minimum duration of a visit to a region). Although as X approaches zero the optimal solutions are no longer implementable, the analysis of the limit properties will prove significant, because the absolute minimum of $\bar{S}$ is always approached as X goes to zero. The limit of $\bar{S}$ as X approaches zero is

$$\lim_{X \to 0} \bar{S} = \frac{MA}{I+F} + \frac{A}{F} + \frac{AIM}{(F+I)(F(K+1)+IK)} - \frac{AI}{F(F(K+1)+I)} \tag{16}$$

The optimal value of K is

$$K \text{ (optimal)} = \frac{+2R(R+1)(M-1) \pm 2(2R+1)M^{1/2}}{2(R+1)^2 - 2MR^2} \tag{17}$$

where $R = F/I$. It is again clear that the optimal value for K does not depend upon the independent values of F and I but only on their ratio, thereby reducing, once again, the number of critical parameters. Table 2 displays the optimal K for a range of M and R. For M equal to one, which means that the two regions generate targets at the same rate, the optimal value of K does not depend upon R but is always one (i.e., equal search in both regions), as was assumed earlier in this paper.

One aspect of the Equation (17) that needs clarification is that since there are two solutions is it possible for both solutions to be positive? As it turns out, this can never happen. For the range of M which results in both values of the numerator being positive, the corresponding denominator will be negative. If the numerator is negative the denominator is positive. This last result provides the key to finding under what conditions the optimal solution is to search only one region.

Search Only One Region

Whenever both solutions are negative, for M greater than one, the optimal solution will be at the upper bound of the feasible region, namely K optimal will be infinite (i.e., search only one region). Consequently, the optimal search focuses on only one region whenever

TABLE 2 — *The Optimal Value of K as a Function of M and R — No Constraint on Minimum Duration of a Visit to RL (X → 0)*

M \ R	.1	.5	1	2	5	10	20	30	40
1	1	1	1	1	1	1	1	1	1
1.1	1.06	1.10	1.15	1.27	1.71	3.00	85.0	∞	∞
1.2	1.12	1.20	1.32	1.59	3.01	45.0	∞	∞	∞
1.3	1.17	1.30	1.49	1.97	6.16	∞	∞	∞	∞
1.4	1.22	1.40	1.67	2.45	25.0	∞	∞	∞	∞
1.5	1.28	1.51	1.87	3.04	∞	∞	∞	∞	∞
2	1.52	2.05	3.12	13.1	∞	∞	∞	∞	∞
2.5	1.74	2.64	5.16	∞	∞	∞	∞	∞	∞
3	1.95	3.31	9.20	∞	∞	∞	∞	∞	∞
3.5	2.15	4.09	21.2	∞	∞	∞	∞	∞	∞
4	2.33	5.00	∞	∞	∞	∞	∞	∞	∞

$$M > \frac{(R+1)^2}{R^2}. \tag{18}$$

To be rigorous, we showed only that in the limit as X approaches zero that the inequality specifies whether or not the low target region should be searched. However, if it does not pay to ever search the low target region for an infinitesimally short duration there should be no incentive to search it if the minimum duration of a visit is instead some number larger than zero.

In order to emphasize the impact of this result we display in Figure 4 the convex region in R, M space for which the optimal strategy limits the search to RH. For example, if R, the ratio of departure to interception, were 10 then if RH generated 21 percent more targets than RL, only RH would be searched. If R is 20, 30, or 40, then if RH generated respectively 10%, 7%, or 5% more targets than RL, only the high target region should be searched. The result is a second strong limitation on the likelihood of searching more than one region.

In the preceding analysis, we showed that for given values of F and I, if the ratio of target arrival rates is above some cutoff value, M, the optimal strategy is to search only RH even for infinitesimally small values of X. However, even for values of M below the cutoff if X, the time to search RL, is 'too large,' the optimal strategy will again restrict the search to RH. Therefore, the next question to resolve is, "When is X too large?"

In Equation (15) only two components are functions of K, with only the negative component a function also of X. The two components are

$$\frac{1}{K+1} \times \left[\frac{AI(M-1)}{F(F+I)}\right] \tag{19}$$

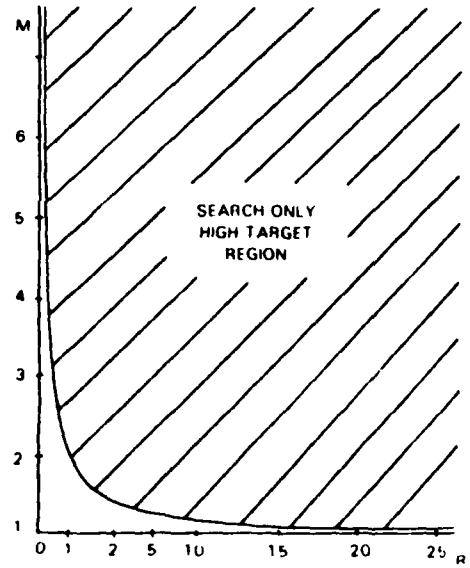

FIGURE 4. A characterization of when to search only high target region as a function of two parameters

and

$$(20)\qquad \frac{-1}{K+1}\times\left[\frac{AI^2}{XF^2(F+I)^2}\right]\times\left[M(1-\exp(-FX))\times\frac{(1-\exp(-(F+I)KX))}{(1-\exp(-(F(K+1)+I)X))}\right.$$
$$\left.+(1-\exp(-(F+I)X))\times\frac{(1-\exp(-FKX))}{(1-\exp(-(F(K+1)+I)X))}\right].$$

If for some value of X the magnitude of the first component were greater than that of the second for all values of K, then their sum would always be positive. The optimal strategy then would be to allow K to go to infinity, since that would drive each component to zero.

In comparing their relative magnitudes, we ignore the common factor $1/(K+1)$; this turns the second expression into a monotonically increasing function of K. Its limit, as K grows without bound, is

$$(21)\qquad \frac{AI^2}{XF^2(F+I)^2}\times[M(1-\exp(-FX))+1-\exp(-(F+I)X)].$$

The problem therefore reduces to finding values of X such that

$$(22)\qquad X\geqslant\frac{I}{(M-1)F(F+I)}\times[M(1-\exp(-FX))+1-\exp(-(F+I)X)].$$

In Figure 5 we have graphed the lower bound on X for a range of M values with F and I set equal to 1. A conservative (sufficient but not necessary) bound on X can be easily found since $[1-\exp(-FX)]$ and $[1-\exp(-(F+I)X)]$ are both bounded by 1. This bound is

$$(23)\qquad X\geqslant\frac{M+1}{M-1}\times\frac{I}{F(F+I)}$$

which approaches the actual bound as M approaches 1. See Figure 5 for a comparison with the actual lower limits.

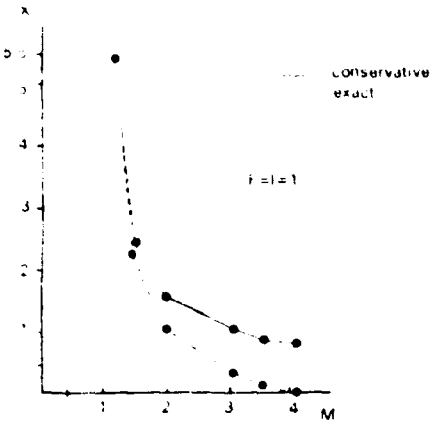

FIGURE 5. Exact and conservative estimates of the cutoff values for X for a range of M. If X is above the cutoff, search is limited to the high target region.

6. EXTENSIONS AND SUMMARY

Multiple Target Types

In order to inclue the multiple target types in the model, it is necessary to add for each target type different values of I and F and a pair of differential equations for each region. To calculate the state of the system, S, under a particular search strategy, each of the targets is initially treated separately. The total average number of targets in progress is then just the sum of all the individual averages. The resultant expression is still a function of only X or K and X, and can be analyzed in the same manner as before. To illustrate this without going into much detail, we generalize our earlier results which characterize when to search only one region.

The first expression we derived involves two identical regions with time lost, L, in traveling between the regions. We associate with each target type, j, three parameters, I_j, F_j, and A_j; the target specific interception rate, departure rate and arrival rate, respectively. The resultant expression for L is now

$$L \geqslant \frac{\sum_j A_j \left[\frac{I_j}{F_j(F_j + I_j)}\right]^2}{\sum_j A_j \left[\frac{I_j}{F_j(F_j + I_j)}\right]} \tag{24}$$

which unlike the earlier expression (13) does depend upon the arrival rates.

The last two expressions, one for M, the other for X, relate to the problem of allocating search between a high and low target region. When we generalized this problem and allowed for multiple target types, we also indexed the ratio of the two region target arrival rates, M_j, and made it target type specific. The expression which corresponds to Equation (18) and specifies when to search only the high target region even for small values of X is

$$\sum_j M_j \left[\frac{A_j I_j}{(F_j + I_j)^2}\right] \geqslant \sum_j \frac{A_j I_j}{F_j^2}. \tag{25}$$

If the above inequality is not satisfied, large values of X will still restrict search to the high target region. The conservative form of this equation with multiple target types is

$$X \geq \frac{\sum_i \frac{A_i I_i^2 (M_i + 1)}{F_i (F_i + I_i)}}{\sum_i \frac{A_i I_i (M_i - 1)}{F_i^2 (F_i + I_i)^2}}. \tag{26}$$

More Than Two Regions

Introducing more than two regions into the problem, significantly increases the complexity of analysis. Although any of the earlier three inequalities can be applied pairwise to all the regions to exclude as many as possible, the complications enter when more than two regions are still left. The heart of the problem lies in the difficulty of defining cycle types to be analyzed. The work by Barnett [1] which tackles problems involving more than two regions can yield no insight here. In his work any search of a region discovers all targets present with probability 1. The consequence of this is that after one unit of search in a region, diminishing return from a continuous search stops, which is not the case here.

Summary

In this paper a differential equation model of a search process was developed which has the potential for capturing the dynamics of a sequential cyclic search strategy. In displaying the application of the model to a number of examples, we have derived a number of independent quantifiable constraints which limit the number of regions to be searched and which are summarized in Table 3. The main thrust of all our results strongly emphasize that with randomly arriving and departing targets and travel times, search should be limited to only the highest target region.

TABLE 3 — *Summary of Conditions Under Which Only One Region is Searched*

Single Target Type	Multiple Target Types
$L \geq \frac{1}{F \times ((F + I) + 1)}$	$L \geq \frac{\sum_i A_i \left\lfloor \frac{I_i}{F_i (F_i + I_i)} \right\rfloor}{\sum_i A_i \left\lfloor \frac{I_i}{F_i (F_i + I_i)} \right\rfloor}$
$M \geq \frac{(R + 1)^2}{R^2}$	$\sum_i M_i \left\lfloor \frac{A_i I_i}{(F_i + I_i)^2} \right\rfloor \geq \sum_i \frac{A_i I_i}{F^2}$
$X \geq \frac{M + 1}{M - 1} \times \frac{I}{F(F + I)}$	$X \geq \frac{\sum_i \frac{A_i I_i^2 (M_i + 1)}{F_i (F_i + I_i)}}{\sum_i \frac{A_i I_i (M_i - 1)}{F_i^2 (F_i + I_i)^2}}$

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OPTIMAL STOPPING PROBLEMS FOR DIFFERENTIAL EQUATIONS PERTURBED BY A POISSON PROCESS

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ABSTRACT

We investigate a class of optimal stopping problems for dynamical systems described by one-dimensional differential equations with an additive Poisson disturbance. The rate of the disturbance may depend upon the current state of the system. A dynamic programming equation for the optimal stopping cost is derived along with conditions which must be met at the boundary of the optimal stopping set. These boundary conditions depend upon whether or not the stopping set may be entered by smooth motion.

1. INTRODUCTION

Differential equations driven by Poisson disturbance terms appear in models of the contents of dams and inventories (Prabhu [7]), viral growth (Bartoszyński [1]), population dynamics in random environments (Hanson and Tuckwell [6]), collective risk (Bühlmann [2]), machine failure (Wickwire [11]), and mass service systems (Prabhu [8]), among others. In some of these models the magnitudes of the disturbances are random variables (the disturbance term is a compound Poisson process), and in others the disturbances have constant magnitude; the latter case is often a useful approximation to the former, and its mathematical treatment via the theory of retarded differential equations differs significantly from the more general approach taken for distributed magnitudes. Many of these models describe changes in resources or populations which lend themselves in a natural way to the application of control theory: dam contents, insurance reserves, and inventories must satisfy demands but not be depleted. Queueing systems should serve as many customers as possible consistent with running costs; latent machine failures ought to be detected early without a large number of false alarms, and malignant viral growth must be checked at acceptable costs for treatment. In this paper we shall assume that Poisson disturbances of a one-dimension[illegible]ynamical system are of constant magnitude and shall also suppose that the allowa[illegible]ontr[illegible] actions, at any time, are of a simple type: one may either stop the system and gai[illegible]ich depends upon the current value of the system, or let it continue to evolve (in the hop[illegible] that the reward, at a subsequent time, will be greater than its current value). Such problems are called *optimal stopping problems* and their general theory has recently attracted much attention (see Chow, Robbins and Siegmund [3] for the case of stopped random sequences and Širyaev [9] for the case of stopped Markov sequences and random processes). A treatment of optimal stopping problems which can be applied to mixed Markov processes (that is, those whose sample paths have jumps superimposed on continuous movement) has been given by Grigelionis [5], but his most important

result, the derivation of conditions satisfied by the payoff function at an optimal stoping boundary, does not, apparently, include the case of a process which can cross such a boundary only by means of a jump. We shall derive here a set of conditions satisfied by the payoff and optimal stopping boundary which applies to both smooth and discontinuous entry into the set of states at which stopping is optimal. We shall limit ourselves to autonomous dynamics defined on a finite interval, because this case exhibits the essential features of a wider class of problems. We remark that optimality conditions for problems with control actions more general than those considered here—actions which alter paths without stopping them—may often be derived in an analogous way.

2. FORMULATION OF THE STOPPING PROBLEM

Let $\{x_t\}$, $t \geq 0$, be a completely observable Markov process on the probability space $\{E, \mathcal{F}, P\}$, where $E \subset R^1$. Since we shall consider E in what follows to be a finite interval, no generality is lost by taking $E = [0, \xi]$, $\xi < \infty$, and we shall do so. Suppose that x_t satisfies the stochastic differential equation (Gihman and Skorohod [4])

$$dx_t = a(x_t)dt + \int_{R^1} \beta(x_t, u)N(dt, du), \quad x_0 = x, \tag{1}$$

where $a(\cdot)$ is continuous on E and nonzero on $E - (\{0\} \cup \{\xi\})$. (The arguments below can be modified without difficulty to account for functions $a(\cdot)$ which vanish inside E.) The function $\beta(x, u)$, the magnitude of the disturbance, is defined as

$$\beta(x, u) = \begin{Bmatrix} \epsilon & \text{if } u \geq \lambda^{-1}(x) \\ 0 & \text{otherwise,} \end{Bmatrix} \tag{2}$$

where $|\epsilon|$ is a bounded constant and $\lambda(x) \geq \lambda_0 > 0$ is the rate at which disturbances occur. $N(t, \cdot)$ is a random Poisson measure in R^1 such that for any Borel set $A \subset R^1$, $EN(t, A) = t\pi(A)$, where π is a finite measure which assigns mass $\lambda(x)$ to the interval $[\lambda^{-1}(x), \infty]$. Solutions of Equation (1) satisfy the ordinary differential equation $dx/dt = a(x)$ between disturbances of magnitude ϵ, and the latter occur according to a Poisson process with rate $\lambda(x_t)$. The infinitesimal generator A of $\{x_t\}$, which acts on bounded, continuous functions $f: E \to R^1$, is defined by $Af(x) = a(x)f'(x) + \lambda(x)[f(x + \epsilon) - f(x)]$. The behavior of $a(x)$ and the sign of ϵ will determine the boundary conditions which may be imposed at $x = 0$ or $x = \xi$ upon solutions of $Af(x) = bf(x)$, an equation which will occur below.

Let $g(\cdot) \geq 0$ be a bounded, continuous function on R^1, and $\mathcal{M}$ the set of stopping times relative to the family of σ − algebras $\mathcal{F} \subset \mathcal{F}$ generated by $\{x_t\}$. The optimal stopping problem we shall consider is to find a stopping time $\hat{\tau} \in \mathcal{M}$ (if one exists) such that for any $x \in E$

$$E_x e^{-b\hat{\tau}} g(x_{\hat{\tau}}) = \sup_{\tau \in \mathcal{M}} E_x e^{-b\tau} g(x_\tau) \equiv s(x), \quad b \geq 0, \tag{3}$$

where E_x denotes the expectation conditional on $x_0 = x$. In the next section we shall derive a dynamic programming equation satisfied by the payoff $s(x)$.

3. A FUNCTIONAL EQUATION SATISFIED BY $s(x)$

Suppose that $x_0 = x$ and we wish to follow an optimal stopping policy for all future time. If we stop the process, we gain $g(x)$ as payoff; if we decide to wait a time $\Delta > 0$ for a higher payoff, our expected gain is $E_x e^{-b\Delta} s(x + \Delta x)$, where Δx is the distance x_t has moved during Δ. Since we may only wait or stop the process, we have

$$s(x) = \max\{g(x),\ E_x\, s(x + \Delta x)e^{-b\Delta}\} \tag{4}$$

and if $s(x) \in D_A$, the domain of A, $E_x e^{-b\Delta} s(x + \Delta x) = s(x) + \Delta A s(x) - b\Delta s(x) + o(\Delta)$, whence

$$s(x) = \max\{g(x),\ s(x) + \Delta A s(x) - b\Delta s(x) + o(\Delta)\},$$

or

$$0 = \max\{g(x) - s(x),\ \Delta(As(x) - bs(x)) + o(\Delta)\}. \tag{5}$$

Since $\Delta > 0$ is arbitrary, we have the dynamic programming equation

$$\max\{g(x) - s(x),\ As(x) - bs(x)\} = 0,\ s \in D_A. \tag{6}$$

If $\mathcal{S}$ is the set of points in E at which it is optimal to stop, and $\mathcal{C} = E - \mathcal{S}$ is the optimal continuation set, we have from Equation (6) that

$$s(x) = g(x) \text{ and } As(x) < bs(x) \Longleftrightarrow x \in \mathcal{S} \tag{7}$$

and

$$As(x) = bs(x) \text{ and } s(x) > g(x) \Longleftrightarrow x \in \mathcal{C}\ .$$

By Theorem 6, p. 108, of Širyaev [9], $\hat{\tau}$, the optimal stopping time, is equal to $\inf\{t \geq 0: x_t \in \mathcal{S}\}$, provided that $P(\hat{\tau} < \infty) = 1$. The equations at (7) are not enough to determine an optimal policy (and therefore $s(x)$) because they do not specify the boundary of $\mathcal{S}$. The next section is devoted to a derivation of condtions for "matching" $s(x)$ to $g(x)$ at the boundary of $\mathcal{S}$. These conditions will serve to determine an optimal policy uniquely in most cases of practical interest.

4. OPTIMAL MATCHING CONDITIONS AT $\partial\mathcal{S}$

The nature of these conditions depends upon whether x_t crosses $\partial\mathcal{S}$ by smooth or discontinuous movement. The following theorem treats the case of continuous movement and is part of the lore of optimal stopping for various continuous processes. It says, in our case, that if x_t can enter $\mathcal{S}$ by smooth movement, the payoff $s(x)$ will be a smooth function on $\partial\mathcal{S}$ inside E.

THEOREM 1: Let $x_0 = \sigma \in \partial\mathcal{S}$, $\sigma \neq 0, \xi$, and suppose that for any small $\Delta > 0$, $x_\Delta \in \mathcal{S}$ if there is no jump during $[0, \Delta]$. Assume further that $D(x) \equiv s(x) - g(x)$ is in D_A at $x = \sigma$, $s'(\sigma)$ and $g'(\sigma)$ exist and are continuous with the derivatives taken for $x \to \sigma$ from within $\mathcal{C}$. Then

$$\frac{ds(\sigma)}{dx} = \frac{dg(\sigma)}{dx}, \tag{8}$$

where d/dx is the left (right) derivative if $a(\sigma) > (<)\ 0$.

PROOF: (A modification of an argument first used by Grigelionis [5]). By hypothesis, $E_\sigma D(x_\Delta) = \Delta A\, D(x) + o(\Delta)$ for small $\Delta > 0$. On the other hand,

$$\begin{aligned} E_\sigma D(x_\Delta) &= D(\sigma + a(\sigma)\Delta)\left\{1 - \int_0^\Delta \lambda(m(s))ds\right\} \\ &\quad + D(\sigma + \epsilon)\int_0^\Delta \lambda(m(s))ds + o(\Delta) \\ &= D(\sigma + \epsilon)\,\lambda(\sigma)\Delta + o(\Delta), \end{aligned}$$

where $dm(s)/ds = a(m(s))$, $m(0) = \sigma$. Hence,

$$AD(\sigma) = \lambda(\sigma) D(\sigma + \epsilon),$$

or

$$a(\sigma)D'(\sigma) + \lambda(\sigma)D(\sigma + \epsilon) = \lambda(\sigma)D(\sigma + \epsilon),$$

so that $(a(\sigma) \neq 0)$

$$s'(\sigma) = g'(\sigma),$$

which was to be proved.

The following theorem treats the case in which x_t can enter $\mathcal{S}$ only by means of a jump. Grigelionis' theorem does not apply here because the expected exit time from a neighborhood of σ into $\mathcal{S}$ is bounded away from zero. The proof of his theorem also requires that s be in the domain of A at σ; since solutions of $Af(x) = bf(x)$, $x \in \mathcal{C}$, $f(x) = g(x)$, $x \in \mathcal{S}$, generally have a discontinuity at σ in this case, it is not a priori clear that s satisfies this requirement. We shall see that the requirement for matching of derivatives is replaced by that of continuity of s across $\partial\mathcal{S}$ when x_t can enter $\mathcal{S}$ only by a jump.

THEOREM 2: Suppose that $x_0 = \sigma \in \partial\mathcal{S}$, $\sigma \neq 0, \xi$, and that for any small $\Delta > 0$, $x_\Delta \in \mathcal{C}$ if x_t has no jump during $[0, \Delta]$. Then the following conclusions hold:

(a) if $\sigma + \epsilon \in \mathcal{C}$ and $\sigma - \epsilon \in \mathcal{S}$, then no optimal boundary condition may be assigned at σ;

(b) if $\sigma + \epsilon \in \mathcal{S}$, then

$$s(\sigma \pm) = g(\sigma) \text{ if } \epsilon \lesseqgtr 0, \tag{9}$$

where $$s(\sigma \pm) \equiv \lim_{\substack{x \to \sigma \\ x \gtrless \sigma}} s(x).$$

PROOF: Under the conditions of (a), σ is not accessible on an optimal path from $\mathcal{C}$. Consider (b). By the definition of s, $s(\sigma \pm) \geqslant g(x)$. Now suppose that instead of stopping at σ (which is optimal) we let x_t evolve for a short time $\Delta > 0$ during which x_t moves a distance Δx. Then

$$\begin{aligned} g(\sigma) &\geqslant E_\sigma\, s(\sigma + \Delta x)e^{-b\Delta} \\ &= [s(\sigma \pm) + a(\sigma)\Delta s'(\sigma)]\,(1 - \lambda(\sigma)\Delta)\,(1 - b\Delta) + s(\sigma + \epsilon)\lambda(\sigma)\Delta + o(\Delta) \\ &= \Delta[a(\sigma)s'(\sigma) + \lambda(\sigma)s(\sigma + \epsilon) - (\lambda(\sigma) + b)s(\sigma \pm)] + s(\sigma \pm) + o(\Delta), \end{aligned}$$

where s' is the appropriate one-sided derivative. Since $\Delta > 0$ is arbitrary, we have $s(\sigma \pm) \leqslant g(\sigma)$, whence the theorem is proved.

5. AN EXAMPLE

Suppose that $a(x) = a$, $\lambda(x) = \lambda > 0$, $g(x) = x^2$ and that 0 and ξ are reflecting barriers if they can be reached by x_t. This is a simple model for a dam or inventory whose input (output) is described by the ordinary differential equation $dm(s)/ds = a$, and whose output (input)

is described by a Poisson process with rate λ and jumps of size ϵ. The reflecting barrier assumption implies that the content of the corresponding system stays at 0 (or ξ) if it reaches or jumps over this level until new input (or output) causes the content to assume a value in $(0, \xi)$. Then by Section 3

$$s(x) = \sup_{\tau \in \mathcal{M}} E_x e^{-b\tau} x_\tau^2$$

satisfies

$$\left.\begin{aligned} As(x) &= bs(x) \text{ in } \mathcal{C} \\ s(x) &= x^2 \text{ in } \mathcal{S} \end{aligned}\right\} \tag{10}$$

where $As(x) = as'(x) + \lambda[s(x+\epsilon) - s(x)]$. Using the fact that $h(x) \equiv E_x e^{-b\Delta} g(x_\Delta)$ is convex if g is convex, one can show, via Equation (4), that for this example the optimal stopping set must be of the form $\mathcal{S} = [\sigma, \xi]$ for some $\sigma \in [0, \xi]$. Conditions which determine the optimal value σ of σ depend upon the signs of a and ϵ.

CASE 1: $a > 0, \epsilon > 0$. Here $\hat{\sigma} + \epsilon \in \mathcal{S}$ and $s(\hat{\sigma} + \epsilon) = g(\hat{\sigma} + \epsilon)$. Applying Theorem 1 and taking Equation (10) into account, we find that if $\hat{\sigma} \in (0, \xi)$, then it must satisfy

$$2a\sigma + \lambda[(\sigma + \epsilon)^2 - \sigma^2] - b\sigma^2 = 0, \tag{11}$$

a quadratic in σ. Only the root

$$\sigma_+ \equiv [a + \lambda\epsilon + \sqrt{(a + \lambda\epsilon)^2 + b\lambda\epsilon^2}]/b$$

can be $\hat{\sigma}$ and one has $\hat{\sigma} = \min(\sigma_+, \xi)$.

CASE 2: $a > 0, \epsilon < 0$. In this case $\hat{\sigma} + \epsilon \in \mathcal{C}$ and Theorem 1 yields

$$2a\sigma + \lambda s(\sigma - |\epsilon|) - (\lambda + b)\sigma^2 = 0 \tag{12}$$

or

$$s(\sigma - |\epsilon|) = \sigma[(\lambda + b)\sigma - 2a]/\lambda$$

as an equation which must be satisfied by $\hat{\sigma}$ if it lies in $(0, \xi)$. Since $s(\sigma - |\epsilon|) \geq g(\sigma - |\epsilon|) = (\sigma - |\epsilon|)^2$, Equation (12) implies that $\hat{\sigma}$ must satisfy $b\sigma^2 - 2(a - \lambda|\epsilon|)\sigma - \lambda\epsilon^2 \geq 0$, or $\hat{\sigma} \geq [a - \lambda|\epsilon| + \sqrt{(a - \lambda|\epsilon|)^2 + b\lambda\epsilon^2}]/b$. To determine $\hat{\sigma}$ from Equation (12), we note that solutions of the functional equation $Af(x) = bf(x)$, $x \in (0, \sigma)$; $f(x) = 0$, $x < 0$; $f(\sigma) = \sigma^2$, $f'(\sigma) = 2\sigma$, will have a discontinuity of magnitude $f_0 \equiv f(0+) > 0$ at $x = 0$, which is determined to satisfy the boundary condition $f(\sigma) = f(\sigma, f_0) = \sigma^2$. The optimal value of σ is then chosen to fulfill $f'(\sigma) = 2\sigma$ or $f(\sigma - |\epsilon|) = \sigma[(\lambda + b)\sigma - 2a]/\lambda$ (cf. Equation (12)).

CASE 3: $a < 0, \epsilon < 0$. Here x_t is decreasing with probability one and we clearly have $\mathcal{S} = E$ and $s(x) = x^2$ for all $x \in E$.

CASE 4: $a < 0, \epsilon > 0$. Theorem 2 applies to yield $s(\hat{\sigma}-) = \hat{\sigma}^2$ as the optimality condition for $\hat{\sigma}$. Since $x = 0$ is reflecting, $s'(0) = 0$, or $s(\epsilon+) = (1 + b/\lambda)s(0+)$. A procedure for determining $\hat{\sigma}$ is then the following: for $\sigma \in (0, \xi)$, solve $Af(x) = bf(x)$, $x \in (0, \sigma)$, $f(x) = x^2$, $x \in [\sigma, \xi]$, $f(x) \equiv \xi$, $x > \xi$. Solutions of this equation will generally have a discontinuity of magnitude $f_\sigma \equiv f(\sigma-)$ at $x = \sigma$, which must be chosen for this σ to satisfy the reflecting boundary condition at $x = 0$. Finally, $\hat{\sigma}$ is chosen such that the optimality condition $f(\hat{\sigma}-) = \hat{\sigma}^2$ is also met. Note that the resulting function $f(x)$ $(= s(x))$ will then be continuous on $(0, \xi)$ and will also have a continuous derivative there.

6. CONCLUDING REMARKS

i. If $a(x) = 0$ in $(0, \xi)$ or $\lambda(x)$ is not bounded away from zero, x_t may not attain a stopping set with probability one, and stopping policies must generally include prescriptions of what actions to take if x_t enters a set in E which is absorbing.

ii. Cost functionals other than (3) may sometimes be treated within the present framework. The costs

$$E_x \int_0^\tau c(x_s)ds, \quad E_x \left[g(x_\tau) - \int_0^\tau c(x_s)ds \right]$$

and

$$\left[k + E_0 \int_0^\tau c(x_s)ds \right] / E_0\tau$$

and their discounted versions are such functionals. For example, in the second case, if $P(\tau < \infty) = 1$, then an equivalent problem is to find an optimal stopping time for the cost

$$f(x) + E_x F(x_\tau), \tag{13}$$

where $F(x) \equiv g(x) - f(x)$ and $f(x)$ satisfies $Af(x) - c(x) = 0$ (cf. Taylor [10]). Nonautonomous costs or dynamics, or both, may also be treated by augmenting x_t with time as a second dimension.

iii. If $|\epsilon| << 1$, machine solution of the retarded or advanced differential equation $Af(x) - bf(x) = 0$ may cause numerical difficulties, especially if there are singularities caused by zeros of $a(x)$. Singular perturbation expansions of $f(x)$ in the form $\sum_{i \geq 0} \epsilon^i F_i(\theta)$, where θ is a suitable stretching transformation (e.g., $\theta = x/\epsilon$), and the F_i are to be found, may be used in this case to derive accurate approximations to $s(x)$ and $\partial\mathcal{S}$.

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DECOMPOSITION ALGORITHMS FOR MINIMAL CUT PROBLEMS

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ABSTRACT

Consider a network $G(N, A)$ with n nodes, where node 1 designates its source node and node n designates its sink node. The cuts $(Z_i, \bar{Z}_i)$, $i = 1, \ldots, n-1$ are called one-node cuts if $1 \in Z_i$, $n \notin Z_i$, $Z_1 = \{1\}$, $Z_i \subset Z_{i+1}$ and Z_i and Z_{i+1} differ by only one node. It is shown that these one-node cuts decompose G into $1 \leqslant m \leqslant n/2$ subnetworks with known minimal cuts. Under certain circumstances, the proposed one-node decomposition can produce a minimal cut for G in $0(n^2)$ machine operations. It is also shown that, under certain conditions, one-node cuts produce no decomposition. An alternative procedure is also introduced to overcome this situation. It is shown that this alternative procedure has the computational complexity of $0(n^3)$.

1. INTRODUCTION

Determining the location of minimal cuts in a flow network is gaining wider importance in recent literature. Jarvis [8, 9] shows that in a communication network, if each arc is assigned a value which represents the effort of disabling that communication link, then the optimal strategy to disable the communication between the message sender and the message receiver is to destroy the links (arcs) on a minimal cut. Phillips and Dessouky [16] indicate that the time/cost tradeoff problem can be converted into a problem of maximal flow where at each iteration the activities on the minimal cut are crashed. Rhys [18] and Pickard [17] indicate that a selection problem of shared fixed costs can be converted into a maximal flow problem where minimum cut identifies the optimum options selection.

Some effort has been made to find a minimal cut or all the minimal cuts in a directed network, regardless of maximal flow (see [7, 15]). However, all the labeling algorithms for finding the maximal flow provide the location of a minimal cut as a by-product at the end of the algorithms [3, 4, 6]. The maximal flow algorithms proposed by Dinic [2] and Karzanov [12], find the maximal flow but they do not provide a minimal cut. However, finding the location of a minimal cut in these algorithms can easily be achieved by applying a labeling procedure on the network once the maximal flow is obtained.

Among the efficient labeling algorithms, the first-labeled first-scanned algorithm proposed by Edmonds and Karp [3] has a theoretical upper bound of $(n^3 - n)/4$ augmentations, where n

is the number of nodes in the flow network. Among the efficient nonlabeling algorithms, Karzanov's [12] algorithm with preflows has the theoretical upper bound of $0(n^3)$ machine operations.

In this paper we propose two decomposition algorithms for locating a minimal cut in a given flow network. The network is decomposed into $1 \leqslant m \leqslant \frac{n}{2}$ subnetworks with known minimal cuts by utilizing a one-node cut concept. Section 2 gives the decomposition of the network into two subnetworks by an arbitrary cut. The relationship between the minimal cuts of the subnetworks and the arbitrary cut is given in Theorem 1. The concept of one-node cuts and some related theorems are introduced in Section 3. Section 4 presents an algorithm for decomposing the original network into three subnetworks where the intermediate network contains a minimal cut. Finally, conclusions are given in Section 5.

2. MINIMAL CUTS AND NETWORK DECOMPOSITION

A network $G(N, A)$ is a collection of nodes and ordered pairs of nodes which are called arcs. N designates the node set and A designates the arc set of the network.

There are two special nodes in an n-node flow network; one is called the source node, denoted by 1 (or s), and the other one is called the sink node and is denoted by n (or t).

With every arc $(i, j) \in A$ of G we associate a positive integer u_{ij}, called the capacity of the arc. A set of nonnegative integers f_{ij} is called a feasible flow in a network if they satisfy the following equations:

$$\Sigma_i f_{ij} - \Sigma_k f_{jk} = \begin{cases} -f & \text{if } j = 1 \\ 0 & \text{if } j \neq 1, n \\ f & \text{if } j = n \end{cases}$$

and

$$0 \leqslant f_{ij} \leqslant u_{ij} \quad \text{for all } (i, j) \in A,$$

where f is a nonnegative integer called the value of the flow. A set of feasible flow, f_{ij}^*, which maximizes f is called a maximal flow.

DEFINITION: Let X be a subset of the nodes in G such that $1 \in X$, $n \notin X$. Also let $\bar{X} = N - X$. Let $(X, \bar{X})$ be a set such that the arc $(i, j) \in (X, \bar{X})$ if $i \in X$, $j \in \bar{X}$ and $(i, j) \in A$ or $j \in X$, $i \in \bar{X}$ and $(i, j) \in A$. Such a set is called a cut (or a cut-set).

The capacity of a cut $(X, \bar{X})$, denoted by $c(X, \bar{X})$, is given by

$$c(X, \bar{X}) = \sum_{\substack{(i,j) \in (X, \bar{X}) \\ i \in X, j \in \bar{X}}} u_{ij}.$$

DEFINITION: A cut with minimum capacity in a network is called a minimal cut (min-cut).

DEFINITION: A cut $(X, \bar{X})$ is said to lie on the left of a cut $(Y, \bar{Y})$ if $X \subseteq Y$. Conversely, a cut $(Y, \bar{Y})$ is said to lie on the right of a cut $(X, \bar{X})$ if $Y \supseteq X$.

DEFINITION: A cut $(X, \bar{X})$ is said to lie between two cuts $(Y, \bar{Y})$ and $(Z, \bar{Z})$ with $Y \subset Z$, if $Y \subseteq X \subseteq Z$.

DEFINITION: Two cuts $(X, \bar{X})$ and $(Y, \bar{Y})$ are said to cross each other if each one of the sets $X \cap Y$, $X \cap \bar{Y}$, $\bar{X} \cap Y$, and $\bar{X} \cap \bar{Y}$ is nonempty.

Ford and Fulkerson [4] show that for single commodity network flow problems the value of a maximal flow, f^*, is equal to the value of the capacity of any minimal cut in that network.

Let $G = (N, A)$ be a network with $N = \{1, \ldots, n\}$. Also, let $(X, \bar{X})$ be an arbitrary cut. Let all the arcs $(i, \bar{X}) \in (X, \bar{X})$ be collected to an artificial node s with

$$u_{is} = \sum_{j \in \bar{X}} u_{ij} \quad \text{for each } i \in X.$$

Similarly, let the arcs $(X, j) \in (X, \bar{X})$ emanate from s with

$$u_{sj} = \sum_{i \in X} u_{ij} \quad \text{for each } j \in \bar{X}.$$

Note that the arcs $(j, i) \in (\bar{X}, X)$ are ignored in this procedure (since they have no effect on cut values).

By this procedure we create two subnetworks connected to each other via a single artificial node s. Let $G_1 = (X \cup \{s\}, A_1)$ be the first subnetwork where the arc set A_1 contains all the arcs $(i, j) \in A$ and $i \in X$, $j \in X$ plus the arcs (i, s), $i \in X$ as described above. Similarly, let $G_2 = (\{s\} \cup \bar{X}, A_2)$ be the other subnetwork, where A_2 contains all the arcs $(i, j) \in A$ and $i \in \bar{X}$, $j \in \bar{X}$ as well as the new arcs (s, j), $j \in \bar{X}$ as described above. Note that $c(X, s) = c(s, \bar{X}) = c(X, \bar{X})$. Also note that the artificial node s serves as a sink node for G_1 and as a source node for G_2.

THEOREM 1: Let a network $G(N, A)$ be partitioned into two subnetworks G_1 and G_2 by an arbitrary cut $(X, \bar{X})$, as described above. Suppose that $(Z_1, \bar{Z}_1)$ is a minimal cut for G_1 and $(Z_2, \bar{Z}_2)$ is a minimal cut for G_2. Then the following statements are true:

(a) $\max\{c(Z_1, \bar{Z}_1), c(Z_2, \bar{Z}_2)\} \leqslant c(X, \bar{X})$,

(b) there exists a minimal cut for G which lies between $(Z_1, \bar{Z}_1)$ and $(Z_2, \bar{Z}_2)$,

(c) if $c(Z_1, \bar{Z}_1) = c(X, \bar{X}) > c(Z_2, \bar{Z}_2)$, then $(Z_2, \bar{Z}_2)$ is a minimal cut for G,

(d) if $c(Z_2, \bar{Z}_2) = c(X, \bar{X}) > c(Z_1, \bar{Z}_1)$, then $(Z_1, \bar{Z}_1)$ is a minimal cut for G,

(e) if $c(Z_1, \bar{Z}_1) = c(Z_2, \bar{Z}_2) = c(X, \bar{X})$, then the cuts $(Z_1, \bar{Z}_1)$, $(Z_2, \bar{Z}_2)$ and $(X, \bar{X})$ are all minimal cuts for G.

PROOF:

(a) If $c(Z_1, \bar{Z}_1) > c(X, \bar{X})$, then we would have chosen $(X, \bar{X})$ as the minimal cut between 1 and s. Thus, $c(Z_1, \bar{Z}_1) \leqslant c(X, \bar{X})$. With the same line of argument we have $c(Z_2, \bar{Z}_2) \leqslant c(X, \bar{X})$.

(b) Suppose that $(Y, \bar{Y})$ is a minimal cut for G which crosses $(Z_1, \bar{Z}_1)$ and $(Z_2, \bar{Z}_2)$ as shown in Figure 1 below.

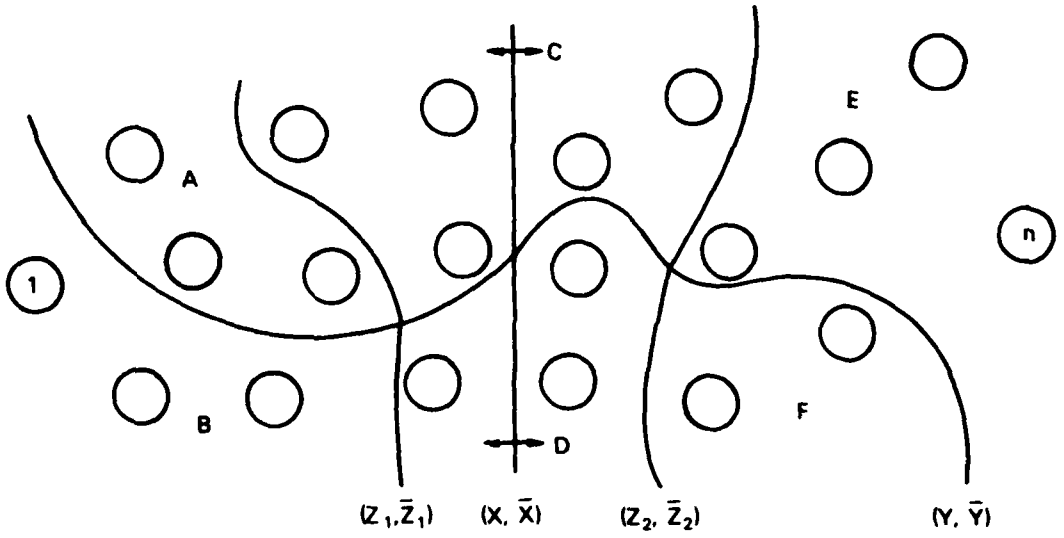


FIGURE 1. A minimal cut $(Y, \bar{Y})$ crossing $(Z_1, \bar{Z}_1)$ and $(Z_2, \bar{Z}_2)$.

Consider the following node-sets:

$A = \bar{Y} \cap Z_1 \cap Z_2$, $B = Y \cap Z_1 \cap Z_2$, $C = \bar{Y} \cap \bar{Z}_1 \cap Z_2$, $D = Y \cap \bar{Z}_1 \cap Z_2$, $E = \bar{Y} \cap \bar{Z}_1 \cap \bar{Z}_2$, $F = Y \cap \bar{Z}_1 \cap \bar{Z}_2$. Note that $A \cup B = Z_1$, $C \cup D \cup E \cup F = \bar{Z}_1$ $A \cup B \cup C \cup D = Z_2$, $E \cup F = \bar{Z}_2$, $B \cup D \cup F = Y$ and $A \cup C \cup E = \bar{Y}$.

Consider the cut $(B, N - B)$. Since $B \subset Z_1$ this cut lies completely in G_1 and thus

$$c(B, N - B) \geqslant c(Z_1, \bar{Z}_1)$$

where $N - B = A \cup C \cup D \cup E \cup F$. Thus, we have

$$(1) \qquad c(B, A) + c(B, C) + c(B, D) + c(B, E) + c(B, F) \geqslant c(A, C) + c(A, D) + c(A, E) + c(A, F) + c(B, C) + c(B, D) + c(B, E) + c(B, F).$$

Similarly, consider the cuts $(N - E, E)$ and $(Z_2, \bar{Z}_2)$ which lie in the subnetwork G_2. With the same line of argument we have

$$(2) \qquad c(A, E) + c(B, E) + c(C, E) + c(D, E) + c(F, E) \geqslant c(A, E) + c(B, E) + c(C, E) + c(D, E) + c(A, F) + c(B, F) + c(C, F) + c(D, F).$$

Adding (1) and (2) together and subtracting $c(B, E)$, $c(C, E)$, $c(B, D)$, $c(A, E)$ and $c(B, F)$ from both sides we get

$$c(B, A) + c(B, C) + c(B, E) + c(D, E) + c(F, E) \geq c(A, C) + c(A, D) + c(A, E) + 2c(A, F) + c(B, C) + c(B, E) + c(B, F) + c(D, E) + c(C, F) + c(D, F). \tag{3}$$

We also have

$$0 \geq -c(A, D) - c(A, F) - c(C, F), \tag{4}$$

$$c(D, C) = c(D, C) \tag{5}$$

and

$$c(D, A) + c(F, A) + c(F, C) \geq 0. \tag{6}$$

By adding (3), (4), (5) and (6) we get

$$c(B, A) + c(B, C) + c(B, E) + c(D, E) + c(F, E) + c(D, C) + c(D, A) + c(F, A) + c(F, C) \geq c(A, C) + c(A, E) + c(A, F) + c(B, C) + c(B, E) + c(B, F) + c(D, C) + c(D, E) + c(D, F). \tag{7}$$

The left-hand side of (7) is the capacity of the minimal cut $(Y, \bar{Y})$ and the right-hand side is the capacity of the cut $(A \cup B \cup D, C \cup E \cup F)$. Thus,

$$c(Y, \bar{Y}) \geq c(A \cup B \cup D, C \cup E \cup F). \tag{8}$$

The inequality in (8) can only be satisfied as an equality. Therefore, we conclude that the cut $(A \cup B \cup D, C \cup E \cup F)$ *is also a minimal cut for* G.

(c) Since $c(Z_1, \bar{Z}_1) = c(X, \bar{X})$ both are minimal cuts for G_1. From the inequality (8) in part (b) we have shown that there exists a minimal cut for G which does not cross a given minimal cut for G_1. Since $(X, \bar{X})$ is a minimal cut for G_1 and since $c(Z_2, \bar{Z}_2) < c(X, \bar{X})$ then $(Z_2, \bar{Z}_2)$ is a minimal cut for G.

(d) With the same line of argument as in part (c), we have $(X, \bar{X})$ as a minimal cut for G_2 and $c(Z_1, \bar{Z}_1) < c(X, \bar{X})$. Therefore, $(Z_1, \bar{Z}_1)$ is a minimal cut for G.

(e) Since $c(Z_1, \bar{Z}_1) = c(X, \bar{X}) = c(Z_2, \bar{Z}_2)$, by considering $(X, \bar{X})$ as a minimal cut for G_1 as in part (c), we conclude that we can find a minimal cut for G which does not cross $(X, \bar{X})$. Therefore, $(Z_1, \bar{Z}_1)$, $(Z_2, \bar{Z}_2)$ and $(X, \bar{X})$ are all minimal cuts for G.

Consider subnetworks G_1 and G_2. Since both are legitimate networks with a source and a sink node, they can similarly be partitioned into two subnetworks G_{11}, G_{12}, and G_{21}, G_{22}, respectively, by arbitrary cuts $(X_1, \bar{X}_1)$ and $(X_2, \bar{X}_2)$ where $X_1 \subset X$ and $X \subset X_2$. Let $(Z_{11}, \bar{Z}_{11})$ and $(Z_{12}, \bar{Z}_{12})$ be minimal cuts for G_{11} and G_{12}, and let $(Z_{21}, \bar{Z}_{21})$ and $(Z_{22}, \bar{Z}_{22})$ be minimal cuts for G_{21} and G_{22}, respectively. This situation is depicted in the figure given below where s_1 is the super node between G_{11} and G_{12}, s_2 is the super node between G_{21} and G_{22} and s is the super node between G_1 and G_2.

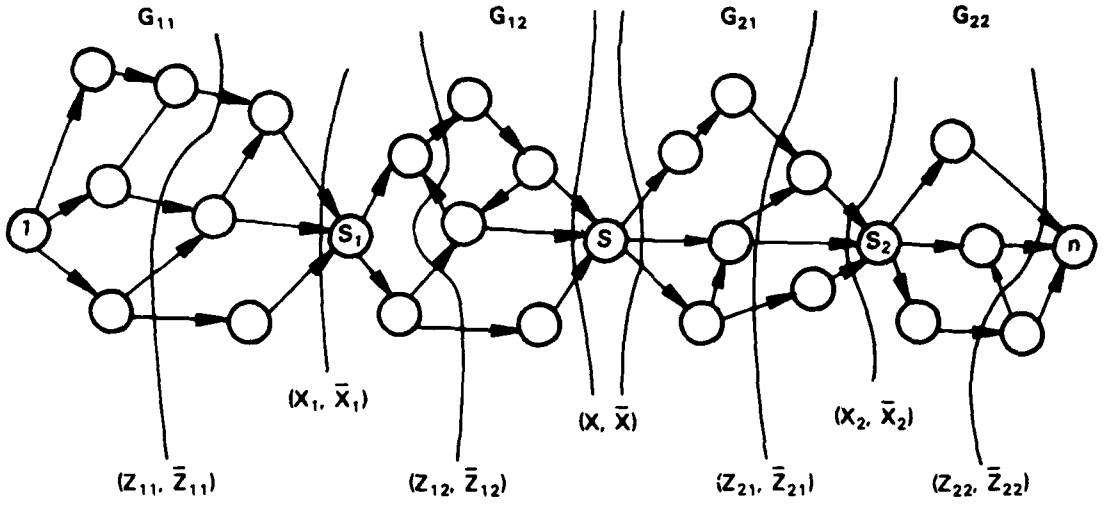


FIGURE 2. G_1 and G_2 are further decomposed into G_{11}, G_{12}, and G_{21}, G_{22}.

Therefore, we can hope to find a minimal cut $(Z_1, \bar{Z}_1)$ for G_1 by considering $(X_1, \bar{X}_1)$ and the minimal cuts $(Z_{11}, \bar{Z}_{11})$ and $(Z_{12}, \bar{Z}_{12})$ for subnetworks G_{11} and G_{12}, respectively. This process leads us to the idea of ultimate decomposition which we will call one-node decomposition. If this partitioning of subnetworks is repeated iteratively, X_i and X_{i+1} will eventually differ from each other by one node. The following section introduces the one-node cuts and some important property.

3. ONE-NODE CUTS

Let a network $G(N, A)$ contain n nodes. Also, let the nodes of the network be numbered from 1 to n, where node 1 is the source node and node n is the sink node.

Consider the following noncrossing cuts:

$$(Z_i, \bar{Z}_i),\ i = 1, 2, \ldots, (n-1), \text{ where}$$
$$Z_i = \cup_{j=1}^{i} \{j\}, \qquad \bar{Z}_i = N - Z_i.$$

Thus, $Z_1 = \{1\},\ Z_2 = \{1, 2\}, \ldots, Z_{n-1} = \{1, 2, \ldots, n-1\},$

and $\bar{Z}_1 = \{2, 3, \ldots, n\},\ \bar{Z}_2 = \{3, 4, \ldots, n\}, \ldots, \bar{Z}_{n-1} = \{n\}.$

We will call these cuts one-node cuts since Z_i differs from Z_{i+1} by one node. The values of the cuts, $c(Z_i, \bar{Z}_i),\ i = 1, \ldots, n-1$ viewed as a sequence of numbers, will produce subsequences of increasing and/or decreasing values.

Graphically, we will have a picture similar to the one given in Figure 3 below. In this picture the downward arrows represent the decreasing subsequences and upward arrows represent the increasing subsequences.

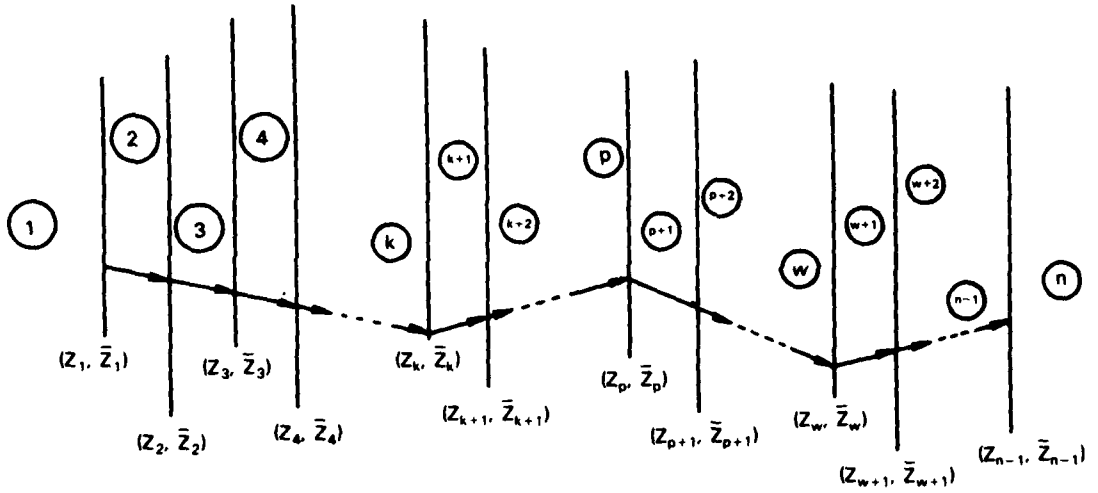


FIGURE 3. A typical picture of one-node cuts and their relative values.

THEOREM 2: If $c(Z_i, \bar{Z}_i) \geqslant c(Z_{i+1}, \bar{Z}_{i+1})$ for $i = 1, 2, \ldots, n-2$, then $(Z_{n-1}, \bar{Z}_{n-1})$ is a minimal cut for G.

PROOF: Let $(Y, \bar{Y})$ be a minimal cut for G which lies on the left of $(Z_{n-1}, \bar{Z}_{n-1})$. Consider the node set Y. We can find an index k such that $Z_j \subseteq Y$ for $j \leqslant k$ and $Z_j \not\subseteq Y$ for $j > k$. (Note that $Z_1 \subseteq Y$.) Let Figure 4 depict this situation.

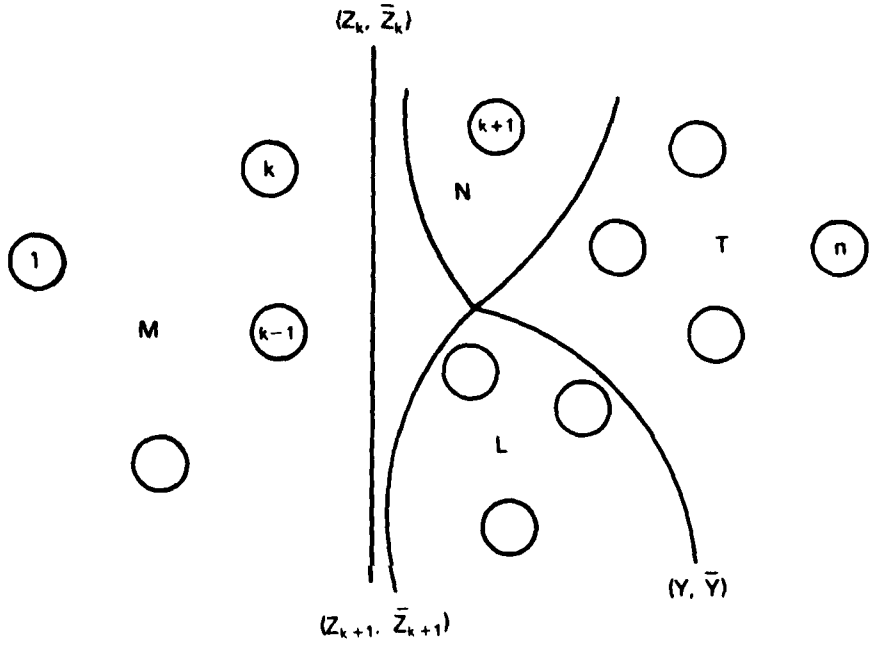


FIGURE 4. The case for $c(Z_i, \bar{Z}_i) \geqslant c(Z_{i+1}, \bar{Z}_{i+1})$, $i = 1, 2, \ldots, n-2$.

Let $Z_{k+1} \cap Y = M$, $Z_{k+1} \cap \bar{Y} - \{k+1\} = N$, $\bar{Z}_{k+1} \cap Y = L$ and $\bar{Z}_{k+1} \cap \bar{Y} = T$.

We have

$$c(Z_k, \bar{Z}_k) \geq c(Z_{k+1}, \bar{Z}_{k+1})$$

or

$$c(M, N \cup L \cup T) \geq c(M \cup N, L \cup T)$$

or

$$c(M, N) + c(M, L) + c(M, T) \geq c(M, L) + c(M, T) + c(N, L) + c(N, T).$$

Since

$$0 \leq c(N, L)$$

we can write

$$c(M, N) + c(M, T) \geq c(N, T) + c(M, T). \tag{9}$$

Since

$$c(L, T) = c(L, T) \tag{10}$$

and

$$c(L, N) \geq 0, \tag{11}$$

by adding (9), (10), and (11), we get

$$c(L, T) + c(L, N) + c(M, T) + c(M, N) \geq c(N, T) + c(M, T) + c(L, T). \tag{12}$$

The left-hand side of (12) is nothing but $c(Y, \bar{Y})$ and the right-hand side is $c(Y \cup \{k+1\}, \bar{Y} - \{k+1\})$. Therefore,

$$c(Y, \bar{Y}) \geq c(Y \cup \{k+1\}, \bar{Y} - \{k+1\}). \tag{13}$$

Hence, the cut $(Y \cup \{k+1\}, \bar{Y} - \{k+1\})$ is as good a cut as $(Y, \bar{Y})$. By assuming the new cut $(Y \cup \{k+1\}, \bar{Y} - \{k+1\})$ as a minimal cut for G we can apply the same reasoning again. Thus, we will eventually reach the conclusion that $(Z_{n-1}, \bar{Z}_{n-1})$ is a minimal cut for G.

THEOREM 3: If $c(Z_i, \bar{Z}_i) \leq c(Z_{i+1}, \bar{Z}_{i+1})$, $i = 1, \ldots, (n-2)$, then $(Z_1, \bar{Z}_1)$ is a minimal cut for G.

PROOF: The proof of this case is very similar to that of Theorem 2. In this case we find an index k such that $Z_j \supseteq Y$ for $j \geq k$ and $Z_j \not\supseteq Y$ for $j < k$. This situation is depicted in Figure 5 below.

We define $M = Z_{k-1} \cap Y$, $N = \bar{Z}_{k-1} \cap Y$, $L = Z_{k-1} \cap \bar{Y}$, and $T = \bar{Z}_{k-1} \cap \bar{Y}$.

We have

$$c(Z_{k-1}, \bar{Z}_{k-1}) \leq c(Z_k, \bar{Z}_k)$$

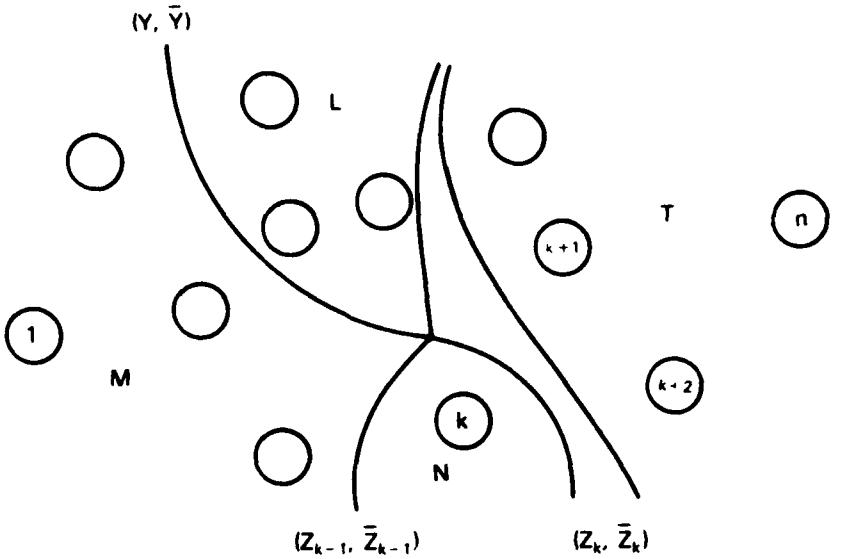


FIGURE 5. The case for $c(Z_i, \bar{Z}_i) \leqslant c(Z_{i+1}, \bar{Z}_{i+1})$, $i = 1, \ldots, n-2$.

or

$$c(M \cup L, N \cup T) \leqslant c(M \cup L \cup N, T)$$

or

$$c(M, N) + c(M, T) + c(L, N) + c(L, T) \leqslant c(M, T) + c(L, T) + c(N, T).$$

Since $c(L, N) \geqslant 0$,

we have

(14) $$c(M, N) + c(M, T) \leqslant c(M, T) + c(N, T)$$

and

(15) $$c(M, L) = c(M, L)$$

and

(16) $$0 \leqslant c(N, L).$$

By adding (14), (15), and (16), we get

$$c(M, N) + c(M, T) + c(M, L) \leqslant c(M, T) + c(N, T) + c(M, L) + c(N, L)$$

or

$$c(Y - \{k\}, \bar{Y} \cup \{k\}) \leqslant c(Y, \bar{Y}).$$

As in Theorem 2 by redefining the minimal cut $(Y, \bar{Y})$ and applying the same argument, we reach the conclusion that $(Z_1, \bar{Z}_1)$ is a minimal cut for G.

THEOREM 4: If the one-node cuts produce only a decreasing sequence followed immediately by an increasing sequence, then the one-node cut with the smallest capacity is a minimal cut for G.

PROOF: Let p be the index where $c(Z_i, \bar{Z}_i) \geqslant c(Z_{i+1}, \bar{Z}_{i+1})$ for $1 \leqslant i \leqslant (p-1)$ and $c(Z_i, \bar{Z}_i) \leqslant c(Z_{i+1}, \bar{Z}_{i+1})$ for $p \leqslant i \leqslant (n-2)$. Let $(Z_p, \bar{Z}_p)$ divide the network into two subnetworks G_1 and G_2 as described before. (Note that $(Z_p, \bar{Z}_p)$ is $(X, \bar{X})$ in Theorem 1.) The subnetwork G_1 contains only a decreasing sequence and the subnetwork G_2 contains only an increasing sequence. From Theorem 2, $(Z_p, \bar{Z}_p)$ is a minimal cut for G_1 and from Theorem 3, $(Z_p, \bar{Z}_p)$ is a minimal cut for G_2. Then from Theorem 1 we conclude that $(Z_p, \bar{Z}_p)$ is a minimal cut for G.

THEOREM 5: If the one-node cuts produce only an increasing sequence followed by a decreasing sequence, then either the one-node cut with minimal capacity is a minimal cut for G, or else there exists a minimal cut for G which crosses the one-node cut with the largest capcity.

PROOF: Let p be an index such that $c(Z_i, \bar{Z}_i) \leqslant c(Z_{i+1}, \bar{Z}_{i+1})$ for $1 \leqslant i \leqslant (p-1)$ and $c(Z_i, \bar{Z}_i) \geqslant c(Z_{i+1}, \bar{Z}_{i+1})$ for $p \leqslant i \leqslant (n-2)$. Let $(Z_p, \bar{Z}_p)$ decompose G into two subnetworks G_1 and G_2 as described before. From Theorem 3 we get $(Z_1, \bar{Z}_1)$ as a minimal cut for G_1 and from Theorem 2 we get $(Z_{n-1}, \bar{Z}_{n-1})$ as a minimal cut for G_2. Therefore, from Theorem 1-b we conclude that there exists a minimal cut for G which lies between $(Z_1, \bar{Z}_1)$ and $(Z_{n-1}, \bar{Z}_{n-1})$. If this minimal cut is not either $(Z_1, \bar{Z}_1)$ or $(Z_{n-1}, \bar{Z}_{n-1})$, then it cannot lie completely in one subnetwork. Hence, it must cross the cut $(Z_p, \bar{Z}_p)$ which separates G_1 and G_2.

COROLLARY 1: For any network $G(N, A)$, the one-node cuts decompose G into $1 \leqslant m \leqslant (n/2)$ subnetworks with known minimal cuts. These subnetworks are characterized as the collection of nodes such that the one-node cuts corresponding to these nodes form a subsequence of consecutive decreasing and increasing values. If the first subsequence is an increasing one then the nodes corresponding to this subsequence constitute the first subnetwork. If the first subsequence is a decreasing one then the nodes corresponding to the first decreasing and increasing subsequences constitute the first subnetwork. After the first subnetwork G_1 is determined, the remaining subnetworks G_i, $i = 2, \ldots, m$ can be established by considering the nodes in each consecutive decreasing and increasing subsequences. If the one-node cuts produce a single sequence of increasing or decreasing values, then a minimal cut for G can be found by choosing the smallest capacity one-node cut.

A minimal cut for each subnetwork is given by the one-node cut in each subnetwork with minimal capacity. (Note that the last subnetwork may contain only a decreasing sequence.)

The computational complexity of decomposition by one-node cuts can easily be given as $(n-1)^2$ additions and/or subtractions and $(n-2)$ comparisons.

Therefore, the one-node cuts can produce a minimal cut for a network in $0(n^2)$ additions and/or subtractions if the one-node cuts produce a single subnetwork (cases in Theorems 2, 3, and 4).

It is clearly obvious that it is to our benefit to end up with a single subnetwork (the network itself) as the result of one-node cuts. The following section exploits this situation. In the approach presented in the next section, the one-node cuts are selected in such a way that either the one-node cuts will produce a single subnetwork or we hope that some nodes will be eliminated from the original network so that a smaller subnetwork can be solved for a minimal cut.

4. THREE-DECOMPOSITION PROCEDURE

The name three-decomposition stems from the fact that we would like to decompose a given network G into three subnetworks G_1, G_2, and G_3 with the property that a minimal cut for G can be found by only considering the intermediate subnetwork G_2.

In order to achieve this goal we will try to obtain a decreasing subsequence of one-node cuts starting from node 1. If this process does not exhaust all the nodes of G then we will try to obtain an increasing subsequence of one-node cuts ending at node n. This process will produce one of the following three outcomes:

(a) $c(Z_i, \bar{Z}_i) \geqslant c(Z_{i+1}, \bar{Z}_{i+1})$, $\quad i = 1, \ldots, n-2$

(b) $c(Z_i, \bar{Z}_i) \leqslant c(Z_{i+1}, \bar{Z}_{i+1})$, $\quad i = 1, 2, \ldots, n-2$

(c) $c(Z_i, \bar{Z}_i) \geqslant c(Z_{i+1}, \bar{Z}_{i+1})$, $\quad i = 1, \ldots, (k-1)$ and
$c(Z_{i-1}, \bar{Z}_{i-1}) \leqslant c(Z_i, \bar{Z}_i)$, $\quad i = (n-1), (n-2), \ldots, p+1$
and $k \leqslant p$.

If the outcome is as given in case (a) then from Theorem 2, $(Z_{n-1}, \bar{Z}_{n-1})$ is a minimal cut for G. If the outcome is as given in case (b) then from Theorem 3, $(Z_1, \bar{Z}_1)$ is a minimal cut for G. We have three possible subcases for an outcome described in case (c):

(i) $k = p$. In this subcase the network contains a decreasing and an increasing block and, hence, $(Z_k, \bar{Z}_k)$ is a minimal cut for G.

(ii) $2 \leqslant p - k \leqslant 3$. In this case $(Z_k, \bar{Z}_k)$ is a minimal cut for G if $c(Z_k, \bar{Z}_k) < c(Z_p, \bar{Z}_p)$ or $(Z_p, \bar{Z}_p)$ is a minimal cut for G if $c(Z_p, \bar{Z}_p) < c(Z_k, \bar{Z}_k)$. If $c(Z_k, \bar{Z}_k) = c(Z_p, \bar{Z}_p)$, then both cuts are minimal cuts for G.

(iii) $p - k \geqslant 4$. In this case either the one-node cut with minimal capacity is a minimal cut for G or else there exists a minimal cut which lies between $(Z_k, \bar{Z}_k)$ and $(Z_p, \bar{Z}_p)$.

The following algorithm formalizes the foregoing discussion.

ALGORITHM:

Step 0. Let $Z_1 = \{1\}$, $\bar{Z}_1 = N - \{1\}$. Also let $c(Z_1, \bar{Z}_1) = \sum_{(1,j) \in A} u_{1j}$, and $k = 1$.

Step 1. Determine

$$c(Z_k \cup \{j^*\}, \bar{Z}_k - \{j^*\}) = \min_{\substack{j \in \bar{Z}_k \\ j \neq n}} \{c(Z_k \cup \{j\}, \bar{Z}_k - \{j\})\}.$$

If $c(Z_k, \bar{Z}_k) \geqslant c(Z_k \cup \{j^*\}, \bar{Z}_k - \{j^*\})$, then set $Z_{k+1} = Z_k \cup \{j^*\}$ and $\bar{Z}_{k+1} = N - Z_{k+1}$, $k = k + 1$, go to Step 1. Otherwise, go to Step 2.

Step 2. Let $\bar{Z}_{n-1} = \{n\}$, $Z_{n-1} = N - \bar{Z}_{n-1}$. Also let

$$c(Z_{n-1}, \bar{Z}_{n-1}) = \sum_{(j,n) \in A} u_{jn}, \quad \text{and } p = n - 1.$$

Step 3. If $p = k$, go to Step 4; otherwise determine

$$c(Z_p - \{j^*\}, \bar{Z}_p \cup \{j^*\}) = \min_{\substack{j \in Z_p \\ j \notin Z_k}} \{c(Z_p - \{j\}, \bar{Z}_p \cup \{j\})\}.$$

If $c(Z_p, \bar{Z}_p) \geqslant c(Z_p - \{j^*\}, \bar{Z}_p \cup \{j^*\})$, then let $Z_{p-1} = Z_p - \{j^*\}$, $\bar{Z}_{p-1} = N - Z_{p-1}$, and $p = p - 1$, go to Step 3. Otherwise, go to Step 4.

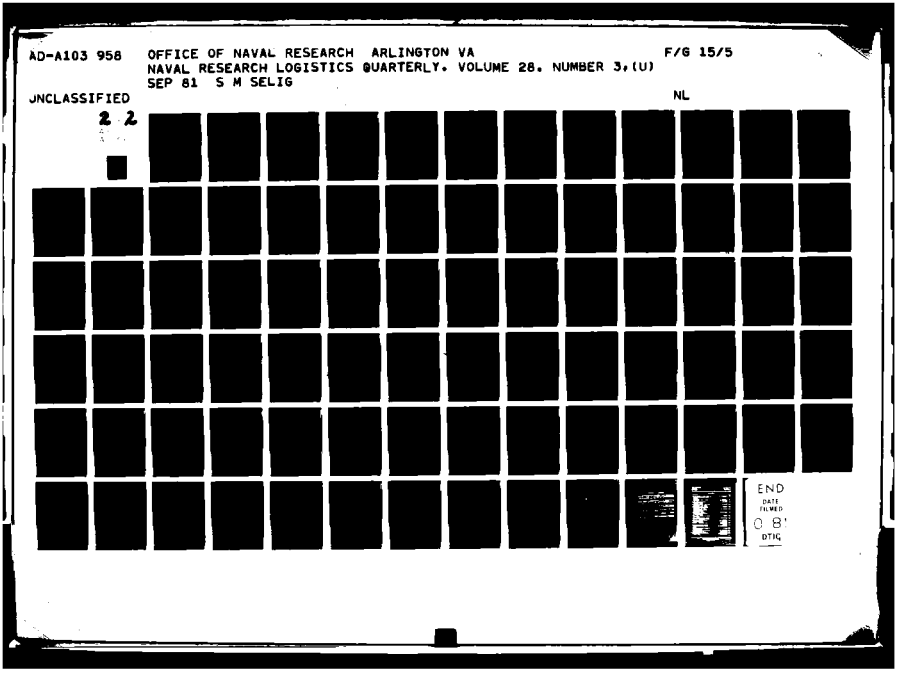
Step 4. If $Z_k = Z_p$ stop, a minimal cut for the original network $G(N, A)$ is at hand. $(Z_k, \bar{Z}_k) = (Z_p, \bar{Z}_p)$ is a minimal cut, with capacity $c(Z_k, \bar{Z}_k)$ already calculated.

If $p - k \leqslant 3$ (i.e., Z_k and Z_p differ by at most 3 nodes) then $(Z_k, \bar{Z}_k)$ is a minimal cut for the original network if $c(Z_k, \bar{Z}_k) < c(Z_p, \bar{Z}_p)$ or $(Z_p, \bar{Z}_p)$ is a minimal cut for G if $c(Z_p, \bar{Z}_p) < c(Z_k, \bar{Z}_k)$, or else both cuts are minimal cuts for G.

If $p - k > 3$, then let $(Z_k, \bar{Z}_k)$ and $(Z_p, \bar{Z}_p)$ divide the node set of the original network into three subsets $N_1 = Z_k$, $N_2 = Z_p - Z_k$, and $N_3 = \bar{Z}_p$, respectively. Construct the intermediate subnetwork $G_2(N_2^*, A_2)$, where $N_2^* = N_2 \cup \{s^*, t^*\}$, where s^* and t^* represent a super source and a super sink, respectively. Determine u_{s^*j} and u_{jt^*}, $j \in N_2$ as described before. Obtain the maximal flow for G_2 by using Karzanov's algorithm [12]. At maximal flow apply Ford and Fulkerson's [4] labeling algorithm once to obtain a minimal cut. The minimal cut obtained by this procedure is a minimal cut for the original network, G.

Assuming Karzanov's [12] algorithm requires $0(n^3)$ operations, we can determine the computational upper bound for the proposed algorithm. Let us assume that after one-node decomposition we have $|Z_k| = k$, $|\bar{Z}_p| = n - p$, also let $x = k + (n - p)$. Thus, the total number of additions and/or subtractions for Steps 0 through 3 can be written as $0(xn^2)$. Similarly, the number of comparisons can be written as $0(xn)$. If $k = 1$ and $p = n - 1 (x = 2)$, then the algorithm requires $0(n^2)$ additions and $0(n)$ comparisons and concludes that either $(Z_1, \bar{Z}_1)$ or $(Z_{n-1}, \bar{Z}_{n-1})$ (whichever has the smallest capacity) is a minimal cut for G or else a minimal cut should be determined on the overall network by utilizing any efficient maximal flow algorithm available.

If $2 \leqslant x \leqslant (n - 4)$ then the algorithm will produce an intermediate network G_2 which will lie between the cuts $(Z_k, \bar{Z}_k)$ and $(Z_p, \bar{Z}_p)$. Including the super source s^* and the super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sink t^*, G_2 will contain $n - x + 2$ nodes. Finding the maximal flow in this network will give us a computational upper bound of $0[(n - x + 2)^3]$ in Step 4, if Karzanov's [12] algorithm is used. Assuming that the upper bound is reached in Karzanov's algorithm, then the computational upper bound of the proposed decomposition algorithm can be written as $0((n - x)^3)$.

5. CONCLUSIONS

Two alternative decomposition procedures have been presented. In the first procedure, one-node cuts were selected in a fixed predetermined order. If we naively assume that the values of one-node cuts are random and coming from the same distribution, then we can find out the expected numbers of runs, $E(X)$, (increasing or decreasing blocks) from the formula $E(X) = (2(N-1) - 1)/3 = (2N - 3)/3$ and its variance $V(X) = [16(N-1) - 29]/90 = (16N - 45)/90$ given by Levene [14], where $(N - 1)$ is the total number of observations (one-node cut values). For $(N - 1) > 20$, the distribution of X can be approximated by a normal distribution with mean $E(X)$ and variance $V(X)$ as given above [14].

Therefore, in a large-scale network, the probability of having $X \leqslant 2$ can be obtained from the statistic

$$Z = \frac{X - E(X)}{(V(X))^{1/2}} = (-2N + 9)/[(16N - 45)/10]^{1/2}.$$

For $N = 22$, $P(X \leqslant 2) = \Phi(-6.317) \cong 1.0 \times 10^{-9}$.

The probability obtained above indicates that the probability of having one or two blocks in $(N - 1)$ cuts is very small. In other words, the probability of hitting the worst case is almost zero (if the two blocks are an increasing and a decreasing one) as well as the probability of finding the minimal cut without further computation (if there is either one block, or two blocks starting with a decreasing one). However, if the arc capacities are identically, uniformly and independently distributed between two integers, a and b, and if the network is complete, then

$$E(u_{ij}) = \mu_{ij} = \mu = (b + a)/2$$

and

$$V(u_{ij}) = \sigma_{ij}^2 = \sigma^2 = (b - a)^2/12.$$

Thus, the expected capacity of the cut $(Z_k, \bar{Z}_k)$ can be written as

$$E\{c(Z_k, \bar{Z}_k)\} = k(n - k)\mu = k(n - k)(b + a)/2$$

and

$$V\{c(Z_k, \bar{Z}_k)\} = [k(n - k)]^2 (b - a)^2/12,$$

where $k(n - k)$ is the number of arcs on the cut $(Z_k, \bar{Z}_k)$ from the node set Z_k to the node set $\bar{Z}_k$. It is clear from the above formulas that the one-node cut $(Z_{n/2}, \bar{Z}_{n/2})$ has the largest expected capacity and also the largest variance.

We have simulated some random networks with arc capacities uniformly and independently distributed over the range (1, 50) and the number of increasing and/or decreasing sequences are given in Table 1 below.

It is clear from the table that a sparse network can be decomposed into more subnetworks than a dense network.

TABLE 1 — *Average Number of Increasing and/or Decreasing Blocks on Random Networks*

Number of Nodes	Number of Replications	Density $\|A\|/n(n-1)$									
		.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
10	50	3.30	3.92	3.82	3.36	3.26	3.02	2.80	2.64	2.44	2.24
20	50	6.5	5.96	5.48	4.48	4.16	3.64	3.76	3.40	2.64	2.80
30	50	10.02	7.88	6.56	5.72	4.96	4.76	4.24	3.76	3.40	3.00
40	50	11.52	8.88	7.20	6.32	5.72	4.52	3.32	4.32	3.60	3.16
50	50	13.78	10.04	8.82	6.88	6.08	5.36	4.36	4.32	3.88	3.72
75	50	18.08	12.6	9.68	8.20	7.12	6.08	5.48	4.84	4.40	4.20

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EXPEDIENTS FOR SOLVING SOME SPECIALLY STRUCTURED MIXED-INTEGER PROGRAMS*

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ABSTRACT

In this paper we consider dual angular and angular structured mixed integer programs which arise in some practical applications. For these problems we describe efficient methods for generating a desirable set of Benders' cuts with which one may initialize the partitioning scheme of Benders. Our research is motivated by the computational experience of McDaniel and Devine who have shown that the set of Benders' cuts which are binding at the optimum to the linear relaxation of the mixed integer program, play an important role in determining an optimal mixed integer solution. As incidental results in our development, we provide some useful remarks regarding Benders' and Dantzig-Wolfe's decomposition procedures. The computational experience reported seems to support the expedients recommended in this paper.

1. INTRODUCTION

In this paper we consider some mixed integer programs with special structures. Our research is motivated by the empirical study of McDaniel and Devine [10] who discovered that when solving a mixed integer program using the decomposition scheme of Benders [2], the set of Benders' cuts or constraints in the master problem which are binding at the linear programming optimum are almost sufficient to obtain an optimal solution to the mixed integer program, in that only a few additional Benders' constraints are required. In fact, working with problems of different structures, McDaniel and Devine [10] found that if they were to solve the linear relaxation of the mixed integer program using the decomposition scheme of Benders [2], that is, solve the master program at each iteration as a linear program, then when they switched over to solving each master program as an integer programming problem after having obtained the linear programming optimum, very few (typically one) additional Benders' inequalities were required to verify optimality.

In the light of this study, we examine two types of mixed integer programs, each with a special structure. For these programs, we describe a means of exploiting their special structures to obtain an initial set of Benders' cuts with which one may initialize the usual partitioning scheme of Benders [2]. This initial set of cuts is desirable in view of McDaniel and Devine's [10] observation, in that the linear relaxation of the master program with these cuts obtains an optimal or near optimal solution to the linear relaxation of the original mixed integer program. As incidental results in our analysis, we provide some interesting remarks regarding Benders' [2] and Dantzig-Wolfe's [4] decomposition procedures.

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2. MIXED INTEGER PROGRAMS WITH DUAL ANGULAR STRUCTURE

In this section, we consider a mixed integer program of the form

$$\text{P1:} \quad \begin{aligned} \text{minimize} \quad & c_1^t x_1 + c_2^t x_2 + d^t y \\ \text{subject to} \quad & A_1 x_1 + D_1 y \geq b_1 \\ & A_2 x_2 + D_2 y \geq b_2 \\ & G y \geq g \\ & x_1,\ x_2,\ y \geq 0 \\ & y \in \Omega \end{aligned}$$

where Ω is the set of integer vectors and where a superscript t will throughout denote a matrix transpose operation. Problem P1 is said to have a dual angular structure with the variables y being coupling variables (see [7] for example). For the sake of notational simplicity, we have considered only two diagonal blocks of x-variables, though of course, the development generalizes to any number of such blocks.

Problem P1 may arise, for example, within the context of discrete stochastic programs with recourse [13] where in addition, the "first stage decision" vector y is restricted to be integer valued (typically binary-valued). The proposed approach in this case becomes particularly attractive if the random right-hand side vector b can only be one of a reasonably few number of vectors $b_1,\ b_2,\ \ldots,\ b_r$ according to some discrete probability distribution.

Now, let LP1 denote the linear relaxation of Problem P1, that is, Problem P1 with the restrictions $y \in \Omega$ relaxed. Further, merely for the sake of simplicity, assume that each set of variables x_1 and x_2 has one component which is an artificial variable with a large positive coefficient in the objective function and a column of ones in its corresponding constraint-coefficient matrix. This assumption guarantees that a feasible solution to Problem LP1 exists for any fixed value of $y \in Y$, where

$$Y = \{y:\ Gy \geq g,\ y \geq 0\} \tag{1}$$

Using the partitioning scheme of Benders [2], Problem LP1 may be decomposed as

$$\min_{y \in Y} d^t y + [\min_{x_1, x_2} c_1^t x_1 + c_2^t x_2 : A_1 x_1 \geq b_1 - D_1 y,\ x_1 \geq 0, \quad A_2 x_2 \geq b_2 - D_2 y,\ x_2 \geq 0]$$

which may be rewritten (using duality) as

$$\min_{y \in Y} d^t y + [\{\max_{\pi_1} (b_1 - D_1 y)^t \pi_1 : A_1^t \pi_1 \leq c_1,\ \pi_1 \geq 0\} + \{\max_{\pi_2} (b_2 - D_2 y)^t \pi_2 : A_2^t \pi_2 \leq c_2,\ \pi_2 \geq 0\}] \tag{2}$$

Finally, denoting

$$S_1 = \{\pi_1 : A_1^t \pi_1 \leq c_1,\ \pi_1 \geq 0\} \tag{3}$$

$$S_2 = \{\pi_2 : A_2^t \pi_2 \leq c_2,\ \pi_2 \geq 0\} \tag{4}$$

and observing that our assumptions imply that S_1 and S_2 are bounded, Problem LP1 may be written as Benders' master problem

BMP: minimize $d'y + z_1 + z_2$

subject to

(5) $z_1 \geqslant (b_1 - D_1y)'\pi_1^j$ for each $j \in J_1$

(6) $z_2 \geqslant (b_2 - D_2y)'\pi_2^j$ for each $j \in J_2$

$y \in Y$

where

(7) $J_1 = \{j : \pi_1^j \text{ is an extreme point of } S_1\}$

(8) $J_2 = \{j : \pi_2^j \text{ is an extreme point of } S_2\}$.

Let this problem yield an optimal vector y^*. Then, the corresponding x_1^* and x_2^* which define an optimal solution (x_1^*, x_2^*, y^*) to Problem LP1 are obtained respectively from (2) as dual optimal vectors to Problems SP1 (y^*) and SP2 (y^*), where

SP1(y): $\max \{(b_1 - D_1y)'\pi_1 \quad : \quad \pi_1 \in S_1\}$

SP2(y): $\max \{(b_2 - D_2y)'\pi_2 \quad : \quad \pi_2 \in S_2\}$.

However, not all the constraints in (5) and (6) may be necessary to define an optimal solution (z_1^*, z_2^*, y^*) to Problem BMP. As shown by Benders [2], the following relaxation of BMP may be sufficient so long as, if $(\hat{z}_1, \hat{z}_2, \hat{y})$ solves BMP (k, l), then $\hat{z}_1$ and $\hat{z}_2$ are respectively the optimal objective values of Problems SP1 $(\hat{y})$ and SP2 $(\hat{y})$:

BMP(k,l): minimize $d'y + z_1 + z_2$

subject to

(9) $z_1 \geqslant (b_1 - D_1y)'\pi_1^j$ for $j = 1, \ldots, k$

(10) $z_2 \geqslant (b_2 - D_2y)'\pi_2^j$ for $j = 1, \ldots, l$

$y \in Y$.

If this is not the case, say, $\hat{z}_1 < (b_1 - D_1\hat{y})'\pi_1^{k+1}$, where π_1^{k+1} solves SP1 $(\hat{y})$, then an additional Benders' cut (or constraint) $z_1 \geqslant (b_1 - D_1y)'\pi_1^{k+1}$ is appended to Problem BMP (k, l). In this manner, Benders' [2] scheme iterates between the master problem BMP (k, l) and the subproblems SP1 (y) and SP2 (y) until for some k and l, the above termination criterion is satisfied. It is our purpose to show how one may construct such a problem BMP (k, l) is a more efficient manner. Thereafter, as indicated by McDaniel and Devine [10], if one commences the application of Benders' scheme on Problem P1 with the master program BMP(k, l), then a minimal amount of additional cuts of the type (9), (10) may be required to solve P1.

The main property we use toward this end is the dual relationship between Benders' [2] scheme and the Dantzig-Wolfe [4] decomposition method (see Lasdon [7], for example). Observe that the dual (LD1) to Problem LP1 has a block diagonal (or angular) structure as shown below

LD1: maximize $b_1^t \pi_1 + b_2^t \pi_2 + g^t \alpha$

subject to

$$D_1^t \pi_1 + D_2^t \pi_2 + G^t \alpha \leqslant d$$

$$\pi_1 \in S_1$$

$$\pi_2 \in S_2$$

$$\alpha \geqslant 0.$$

This problem is most amenable to the Dantzig-Wolfe decomposition method [4] which writes each $\pi_1 \in S_1$ as a convex combination $\pi_1 = \sum_{j \in J} \lambda_{1j} \pi_1^j$, where $\sum_{j \in J_1} \lambda_{1j} = 1$, $\lambda_{1j} \geqslant 0$, and similarly expresses each $\pi_2 \in S_2$. In this manner, Problem LD1 is written as the Dantzig-Wolfe master program.

DWMP: maximize $\sum_{j \in J_1} \lambda_{1j} (b_1^t \pi_1^j) + \sum_{j \in J_2} \lambda_{2j} (b_2^t \pi_2^j) + \alpha^t g$

subject to

$$\sum_{j \in J_1} \lambda_{1j} (D_1^t \pi_1^j) + \sum_{j \in J_2} \lambda_{2j} (D_2^t \pi_2^j) + G^t \alpha \leqslant d$$

$$\sum_{j \in J_1} \lambda_{1j} = 1, \qquad \sum_{j \in J_2} \lambda_{2j} = 1$$

$$\lambda_{1j} \geqslant 0,\ j \in J_1;\ \lambda_{2j} \geqslant 0,\ j \in J_2;\ \alpha \geqslant 0.$$

Again, not all the columns for λ_{1j}, $j \in J_1$ and λ_{2j}, $j \in J_2$ may be necessary to solve LD1 through DWMP. In particular, the following restriction of DWMP may be sufficient.

DWMP(k, l): maximize $\sum_{j=1}^{k} \lambda_{1j} (b_1^t \pi_1^j) + \sum_{j=1}^{l} \lambda_{2j} (b_2^t \pi_2^j) + \alpha^t g$

subject to

(11) $$\sum_{j=1}^{k} \lambda_{1j} (D_1^t \pi_1^j) + \sum_{j=1}^{l} \lambda_{2j} (D_2^t \pi_2^j) + G^t \alpha \leqslant d$$

(12) $$\sum_{j=1}^{k} \lambda_{1j} = 1$$

(13) $$\sum_{j=1}^{l} \lambda_{2j} = 1;\ \lambda_{1j} \geqslant 0,\ j = 1, \ldots, k;\ \lambda_{2j} \geqslant 0,\ j = 1, \ldots, l;\ \alpha \geqslant 0$$

so long as, if $\hat{y}$, $\hat{z}_1$, and $\hat{z}_2$ are optimal dual variables associated with constraints (11), (12) and (13), respectively, then $\hat{z}_1$ and $\hat{z}_2$ are respectively the optimal solution values of Problems SP1 ($\hat{y}$) and SP2 ($\hat{y}$). If this is not the case, say $\hat{z}_1 < (b_1 - D_1 \hat{y})^t \pi_1^{k+1}$, where π_1^{k+1} solves SP1 ($\hat{y}$), then an additional column corresponding to π_1^{k+1} (with variable $\lambda_{1(k+1)}$) is appended to Problem DWMP (k, l). In this manner, Dantzig-Wolfe's [4] method iterates between the master problem DWMP (k, l) and the subproblems SP1 (y) and SP2 (y) until for some k and l, the above termination criterion is satisfied.

Note the dual nature of the foregoing remarks [7]. In fact, Problems DWMP and BMP are duals of each other and so are Problems DWMP (k, l) and BMP (k, l). Thus, once we have solved LD1 and terminated with some DWMP (k, l), one can easily write out the Problem BMP (k, l) which solves LP1. The reason why it's preferable to solve LD1 is of course the savings in effort whereby the Dantzig-Wolfe decomposition scheme maintains a basis of a fixed size as opposed to one of an ever increasing size when using Benders' scheme. This may be more pronounced if y is a vector of a few variables and Y has several constraints and less pronounced otherwise.

We will consider next a mixed integer program with an angular structure. Although this structure does not lend itself to as straightforward an analysis as above, we are nonetheless able to make some pertinent observations and remarks which are supported by our computational experience.

3. MIXED INTEGER PROGRAMS WITH ANGULAR STRUCTURE

Consider a mixed integer program P2 with the following angular structure.

P2: minimize $c^t x + d^t y$

subject to

(14) $Cx + Dy \geqslant f$

(15) $Ax \geqslant b$

$Gy \geqslant g$

$x \geqslant 0,\ y \geqslant 0$ and $y \in \Omega$

where Ω is the set of integer vectors. Problem P2 may be used to model fixed charge production and location, and other related problems, where the vector y is constrained to be binary valued, denoting a yes/no type of decision, and the vector x represents production or allocation variables. As before, let LP2 be the linear relaxation of P2, that is, with the constraint $y \in \Omega$ relaxed, let Y be as defined in Equation (1), and assume that the set

(17) $X = \{x: Ax \geqslant b,\ x \geqslant 0\}$

is nonempty. For simplicity, we assume that both X and Y are bounded in the following development. Also, we have considered just one block of constraints in (15), although it may be that $x^t = (x_1^t, \ldots, x_r^t)$, $c^t = (c_1^t, \ldots, c_r^t)$, $b^t = (b_1^t \ldots, b_r^t)$, $C = [C_1, \ldots, C_r]$, and

$$A = \begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_r \end{bmatrix}.$$

Again, we will assume that the vector x has an artificial variable as one of its components, with a large positive objective function coefficient, and a column of ones in the rows (14) and zeroes in the rows (15). This will then guarantee that for any fixed $y \in Y$, there exists a vector x such that (x, y) is feasible to LP2.

Now, one may partition Problem LP2 using Benders' decomposition [2] scheme as follows:

$$\min_{y \in Y} d^t y + \{\min_x c^t x : Cx \geqslant f - Dy,\ Ax \geqslant b,\ x \geqslant 0\}$$

$$\equiv \min_{y \in Y} d^t y + \{\max_{\pi, \alpha} (f - Dy)^t \pi + b^t \alpha : C^t \pi + A^t \alpha \leqslant c,\ \pi \geqslant 0,\ \alpha \geqslant 0\}.$$

Defining the sets

$$S = \{(\pi, \alpha) : C^t \pi + A^t \alpha \leqslant c,\ \pi \geqslant 0,\ \alpha \geqslant 0\} \tag{18}$$

$$J = \{j : (\pi^j, \alpha^j) \text{ are extreme points of } S\} \tag{19}$$

a relaxed Benders' master problem with k Benders' cuts may be written as

MP(K): minimize z

subject to

$$z \geqslant d^t y + (f - Dy)^t \pi^p + b^t \alpha^p \text{ for each } p \in K \subseteq J \tag{20}$$

$$y \in Y$$

where $K = \{1, \ldots, k\}$. Then if $(\hat{z}, \hat{y})$ solves Problem MP(K), one would solve the subproblem SP $(\hat{y})$ or its dual SD $(\hat{y})$, where

SP(y): maximize $\{d^t y + (f - Dy)^t \pi + b^t \alpha : (\pi, \alpha) \in S\}$

and its dual is

SD(y): minimize $\{d^t y + c^t x : Cx \geqslant f - Dy,\ Ax \geqslant b,\ x \geqslant 0\}$.

If it turns out that $\hat{z} = d^t \hat{y} + c^t \hat{x}$ where $\hat{x}$ solves SD $(\hat{y})$, then $(\hat{x}, \hat{y})$ solves LP2. Otherwise, $\hat{z} < d^t \hat{y} + c^t \hat{x}$, and a Benders' cut $z \geqslant d^t y + (f - Dy)^t \pi^{k+1} + b^t \alpha^{k+1}$ is appended to rows (20), where $(\pi^{k+1}, \alpha^{k+1})$ solves SP$(\hat{y})$. The process now continues in this manner.

Note that every time a new row is added to (20), a solution to Problem SD(y) is called for, a problem with an angular structure. However, Problem LP2 itself has just one more diagonal block than Problem SD(y). Therefore, the application of Dantzig-Wolfe's [4] decomposition to the former problem has a basis of dimension one more than that applied to the latter problem. Moreover, while using Dantzig-Wolfe's decomposition, we would be able to exploit any special structures which the set Y may have. This advantage may be lost in the solution of problems MP(K) for $|K| \geqslant 2$. Hence, we propose to do the following. We will solve Problem LP2 using Dantzig-Wolfe's decomposition method [4]. From this solution process, we will show how one may obtain a desirable set of cuts (20) so as to avoid expending effort in generating cuts which may not be binding at the linear programming optimum. From this point onwards, one may continue with the solution of Problem LP2 using Benders' decomposition and then switch over to solve Problem P2 as advocated by McDaniel and Devine [10]. In conclusion, we will also make a remark which may be a further expedient in the latter process.

First of all, let us present a pertinent observation afforded by Balas and Bergthaller [1]. Note that if (x^*, y^*) solves LP2, then the set of Benders' cuts which are binding at optimality are indexed by the following subset of the set J of Equation (19)

$$\bar{J} = \{j \in J : (\pi^j, \alpha^j) \text{ is optimal to Problem SP}(y^*)\}. \tag{21}$$

However, not all the cuts indexed by $j \in \bar{J}$ may be necessary to determine y^* in the Benders' master problem at optimality. In particular, if y^* has δ positive components, then $SD(y^*)$ has δ degenerate basic variables at optimality, and so $SP(y^*)$ has up to 2^δ alternative optimal solutions. Thus, $|\bar{J}| \leqslant 2^\delta$ with the equality often holding [1]. But the dimension of Benders' master program is $n + 1$, where y has n components. Further, we know that $n - p$ of the hyperplanes $y_i \geqslant 0 \quad i \in \{1, \ldots, n\}$ hold as an equality at y^*. Thus, at most, some $(n + 1) - (n - p) = p + 1$ Benders' cuts indexed by $\bar{J}$ may be needed to define the linear programming optimum y^*. We now proceed to demonstrate how one may generate a desirable subset of cuts indexed by $\bar{J}$. Our discussion will motivate the choice of this subset. In addition, we will show how some more Benders' cuts which are binding at optimality may be recovered during the application of Dantzing-Wolfe's decomposition procedure to Problem LP2, without any extra effort.

Toward this end, suppose that we have solved Problem LP2 using Dantzig-Wolfe's decomposition method. Let the Dantzig-Wolfe master program at optimality be

$$\text{DWMP2:} \quad \text{minimize} \quad \sum_{p=1}^{k} \lambda_p (c^t x_p) + \sum_{q=1}^{l} \gamma_q (d^t y_q)$$

$$\sum_{p=1}^{k} \lambda_p (C x_p) + \sum_{q=1}^{l} \gamma_q (D y_q) \geqslant f \tag{22}$$

$$\sum_{p=1}^{k} \lambda_p = 1, \ \lambda_p \geqslant 0 \text{ for } p = 1, \ldots, k \tag{23}$$

$$\sum_{q=1}^{l} \gamma_q = 1, \ \gamma_q \geqslant 0 \text{ for } q = 1, \ldots, l \tag{24}$$

Suppose (λ^*, γ^*) solves DWMP2 with $\lambda_p^* > 0$ $p = 1, \ldots, k$, $\gamma_q^* > 0$ $q = 1, \ldots, l$. Then (x^*, y^*) solves LP2 where $x^* = \sum_{p=1}^{k} \lambda_p^* (x_p)$, $y^* = \sum_{q=1}^{l} \gamma_q^* (y_q)$. Here, $x_p \in$ vert X (Equation (17)) and $y_q \in$ vert Y (Equation (11)). Further, if π^*, θ^* and ϕ^* are optimal dual variables associated with the constraints (22), (23) and (24) respectively, then θ^* and ϕ^* are respectively the optimal solution values of Problems $SPx(\pi^*)$ and $SPy(\pi^*)$, where

$$SPx(\pi): \quad \text{minimize} \quad \{(c - C^t\pi)^t x : x \in X\}$$

$$SPy(\pi): \quad \text{minimize} \quad \{(d - D^t\pi)^t y : y \in Y\}.$$

Lemma 1 below recovers a dual optimal solution to Problem LP2 from the Dantzig-Wolfe master and subproblems at optimality. This result may be viewed as an extended special case of the more general result due to Magnanti, Shapiro, and Wagner [8].

LEMMA 1: Let Problems DWMP2, SP$x(\pi)$ and SP$y(\pi)$ and the vector π^* be as defined above. Further, let α^* and β^* be optimal dual variables in Problems SP$x(\pi^*)$ and SP$y(\pi^*)$, respectively. Then, $(\pi^*, \alpha^*, \beta^*)$ is an optimal dual solution to Problem LP2.

PROOF: From the duals to Problems SP$x(\pi^*)$ and SP$y(\pi^*)$, it follows that $(\pi^*, \alpha^*, \beta^*)$ is dual feasible to LP2. Further, with x^*, y^*, θ^* and ϕ^* as defined above, the Dantzig-Wolfe termination criterion implies that $\theta^* = b^t\alpha^*$ and $\phi^* = g^t\beta^*$. Thus, from DWMP2, $c^tx^* + d^ty^* = f^t\pi^* + \theta^* + \phi^* = f^t\pi^* + b^t\alpha^* + g^t\beta^*$. This completes the proof.

Noting that $(\pi^*, \alpha^*) \in S$, let us construct the following Benders' master problem with a single cut constraint.

$$\begin{array}{lll} \text{MP}(^*): & \text{minimize} & z \\ (25) & \text{subject to} & z \geq d^ty + (f - Dy)^t\pi^* + b^t\alpha^* \\ & & y \in Y. \end{array}$$

Now, consider the following result. In view of Lemma 1, this result is closely associated with the concept of "strongest surrogate constraints," a name given to constraints of type (25). (See Geoffrion [5], Glover [6] and Rardin and Unger [11]).

LEMMA 2: Let (x^*, y^*) and $(\pi^*, \alpha^*, \beta^*)$ be optimal primal and dual solutions respectively to Problem LP2. Then y^* is optimal to MP$(^*)$ with $z^* = c^tx^* + d^ty^*$.

PROOF: A Lagrangian dual to Problem LP2 is

$$\text{LD:} \quad \text{maximize} \quad \{h(\pi, \alpha) : \pi \geq 0, \alpha \geq 0\}$$

$$\text{where} \quad h(\pi, \alpha) = \min_{y \in Y} \{d^ty + (f - Dy)^t\pi + b^t\alpha\} + \min_{x \geq 0} \{(c - C^t\pi - A^t\alpha)^tx\}.$$

Since there is no duality gap between LP2 and LD, we know that (π^*, α^*) solves *LD* with objective value $h(\pi^*, \alpha^*)$ equal to $c^tx^* + d^ty^*$. Further, (x^*, y^*) evaluates $h(\pi^*, \alpha^*)$. But by complementary slackness in LP2, the second problem in $h(\pi^*, \alpha^*)$ has a value of zero at $x = x^*$. Thus, the first problem in $h(\pi^*, \alpha^*)$, which is Problem MP$(^*)$, has an optimal solution y^* and an optimal value of $c^tx^* + d^ty^*$. This completes the proof.

Let us digress momentarily to discuss a certain issue related to the strongest surrogate constraint (25). Often, such a constraint is prescribed as a device for somehow fathoming partial solutions, or selecting branching variables in a branch and bound or implicit enumeration context, implemented within the framework of an algorithm which solves Problem P2 using Benders' decomposition. Lemma 3 below addresses the case when such a strategy may not be very meaningful, that is, when (25) degenerates to simply $z \geq z^*$.

LEMMA3: Let (x^*, y^*) and $(\pi^*, \alpha^*, \beta^*)$, respectively, be optimal primal and dual solutions (of value z^*) for Problem LP2. Then, if $y^* \in \text{int } Y$, the strongest surrogate constraint (25) degenerates to $z \geq z^*$.

PROOF: If $y^* \in \text{int } Y$, then $\beta^* = 0$ and $y^* > 0$. Thus, $z^* = f'\pi^* + b'\alpha^*$, and the dual constraints $D'\pi + G'\beta \leqslant d$ are binding at $(\pi, \beta) = (\pi^*, \beta^*)$. In other words, $D'\pi^* = d$. Noting that (25) is $z \geqslant (f'\pi^* + b'\alpha^*) + (d - D'\pi^*)'y$, the proof is complete.

COROLLARY: In case (25) does not degenerate to $z \geqslant z^*$, then y^* and hence each $y_q \in \text{vert } Y$, where $y^* = \sum_{q=1}^{l} \gamma_q^* y_q$, belong to the face of Y defined by its intersection with the hyperplane $(d - D'\pi^*)'y = g'\beta^*$.

The above result holds trivially from Lemma 2 and Equation (25). Its significance is the subject of our next discussion.

Now, Lemma 2 seems to indicate that a very desirable starting point at which to commence the solution of Problem LP2 by Benders' decomposition is with Problem MP(*). In addition to this single cut, one may also choose to generate the following desirable set of independent cuts which will be binding at the linear programming optimum, i.e., which are indexed by the set $\bar{J}$.

The reader may note from the corollary to Lemma 3 that each y_q, $q = 1, \ldots, l$ is an alternative optimal solution to MP(*). Therefore, it is intuitively appealing to generate l Benders' cuts in addition, one for each y_q through Problem SP(y_q) or $SD(y_q)$ for $q = 1, \ldots, l$. However, these cuts may not be binding at the linear programming optimum, that is, they may not be indexed by the set $\bar{J}$. To overcome this, consider the following subproblem which differs from SP(y_q) by the additional constraint (26)

$$\begin{aligned} \text{SP}'(y_q)\text{:} \qquad & \text{maximize} \quad d'y_q + (f - Dy_q)'\pi + b'\alpha \\ & \text{subject to} \quad (\pi, \alpha) \in S \\ & \qquad\qquad (f - Dy^*)'\pi + b'\alpha \geqslant z^* - d'y^* = c'x^*. \end{aligned}$$

Lemma 4 below characterizes a Benders' cut obtained through Problem SP'(y_q). Thereafter, we show how to solve SP'(y_q) while still exploiting the angular structure which SD(y_q) possesses.

LEMMA 4: Consider any y_q, $q \in \{1, \ldots, l\}$ and suppose that (π^q, α^q) solves Problem SP'(y_q). Consider the Benders' cut

$$z \geqslant d'y + (f - Dy)'\pi^q + b'\alpha^q. \tag{27}$$

Then the cut (27) satisfies $q \in \bar{J}$. Moreover, when this cut is appended to Problem MP(*), it deletes y_q provided there does not exist an $x_q \in X$ such that (x_q, y_q) solves LP2.

PROOF: Since (z^*, y^*) is an optimal solution to MP(J) where J is defined in Equation (19), it follows that

$$z^* \geqslant d'y^* + (f - Dy^*)'\pi + b'\alpha \tag{28}$$

for any $(\pi, \alpha) \in S$. Noting Equations (26) and (28), we may assert that the cut (27) is indexed by $q \in \bar{J}$. Now, y_q is an alternative optimal solution to MP(*). Thus, if (x_q, y_q) does not represent an optimal solution to LP2 for some $x_q \in X$, then there must exist some $u \in \bar{J}$ such that $z^* < d'y_q + (f - Dy_q)'\pi^u + b'\alpha^u$. But (π^u, α^u) is feasible to SP'(y_q). Hence, we must have $z^* < d'y_q + (f - Dy_q)'\pi^q + b'\alpha^q$. This completes the proof.

We will now proceed to show how one may solve $SP'(y_q)$ with no more effort than that required to solve $SP(y_q)$ or $SD(y_q)$. Observe that for a fixed $y = y^*$ in LP2, we obtain $z^* - d'y^* = c'x^*$ as the maximum value of $(f - Dy^*)'\pi + b'\alpha$ over $(\pi, \alpha) \in S$. Noting Equation (26), Problem $SP'(y_q)$ may be viewed as a preemptive priority multiobjective program over the set S, with an objective $(f - Dy^*)'\pi + b'\alpha$ having first priority and an objective $(f - Dy_q)'\pi + b'\alpha$ having a lower priority. According to Sherali and Soyster [12], there exists a positive scalar ξ^* such that for any $\xi \geq \xi^*$, an optimal solution to the following problem is also optimal to $SP'(y_q)$.

$$d'y_q + \text{maximum } \{[(1 + \xi)f - Dy_q - \xi Dy^*]'\pi + (1 + \xi)b'\alpha : (\pi, \alpha) \in S\}.$$

The dual to this problem is the following angular structured linear program.

$$SD'(y_q, \xi): \quad d'y_q + \text{minimum } \{c'x: \; Cx \geq (1 + \xi)f - Dy_q - \xi Dy^* \; Ax \geq (1 + \xi)b, \; x \geq 0\}.$$

Note that finding a ξ large enough ($\xi \geq \xi^*$) should be no problem since it is easy to show [12] that ξ^* in our case is simply the negative of the optimal dual variable associated with constraint (26). Moreover, we know that we have chosen ξ large enough after solving $SD'(y_q, \xi)$ by verifying that (26) is binding at the optimal solution (π^q, α^q).

Summarizing, one may first solve LP2 using Dantzig-Wolfe's decomposition and retrieve from that (through Lemma 1) the strongest surrogate constraint (25). Then the master program MP(*) with this single cut constraint determines the optimal value of LP2. However, since any point in the convex hull of $\{y_1, \ldots, y_l\}$ is also optimal to MP(*), one may determine for each $q = 1, \ldots, l$ a Benders' cut which will be binding at optimality and will also delete y_q by solving $SD'(y_q, \xi)$ for a large enough value of ξ as indicated above. In concluding this section, we provide below two remarks. The first of these pertains to an alternative strategy for generating an initial set of Benders' cuts and the second remark further addresses the issue of strongest surrogate constraints.

REMARK 1: This remark is an alternative to generating an initial set of Benders' cuts via Problems $SD'(y_q, \xi)$. Note that without any extra effort, during the solution of Problem LP2 by Dantzig-Wolfe's decomposition method, certain Benders' cuts are automatically available. As the following discussion motivates, these cuts are quite likely to be binding at optimality. Suppose we execute the Dantzig-Wolfe procedure as follows. At any stage, having generated extreme points $y_1, \ldots, y_v$ of Y in the master program, we continue to iterate between the master program and the subproblem $SPx(\pi)$ only until no more extreme points of X need to be generated. At this point in time, the current solution is clearly optimal to the problem

$$\text{DW(v)}: \quad \text{minimize} \quad c'x + \sum_{q=1}^{v} \gamma_q (d'y_q)$$

$$(29) \quad \text{subject to} \quad Cx + \sum_{q=1}^{v} \gamma_q (Dy_q) \geq f$$

$$(30) \quad Ax \geq b$$

$$(31) \quad \sum_{q=1}^{v} \gamma_q = 1$$

$$x \geq 0, \gamma_q \geq 0 \text{ for } q = 1, \ldots, v.$$

Currently, suppose that π^v, θ^v and ϕ^v are optimal dual variables associated with the constraints of the type (22), (23) and (24), respectively, and that α^v is an optimal dual solution to the subproblem $SPx(\pi^v)$. Then following Lemma 1, π^v, α^v and ϕ^v may be shown to be optimal dual variables associated with constraints (29), (30) and (31), respectively. But this means that if $(\bar{\gamma}, \bar{x})$ solves $DW(v)$, then for a fixed $\bar{y} = \sum_{q=1}^{v} \bar{\gamma}_q y_q$, $\bar{x}$ and (π^v, α^v) are, respectively, optimal solutions to $SD(\bar{y})$ and $SP(\bar{y})$. Therefore, a Benders' cut

$$z \geq d'y + (f - Dy)'\pi^v + b'\alpha^v \tag{32}$$

is readily available.

Hence, if we execute Dantzig-Wolfe's procedure by generating as many extreme points of X as required before we generate an additional extreme point of Y, then for each optimal combination of extreme points of Y which have been generated at any stage, we have a Benders' cut (32) available from the current dual variables. It is intuitive that these cuts are desirable. Of course, at termination, (32) is precisely (25). Also, one may finally choose to keep those cuts (32) which turn out to be binding at the linear programming optimum. We provide some computational experience in Section 5 regarding this remark.

REMARK 2: This note is related to the strongest surrogate constraint (25) and to our comments associated with Lemma 3. Observe that by also accomodating the constraints $Gy \geq g$ into the objective function of Problem LD in the proof of Lemma 2, it is easy to show that the following cut may be used in lieu of (25) in Problem MP(*).

$$z \geq d'y + (f - Dy)'\pi^* + b'\alpha^* + (g - Gy)\beta^*$$

or

$$z \geq (f'\pi^* + b'\alpha^* + g'\beta^*) + (d - D'\pi^* - G'\beta^*)'y.$$

This is again the strongest surrogate constraint of Rardin and Unger [11]. But $f'\pi^* + b'\alpha^* + g'\beta^* = z^*$, the optimal objective value of Problem LP2, and $(d - D'\pi^* - G'\beta^*) = \bar{d} \geq 0$, the reduced cost coefficient vector of the y-variables at the linear programming optimum. Thus, the above cut is

$$z \geq z^* + \bar{d}'y. \tag{33}$$

Observe that if y^* has all positive components, then $\bar{d} = 0$, whence (33) is simply $z \geq z^*$. Also, it is easy to see that if (25) degenerates to $z \geq z^*$, then so does (33), though not necessarily vice versa. For example, $g\beta^* \neq 0$ is sufficient for (25) not to generate to $z \geq z^*$, though not for (33). Moreover, since our master program treats the constraint set $y \in Y$ explicitly, we find (25) more appropriate.

4. ILLUSTRATIVE EXAMPLE

Consider the following problem adapted from Lasdon [7].

P2: $$\text{minimize}\{-2x_1 - x_2 - y_1 - y_2: -2x_1 - x_2 - y_1 - 2y_2 \geq -40,\ x \in X,\ y \in Y \text{ and } y \text{ integer}\}$$

$$X = \{x: -x_1 \geq -10,\ -x_2 \geq -10,\ -x_1 - x_2 \geq -15,\ x_1,\ x_2 \geq 0\}$$

$$Y = \{y: -y_1 - 3y_2 \geq -30,\ -2y_1 - y_2 \geq -20,\ y_1,\ y_2 \geq 0\}.$$

Using Dantzig-Wolfe's decomposition to solve LP2, we first generate a vector y by solving the problem: $\min\{-y_1 - y_2: y \in Y\}$. This yields $y = (6, 8)$. Continuing with the usual procedure but generating only vertices of X as needed, that is, solving DW(v = 1), yields $\bar{x} = (4, 10)$ as a convex combination of the vertices (0, 10) and (5, 10) of X. For this solution, $\pi^v = 1$, $b'\alpha^v = 0$ and hence the cut (32) is

$$z \geq y_2 - 40. \tag{34}$$

The next vertex of Y generated through $SPy(\pi^v)$ is $y = (10, 0)$. Now, v = 2 and the solution to DW(v = 2) through the continued generation of extreme points of X yields $\bar{x} = (10, 5)$, a single vertex of X, and $\bar{y} = \frac{5}{12}(6, 8) + \frac{7}{12}(10, 0) = \left(\frac{25}{3}, \frac{10}{3}\right)$. At this stage, $\pi^v = \frac{1}{3}$, $b'\alpha^v = \frac{-50}{3}$ and hence the cut (32) is

$$z \geq -\frac{2}{3}y_1 - \frac{1}{3}y_2 - 30. \tag{35}$$

However, no more vertices of Y need to be generated and so $x^* = (10, 5)$ and $y^* = \left(\frac{25}{3}, \frac{10}{3}\right)$ solves LP2 with optimal value $z^* = -\frac{110}{3}$. Thus, (35) is actually the strongest surrogate constraint (25). Note that according to Remark 2, that cut (33) is simply $z \geq -\frac{110}{3}$. Further, we obtain

$$\text{MP(*)}: \qquad \min\{z: z \geq -\frac{2}{3}y_1 - \frac{1}{3}y_2 - 30, y \in Y\}.$$

This problem has (6, 8) and (10, 0) as alternative optimal extreme point solutions with a value of $z^* = -\frac{110}{3}$. Note that the incidental cut (34) automatically obtained during the solution of LP2 is binding at the linear programming optimum. Indeed, this is also the cut which would be obtained through $SD'(y_q, 0)$ with $y_q \equiv (6, 8)$. The reader may easily verify that the problem $SD'(y_q, \xi)$, for $y_q = (10, 0)$ and for any $\xi \geq 0$, yields an optimal dual solution (π^q, α^q) with $\pi^q = 0$ and $b'\alpha^q = 25$. Hence, corresponding to $y_q = (10, 0)$, the Benders' cut which is binding at optimality is

$$z \geq -y_1 - y_2 - 25. \tag{36}$$

It turns out here that the cuts (34) and (36) imply the strongest surrogate constraint (35). This may be verified by using the surrogate multipliers $\frac{1}{3}$ and $\frac{2}{3}$ on (34) and (36) respectively. Thus, the master program MP (K) with the cuts (34) and (36) yields in this case an optimal solution to LP2 using the fewest number of Benders' cuts. In general one would have continued solving LP2 with the Benders' cuts (34), (35) and (36).

Now, continuing with the solution to Problem P2, we obtain an optimal solution to MP(K) with y restricted to be integer valued as $\bar{y} = (8, 4)$ with $\bar{z} = -36$. Solving the subproblem $SD(\bar{y})$, we obtain an optimal solution $\bar{x} = (10, 4)$ also with value -36. Thus, $(\bar{x}, \bar{y})$ solves P2 and we terminate without having to generate any further Benders' cuts. Indeed, such a propensity is borne out by the computational experience of McDaniel and Devine [10].

5. COMPUTATIONAL EXPERIENCE

In this section, we will attempt to support the ideas recommended in this paper through some computational experience. Toward this end, recall the principal thrust of this paper. McDaniel and Devine [10] have amply demonstrated that in solving a mixed integer program via Benders' [2] procedure, an attractive strategy is to solve its linear relaxation first via Benders' partitioning method, and use the resulting cuts as an initial set of cuts for the former problem. Hence, we have attempted in this paper to demonstrate how one may solve the linear relaxation via Benders' decomposition method more efficiently, for some specially structured mixed integer programs.

The discussion in Section 2 clearly demonstrates the value of the suggestions for dual angular structured problems, particularly when there are few y-variables and several more constraints in Y. On the other hand, the suggestions in Section 3 need computational support. More specifically, one needs to demonstrate how well the proposed initial set of Benders' cuts represents the linear programming solution, as well as the savings in effort which accrues from their use. Furthermore, the comments in Remark 1 need to be tested.

Hence, we performed the following brief computational experiment on an IBM 370 Model 158 computer with coding in Fortran. We generated linear programs of the form to minimize $c'x + d'y$, subject to coupling constraints $Cx + Dy \geqslant f$, $x \geqslant 0$, $y \geqslant 0$ and block constraints $Gy \geqslant g$, $y \geqslant 0$, with c and d generated uniformly on $[-11, -1]$, C and D generated uniformly on $[-50, 0]$, f generated uniformly on $[-251, -1]$, g generated uniformly on $[-6, -1]$ and with G having components 0 or -1 with an expected density of 0.5. All generated coefficients were rounded off to the nearest integer.

Table 1 summarizes the results. The "Usual Benders' Method" is being performed by treating the set $Y = \{y: Gy \geqslant g, y \geqslant 0\}$ as the set of complicating variables. The "Section -3-Method" generates an initial set of cuts via problems $SD'(\cdot)$ and then continues as usual, whereas the "Remark-1-Method" employs the recommendations embodied in Remark 1 of Section 3 to generate an initial set of cuts.

Essentially, both the methods of Section 3 perform better than the usual Benders' approach, with that of Remark 1 appearing somewhat superior. However, as in Problems 1 and 2, the Section-3-Method seemed better than the Remark-1-Method in some other larger sized problems we attempted in which NB >> NC. The reason being that the Remark-1-Method generates cuts from $SP(\bar{y})$ where $\bar{y}$ is a combination of some extreme points of Y, and whenever the Dantzig-Wolfe algorithm generates and drops off several y-vectors while solving LP2, the Remark-1-Method cuts are not as effective.

In conclusion, we reiterate that the savings in effort as reflected through columns f would have been more enhanced if Y had a special structure, since this could have been exploited in solving problems $SP_Y(\cdot)$ though not necessarily in solving $MP(K)$ for $|K| \geqslant 2$.

6. SOME CONCLUDING REMARKS

In conclusion, we present two further expeditious remarks. The first of these pertains to iteratively updating solutions to the subproblems $SD(y)$ rather than resolving each subproblem independently. Essentially, this remark in general contributes towards performing sensitivity

TABLE 1 — *Computational Results*

Problem Number	m	n	NC	NB	Usual-Benders Method		Section-3-Method				Remark-1-Method			
					a	b	c	d	e	f	c	d	e	f
1	50	50	30	40	12	3	4	4	4	0.427	5	7	3	0.445
2	50	50	10	40	5	3	4	4	4	0.760	4	5	4	0.900
3	40	40	30	40	6	3	4	4	4	0.850	4	4	4	0.627
4	30	30	10	40	6	3	4	5	5	0.973	5	5	4	0.757
5	30	30	30	40	9	3	4	5	5	0.648	4	4	4	0.396
6	20	20	10	40	4	3	4	4	4	0.768	3	3	3	0.638
7	50	50	40	30	8	2	3	3	3	0.626	4	4	2	0.431
8	50	50	40	10	8	3	4	4	4	0.835	5	5	4	0.464
9	40	40	40	30	7	2	3	3	3	0.537	2	2	2	0.326
10	30	30	40	10	5	3	4	5	5	1.379	3	3	3	0.621
11	30	30	40	30	5	3	4	5	5	1.270	3	3	3	0.622
12	20	20	40	10	10	3	4	4	4	0.625	3	3	3	0.286
13	50	50	40	40	8	2	3	3	3	0.604	4	4	2	0.432
14	40	40	40	40	7	2	3	3	3	0.477	2	2	2	0.297
15	30	30	40	40	5	3	4	5	5	1.190	3	3	3	0.619
16	20	20	40	40	10	3	4	4	4	0.494	3	3	3	0.271

(i) m = # of x variables; n = # of y variables; NC = # of coupling constraints; NB = # of block constraints (in Y).

(ii) $a \equiv$ total number of Benders' cuts required; $b \equiv$ # of cuts in "d" which were binding at optimality.

(iii) $c \equiv$ # of initial cuts (including strongest surrogate constraint); $d \equiv$ total number of cuts required; $e \equiv$ # of cuts in "d" which were binding at optimality; $f \equiv$ ratio of total solution time to the total solution time for the Usual-Benders-Method.

analysis in the context of Dantzig-Wolfe's decomposition procedure. However, the question regarding its computational efficiency is open to further research. The second remark briefly addresses the issue of effectiveness of Benders' partitioning scheme.

REMARK A: This remark concerns the iterative solutions of the subproblems SD(y), both, while continuing to solve LP2 as well as while subsequently solving P2 using Benders' decomposition. Suppose that we have solved SD($\bar{y}$) for some $\bar{y}$ using Dantzig-Wolfe's decomposition. (Here, $Ax \geq b$ may be one of several angular blocks of constraints). Let $\bar{\pi}$ and $\bar{\theta}$ be optimal dual multipliers for the constraints $Cx \geq f - D\bar{y}$ and the convexity constraint for the extreme points of X, respectively. Then, *on replacing* $\bar{y}$ *by some* $\hat{y}$ for the subsequent solution of Problem $SD(\hat{y})$, we have a dual feasible but primal infeasible Dantzig-Wolfe tableau. Let the ith row in this tableau have a negative right-hand-side value and denote the first m columns of the ith row of the current basis inverse by B_i, where C has m rows. Further, let the element of the basis inverse in the ith row and the same column as the row of the convexity constraint for the x-variables be ψ_i. Then, it follows that the reduced cost coefficient $(c - C\bar{\pi})'x - \bar{\theta}$ is nonnegative for each $x \in$ vert X. Now, in order to perform a dual simplex iteration in row i, we need to generate a column corresponding to a vertex x of X such that its updated value $B_iCx + \psi_i$ in row i is negative and yields the smallest value for the ratio $\{(c - C\bar{\pi})'x - \bar{\theta}\}/-\{B_iCx + \psi_i\}$ among all such columns. To accomplish this, consider the following linear fractional program.

$$\text{LFP:} \qquad \text{minimize} \left\{ \frac{(c - C\bar{\pi})' x - \bar{\theta}}{-(B_i C x + \psi_i)} : Ax \geq b,\ x \geq 0 \right\}.$$

Note that if one solves this problem using Charnes and Cooper's [3] method by letting $u = (1/-(B_i Cx + \psi_i)) \geq 0$ and $v = ux$, then at optimality, we will have $u > 0$ since $SD(\hat{y})$ is feasible by our original assumption. This transformation converts LFP to the linear program

$$\text{minimize } \{(c - C\bar{\pi})' v - \bar{\theta} u : Av - bu \geq 0,\ B_i C v + \psi_i u = -1,\ u \geq 0,\ v \geq 0\}.$$

Thus, instead of solving each subproblem from scratch using Dantzig-Wolfe's decomposition, one may update the solutions to subsequent subproblems through dual simplex iterations as indicated above. Note that this is a general technique which one may adopt to perform a sensitivity analysis on right-hand-side perturbations within the context of Dantzig-Wolfe's decomposition method.

REMARK B: It has been conjectured that the "nearer" the solution to the linear relaxation of a mixed integer program is to a solution of the orginal mixed integer program, the fewer the number of Benders' cuts that will be required to solve the mixed integer program. The discussion is this paper tends to support this statement in the following manner.

Suppose that it turns out that LP2 has an optimal solution (x^*, y^*) satisfying $y^* \in Y \cap \Omega$. Let $z^* = c'x^* + d'y^*$ and let $(\pi^*, \alpha^*, \beta^*)$ be a dual optimal solution to Problem LP2. Then, the strongest surrogate constraint (25) is also the Benders' cut which would be generated through $SP(y^*)$. Hence, using Lemma 2, it follows that the Benders' master problem

$$\begin{aligned} &\text{minimize} && z \\ &\text{subject to} && z \geq d'y + (f - Dy)'\pi^* + b'\alpha^* \\ &&& y \in Y \cap \Omega \end{aligned}$$

with the single cut (25) has (z^*, y^*) as an optimal solution. Thus, it is possible for the procedure to terminate after generating a single Benders' cut. In any case, if one is applying the usual Benders' scheme to solve Problem P2, then the procedure will terminate at the particular iteration at which the current master problem yields y^* (of value z^*) as an optimal solution.

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OPTIMAL CONTROL OF PRICE THROUGH RESTRICTED PRODUCTION

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ABSTRACT

Consider a regulated monopolist whose current profits would be maximized if they could charge a price $\bar{p}$, where $\bar{p}$ exceeds the current market price. By reducing production below current consumer demand the monopolist can create an illusion of a shortage and induce the regulator to allow a price increase. Conditions are given for which the production rate that maximizes the monopolist's expected discounted profits over an infinite horizon will have the property that the amount of unsatisfied consumer demand will be a nonincreasing function of current market price.

1. INTRODUCTION

Consider a monopolist facing a demand curve $D(p)$. If the cost of producing x units is cx then the firm would like to charge the profit maximizing price $\bar{p}$, maximizing the revenue $(p - c)D(p)$. Due to regulation the monopolist cannot immediately attain a price of $\bar{p}$. However, by producing less than $D(p)$, the demand associated with the price p, the monopolist can create the illusion of a shortage; this shortage induces the regulatory agency to increase the price. Although the economic grounds for the regulator to base his pricing decisions on the observed level of excess market demand may be scant, regulatory agencies in several industries behave in this fashion. For example, the degree of market disequilibrium in the natural gas industry is a crucial factor in the determination of the regulated price. Owen and Brautigam note [9], that "Producers are sometimes alleged to 'hold back' or delay production until regulators raise prices in the future."

Assuming the current price is p with $p < \bar{p}$ and $p > c$, the monopolist must trade off the profits lost by not satisfying a portion of current demand against the future profits that can be gained by holding back production in order to induce a price increase. Our approach utilizes the concept of continuous time and semiMarkov decision processes. (Markov decision models in which the firm controls price and not production have been analyzed elsewhere (see [3], [4], and [8].) At each point in time the state of the process is the (regulated) price at which the

monopolist's production can be sold; the monopolist's action is the rate at which the product is produced or, equivalently, the amount of consumer demand the firm chooses to leave unsatisfied. The regulator's willingness to increase price depends on the current price as well as the consumer demand that the firm leaves unsatisfied. Thus, it is natural to attempt to determine whether the optimal production strategy of the regulated monopolist increases in the regulated price p and, more importantly, whether the excess or unsatisfied consumer demand associated with its production strategy decreases as the price increases. Our goal is to derive conditions which ensure that the amount of unsatisfied consumer demand is a nonincreasing function of the price.

In Section 2 we describe the two models analyzed in this paper. Sections 3 and 4 are devoted to a characterization of the manner in which the firm's optimal production strategy depends on parameters such as the current price, the discount factor, and the length of the planning horizon. Finally, Section 5 suggests an extension of our model.

2. MODEL DESCRIPTION

In both models to be considered the monopolist observes the price p at which its product can be sold. This price is set by the regulatory agency. When the price is p, demand for the product occurs at an instantaneous rate $D(p)$. While the price is p and the monopolist's output rate is $y \leqslant D(p)$, the monopolist earns profit at the rate $(p - c)y$, where c is the true constant marginal cost of production.

Our two models differ in the manner in which the monopolist's production decision affects the evolution of the product's price. In model 1 we assume that at any instant during which the price is p and the amount of unsatisfied consumer demand is x, the price increases to $p + 1$ in a time that is governed by a Poisson process with rate $\lambda(p, x)$. Naturally, that amount x of unsatisfied consumer demand satisfies $x = D(p) - y$. In model 2 the price increases from p to $p + k$, $k = 1, 2, 3 \ldots$, with probability $f(k)$ after a time that is governed by a Poisson process with rates $\lambda(x)$. For both models we assume profits are continuously discounted at a rate α and the monopolist wishes to maximize the expected discounted profit earned over an infinite horizon.

The firm is not allowed to select the price but rather is restrained by the regulatory agency. Determining the firm's marginal cost curve is no easy task for the regulatory agency and in this case the agency will make its inferences based upon the firm's response to the agency's actions.*

The agency presumes that the firm has an increasing marginal cost curve as depicted in Figure 1. With the price given and $p \leqslant P_R$ the firm will set its production so as to equate marginal cost and price. Moreover, it is clear from Figure 1 that if the firm's actual marginal cost curve were as shown, then excess demand $(D(p) - y)$ decreases with price while production increases. From the agency's perspective, the desired price is p_R, for increasing the price beyond p_R will not cause production to increase and excess demand vanishes at p_R. Thus, the agency will allow price to increase until excess demand vanishes and market is cleared.

*See D.P. Baron, [1], for another approach.

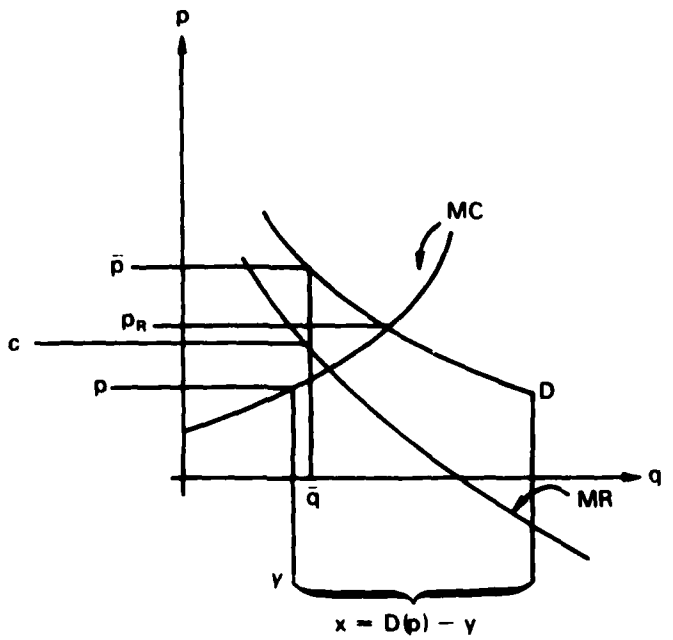

FIGURE 1. Agency's perspective

In fact, the firm has constant marginal cost (and $< p_R$ for otherwise the market would not clear and the agency would revise its estimate of p_R upward). Consequently, the agency *eventually* allows the price to reach the monopolistic price.

Finally, the speed of the price adjustment varies directly with the disequilibrium as measured by the excess demand x (see assumption 2 below).

3. ANALYSIS OF MODEL 1

We begin by anlyzing the dependence of the optimal production strategy on the current price. Defining $I = \{0, 1, 2, \ldots\}$ and $\pi(p) = (p - c)D(p)$, our results require the following assumptions:

1. $\lambda(p, x)$ is a differentiable function of x.
2. $\lambda(p, x_2) \geqslant \lambda(p, x_1)$, $x_2 \geqslant x_1$.
3. $\lambda(p, 0) = 0$, $p \in I$.
4. $\lambda(p + 1, x) \leqslant \lambda(p, x)$, $p \in I$.
5. $\lambda(p + 1, x_2) - \lambda(p + 1, x_1) \leqslant \lambda(p, x_2) - \lambda(p, x_1)$, $p \in I$, $x_2 \geqslant x_1$.
6. $\lambda(p + 1, x_2)/\lambda(p + 1, x_1) \leqslant \lambda(p, x_2)/\lambda(p, x_1)$, $p \in I$, $x_2 \geqslant x_1$.

7. $D(p+1) \leqslant D(p)$, $p \in I$.

8. There is a $\bar{p} \in I$ such that $\pi(\bar{p}) \geqslant \pi(p)$ for all $p \in I$.

9. $\pi(p+1) - \pi(p) \leqslant \pi(p) - \pi(p-1)$, $p = c+2, \ldots, \bar{p}$.

10. $\pi(p+1) \leqslant \pi(p)$, $p = \bar{p}, \bar{p}+1, \ldots$.

Assumption 1 is made for the sake of convenience. Assumption 2 reveals that the larger the excess demand the greater the tendency for the price to rise. If the likelihood that the regulator will grant a price increase increases with the observed level of excess demand then Assumption 3 will be true. This is true in the natural gas industry (see [9]). Assumption 3 states that the firm may maintain the current price if it so desires by producing exactly what the consumers demand. Assumption 4 implies that at higher price levels it is more difficult for the firm to induce price increases, whereas assumptions 5 and 6 imply that at a higher price an increase in unsatisfied demand has a smaller effect on the rate at which the price increases. We note that Assumptions 5 and 6 are trivially satisfied if $\lambda(p, x)$ is independent of p. Assumption 7 merely asserts that the demand curve for the monopolist's product is downward sloping. Assumptions 8-10 are valid for a variety of reasonable demand curves including $D(p) = ae^{-bp}$ $(a, b > 0)$, $D(p) = p^{-k}(k > 1)$, and $D(p) = a - bp$ $(a, b > 0)$. Finally, we make the reasonable assumption that whenever the current price is p, the monopolist will never produce at a rate exeeding $D(p)$. This is reasonable because any production in excess of $D(p)$ will earn no revenues and (by Assumptions 2 and 3) will have no effect on the future price. This assumption (coupled with Assumption 7) allows us to assume that the set of possible actions for each price is a finite set.

We now define $V(p)$ to be the maximum expected discounted profit earned over an infinite horizon when the current market price is p. It follows that

$$V(p) = \max_{0 \leqslant x \leqslant D(p)} J(p, x), \tag{1}$$

where $$J(p, x) = \{(p-c)(D(p)-x) + \lambda(p, x) V(p+1)\}/(\lambda(p, x) + \alpha).$$

Let $x(p)$ be the largest value of p attaining the maximum in (1). Our goal is to derive conditions which ensure that $x(p)$ is a nonincreasing function of p, that is, the higher the price the smaller the amount of consumer demand that will be left unsatisfied by the firm.

Our analysis of the behavior of $x(p)$ will be facilitated by the study of a modified version of the infinite horizon problem described above. Towards this end we let $V_n(p)$ be the maximum expected discounted profit earned by the firm over an infinite horizon when the current price is p and at most n price increases are permitted. It follows that ($V_0(p)$ equals $\pi(p)/\alpha$ if $p \geqslant c$ and 0 if $p < c$)

$$V_n(p) = \max_{0 \leqslant x \leqslant D(p)} J_n(p, x) \quad n \geqslant 1, \tag{2}$$

where $$J_n(p, x) = \{(p-c)(D(p)-x) + \lambda(p, x) V_{n-1}(p+1)\}/(\lambda(p, x) + \alpha).$$

Our main result will require the following lemmas. The trivial proof of Lemma 1 will be omitted.

LEMMA 1: If Assumptions 8 and 9 are valid, then for $n \geqslant 0$ and $c \leqslant p \leqslant \bar{p}$, $V_n(p) \geqslant \pi(p-1)/\alpha$.

LEMMA 2: If Assumption 10 holds, then for $p \geqslant \bar{p}$ and $n \geqslant 0$ $V_n(p) = J_n(p, 0)$.

PROOF: The result is clearly true for $n = 0$. We therefore assume the result to be valid for $n-1$ and verify that it remains valid for n. To prove this observe that

$$\begin{aligned}(\alpha + \lambda(p, x))J_n(p, 0) &= (\alpha + \lambda(p, x))\pi(p)/\alpha \qquad \text{[by (2)]}\\ &\geqslant (p-c)(D(p)-x) + \lambda(p, x)\pi(p+1)/\alpha \qquad \text{[by Assumption 10]}\\ &= (\alpha + \lambda(p, x))J_n(p, x) \qquad \text{[by the induction hypothesis].}\end{aligned}$$

Thus, the induction hypothesis implies that $J_n(p, 0) = \pi(p)/\alpha$ so the above inequalities imply that $J_n(p, 0) \geqslant J_n(p, x)$ which is the desired result.

LEMMA 3: If Assumptions 4, 7, 8, 9, and 10 are valid, then for $n \geqslant 0$ and $c \leqslant p \leqslant \bar{p} - 1$.

$$\alpha V_n(p+1) - \pi(p) \leqslant \alpha V_n(p) - \pi(p-1). \tag{3}$$

PROOF: We prove (3) by induction. For $n = 0$ (3) reduces to Assumption 9. We therefore assume that (3) is valid for $n-1$ and verify that it remains valid for n. Define $x(n, p)$ to be the largest value of x satisfying $V_n(p) = J_n(p, x(n, p))$. Let $x = x(n, p+1)$. Since $D(p+1) \leqslant D(p)$, x is a feasible action when the current price is p. This implies that

$$\alpha V_n(p) \geqslant \alpha\{(p-c)(D(p)-x) + \lambda(p, X)V_{n-1}(p+1)\}/(\lambda(p, x) + \alpha). \tag{4}$$

$$\begin{aligned}\alpha V_n(p+1) &= \alpha\{(p+1-c)(D(p+1)-x)\\ &\quad + \lambda(p+1, x)V_{n-1}(p+2)\}/(\lambda(p+1, x+\alpha) \text{ (by (2))}\\ &\leqslant \alpha\{(p+1-c)(D(p+1)-x) + \lambda(p, x)V_{n-1}(p+2)\}/(\lambda(p, x)+\alpha)\\ &\quad \text{[by Lemma 1 and Assumption 4].}\end{aligned} \tag{5}$$

Together (4) and (5) imply

$$\begin{aligned}\alpha(V_n(p+1) - V_n(p)) \leqslant \alpha\{&(p+1-c)(D(p+1)-x) - (p-c)(D(p)-x)\\ &+ \lambda(p, x)[V_{n-1}(p+2) - V_{n-1}(p+1)]\}/(\lambda(p, x)+\alpha).\end{aligned} \tag{6}$$

Note that

$$\begin{aligned}(p+1-c)(D(p+1)-x) - (p-c)(D(p)-x) &= \pi(p+1) - \pi(p) - x\\ &\leqslant \pi(p) - \pi(p-1) \text{ [by (9) and } p+1 \leqslant \bar{p}].\end{aligned} \tag{7}$$

Also observe that for $c \leqslant p < \bar{p} - 1$ the induction hypothesis implies that $\alpha(V_{n-1}(p+2) - V_{n-1}(p+1)) \leqslant \pi(p+1) - \pi(p)$ while Lemma 2 and Assumption 8 imply

$$\alpha(V_{n-1}(\bar{p}+1) - V_{n-1}(\bar{p})) = \pi(\bar{p}+1) - \pi(\bar{p}) \leqslant \pi(\bar{p}) - \pi(\bar{p}-1).$$

Thus, for $c \leqslant p \leqslant \bar{p} - 1$ it follows that

$$\alpha(V_{n-1}(p+2) - V_{n-1}(p+1)) \leqslant \pi(p+1) - \pi(p) \leqslant \pi(p) - \pi(p-1). \tag{8}$$

Together (6)-(8) imply that

$$\alpha(V_n(p+1) - V_n(p)) \leqslant \pi(p) - \pi(p-1) \quad c \leqslant p \leqslant \bar{p} - 1$$

which is the desired result.

Lemma 2 implies that for $p \geqslant \bar{p}$ the firm's optimal policy is to satisfy all consumer demand by producing at a rate $D(p)$. We now show that for $n \geqslant 0$ and $c \leqslant p \leqslant \bar{p} - 2$, $x(n, p) \geqslant x(n, p+1)$. Coupling this result with Lemma 2 we see that for all n and $p \geqslant c$ the firm's optimal level of unsatisfied demand (the amount of market disequilibrium) is a nonincreasing function of p.

THEOREM 1: If (1)-(10) are valid, then for $n \geqslant 0$ and $c \leqslant p \leqslant \bar{p} - 2$.

$$x(n, p+1) \leqslant x(n, p). \tag{9}$$

PROOF: Observe that (9) will follow if we can prove that

$$J_n(p+1, x_2) - J_n(p+1, x_1) \geqslant 0 \Rightarrow J_n(p, x_2) - J_n(p, x_1) \geqslant 0 \tag{10}$$

holds for $n \geqslant 0$, $p + 1 \leqslant \bar{p} - 1$, and $D(p+1) \geqslant x_2 \geqslant x_1$.

We now proceed to verify (10). After some algebraic manipulations $J_n(p+1, x_2) - J_n(p+1, x_1) \geqslant 0$ can be shown to be equivalent to

$$\begin{aligned} \alpha V_{n-1}(p+2) - \pi(p+1) \geqslant (p+1-c)\{&\alpha(x_2 - x_1) \\ &+ x_2\lambda(p+1, x_1) - x_1\lambda(p+1, x_2)\}/(\lambda(p+1, x_2) - \lambda(p+1, x_1)). \end{aligned} \tag{11}$$

By (11), (10) will hold if the left side of (11) is a nonincreasing function of p and the right side of (11) is a nondecreasing function of p. Since $p + 2 \leqslant \bar{p}$, Lemma 3 implies that the left side of (11) is nonincreasing in p. Since Assumptions 2 and 5 imply that $\alpha(p + 1 - c)(x_2 - x_1)/(\lambda(p+1, x_2) - \lambda(p+1, x_1))$ is nondecreasing in p, the right side of (11) will be nondecreasing in p if

$$\frac{x_2\lambda(p+1, x_1) - x_1\lambda(p+1, x_2)}{\lambda(p+1, x_2) - \lambda(p+1, x_1)} \geqslant \frac{x_2\lambda(p, x_1) - x_1\lambda(p_2, x_2)}{\lambda(p, x_2) - \lambda(p, x_1)}. \tag{12}$$

Upon simplification (12) is seen to be equivalent to $(x_2 - x_1)\{\lambda(p+1, x_1)\lambda(p, x_2) - \lambda(p+1, x_2)\lambda(p, x_1)\} \geqslant 0$. Since $x_2 \geqslant x_1$, the last inequality follows from Assumption 6. This completes the verification of (11), and the proof of Theorem 1 is complete.

The following Lemma relates $V_n(p)$ to $V(p)$.

LEMMA 4: For all p $\lim_{n\to\infty} V_n(p) = V(p)$.

PROOF: Since $\pi(p)$ is bounded above by $\pi(\bar{p})$, the result follows directly from Lemma 1 of [2].

Lemmas 2 and 4 imply that for $n \geq \bar{p}$, $V_n(p) = V(p)$. Theorem 1 plus Lemmas 2 and 4 now imply that $V(p) = J(p, 0)$ $(p \geq \bar{p})$, and $x(p+1) \leq x(p)$, $(c \leq p \leq \bar{p} - 2)$.

Let $x_\alpha(p)$ and $V_\alpha(p)$ denote the dependence of these quantities on the discount factor α. We now show that $x_\alpha(p)$ is a nonincreasing function of the discount factor α. In order to prove this result, we first consider a finite horizon semiMarkov decision process formulation of the problem. We use the technique of Lippman [6] in which the system is observed and the production rate can be changed only at discrete points in time, namely those corresponding to price increases and certain null events. These null events, which do not change the current price, are introduced in order to make the time between observations independent of the state and action and they render the problem equivalent to a discrete time Markov decision process. After analyzing the discrete time Markov process we will let the horizon length go to infinity and exploit the equivalence of this formulation to the original continuous time problem to get results for our original problem, that is, the inifinite horizon problem in which the production rate can be changed at any point in time.

We now consider the system to be observed only at times when price increases or null events are observed: The time between observations is exponentially distributed with parameter $\Lambda = \sup_{\substack{p \in I \\ x \in D(p)}} \lambda(p, x) = \lambda(0, D(0))$, where the last equality follows from Assumptions 2 and 4. The state of the system is p, the current price, and the action is x if production is occurring at a rate $D(p) - x$. The probability that the next observation point is occasioned by a price increase is $\lambda(p, x)/\Lambda$ while the probability that the next observation point is a null event is $1 - \lambda(p, x)/\Lambda$.

Let $W_{n,\alpha}(p)$ be the maximum expected profit earned by the firm when n observation points remain, the current price is i, and the discount factor is α. Then

$$(W_{0,\alpha}(p) = 0)$$

$$W_{n,\alpha}(p) = \max_{0 \leq x \leq D(p)} G_{n,\alpha}(p, x)/(\Lambda + \alpha), \quad n \geq 1, \tag{13}$$

where

$$G_{n,\alpha}(p, x) = p(D(p) - x) + \lambda(p, x)\Delta W_{n-1,\alpha}(p) + \Lambda W_{n-1,\alpha}(p)$$

and

$$\Delta W_{n,\alpha}(p) = W_{n,\alpha}(p+1) - W_{n,\alpha}(p).$$

Let $x_{n,\alpha}(p)$ be the largest value of x attaining the maximum in (12). We now prove that if Assumptions 1, 2, 3, 7, 8, and 10, and $\pi(p+1) \geq \pi(p)(p = c, c+1, \ldots \bar{p})$, are valid then $x_{n,\alpha}(p)$ is a nonincreasing function of α. By Theorem 1 of [5] and Theorem 1.1 of [10] this will imply that $\bar{x}_{\alpha_2}(p) \leq \bar{x}_{\alpha_1}(p)$ $(\alpha_2 \geq \alpha_1)$ which is the desired result. Before proving this result we require the following lemma.

LEMMA 5: If Assumption 10 holds, then for $n \geq 0$ and $p \geq \bar{p}$, $W_{n,\alpha}(p) = G_{n,\alpha}(p, 0)$.

PROOF: The proof is virtually identical to the proof of Lemma 2 and is therefore omitted.

LEMMA 6: If Assumptions 1, 2, 3, 7, 8, and 10 are valid and $\pi(p+1) \geq \pi(p)$ is valid for $c \leq p \leq \bar{p}$, then $\Delta W_{n,\alpha}(p) \geq 0$ $(c \leq p \leq \bar{p}-1)$.

PROOF: We prove the result by induction on n. For $n = 1$ the result follows immediately from $\pi(p+1) \geq \pi(p)$. Assuming the result for $n-1$ and letting $x = x_{n,\alpha}(p+1)$, (13) and $x - D(p+1) + D(p) \leq D(p)$ imply

$$(\Lambda+\alpha)\Delta W_{n,\alpha}(p) \geq G_{n,\alpha}(p+1, x) - G_{n,\alpha}(p, x - D(p+1) + D(p)) \tag{14}$$

$$= (\Lambda - \lambda(p, x - D(p+1) + D(p))\Delta W_{n-1,\alpha}(p) + \lambda(p+1, x)\Delta W_{n-1,\alpha}(p+1).$$

For $c \leq p < \bar{p} - 1$ the nonnegativity of the last expression follows from the induction hypothesis. To complete the proof it therefore suffices to show that $W_{n,\alpha}(\bar{p}) \geq W_{n,\alpha}(\bar{p}-1)$. To prove this, note that $\pi(\bar{p}) \geq \pi(\bar{p}-1)$ the induction hypothesis, and Lemma 5 imply

$$\begin{aligned} W_{n,\alpha}(\bar{p}-1) &= (\bar{p}-c-1)(D(\bar{p}-1) - \bar{x}_n(\bar{p}-1)) \\ &\quad + \lambda(\bar{p}-1, \bar{x}_{n,\alpha}(\bar{p}-1))W_{n-1,\alpha}(\bar{p}) \\ &\quad + (\Lambda - \lambda(\bar{p}-1, x_{n,\alpha}(p-1)))\ W_{n-1,\alpha}(\bar{p}-1)\}/(\Lambda+\alpha) \\ &\leq \{\pi(\bar{p}) + \Lambda W_{n-1,\alpha}(\bar{p})\}/(\Lambda+\alpha) = W_{n,\alpha}(\bar{p}) \end{aligned}$$

This completes the proof of Lemma 6.

We can now prove

THEOREM 2: If Assumptions 1, 2, 3, 7, 8, and 10 are valid and $\pi(p+1) \geq \pi(p)$ is valid for $c \leq p \leq p-1$, then for $\alpha_2 > \alpha_1$ and $c \leq p \leq \bar{p} - 1$ $x_{n,\alpha_2}(p) \leq x_{n,\alpha_1}(p)$.

PROOF: By reasoning analogous to that used to justify (10) it suffices to prove that

$$G_{n,\alpha_2}(p, x_2) - G_{n,\alpha_2}(p, x_1) \leq G_{n,\alpha_1}(p, x_2) - G_{n,\alpha_1}(p, x_1)$$

holds for $\alpha_2 > \alpha_1$, and $x_2 > x_1$, and $c \leq p \leq \bar{p} - 1$. By (13) this inequality is equivalent to

$$(\lambda(p, x_2) - \lambda(p, x_1))(\Delta W_{n-1,\alpha_2}(p) - \Delta W_{n-1,\alpha_1}(p)) \leq 0, \tag{15}$$

By Assumption 2, inequality (15) will hold if

$$\Delta V_{n-1,\alpha_2}(p) \leq \Delta V_{n-1,\alpha_2}(p),\ \alpha_2 > \alpha_1,\ c \leq p \leq \bar{p} - 1. \tag{16}$$

To prove (16) by induction note that for $n = 1$ (16) follows from $1/(\Lambda+\alpha_2) \leq 1/(\Lambda+\alpha_1)$. Assuming that (16) is valid for $n-1$ we can verify it for n by observing that

$$(\Lambda+\alpha)W_{n,\alpha_1}(p) \geq G_{n,\alpha_1}(p+1, x_1) - G_{n,\alpha_1}(p, x_1)$$

and

$$(\Lambda+\alpha)W_{n,\alpha_2}(p) \leq G_{n,\alpha_2}(p+1, x_1) - G_{n,\alpha_2}(p+1, x_2),$$

where $x_1 = x_{n,\alpha}(p+1)$ and $x_2 = x_{n,\alpha}(p)$.

These inequalities imply that

$$\begin{aligned}(\Lambda + \alpha)[\Delta W_{n,\alpha_1}(p) - \Delta W_{n,\alpha_2}(p)] &\geqslant [\lambda(p+1, x_1)(\Delta W_{n-1,\alpha_1}(p+1) \\ &\quad - \Delta W_{n-1,\alpha_2}(p+1)) \\ &\quad + (\Lambda - \lambda(p, x_2)(\Delta W_{n-1,\alpha_1}(p) - \Delta W_{n-1,\alpha_2}(p))] \\ &\geqslant 0,\end{aligned}$$

where the last inequality follows from the induction hypothesis. This completes the proof of Theorem 2. By the remarks following Lemma 4 the hypotheses of Theorem 2 also imply $x_{\alpha_2}(p) \leqslant x_{\alpha_1}(p)$.

We now characterize the dependence of $x_{n,\alpha}(p)$ on n. More specifically we show that the hypotheses of Theorem 2 imply that $x_{n+1,\alpha}(p) \geqslant x_{n,\alpha}(p)$ is valid for $c \leqslant p \leqslant \bar{p} - 1$. If we assume that the firm wishes to maximize its expected discounted profits over a finite horizon of length $T < \infty$ and we define $\bar{x}_{t,\alpha}(p)$ to be an optimal action when a time t remains, Theorem 4 of [6] and the above result will enable us to conclude that $x_{t_2,\alpha}(p) \geqslant x_{t_1,\alpha}(p)$ for $t_2 \geqslant t_1$.

THEOREM 3: If the hypotheses of Theorem 2 are valid, then for $n \geqslant 1$ and $c \leqslant p \leqslant \bar{p} - 1$ $x_{n+1,\alpha}(p) \geqslant x_{n,\alpha}(p)$.

PROOF: The result will follow if we can prove that

$$G_{n+1,\alpha}(p, x_2) - G_{n+1,\alpha}(p, x_1) \geqslant G_{n,\alpha}(p, x_2) - G_{n,\alpha}(p, x_1)$$

holds for $n \geqslant 1$, $x_2 > x_1$, and $c \leqslant p \leqslant \bar{p} - 1$. By (13) this inequality is equivalent to $(\lambda(p, x_2) - \lambda(p, x_1))(\Delta W_{n,\alpha}(p) - \Delta W_{n-1,\alpha}(p)) \geqslant 0$. By Assumption 2, the last inequality will follow if

$$\Delta W_{n,\alpha}(p) \geqslant \Delta W_{n-1,\alpha}(p), \quad n \geqslant 1, \ c \leqslant p \leqslant \bar{p} - 1. \tag{17}$$

The proof of (17) is similar to the proof of (16), and is therefore omitted.

3. ANALYSIS OF MODEL 2

We now characterize the dependence of the optimal production strategy for Model 2 on the current price. Our results require the following assumptions:

11. $\lambda(x_2) \geqslant \lambda(x_1)$, $x_2 \geqslant x_1$
12. $\lambda(x)$ is differentiable
13. $\lambda(0) = 0$
14. $x_2/(\lambda(x_2) + \alpha) \geqslant x_1/(\lambda(x_1) + \alpha)$, $x_2 > x_1$
15. There is a $\bar{p} \in I$ such that $\pi(\bar{p}) \geqslant \pi(p)$ for all $p \in I$.
16. For $p \in I$, $\pi(p)$ is a concave function of p, that is $\pi(p+1) - \pi(p) \leqslant \pi(p) - \pi(p-1)$.

Assumption 14 is the only condition which requires any explanation. If $\lambda(x)$ is concave, then Assumption 14 will be valid. To show this define $h(x) = x/(\lambda(x) + \alpha)$. Then for any $x_0 \geq 0$ $h'(x_0) = \lambda(x_0) + \alpha - x_0\lambda'(x_0)/(\lambda(x_0) + \alpha)^2$. Together Assumption 13 and the concavity of $\lambda(x)$ imply that $\lambda(x_0) \geq \lambda(0) + x_0\lambda'(x_0) = x_0\lambda'(x_0)$. This shows that $h'(x_0) \geq 0$, which implies Assumption 14.

We now define $T(p)$ and $T_n(p)$ to be the analogs (for Model 2) of $V(p)$ and $V_n(p)$. Then

$$T(p) = \max_{0 \leq x \leq D(p)} H(p, x), \tag{18}$$

where

$$H(p, x) = \{(p - c)(D(p) - x) + \lambda(x) \sum_{k=1}^{k=\infty} f(k)T(p + k)/(\lambda(x) + \alpha)$$

and

$$T_n(p) = \max_{0 \leq x \leq D(p)} H_n(p, x), \tag{19}$$

where

$$H_n(p, x) = \{(p - c)(D(p) - x) + \lambda(x) \sum_{k=1}^{k=\infty} f(k)T_{n-1}(p + k)\}/(\lambda(x) + \alpha).$$

We also define $x(p)$ and $x_n(p)$ to be the largest values of x attaining the maximum in (18) and (19), respectively. The proof of the following lemma is virtually identical to the proof of Lemma 2 and is therefore omitted.

LEMMA 7: If Assumptions 15 and 16 are valid, then for $n \geq 0$ and $p \geq \bar{p}$ $T_n(p) = H_n(p, 0)$.

As in Section 3, we now focus our attention on showing that for $c \leq p \leq \bar{p} - 1$, $x_n(p + 1) \leq x_n(p)$. Before proving this result we require the following Lemma:

LEMMA 8: If Assumptions 15 and 16 are valid, then for $p \geq c$ and $n \geq 0$.

$$\alpha \sum_{k=1}^{k=\infty} f(k)T_{n-1}(p + 1 + k) - \pi(p + 1) \leq \alpha \sum_{k=1}^{k=\infty} f(k)T_{n-1}(p + k) - \pi(p). \tag{20}$$

PROOF: It sufficies to show that for each $k \geq 1$

$$\alpha T_{n-1}(p + 1 + k) - \pi(p + 1) \leq \alpha T_{n-1}(p + k) - \pi(p). \tag{21}$$

For $n = 1$ (21) reduces to Assumption 16. We therefore assume that (21) holds for $n - 1$ and verify that (21) holds for n. Letting $x = x_n(p + 1 + k)$ we see that $\alpha T_n(p + 1 + k) = H_n(p + 1 + k, x)$ and

$$\alpha T_n(p + k) \geq \alpha H_n(p + k, x).$$

The last two statements imply

$$\alpha[T_n(p+1+k) - T_n(p+k)] \leqslant \alpha\{(p+k+1-c)(D(p+k+1)-x) \quad (22)$$
$$(p+k-c)(D(p+k)-x)$$
$$+\lambda(x)\sum_{j=1}^{j=\infty} f(j)[T_{n-1}(p+k+j+1) - T_{n-1}(p+k+j]\}/(\lambda(x)+\alpha).$$

Since

$$(p+k+1-c)(D(p+k+1)-x) - (p+k-c)(D(p+k)-x)$$
$$= \pi(p+k+1) - \pi(k) - x \leqslant \pi(p+1) - \pi(p) \quad \text{[by Assumption 16]}$$

and

$$\alpha(T_{n-1}(p+k+j+1) - T_{n-1}(p+k+j)) \leqslant \pi(p+1) - \pi(p),$$

[by the induction hypothesis]

(22) implies that

$$\alpha(T_{n-1}(p+1+k) - T_{n-1}(p+k)) \leqslant \pi(p+1) - \pi(p),$$

which is equivalent to (21).

THEOREM 4: For $n \geqslant 0$ and $c \leqslant p \leqslant \bar{p} - 2$

$$x_n(p+1) \leqslant x_n(p). \quad (23)$$

PROOF: The proof is similar to the proof of Theorem 1. As before, it suffices to demonstrate that for $D(p+1) \geqslant x_2 \geqslant x_1$, $p \leqslant \bar{p} - 2$, and $n \geqslant 0$

$$H_n(p+1, x_2) - H_n(p+1, x_1) \geqslant 0 \Rightarrow H_n(p, x_2) - H_n(p, x_1) \geqslant 0.$$

To show this, note that $H_n(p+1, x_2) - H_n(p+1, x_1) \geqslant 0$ may be shown to be equivalent to

$$(\lambda(x_2) - \lambda(x_1))\left\{\alpha\sum_{k=1}^{k=\infty} f(k)\,T_{n-1}(p+k+1) - \pi(p+1)\right\} \quad (24)$$
$$\geqslant (p+1-c)[x_2(\lambda(x_1)+\alpha) - x_1(\lambda(x_2)+\alpha)].$$

By Lemma 8 the left side of (24) is a nonincreasing function of p. Since Assumption 14 implies that the right side of (24) is a nondecreasing function of p it follows that (24) implies $H_n(p, x_2) - H_n(p, x_1) \geqslant 0$. This completes the proof of the theorem.

As in Section 2, it easily follows that $\lim_{n\to\infty} W_n(p) = W(p)$ and that for $n \geqslant \bar{p}$, $W_n(p) = W(p)$. Together with Theorem 4 plus Lemma 8, these facts imply that $x(p) \geqslant x(p+1)$ $(c \leqslant p \leqslant \bar{p} - 2)$ and that in the unmodified infinite horizon version of Model 2 the firm should produce at a rate $D(p)$ for $p \geqslant \bar{p}$.

Unfortunately, all efforts to determine how the optimal production policy for Model 2 depends on the discount rate and the length of the horizon have been unsuccessful.

5. EXTENSIONS AND SUGGESTIONS FOR FUTURE RESEARCH

One obvious defect of our model is that decreases in price are not allowed. A model which allowed for the price to decrease from p to $p - 1$ in a time governed by a Poisson process with rate $\mu(p, x)$ (with $\mu(p, x_2) \leqslant \mu(p, x_1)$) as well as increase from p to $p + 1$ in a time governed by a Poisson process with rate $\lambda(p, x)$ ($\lambda(p, x_2) \geqslant \lambda(p, x_1)$) would be of obvious interest. Unfortunately, we have been unable to obtain characterizations of the optimal production policy for such a model.

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ESTIMATION IN SINGLE SERVER QUEUES

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ABSTRACT

Moment and maximum likelihood estimates (m.l.e.'s) are investigated for nonparametric and parametric models for a single server queue observed over a random time horizon, namely, up to the nth departure epoch. Also, m.l.e.'s of the mean interarrival time and mean service time in an $M/M/1$ queue observed over a fixed time-interval are studied. Limit distributions of these estimates are obtained without imposing steady state assumptions on the queue-size or waiting time processes.

1. INTRODUCTION

The theory of queues as developed so far is largely a descriptive theory, namely, it is concerned with the probabilistic structure of the models and the behaviour in finite time t as well as in the limit as $t \to \infty$ of the processes arising from these models. Relatively less has been done on a prescriptive theory dealing with the statistical analysis, design and control of queueing systems. Yet from the practical point of view these latter aspects are very important. Thus, for example, the management of a service facility subject to congestion and wishing to design an efficient queueing system should be in a position to estimate the various parameters of the model on the basis of data collected at the facility.

The earliest paper on problems of statistical inference from queueing models seems to be that of Clark [4] who investigated the problem of maximum likelihood estimation of the parameters of an $M/M/1$ queue in equilibrium. While this paper is of historic importance, the more substantial investigation of these problems was carried out by Cox [5] and Wolff [14]. The work of Cox contains several ideas, but no major results seem to emerge. Wolff discussed maximum likelihood estimation and likelihood ratio tests for a class of ergodic queueing models which give rise to birth and death processes (including the queue-length process in $M/M/s$ and related systems). Since these processes are continuous time Markov processes, Wolff was able to derive limit distributions of the estimates and test statistics as a direct application of Billingsley's [3] results for Markov processes. The Markov property of the queue-length process and the assumption of steady state are the essential features in Wolff's study as they are in

Billingsley's. For further work on the $M/M/1$ queue see Harris [8], Jenkins [10], and Samaan and Tracy [13]. Benes [2] treated the estimation problem for the $M/M/\infty$ system. Goyal and Harris [7] obtained maximum likelihood estimates of the parameters of a queueing system with Poisson arrivals and state-dependent service, again assuming steady state. Henningsen [9] investigated the $M/G/1$ queue in equilibrium and obtained estimates for the arrival rate and the parameters of the service time density. Crane and Lemoine [6] have applied simulation techniques to the problem of estimating the steady state mean waiting time in a single server queue. In their recent monograph, Basawa and Prakasa Rao [1] discuss examples of inference for systems such as $M/E_k/1$ and $E_k/M/1$, again within the framework of Billingsley [3].

The objective of this paper is to consider a single server queueing model and derive 'reasonable' estimates for the interarrival and service time distribution functions (d.f.'s) and their means. The processes arising from this model are not always Markovian, nor do we assume the existence of steady state. Our estimates are not necessarily maximum likelihood estimates, but have a simple structure of intuitive appeal and reduce to m.l.e.'s in special cases. The estimates of the means are moment estimates and those of the d.f.'s are empirical d.f.'s. Limit distributions of the estimates are obtained using direct and simple arguments.

In Section 2 we study the properties of moment estimates of the interarrival and service time means, and also the estimates of the corresponding d.f.'s in a $G/G/1$ queueing system observed over a random time horizon $(0, D_n]$, where D_n is the nth departure epoch. The limit distributions of these estimates are obtained without imposing any restrictions on the traffic intensity. Section 3 is concerned with the maximum likelihood estimation of the parameters of a $G/G/1$ queue, using the same sampling plan as in Section 2. In Section 4 we discuss m.l.e.'s of the interarrival and service time means in an $M/M/1$ queue observed over a nonrandom time-interval $(0, t]$, and study their properties as $t \to \infty$, again without any restrictions on the traffic intensity. The problem of hypotheses testing will be considered in a forthcoming paper.

NOTATION: We shall use $\Rightarrow$ to denote convergence in distribution. The normal random variable with mean μ and variance σ^2 will be denoted by $N_1(\mu, \sigma^2)$, and the random vector having the bivariate normal density with means μ_1, μ_2 and variance-covariance matrix Σ will be denoted by $N_2\left\{\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}, \Sigma\right\}$. As usual, a.s. means almost surely.

2. THE G/G/1 QUEUE—MOMENT ESTIMATES

Consider a single server queueing system in which the interarrival times $\{u_k, k \geq 1\}$ and the service times $\{v_k, k \geq 1\}$ are two independent sequences of independent and identically distributed nonnegative random variables with d.f.'s F and G respectively. Assume that the moments

$$E(u_k) = a, \quad \mathrm{Var}(u_k) = \sigma_1^2 \tag{1}$$

$$E(v_k) = b, \quad \mathrm{Var}(v_k) = \sigma_2^2 \tag{2}$$

are all finite. The traffic intensity of the system is then $\rho = b/a$; also, let $\eta = \max(1, \rho)$. We assume that the initial customer arrives at $t = 0$. Our sampling scheme is to observe the phenomenon until the first n customers have departed from the system and note the service times of these n customers, say $(v_1, v_2, \ldots, v_n)$. Let the nth departure epoch be D_n, and also observe the interarrival times of all customers who arrive during $(0, D_n]$; thus, we obtain $(u_1, u_2, \ldots, u_{\nu_n})$, where

$$N_A \equiv N_A(D_n) = \max\{k: u_1 + u_2 + \ldots + u_k \leqslant D_n\}. \tag{3}$$

Clearly, $N_A \geqslant n$. For the means a, b we propose the estimates

$$\hat{a}_n = \frac{1}{N_A}\sum_1^{N_A} u_i,\ \hat{b}_n = \frac{1}{n}\sum_1^n v_i. \tag{4}$$

It should be observed that the estimate $\hat{b}_n$ is the usual sample mean, whereas $\hat{a}_n$ is based on a random number of observations. It would be interesting to compare $\hat{a}_n$ with the ordinary sample mean

$$\hat{a}_n^* = \frac{1}{n}\sum_1^n u_i. \tag{5}$$

It turns out that asymptotically $\hat{a}_n$ and $\hat{a}_n^*$ display similar behaviour, namely, they are both consistent estimates of a and asymptotically normal, with $\hat{a}_n$ having a smaller variance. The obvious independence of $\hat{a}_n^*$ and $\hat{b}_n$ is used to establish the asymptotic independence of the estimates (4). These properties are proved in Theorem 1 below. We need the following preliminary result, which will be proved in the appendix.

LEMMA 1: As $n \to \infty$,

$$\frac{1}{n} N_A \to \eta \text{ in probability.} \tag{6}$$

THEOREM 1: As $n \to \infty$,

(i) $\hat{a}_n \to a$, $\hat{b}_n \to b$ a.s., and

$$\text{(ii)}\quad \begin{pmatrix} \sqrt{n}(\hat{a}_n - a) \\ \sqrt{n}(\hat{b}_n - b) \end{pmatrix} \Rightarrow N_2\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\sigma_1^2/\eta & 0 \\ 0 & \sigma_2^2\end{pmatrix}\right\}.$$

PROOF: (i) The strong law of large numbers gives

$$\frac{1}{n}\sum_1^n u_i \to a,\ \frac{1}{n}\sum_1^n v_i \to b \text{ a.s.} \tag{7}$$

Since (6) implies that $N_A(D_n) \uparrow \infty$ as $n \to \infty$, we also obtain

$$\frac{1}{N_A}\sum_1^{N_A} u_i \to a \quad \text{a.s.} \tag{8}$$

as $n \to \infty$. We have thus proved both statements in (i).

(ii) We first consider the limit distribution of $\sqrt{n}(\hat{a}_n^* - a)$, $\sqrt{n}(\hat{b}_n - b)$, where $\hat{a}_n^*$ is the sample mean (5). On account of the independence of the sequences $\{u_k\}$ and $\{v_k\}$ we have

$$\begin{pmatrix} \sqrt{n}(\hat{a}_n^* - a) \\ \sqrt{n}(\hat{b}_n - b) \end{pmatrix} \Rightarrow N_2\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\sigma_1^2 & 0 \\ 0 & \sigma_2^2\end{pmatrix}\right\} \tag{9}$$

by the central limit theorem. The difficulty is that our estimate $\hat{a}_n$ is based on N_A observations where (3) shows that N_A depends on D_n and hence on $\{v_k\}$. However, we shall show that this

dependence of $\hat{a}_n$ and $\hat{b}_n$ disappears in the limit as $n \to \infty$. The desired result (ii) will be proved via the Cramér-Wold device, if we show that for arbitrary real numbers α, β,

$$\alpha\sqrt{n}\,(\hat{a}_n - a) + \beta\sqrt{n}\,(\hat{b}_n - b) \Rightarrow N_1\left(0,\ \alpha^2\,\frac{\sigma_1^2}{\eta} + \beta^2\sigma_2^2\right). \tag{10}$$

To this end, we write the left side of (10) as

$$\left\{\alpha\left[\frac{n}{\eta}\right]^{1/2}(\hat{a}_n^* - a) + \beta\sqrt{n}\,(\hat{b}_n - b)\right\} + \alpha\left\{\sqrt{n}\,(\hat{a}_n - a) - \left[\frac{n}{\eta}\right]^{1/2}(\hat{a}_n^* - a)\right\}. \tag{11}$$

The limit distribution of the term within the first pair of brackets in (11) is given by (10) on account of (9). Therefore, the desired result (10) will follow if we show that the term within the second pair of brackets in (11) converges to zero in probability. Now

$$\sqrt{n}\,(\hat{a}_n - a) - \left[\frac{n}{\eta}\right]^{1/2}(\hat{a}_n^* - a) = \left[\left[\frac{n}{N_4}\right]^{1/2} - \left[\frac{1}{\eta}\right]^{1/2}\right]\frac{1}{N_4^{1/2}}\sum_1^{N_4}(u_i - a) + \left[\frac{1}{\eta}\right]^{1/2}\left[\frac{1}{N_4^{1/2}}\sum_1^{N_4}(u_i - a) - \frac{1}{n^{1/2}}\sum_1^{n}(u_i - a)\right]. \tag{12}$$

First, we note that by the Central Limit theorem applicable to random sums (see Billingsley [3]) it follows, in view of (6), that as $n \to \infty$,

$$\frac{1}{N_4^{1/2}}\sum_1^{N_4}(u_i - a) \Rightarrow N_1(0, \sigma_1^2) \tag{13}$$

and that

$$\frac{1}{N_4^{1/2}}\sum_1^{N_4}(u_i - a) - \frac{1}{n^{1/2}}\sum_1^{n}(u_i - a) \to 0 \tag{14}$$

in probability. Using (13), (14) and (6) we see that the right side of (12) converges to zero in probability as $n \to \infty$. Thus, (10) is proved and the proof of Theorem 1 is complete.

For the d.f.'s F and G the natural estimates are the empirical d.f.'s

$$\hat{F}_n(u) = \frac{1}{N_4}\sum_1^{N_4}\delta(u - u_i),\quad \hat{G}_n(v) = \frac{1}{n}\sum_1^{n}\delta(v - v_i) \tag{15}$$

where $\delta(x) = 0$ for $x < 0$ and $= 1$ for $x \geq 0$. The asymptotic properties of these estimates are given by the following theorem. Its proof is similar to that of Theorem 1 and is therefore omitted.

THEOREM 2: For all $u > 0$, $v > 0$, we have as $n \to \infty$

(i) $\hat{F}_n(u) \to F(u)$, $\hat{G}_n(v) \to G(v)$ in probability, and

$$\text{(ii)}\quad \begin{bmatrix}\sqrt{n}\,(\hat{F}_n(u) - F(u))\\ \sqrt{n}\,(\hat{G}_n(v) - G(v))\end{bmatrix} \Rightarrow N_2\left\{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}\sigma_u^2/\eta & 0\\ 0 & \sigma_v^2\end{bmatrix}\right\}.$$

where

$$\sigma_u^2 = F(u)[1 - F(u)],\ \sigma_v^2 = G(v)[1 - G(v)]. \tag{16}$$

3. THE G/G/1 QUEUE—MAXIMUM LIKELIHOOD ESTIMATES

In this section we assume that the d.f.'s F and G of the interarrival times and service times are absolutely continuous with continuous densities $f(u;\theta)$ and $g(v;\phi)$, where f and g are known functions of unknown real parameters θ and ϕ. For simplicity of presentation we assume that θ and ϕ are scalars; the case of vector parameters can be treated in an analogous manner. Consider the problem of maximum likelihood estimation of θ and ϕ. Under the sampling scheme of Section 2 the likelihood function is given by

$$L_n(f, g) = \left\{\prod_1^{N_A} f(u_i;\theta)\right\}\left\{\prod_1^{n} g(v_j;\phi)\right\}[1 - F(X_n;\theta)] \tag{17}$$

where $X_n \equiv X_n(D_n) = D_n - \sum_1^{N_A} u_i$. The only factor that causes any difficulty (in the sense of yielding simple estimates) in (17) is $1 - F(X_n;\theta)$, which corresponds to the incomplete arrival interval when sampling is terminated at the epoch D_n. Consider instead

$$L_n^a(f, g) = \left\{\prod_1^{N_A} f(u_i;\theta)\right\}\left\{\prod_1^{n} g(v_j;\phi)\right\}, \tag{18}$$

which can be viewed as an approximation to $L_n(f, g)$ in a sense to be explained later. Let $\hat\theta_n^a$ and $\hat\phi_n^a$ denote the likelihood equation estimators (see Rao [12]) of θ and ϕ based on $L_n^a(f, g)$; thus $\hat\theta_n^a$, $\hat\phi_n^a$ are the roots of the equations

$$\sum_1^{N_A} \frac{\partial}{\partial\theta} \log f(u_i;\theta) = 0,\ \sum_1^{n} \frac{\partial}{\partial\phi} \log g(v_j;\phi) = 0. \tag{19}$$

The following theorem states the asymptotic properties of $\hat\theta_n^a$, $\hat\phi_n^a$.

THEOREM 3: Under appropriate regularity conditions of f and g (see Rao [12], for example) we have as $n \to \infty$

(i) $\hat\theta_n^a \to \theta$, $\hat\phi_n^a \to \phi$ in probability, and

$$\text{(ii)}\quad \begin{bmatrix}\sqrt{n}\,(\hat\theta_n^a - \theta)\\ \sqrt{n}\,(\hat\phi_n^a - \phi)\end{bmatrix} \Rightarrow N_2\left\{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}\sigma_\theta^2/\eta & 0\\ 0 & \sigma_\phi^2\end{bmatrix}\right\}$$

where

$$\sigma_\theta^2 = \left[E\left(\frac{\partial}{\partial\theta}\log f\right)^2\right]^{-1},\ \sigma_\phi^2 = \left[E\left(\frac{\partial}{\partial\phi}\log g\right)^2\right]^{-1}. \tag{20}$$

PROOF: The stated properties of ϕ_n^a follow from the classical maximum likelihood theory (Rao [12]), while the results for $\hat\theta_n^a$ can be deduced from the classical results using the strong Law of Large Numbers and Central limit theorem for random sum involved, namely, as $n \to \infty$,

$$\frac{1}{N_A}\sum_1^{N_A}\frac{\partial}{\partial\theta}\log f(u_j;\theta)\to 0 \text{ in probability}$$

$$\frac{1}{\sqrt{N_A}}\sum_1^{N_A}\frac{\partial}{\partial\theta}\log f(u_j;\theta)\Rightarrow N_1(0,\sigma_u^{-2}).$$

The use of Lemma 1 then allows us to replace the random scale N_A by the usual scale n to get the desired properties of $\hat{\theta}_n^a$. The asymptotic independence of $\hat{\theta}_n^a$ and $\hat{\phi}_n^a$ can be verified via the Cramér-Wold device as in the proof of Theorem 1. Details are omitted.

Returning now to the full likelihood function (17), let us denote by $\hat{\theta}_n$ and $\hat{\phi}_n$ the likelihood equation estimators based on it; thus, $\hat{\theta}_n$ and $\hat{\phi}_n$ satisfy the equations

$$\sum_1^{N_1}\frac{\partial}{\partial\theta}\log f(u_j;\theta)+H(X_n;\theta)=0 \tag{21}$$

$$\sum_1^{n}\frac{\partial}{\partial\phi}\log g(v_j;\phi)=0 \tag{22}$$

where

$$H(x;\theta)=\frac{\partial}{\partial\theta}\log[1-F(x;\theta)]. \tag{23}$$

Comparing (21)-(22) with (19) it is seen that $\hat{\phi}_n=\hat{\phi}_n^a$, while $\hat{\theta}_n$ differs from $\hat{\theta}_n^a$. We seek conditions under which the estimates $\hat{\theta}_n$ and $\hat{\theta}_n^a$ are asymptotically equivalent. For the purpose of motivation we first consider the important special case of Poisson arrivals. We have the following:

THEOREM 4: For the $M/G/1$ queue with mean interarrival time θ and service time density $g(v;\phi)$ we have

(i) $\hat{\theta}_n=\dfrac{D_n}{N_A}$, $\hat{\theta}_n^a=\dfrac{1}{N_A}\displaystyle\sum_1^{N_A}u_j$

and

(ii) $\sqrt{n}\,(\hat{\theta}_n-\theta)$ and $\sqrt{n}\,(\hat{\theta}_n^a-\theta)$ have the same limit distribution, namely $N_1(0,\theta^2/\eta)$.

PROOF: (i) We have $f(u;\theta)=\theta^{-1}e^{-u/\theta}$, so that $1-F(u;\theta)=e^{-u/\theta}$ and

$$H(x;\theta)=x\theta^{-2}.$$

The first of the Equations (19) reduces to

$$-\frac{N_A}{\theta}+\frac{1}{\theta^2}\sum_1^{N_A}u_j=0,$$

while (21) reduces to

$$-\frac{N_A}{\theta}+\frac{1}{\theta^2}\sum_1^{N_A}u_j+\frac{X_n}{\theta^2}=0.$$

These equations yield the estimates $\hat{\theta}_n^a$ and $\hat{\theta}_n$, respectively, as given by (i).

(ii) In view of Theorem 3 (ii) it suffices to show that as $n\to\infty$

$$\sqrt{n}\,(\hat{\theta}_n - \hat{\theta}_n^a) \to 0 \text{ in probability.}$$

Now

$$\sqrt{n}\,(\hat{\theta}_n - \hat{\theta}_n^a) = \frac{\sqrt{n}}{N_A} X_n = \frac{n}{N_A} \cdot \frac{X_n}{\sqrt{n}} \to 0$$

since $N_A n^{-1} \to \eta$ in probability by Lemma 1, and X_n has a limit distribution as $n \to \infty$.

COROLLARY 1: In the $M/M/1$ queue with mean interarrival time θ and mean service time ϕ we have

(i) $\hat{\theta}_n = \dfrac{D_n}{N_A}$, $\hat{\theta}_n^a = \dfrac{1}{N_A} \sum_1^{N_A} u_i$

$\hat{\phi}_n = \hat{\phi}_n^a = \dfrac{1}{n} \sum_1^n v_i$, and

(ii) $\sqrt{n}\,(\hat{\theta}_n - \theta)$ and $\sqrt{n}\,(\hat{\theta}_n^a - \theta)$ have the same limit distribution namely $N_1(0, \theta^2/\eta)$.

PROOF: The only new result is the one concerning the estimate of ϕ, which is obtained from (22).

The proof of Theorem 4 indicates that the appropriate condition for the asymptotic equivalence of the estimates $\hat{\theta}_n$ and $\hat{\theta}_n^a$ is that as $n \to \infty$

$$\frac{1}{\sqrt{n}} H(X_n; \theta) \to 0 \text{ in probability.} \tag{24}$$

This condition is satisfied in the case of Erlangian arrivals, as shown in the following:

EXAMPLE 1: For the queue $E_k/G/1$ we have

$$f(u; \theta) = \left(\frac{k}{\theta}\right)^k e^{-ku/\theta}\, u^{k-1}/(k-1)!$$

so that

$$1 - F(u; \theta) = \sum_0^{k-1} e^{-ku/\theta} \left(\frac{ku}{\theta}\right)^r / r!$$

and

$$\frac{\partial}{\partial \theta}\,[1 - F(u; \theta)] = \frac{ku}{\theta^2}\, e^{-ku/\theta} \left(\frac{ku}{\theta}\right)^{k-1} / (k-1)!.$$

It follows that

$$H(u; \theta) \leqslant \frac{ku}{\theta^2}$$

and so

$$\frac{1}{\sqrt{n}} H(X_n; \theta) \leqslant \frac{k}{\theta^2} \cdot \frac{X_n}{\sqrt{n}} \to 0 \text{ in probability}$$

since X_n converges in distribution as $n \to \infty$.

Under the condition (24) we are now able to establish the desired asymptotic equivalence of the estimates $\hat{\theta}_n$ and $\hat{\theta}_n^a$. We have the following:

THEOREM 5: Under the condition (24), $\sqrt{n}\,(\hat{\theta}_n - \theta)$ and $\sqrt{n}\,(\hat{\theta}_n^a - \theta)$ have the same limit distribution.

PROOF: We first approximate

$$\frac{1}{\sqrt{n}} \frac{\partial}{\partial \theta} \log L_n(f, g) \text{ and } \frac{1}{\sqrt{n}} \frac{\partial}{\partial \theta} \log L_n^a(f, g)$$

by linear functions of $\sqrt{n}\,(\hat{\theta}_n - \theta)$ and $\sqrt{n}\,(\hat{\theta}_n^a - \theta)$ in the usual manner (Rao [12] and then use condition (24), namely,

$$\frac{1}{\sqrt{n}} \left[\frac{\partial}{\partial \theta} \log L_n(f, g) - \frac{\partial}{\partial \theta} \log L_n^a(f, g) \right] \to 0 \tag{24'}$$

in probability.

4. THE M/M/1 QUEUE — m.l.e.'s BASED ON A SAMPLE FUNCTION OBSERVED OVER A FIXED INTERVAL (0, t]

In Sections 2 and 3 we observed the queueing phenomenon over a random time horizon $(0, D_n)$, where D_n is the nth departure epoch. This choice of observation period resulted in a considerable simplification in the forms of the estimates and their limit distributions. Corollary 1 shows that in the case of the $M/M/1$ queue the moment estimates coincide with the approximate m.l.e.'s of the interarrival and service time means, these latter being asymptotically equivalent to the full m.l.e.'s. However, due to the simplicity of the $M/M/1$ model we can consider a *likelihood function based on a continuous* observation of the phenomenon over a fixed interval $(0, t]$ and study the asymptotic properties of the m.l.e.'s as $t \to \infty$. This latter sampling plan was used by Wolff [14] who derived the asymptotic distributions for the case $\rho < 1$ directly from the known results of Billingsley [3] for Markov processes. In this section we obtain the limit distributions of the m.l.e.'s without any restriction on ρ, using elementary methods similar to those used in Section 2.

Let $Q(t)$ be the number of customers present in the system (including the one being served, if any) at time t. We formulate a sample function representation for $Q(t)$ as follows. Let $A(t)$ be the number of arrivals during a time-interval $(0, t]$; then $A(t)$ is a Poisson process with parameter λ. Let $D(t)$ be a Poisson process with parameter μ and independent of $A(t)$. Then clearly

$$Q(t) = Q(0) + A(t) - \int_0^t 1_{Q(s-)>0}\, dD(s) \tag{25}$$

where 1_A is the indicator function of the even tA. Here the integral represents $\bar{D}(t)$, the number of departures from the system during $(0, t]$; thus,

$$\bar{D}(t) = \int_0^t 1_{Q(s-)>0}\, dD(s). \tag{26}$$

The total time during $(0, t]$ that the server was busy is given by

$$B(t) = \int_0^t 1_{Q(s-)>0}\, ds. \tag{27}$$

The interarrival and service time means are given by $a = \lambda^{-1}$ and $b = \mu^{-1}$, respectively. The traffic intensity is $\rho = \lambda/\mu$. Also, we denote $\xi = \min(1, \rho)$. We need the following preliminary results, which will be proved in the appendix.

LEMMA 2: As $t \to \infty$

(i) $\dfrac{\bar{D}(t)}{t} \to \mu\xi$ a.s.

(ii) $\dfrac{B(t)}{t} \to \xi$ in probability.

Since $\{Q(t),\ t \geqslant 0\}$ is a Markov process with transition intensities

$$q_{i,i+1} = a^{-1}\ (i \geqslant 0),\ q_{i,i-1} = b^{-1}\ (i \geqslant 1)$$

it follows from Billingsley [3] that the likelihood function based on the sample function $\{Q(s),\ 0 \leqslant s \leqslant t\}$ is given by

$$L_t(a, b) \propto a^{-A(t)}\, e^{-t/a}\, b^{-\bar{D}(t)}\, e^{-B(t)/b}. \tag{28}$$

The m.l.e.'s of a and b obtained from (28) are seen to be

$$\hat{a}_t = \frac{t}{A(t)} \quad \text{and} \quad \hat{b}_t = \frac{B(t)}{\bar{D}(t)}. \tag{29}$$

We have the following.

THEOREM 6: As $t \to \infty$,

(i) $\hat{a}_t \to a$ a.s., and $\hat{b}_t \to b$ in probability,

(ii) $$\begin{pmatrix} \sqrt{t}\,(\hat{a}_t - a) \\ \sqrt{t}\,(\hat{b}_t - b) \end{pmatrix} \Rightarrow N_2\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} a^3 & 0 \\ 0 & b^3/\xi \end{pmatrix} \right\}.$$

PROOF: (i) By the strong law of large numbers applied to $A(t)$ we obtain

$$\hat{a}_t \to \lambda^{-1} = a \quad \text{a.s.}$$

Lemma 2 yields

$$\hat{b}_t = \frac{B(t)/t}{\bar{D}(t)/t} \to \frac{\xi}{\mu\xi} = \frac{1}{\mu} = b$$

in probability.

(ii) We write

$$\sqrt{t}\,(\hat{a}_t - a) = a^{3/2} \cdot \frac{\lambda t - A(t)}{\sqrt{\lambda t}} \cdot \frac{\lambda t}{A(t)}. \tag{30}$$

Here $A(t)/t \to \lambda$ a.s. by the strong Law of Large Numbers. Also, as $t \to \infty$ the distribution of $A(t)$ is asymptotically normal with mean λt and variance λt. Therefore, the right side of (30) converges to $N_1(0, a^3)$ in distribution.

Next consider

$$\sqrt{t}\,(\hat{b}_t - b) = \sqrt{t}\,\frac{B(t) - b\bar{D}(t)}{\bar{D}(t)} \tag{31}$$

$$= \sqrt{t}\cdot\frac{\sum_{1}^{\bar{D}(t)+1}(v_i - b)}{\bar{D}(t)} - \sqrt{t}\cdot\frac{(v_t' - b)}{\bar{D}(t)}$$

where v_t' is the residual service time of the customer in service (if any) at time t. Due to the lack of memory property of the exponential density, the distribution of v_t' is free of t. Therefore, writing

$$\sqrt{t}\cdot\frac{v_t' - b}{\bar{D}(t)} = \frac{t}{\bar{D}(t)}\cdot\frac{v_t' - b}{\sqrt{t}}$$

and using Lemma 2(i) we see that the second term on the right side of (31) converges to zero as $t \to \infty$. The first term can be written as

$$\sqrt{\frac{t}{\bar{D}(t)}}\cdot\sqrt{\frac{\bar{D}(t)+1}{\bar{D}(t)}}\cdot\frac{1}{\sqrt{\bar{D}(t)+1}}\sum_{1}^{\bar{D}(t)+1}(v_i - b). \tag{32}$$

By the central limit theorem applied to random sums we obtain

$$\frac{1}{\sqrt{\bar{D}(t)+1}}\sum_{1}^{\bar{D}(t)+1}(v_i - b) \Rightarrow N_1(0,\ b^2).$$

Again, using Lemma 2(i), we find that (32) converges in distribution to

$$\frac{1}{\sqrt{\mu\xi}}N_1(0,\ b^2) = N_1(0,\ b^3/\xi).$$

Thus,

$$\sqrt{t}\,(\hat{b}_t - b) \Rightarrow N_1(0,\ b^3/\xi). \tag{33}$$

The asymptotic independence of $\hat{a}_t$ and $\hat{b}_t$ is not immediately evident because of the dependence of $\bar{D}(t)$ on both the sequences $\{u_k\}$ and $\{v_k\}$. However, this independence can be established as in the proof of Theorem 1(ii) by first considering the random variable

$$\hat{b}_t^* = \frac{1}{D(t)}\sum_{1}^{D(t)} v_i \tag{34}$$

which is independent of $\hat{a}_t$, since the processes $A(t)$ and $D(t)$ are independent. We have $\sqrt{t}\,(\hat{b}_t^* - b) \Rightarrow N_1(0,\ b^3)$ as before, and

$$\begin{bmatrix}\sqrt{t}\,(\hat{a}_t - a)\\ \sqrt{t}\,(\hat{b}_t^* - b)\end{bmatrix} \Rightarrow N_2\left\{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}a^3 & 0\\ 0 & b^3\end{bmatrix}\right\}. \tag{35}$$

Using (35) we shall prove that

$$\alpha\sqrt{t}\,(\hat{a}_t - a) + \beta\sqrt{t}\,(\hat{b}_t - b) \Rightarrow N_1(0,\ \alpha^2a^3 + \beta^2b^3/\xi) \tag{36}$$

for arbitrary real numbers α, β. The desired result (ii) will then follow via the Cramér-Wold device.

Now the left side of (36) can be written as

$$(37) \qquad \left\{\alpha\sqrt{t}\,(\hat{a}_t - a) + \beta\frac{\sqrt{t}}{\xi}\,(\hat{b}_t^* - b)\right\} + \beta\left\{\sqrt{t}\,(\hat{b}_t - b) - \sqrt{\frac{t}{\xi}}\,(\hat{b}_t^* - b)\right\}.$$

The limit distribution of ther term within the first pair of brackets in (37) is $N_1(0,\ \alpha^2a^3 + \beta^2b^3/\xi)$ on account of (35). The term within the second pair of brackets in (37) can be written as

$$\left\{\sqrt{\frac{t}{\bar{D}(t)}} - \sqrt{\frac{b}{\xi}}\right\}\frac{1}{\sqrt{\bar{D}(t)}}\sum_1^{\bar{D}(t)}(v_i - b) + \sqrt{\frac{b}{\xi}}\left\{1 - \sqrt{\frac{t}{D(t)b}}\right\}\frac{1}{\sqrt{D(t)}}\sum_1^{D(t)}(v_i - b)$$
$$+ \sqrt{\frac{b}{\xi}}\left\{\frac{1}{\sqrt{\bar{D}(t)}}\sum_1^{\bar{D}(t)}(v_i - b) - \frac{1}{\sqrt{D(t)}}\sum_1^{D(t)}(v_i - b)\right\}$$

where we have used the fact that $v_i' - b$ plays the same role as $v_i - b$ $(1 \leqslant i \leqslant \bar{D}(t))$—see comments following (31). Using arguments similar to the ones used in the proof of Theorem 1(i) we find that all three terms in the last expression converges to zero in probability. This completes the proof.

5. APPENDIX

Only the outlines of the proofs are given below; for details see the monograph by Prabhu [11].

Lemma 1 is concerned with a single server queue, with notations as in Section 2, while Lemma 2 is concerned with the *M/M/1 queue described in Section 4.*

PROOF OF LEMMA 1: Let W_n be the nth customer's waiting time, and I_n the idle time (if any) preceding this customer. Then

$$(38) \qquad W_{n+1} = \max(0,\ W_n + X_{n+1}),\ I_{n+1} = -\min(0,\ W_n + X_{n+1})$$

where $X_k = v_k - u_k (k \geqslant 1)$. Let $S_0 \equiv 0$, $S_n = X_1 + X_2 + \ldots + X_n (n \geqslant 1)$. Then since $W_0 \equiv 0$, as has been assumed, we obtain from (38)

$$(39) \qquad W_n = S_n + \mathcal{I}_n,\ \mathcal{I}_n = I_1 + I_2 + \ldots + I_n = -\min_{0 \leqslant k \leqslant n} S_k.$$

It is clear that the successive departure epochs D_n are given by

$$D_1 = v_1,\ D_n = u_1 + u_2 + \ldots + u_{n-1} + W_{n-1} + v_n (n \geqslant 2).$$

In view of (39) we can write

$$(40) \qquad D_n = v_1 + v_2 + \ldots + v_n + \mathcal{I}_{n-1}\ (n \geqslant 1).$$

Now it is known that (i) if $\rho > 1$, then $\mathcal{I}_n \to \mathcal{I}$ a.s., while (ii) if $\rho \leqslant 1$, $\mathcal{I}_n \to \infty$ a.s., but $n^{-1}\mathcal{I}_n \to \max(0,\ a - b)$. Therefore,

$$(41) \qquad \frac{D_n}{n} = \frac{v_1 + v_2 + \ldots + v_n}{n} + \frac{\mathcal{I}_{n-1}}{n}$$
$$\to b + \max(0,\ a - b) = \max(a,\ b).$$

Next let $N_A(t) = \max\{k: u_1 + u_2 + \ldots + u_k \leqslant t\}$. Then by the elementary renewal theorem, $t^{-1}N_A(t) \to a^{-1}$ a.s. Since $D_n \to \infty$ by (41), we obtain

$$\frac{N_A(D_n)}{D_n} \to a^{-1} \quad \text{a.s.} \tag{42}$$

From (41) and (42) it follows that

$$\frac{N_A}{n} = \frac{N_A(D_n)}{D_n} \cdot \frac{D_n}{n} \to a^{-1} \max(a, b) = \eta \tag{43}$$

as required.

PROOF OF LEMMA 2: (i) We can write

$$\bar{D}(t) = D(t) - \int_0^t 1_{Q(s-)=0}\, dD(s) \tag{44}$$

where

$$\int_0^t 1_{Q(s-)=0}\, dD(s) = -m(t) = -\inf_{0 \leqslant s \leqslant t} [A(s) - D(s)] \tag{45}$$

and it is known that

$$\frac{-m(t)}{t} \to \max(0, \mu - \lambda) \quad \text{a.s.} \tag{46}$$

Therefore,

$$\frac{\bar{D}(t)}{t} = \frac{D(t)}{t} + \frac{m(t)}{t} \to \mu - \max(0, \mu - \lambda) = \mu\xi \quad \text{a.s.} \tag{47}$$

(ii) We can write $B(t) = t - I(t)$, where $I(t)$ is the idel time during $(0, t]$ and it is known that

$$\frac{I(t)}{t} \to \max(0, 1 - \rho) \quad \text{in probability.} \tag{48}$$

Therefore,

$$\frac{B(t)}{t} = 1 - \frac{I(t)}{t} \to 1 - \max(0, 1 - \rho) = \xi. \tag{49}$$

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OPTIMAL CONTROL FOR ENTRY OF MANY CLASSES OF CUSTOMERS TO AN $M/M/1$ QUEUE

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ABSTRACT

A method is developed for determining the optimal policy for entry of customers from many independent classes of Poisson arrivals to a first-come, first-serve (for customers admitted to the queue) single-server queue with exponential service times. The solution technique utilizes a semi-Markov formulation of the decision problem.

INTRODUCTION

A small but substantial collection of literature has developed on optimal control of entry to a queue starting with Naor's work [10] in 1969 and continuing to the present. Most of this work has been concerned with optimal policies and their properties for queues serving a single class of customers. In these models the reward and cost structure is the same for each customer although a customer's reward may be the value of a random variable.

In this paper, a model is considered in which several classes of customers arrive in independent Poisson streams. Each class has its own reward structure in that each member of class m receives a reward R_m for service and pays C_m per unit time spent in the system. This generalization has obvious applications in queueing systems. For example, consider an airport serving many classes of commercial aircraft (jumbo jets, wide-bodies, four engine jets, three engine jets, two engine jets) in which each class has its own reward for service and cost of waiting. It does not appear that the optimal control policy for such cases has been determined.

Two types of optimum control policies, the individual optimum and the social optimum, are considered in this paper. For an individual optimum policy, join or balk decisions are based on the expected gain of each individual customer at the time of his arrival. If his expected gain is nonnegative he joins; otherwise he balks. Under a social optimum policy, the join or balk decision is made based on maximizing the expected gain per unit time considering arrivals from all classes of customers.

A review of the literature indicates that "control-limit" policies are optimal for a single class of customers. This type of policy specifies that a customer be admitted if the number of customers already in the system is less than or equal to a given threshold value (the "control limit"). If the control limit is exceeded, the customer is turned away. Naor assumed that a control-limit policy is optimal for a single class of customers and then compared the control limits for individual and social optima. Yechiali [13] proved that a control-limit policy is indeed individually and socially optimum for a $GI/M/1$ queueing system serving a single class of customers. Subsequent work by several authors (for example, see [6], [7], [8], [9], [12], and [14]) demonstrated that control-limit policies are optimal for single class models under even more general assumptions.

When considering several classes of customers, the argument that the individual optimum policy is a control-limit policy carries over directly, as shown later, from the argument for a single class model. However, it is not possible to carry over Yechiali's argument for the social optimum policy and a different approach is used.

Previous work with several classes of customers appears to be quite limited. Balachandran and Schaefer [1, 2] deal with several classes of arrivals to $M/G/1$ queues. However, each class has the ability to adjust its arrival rate so that the class with the most favorable net benefit (reward minus expected cost) dominates the arrivals to the exclusion of all other classes of customers. Furthermore, this work is based entirely on long term expected values and no customers are able to take advantage of short-lived phenomena, e.g., a customer who does not belong to the single class which dominates arriving to find the system empty will be denied service. Edelson and Hildebrand [4] come closer to the problem dealt with in this paper. They consider the $M/M/1$ model for classes with different reward and cost parameters and balking. The claim to have considered more than two classes of customers, but no results for more than two classes are presented, and in what seems to be a contradiction they indicate in their conclusion that the solution procedure they have used (which is not specified in their paper) does not extend to cases where "dichotomous classification of customers is not acceptable." While they are interested in comparing revenue maximizing policies with the social optimum, it should be noted that the solution given for the two class social optimum policy in their first example, after appropriate adjustments for different definitions, was checked and found to yield the same policy that would be obtained by the techniques established in this paper. Furthermore, this check provides an indication that the general techniques set forth in this paper are, in addition to being workable, correct.

In this paper, after defining the model, it will be shown that Naor's solution for the individual optimum for a single class of customers easily extends to M classes of customers. The extension of his social optimum solution to many classes of customers is then made by adapting the semi-Markov decision process technique used by Yechiali for a single class of customers. The optimal policy is shown to be a control-limit policy and simple bounds on this policy are presented. An example of the application of the technique to three classes of customers is given.

THE MODEL

M classes of customers are considered. Customers of class m, $m = 1, 2, \ldots, M$, arrive in a Poisson stream with mean rate λ_m. The service times of the single server are independent, identically distributed, exponential random variables with mean $1/\mu$. Thus, the model

represents an $M/M/1$ queueing system with customers from M separate groups or classes. Each class has its own reward and cost structure. Once a customer enters the system, service is first-come, first-served, regardless of class. A member of class m receives a reward for service, R_m, and pays C_m per unit time spent in the system. R_m is required to be greater than or equal to C_m/μ for all m. (Customers from any class for which this is not true would balk under all circumstances.) For determining the individual optimal policy, each customer is assumed to be able to determine the number of customers in the system at the time of his arrival. Also, for determining the social optimal policy, an administrator is assumed who knows the number of customers in the system at the time of each arrival. An optimal policy is sought; that is, a set of join or balk decisions is sought for each class for each possible state (the total number of customers in the system) to maximize (i) the expected return to the self-optimizing (individual) customer or (ii) the gain per unit time of all customers (social optimum).

SOLUTION FOR THE INDIVIDUAL OPTIMUM

If an arrival from class m finds i customers ahead of him, his expected net benefit for joining the system is $R_m - (i+1)\,C_m/\mu$. Since the net benefit for balking is zero, the customer will join the system if the number of customers in the system is less than n_{s_m}, where n_{s_m} is such that

$$R_m - (n_{s_m}+1)\,C_m/\mu < 0 \leqslant R_m - n_{s_m}\,C_m/\mu. \tag{1}$$

(Customers are assumed to join the system if their expected net benefit for joining is zero.) Thus, self-optimizing customers of class m determine a balking point n_{s_m} such that

$$n_{s_m} = [R_m\mu/C_m], \tag{2}$$

where the brackets indicate the greatest integer function. (2) is a straight-forward extension of Naor's result for a single class of customers.

SOLUTION FOR THE SOCIAL OPTIMUM

For the social optimum problem, an administrator decides whether or not a customer of class m can join when i customers are ahead of him. The sum of the expected net benefits per unit time of all arrivals of all classes is shown to be maximized by a policy that imposes a vector $\bar{n}_0$ of forced balking points on the customers. $\bar{n}_0 = (n_{0_1}, n_{0_2}, \ldots, n_{0_M})$, where a member of class m is allowed to join if the state of the system is less than n_{0_m} but must balk otherwise.

Let $\bar{n} = (n_1, n_2, \ldots, n_M)$ be a vector of balking points. The sum of the expected net benefits per unit time of all arrivals when the balking points given by $\bar{n}$ are chosen, is given by

$$g(\bar{n}) = \sum_{m=1}^{M} \lambda'_m(\bar{n})\,R_m - \sum_{m=1}^{M} C_m\,L_m(\bar{n}). \tag{3}$$

$\lambda'_m(\bar{n})$ and $L_m(\bar{n})$ are, respectively, the effective arrival rate for class m and the contribution of class m to the expected number of customers in the system when balking points $\bar{n}$ are employed. $\lambda'_m(\bar{n})$ and $L_m(\bar{n})$ depend on $\bar{n}$ because the stationary probability that i customers are in the system, $\phi_i(\bar{n})$, depends on $\bar{n}$.

Rue [11] shows that the following semi-Markov decision process formulation of $g(\bar{n})$ is equivalent to (3):

$$g(\bar{n}) = \sum_{i=0}^{n^*} \phi_i(\bar{n}) \sum_{m=1}^{M} k_m(i)\lambda_m\{R_m - (i+1)C_m/\mu\}, \tag{4}$$

where $n^* = \max_m \{n_m\}$, $\phi_i(\bar{n})$ is the stationary probability that i customers are in the system given that the balking points $\bar{n}$ are employed, and

$$k_m(i) = \begin{cases} 1 \text{ if } i < n_m \\ 0 \text{ if } i \geq n_m \end{cases}$$

(It may be helpful to note at this point that the term $\lambda_m\{R_m - (i+1)C_m/\mu\}$ is the gain rate in state i due to class m customers joining the queue.) The work of Derman [3] can be used to limit the semi-Markov decision process formulation to nonrandomized policies as in (4). Policy iteration can be used to obtain from (4) the maximum gain rate and socially optimal joining policy.

Theorem 1 justifies restricting the consideration of optimal policies to control-limit policies as in (3). However, lemma 1 must be proved first.

LEMMA 1: *The social optimum policy will not allow reneging.*

PROOF: Suppose customer A of class m arrives at time T_A and joins the system. Later, at time T_D, customer A departs the system before he is served. The actions of customer A affect no customer who arrived before him. The contribution of customer A to the net gain of the system over the interval $[T_A, T_D]$ is $-C_m(T_D - T_A) < 0$, his holding cost for the time he is in the system. If no other customer arrives before customer A departs, his actions have no effect on those customers arriving after him. If other customers arrive during $[T_A, T_D]$, the only effect of customer A's temporary presence in the system is to possibly cause a customer to balk when, in fact, without the presence of A this customer would be profitable (and thus join the system). Thus, the contribution of customer A to the system is negative when compared with his not joining the system at all.

THEOREM 1: A control-limit policy is optimal for each class.

PROOF: Suppose that there exists a class m and a state i such that in the optimal policy, p, $k_m(i) = 1$, but $k_m(i-1) = 0$; that is, the optimal policy for class m is not a control-limit policy. The optimal gain rate is denoted by g^*. If the stationary probability of the system occupying state i under policy p, ϕ_i^p is equal to zero, then a policy which is the same as the optimal except that $k_m(i) = 0$ also yields g^*. To prove Theorem 1 for the case $\phi_i^p \neq 0$, consider the following modification to the model. At the completion of a service, each customer returns his expected net benefit for recalculation based on the current state of the system. Likewise, his occupation costs incurred during the service just completed are also returned. The customers remain in the order they arrived, but the administrator recomputes the expected net benefit of each customer based on the number now ahead of that customer and uses the given policy, p, to decide whether or not each customer can re-enter (stay in) the system. Because there is no discounting, the gain rate of this model is equivalent to that of the original model for the same policy. Since the transitions are Markovian, all relevant information about the future of the system is contained in the current state of the system. (The state of the system must give the position of every customer to accommodate the new method of assessing costs.) A customer of class m who had joined when i customers were ahead of him would be forced to depart when

the number of customers ahead of him dropped to $i - 1$, since $k_m(i - 1) = 0$. Thus, the customer would be forced to renege. By Lemma 1, the given policy cannot be optimal for the modified or original model.

Since the expected net benefit for members of any class joining the system decreases as the number of customers in the system increases, the optimality of a control limit policy is a reasonable result.

It is now shown (Theorem 2) that for each class m, $n_{0_m} \leq n_{s_m}$, where n_{0_m} is the socially optimal balking point for class m customers and n_{s_m} is the individual optimum balking point for class m. Thus, $n_m^* = \max_m \{n_{s_m}\}$ can serve as a bound on the state space required for the policy iteration solution of (4).

THEOREM 2: For each class m, $n_{0_m} \leq n_{s_m}$.

PROOF: First, the decision made by the administrator for a given arrival does not affect the time of arrival of any customer yet to arrive. As previously shown, n_{s_m} satisfies (1)

$$R_m - (n_{s_m} + 1)\, C_m/\mu < 0 \leq R_m - n_{s_m} C_m/\mu .$$

Suppose a customer, customer A, of class m arrives to find the state of the system $i \geq n_{s_m}$. Let T_A be the time of arrival of customer A, and T_B be the time of arrival of the next customer, customer B. In view of (1), let $\alpha < 0$ be the expected net benefit of customer A joining the system. If all expected costs and rewards are assigned to a customer upon arrival (as in (4)), the contribution of the interval $[T_A, T_B]$ to the expected net gain is $\alpha < 0$ if customer A is allowed to join. The decision regarding customer A does not affect the expected net gain of customers who arrived before him, but it does affect the expected net gain of those arriving after him. If customer A does not join, the state of the system found by customer B and all others after him is less than or equal to the state of the system if customer A joins. Since $R_m - (i + 1)\, C_m/\mu$, the expected gain for a customer of class m joining when i are in the system, is a strictly decreasing function of i, the contribution to the expected net gain of the interval $[T_B, T_C]$ for any $T_C > T_B$ is at least as large when customer A balks as when he joins. Thus, if $i \geq n_{s_m}$, forcing a customer of class m to balk yields a larger expected net gain or gain rate than allowing him to join.

The two theorems proved above make it possible to solve for the social optimum policy using policy iteration or linear programming. For M classes of customers, we seek to determine the $\bar{n}$ which maximizes the gain rate given in Equation (4). The solution technique will be illustrated by an example. Suppose that three classes of customers arrive at a single exponential server who has a service rate capability of four customers per unit time. The reward for service, cost per unit time in the system, and mean arrival rate of each class are given in Table 1.

TABLE 1 — *Model Parameters for the Three-Class Example*

Class	R_m	C_m	λ_m
1	3	4	2
2	2	3	4
3	4	7	6

To see what $g(\bar{n})$ looks like for a particular policy $\bar{n}$, consider $g(\bar{n}_s)$ where $\bar{n}_s$ is the individual optimum policy. From (2)

$$\bar{n}_s = (n_{s_1}, n_{s_2}, n_{s_3}) = (3, 2, 2).$$

Since $n_s^* = \max_m (n_{s_m}) = 3$ at most states 0 through 3 are required to evaluate $g(\bar{n})$ for any $\bar{n}$, e.g.,

$$g(3, 2, 2) = g(\bar{n}_s) = \sum_{i=0}^{2} \phi_i(\bar{n}_s)\lambda_1 \{R_1 - (i+1)C_1/\mu\} + \sum_{i=0}^{1} \phi_i(\bar{n}_s) \sum_{m=2}^{3} \lambda_m \{R_m - (i+1)C_m/\mu\}$$

where the $\phi_i(\bar{n}_s)$, $i = 0, 1, 2, 3$ can be obtained from the simultaneous solution of the three rate equations

$$\sum_{m=1}^{3} \lambda_m \phi_0(\bar{n}_s) = \mu\phi_1(\bar{n}_s)$$

$$\left[\sum_{m=1}^{3} \lambda_m + \mu\right] \phi_1(\bar{n}_s) = \sum_{m=1}^{3} \lambda_m \phi_0(\bar{n}_s) + \mu\phi_2(\bar{n}_s)$$

$$(\lambda_1 + \mu)\phi_2(\bar{n}_s) = \sum_{m=1}^{3} \lambda_m \phi_1(\bar{n}_s) + \mu\phi_3(\bar{n}_s)$$

and the normalizing equation

$$\sum_{i=0}^{3} \phi_i(\bar{n}_s) = 1.$$

The first iteration of Howard's [5] policy iteration algorithm yields the individually optimal policy, $\bar{n}_s$. Because the relative values of occupying the various states (Howard's v_i's) are all set to zero, a social optimizing customer cannot determine the effect his joining has on later arriving customers and, thus, can do no better than the individually optimal policy. Thus, the first iteration yields a control-limit policy with balking points $\bar{n} = (3, 2, 2)$ and the expected gain rate associated with the policy is $g = 2.486$. As the policy iteration algorithm proceeds, better and better knowledge of the v_i's will result in better and better policies until an optimal policy is found.

The second iteration of Howard's algorithm leads to another control-limit policy with balking points $\bar{n} = (2, 0, 1)$ and an expected gain rate of $g = 5.375$. This policy excludes class two from the system.

The socially optimal policy is found on the third iteration and is, of course, a control-limit policy. The balking points are $\bar{n}_0 = (1, 0, 1)$ and the expected gain rate is 5.833. Again, class two is excluded from the system. The stationary probabilities of the system occupying the various states under the optimal policy are $\bar{\phi}(\bar{n}_0) = (1/3, 2/3, 0, 0)$.

SUMMARY

The single class of customers model of Naor is extended in this paper to several customer classes and then solved for the individual optimum policy and the social optimum policy and

associated gain rate. Both the individual and social optimum policies are shown to be control-limit policies. In addition, the socially optimal control limit for each class is shown to be no greater than the individually optimal control limit for the same class. The solution techniques are illustrated by an example.

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ON QUEUES WITH DEPENDENT INTERARRIVAL AND SERVICE TIMES

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ABSTRACT

An $M/G/1$ queueing system is studied in which the service time required by a customer is dependent on the interarrival time between his arrival and that of his predecessor. Assuming the two variables are "associated," we prove that the expected delay in this system is less than or equal to that of a conventional $M/G/1$ queue. This conclusion has been verified via simulation by Mitchell and Paulson [9] for a special class of dependent $M/M/1$ queue. Their model is a special case of the one we consider here. We also study another modified $GI/G/1$ queue, where the arrival process and/or the service process are individually "associated."

1. INTRODUCTION

The conventional $GI/G/1$ queueing model considered in most papers in the literature assumes that the sequences of interarrival times and service times are i.i.d. (independent *and* identically distributed) random variables. However, the independence assumption may not be realistic for many real world problems. The purpose of this note is to investigate the effect on average delay (queueing time only) of customers in systems whose interarrival and service times are *associated,* a concept of positive dependence developed by Esary, Proschan and Walkup [6].

In a recent paper, Mitchell and Paulson [9] studied via simulation an $M/M/1$ queue with the modification that a customer's service time and the interarrival time between his arrival and that of his predecessor are positively correlated random variables having a bivariate exponential distribution. Their simulation results indicate that this type of dependency reduces the mean waiting time of customers as compared to the usual $M/M/1$ queue. Some related results also appeared in Conolly [2], Conolly and Hadidi [3, 4]. Motivated by Mitchell and Paulson's simulation results, we shall show here in Section 3 that their conclusion can be proven analytically under weaker conditions. In Section 2, we briefly summarize the useful concept of association of random variables. Another variant of $GI/G/1$ queues with dependent interarrival times or service times will also be discussed in Section 4.

2. ASSOCIATION OF RANDOM VARIABLES

The concept of association of random variables was developed by Esary, Proschan and Walkup [6]. It is a very useful tool in the study of reliability systems (see Barlow and Proschan

[1], Esary and Proschan [5]), queueing systems (see Niu [10]), and simulation (see Heidelberger and Iglehart [7]).

DEFINITION 1: Random variables $\underline{X} = (X_1, \ldots, X_n)$ are called *associated* if cov $[f(\underline{X}), g(\underline{X})] \geq 0$ for all pairs of nondecreasing functions f and g.

The following is a partial list of properties enjoyed by associated random variables:

(P1): The set consisting of a single random variable is associated.

(P2): Nondecreasing functions of associated random variables are associated.

(P3): Any subset of associated random variables are associated.

(P4): If two sets of associated random variables are independent of each other, then their union is a set of associated random variables.

(P5): A set of independent random variables are associated.

The last property, P5, is a direct consequence of P1 and P4. As an illustration, we will next show that the bivariate exponential distribution discussed in Mitchell and Paulson [9] is generated from a pair of associated random variables. Consider the random vector (X, Y) defined by

$$(X, Y) = \left(\sum_{i=1}^{N} X_i, \sum_{i=1}^{N} Y_i\right),$$

where X_i, $i = 1, 2, \ldots$, are i.i.d. random variables, Y_i, $i = 1, 2, \ldots$, are i.i.d. random variables, and N is an integer-valued positive random variable.

PROPOSITION 1: (X, Y) is associated.

PROOF: Let f and g be an arbitrary pair of nondecreasing functions. Conditioning on $N = n$, we have

$$\begin{aligned}
&\operatorname{cov}[f(X, Y), g(X, Y)] \\
&= E\left\{\operatorname{cov}\left[f\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i\right), g\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i\right) | N = n\right]\right\} \\
&+ \operatorname{cov}\left\{E\left[f\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i\right) | N = n\right], E\left[g\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i\right) | N = n\right]\right\}.
\end{aligned}$$

Now, given $N = n$, the vector $(X_1, \ldots, X_n, Y_1, \ldots, Y_n)$ is a collection of independent random variables and hence is associated (by P5). By P2, this implies $\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i\right)$ is associated. Therefore, the first term above is nonnegative. The conclusion of the proposition then follows by observing that both $E\left[f\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i\right) | N = n\right]$ and $E\left[g\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i\right) | N = n\right]$ are nondecreasing functions of n.

Q.E.D.

REMARK: When both X_i, $i = 1, 2, \ldots$, and Y_i, $i = 1, 2, \ldots$, have exponential distributions and N is geometric, then (X, Y) has the bivariate exponential distribution described in [9].

3. AN M/G/1 QUEUE WITH DEPENDENT ARRIVAL AND SERVICE

Consider the following model of a single server queue with infinite waiting room. Denote by T_n the interarrival time between customers C_n and C_{n+1} with $E(T_n) = 1/\lambda$. Let S_n be the service time of customer C_n with $E(S_n) = 1/\mu$ and $\lambda/\mu < 1$. We assume (T_n, S_{n+1}), $n = 0, 1, \ldots$, is a sequence of i.i.d. random vectors and (T_n, S_{n+1}) is associated for each n. It should be noted that this model includes both the conventional $GI/G/1$ queue (where T_n and S_{n+1} are independent) and the $M/M/1$ queue considered by Mitchell and Paulson [9] by letting (T_n, S_{n+1}) to have their bivariate exponential distribution (see remark after Proposition 1).

Denoting the delay of customer C_n by D_n, it is well-known that

$$D_{n+1} = \max[0, D_n + S_n - T_n], \tag{1}$$

or equivalently,

$$D_{n+1} - A_n = D_n + S_n - T_n, \tag{2}$$

where $A_n = -\min[0, D_n + S_n - T_n]$. We will assume that the first customer arrives at time 0 and the system is initially empty, i.e., $D_1 \equiv 0$. The following key lemma will be needed:

LEMMA 1: $\operatorname{cov}[D_n, S_n] \leqslant 0$ for all $n \geqslant 1$.

PROOF: The assertion is clearly true for $n = 1$ since $D_1 \equiv 0$. For $n \geqslant 2$, (T_{n-1}, S_n) is, by assumption, associated and independent of $-(D_{n-1} + S_{n-1})$. Therefore, $(-D_{n-1} - S_{n-1}, T_{n-1}, S_n)$ is associated by P4. Observe that $-D_n$ and S_n are nondecreasing functions of $(-D_{n-1} - S_{n-1}, T_{n-1}, S_n)$. Hence, by P2, $(-D_n, S_n)$ is associated and consequently $\operatorname{cov}[D_n, S_n] \leqslant 0$.

Q.E.D.

Now, squaring both sides of (2) and taking expectations, we have

$$E(D_{n+1}^2) + E(A_n^2) = E(D_n^2) + E(S_n - T_n)^2 + 2E(D_nS_n) - 2E(D_nT_n), \tag{3}$$

where we have used the fact that $D_{n+1}A_n \equiv 0$. Letting n go to infinity in (3), we can cancel $E(D_{n+1}^2)$ and $E(D_n^2)$ from both sides assuming they are finite. Since D_n and T_n are independent and, by Lemma 1, $\operatorname{cov}[D_n, S_n] = E(D_nS_n) - E(D_n)\,E(S_n) \leqslant 0$, the right-hand-side of (3) is less than or equal to

$$E(S_n - T_n)^2 + 2E(D_n)\,E(S_n) - 2E(D_n)\,E(T_n).$$

Noting that $E(T_n - S_n) = E(A_n)$, (3) simplifies to

$$E(D_n) \leqslant \frac{E(T_n - S_n)^2}{2E(T_n - S_n)} - \frac{E(A_n^2)}{2E(A_n)}. \tag{4}$$

We are now ready for the main result in this section.

THEOREM 1: If the arrival process is Poisson and the system is in equilibrium, then

$$E(D_n) \leqslant \frac{\lambda E(S_n^2)}{2(1-\lambda E(S_n))}, \tag{5}$$

where the right-hand-side term is the expected delay of a conventional $M/G/1$ queue.

PROOF: Note that $(A_n|A_n > 0)$ is the idle period ended by the arrival of customer C_{n+1}. Therefore, by the memoryless property of exponential distributions, $(A_n|A_n > 0)$ is also exponential with rate λ. Hence,

$$\begin{aligned}\frac{E(A_n^2)}{2E(A_n)} &= \frac{E(A_n^2|A_n > 0) \cdot P\{A_n > 0\}}{2E(A_n|A_n > 0) \cdot P\{A_n > 0\}} \\ &= \frac{E(A_n^2|A_n > 0)}{2E(A_n|A_n > 0)} \\ &= 1/\lambda\end{aligned}$$

and from which (5) follows. Q.E.D.

For arrival processes other than Poisson, the analysis of (4) becomes more difficult because we do not know the distribution of $(A_n|A_n > 0)$. However, various bounds for it may be obtained by considering special classes of arrival processes (see Marshall [8]). We shall not pursue this further here except for mentioning that all upper bounds for expected delay in special classes of $GI/G/1$ queues obtained by Marshall [8] can also be applied to our modified $GI/G/1$ queues.

4. SOME RELATED RESULTS

In this section, we will consider another modification of $GI/G/1$ queues which is different from the one discussed in Section 3. We shall assume that the sequences of service times and interarrival times, $\{S_n, n \geqslant 1\}$ and $\{T_n, n \geqslant 1\}$, are identically distributed *and* associated, i.e., $(S_1, S_2, \ldots, S_n)$ and $(T_1, T_2, \ldots, T_n)$ are associated vectors for all $n \geqslant 1$. Of course, this is again a generalization of the conventional $GI/G/1$ queue because independent random variables are associated. Now, it is easy to see that

$$D_n = \max\left[0, S_{n-1} - T_{n-1}, \ldots, \sum_{i=1}^{n-1}(S_i - T_i)\right].$$

Hence, (D_n, S_n) and $(D_n, -T_n)$ are associated vectors for all $n \geqslant 1$. It follows that $E(D_n S_n) \geqslant E(D_n)E(S_n)$ and $E(D_n T_n) \leqslant E(D_n)E(T_n)$. An argument similar to Section 3 will then lead to

$$E(D_n) \geqslant \frac{E(T_n - S_n)^2}{2E(T_n - S_n)} - \frac{E(A_n^2)}{2E(A_n)}. \tag{6}$$

Again, the second term on the right-hand-side of (6) is difficult to analyze. However, we have

THEOREM 2: If $\{T_n, n \geqslant 1\}$ are i.i.d. exponential random variables, i.e., the arrival process is Poisson, then, under stationary conditions, we have

$$E(D_n) \geq \frac{\lambda E(S_n^2)}{2(1-\lambda E(S_n))}.$$

In other words, associated service times in an $M/G/1$ queue tend to *increase* the expected delay of a customer. This is a somewhat expected result.

When $\{T_n, n \geq 1\}$ is a renewal process (not necessarily Poisson), lower bounds for $E(D_n)$ may be found through (6) for several special classes of interarrival times as in Marshall [8]. As a final remark, we note that the departure process of a conventional $GI/D/1$ queue (the service times are deterministic) is an associated process. So, (6) may be used to find a lower bound for the expected delay of a subsequent station.

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ON THE OPTIMAL ALLOCATION OF CREWMEN ABOARD A WARSHIP

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ABSTRACT

In the temporarily isolated situation in which a warship finds itself during a mission, not only spare parts, but also "spare" crewmen in various categories of specialization must be on board.

Mathematical models for the probabilities of mission survival for personnel and for personnel and materiel jointly are proposed. A practical example is worked out: the optimal allocation of spare crewmen to different categories of specialization is calculated.

1. INTRODUCTION

In relation to a system on board a warship (weapons, communications etc.) a meaningful figure of merit is the mission reliability, defined as: "the probability that the system will operate in the mode for which it was designed for the duration of the mission" [1]. Loosely described it is the probability that at the end of the planned mission the system is still in operation in an acceptable manner.

Most of the equipment on board, apart from normal control, needs service, checkup, adjustments, replacement of failing units, and other forms of maintenance and short-term repair during the mission. In terms of reliability theory, materiel and personnel are "in series": the unavailability of one invalidates the other. The mission reliability of a system on board is the joint probability that at the end of the mission the system is still in an operable state or on short term repairable while at the same time at least the minimal human attendance is on hand.

By way of precaution against nonoperation of an entire system as a result of failure of a vital part, spare parts are taken along. This is one of the possible forms of materiel redundancy. On the effect of materiel redundancy on the reliability, a reasonably well-developed theory is available [3]. In [4], the composition of optimal spare-part kits is dealt with. Such a kit, of all kits of the same cost , guarantees the highest reliability of the system that it pertains to.

In the situation of temporary isolation in which a warship finds itself during a mission, there are no easy means to replace operators and maintenance men who have become "hors de combat." Aboard a modern and automated warship, a growing percentage of the crew is highly specialized and classified in a considerable variety of specializations, which makes it increasingly

difficult for one to stand in for any other outside his own category. This leads to the necessity for a relatively large number of "spare" crewmen in "expensive" categories to be taken along. As a result of this personnel redundancy it may be found that a high degree of automation does not automatically yield the savings in personnel costs that are generally expected.

In view of these tendencies a modern management demands, within operational and financial constraints, a careful optimization in the spending of funds on "spares," where materiel and personnel aspects are not separable. This paper is primarily intended to be a contribution to the development of methods for the modeling and solution of this optimization problem. Related problems were treated in [5].

2. A MATHEMATICAL MODEL FOR THE SURVIVAL OF CREWMEN

In this section we introduce a mathematical model for the probability for crewmen to survive a mission.

The key parameter in this model is the expected number of hits, t, that the ships incurs during the mission. This number is assumed to be an integer-valued Poisson-distributed random variable with mean g, so

$$P(t) = \exp(-g) \cdot \frac{g^t}{t!}. \tag{2.1}$$

The probability for one particular crewman to survive a hit is p. In consequence, the probability for him to survive t hits is p^t, and the probability r that he survives the mission is

$$r = \sum_{t=0}^{\infty} p^t P(t) = \sum_{t=0}^{\infty} \exp(-g) \cdot \frac{(pg)^t}{t!} = \exp(p-1)g. \tag{2.2}$$

If A and B are two particular crewmen, then at each hit, their survivals are assumed to be independent events. This implies that the probability that both A and B survive t hits is p^{2t}, and the probability $r(2/2)$ that both of them survive the mission is

$$r(2/2) = \sum_{t=0}^{\infty} p^{2t} P(t) = \sum_{t=0}^{\infty} \exp(-g) \cdot \frac{(p^2 g)^t}{t!} = \exp(p^2-1)g. \tag{2.3}$$

Because of their common dependence on t, the survivals of the mission by A and by B are no independent events. A useful measure of interdependence is the (pairwise) correlation coefficient C_{corr}, which is defined in the following way. Consider the two random variables x_A and x_B, where $x_A = 1$ if A survives the mission and $x_A = 0$ if he does not, while x_B is defined similarly. We now put

$$C_{\text{corr}} \overset{def}{=} \frac{\text{cov}(x_A, x_B)}{\sqrt{\text{var}(x_A) \cdot \text{var}(x_B)}} \tag{2.4}$$

$$= \frac{E(x_A x_B) - E(x_A) \cdot E(x_B)}{E(x_A^2) - E^2(x_A)} \tag{2.5}$$

which is found to be

$$C_{\text{corr}} = \frac{\exp(p^2 - 1)g - \exp 2(p-1)g}{\exp(p-1)g - \exp 2(p-1)g}. \tag{2.6}$$

If $|(p-1)g| \ll 1$, we can use the approximations

$$C_{\text{corr}} \approx 1 - p = \frac{-\ln r}{g} \tag{2.7}$$

$$p \approx 1 - C_{\text{corr}} \text{ and } g \approx \frac{-\ln r}{C_{\text{corr}}}. \tag{2.8}$$

If g and p are varied while $(1-p)g$ and thus r remain constant, the extreme cases are

$g = -\ln r$ and $p = 0$: then $C_{\text{corr}} = 1$

$g \to \infty$ and $p \to 1$: then $C_{\text{corr}} = 0$.

For calculations in the latter (uncorrelated) case, the following property is useful

$$\exp(p^i - 1)g \to \exp i(p-1)g = r^i \text{ for } i = 2, 3, \ldots. \tag{2.9}$$

3. A MATHEMATICAL MODEL FOR THE MISSION RELIABILITY OF A MANNED SYSTEM

For the case where materiel and personnel are "in series," we propose an extension of the mathematical model of the previous section as will be demonstrated by the following example.

We are dealing with a system that must be manned by a crew of minimally z. At the beginning of the mission, n crewmen qualified to operate/maintain the system are on board. During the mission, the ship incurs t hits where t is integer-valued Poisson-distributed with mean g. At each hit, an individual has a probability of survival p, while the materiel of the system has a probability of survival a.

Given the number of hits, t, the probability that a sufficient number of crew survives is (see appendix):

$$\binom{n}{z} \sum_{u=0}^{n-z} (-1)^u \binom{n-z}{u} \frac{z}{z+u} p^{(z+u)t} \tag{3.1}$$

while the probability that the materiel survives is

$$a^t. \tag{3.2}$$

Hence, the probability that the materiel as well as enough personnel survive t hits is

$$\binom{n}{z} \sum_{u=0}^{n-z} (-1)^u \binom{n-z}{u} \frac{z}{z+u} (ap^{z+u})^t. \tag{3.3}$$

The mission reliability R_{ap} of the system, which is the probability that the materiel as well as enough crew survive the mission, is obtained by multiplying (3.3) by $P(t)$ and summing with respect to t. The result is

$$R_{ap} = \binom{n}{z} \sum_{u=0}^{n-z} (-1)^u \binom{n-z}{u} \frac{z}{z+u} \exp(ap^{z+u} - 1)g. \tag{3.4}$$

In the uncorrelated case, by invoking (2.9), we obtain for the mission reliability

$$R_{ap} = \binom{n}{z} \sum_{u=0}^{n-z} (-1)^u \binom{n-z}{u} \frac{z}{z+u} r_a r_p^{z+u} \tag{3.5}$$

where r_p is the probability that an individual survives the mission and r_a is the probability that the materiel survives the mission.

A Practical Example

With $z = 1$ we have for $n = 1, 2, 3$, respectively, in the correlated case

$$R_{ap} = \exp(ap - 1)g$$

$$R_{ap} = 2\exp(ap - 1)g - \exp(ap^2 - 1)g$$

$$R_{ap} = 3\exp(ap - 1)g - 3\exp(ap^2 - 1)g + \exp(ap^3 - 1)g$$

and in the uncorrelated case

$$R_{ap} = r_a r_p$$

$$R_{ap} = 2r_a r_p - r_a r_p^2$$

$$R_{ap} = 3r_a r_p - 3r_a r_p^2 + r_a r_p^3.$$

We vary p and g in such a way that a

$$(p - 1)g = \text{constant} = -0.0305$$

keeping

$$a = p$$

which means that the probability of mission survival for an individual as well as for the materiel is fixed at

$$r_p = r_a = 0.970.$$

In Figure 1 we plot $1 - R_{ap}$ as a function of p. In order to show the influence of the vulnerability of the materiel, we also plot $1 - R_p$, which is the probability that enough crewmen survive the mission, regardless of materiel survival. The quantity R_p derived from R_{ap} by putting $a = 1$ or $r_a = 1$.

In Table 1 we list $1 - R_p$ and $1 - R_{ap}$ as functions of n in the uncorrelated case. For $n \to \infty$ we have $1 - R_p \to 0$ and $1 - R_{ap} \to 1 - r_a$.

① $1 - R_{ap}$ for $n = 1$
② idem for $n = 2$
③ idem for $n = 3$
④ $1 - R_p$ for $n = 1$; also $1 - R_{ap}$ for $n \to \infty$
⑤ $1 - R_p$ for $n = 2$
⑥ idem for $n = 3$

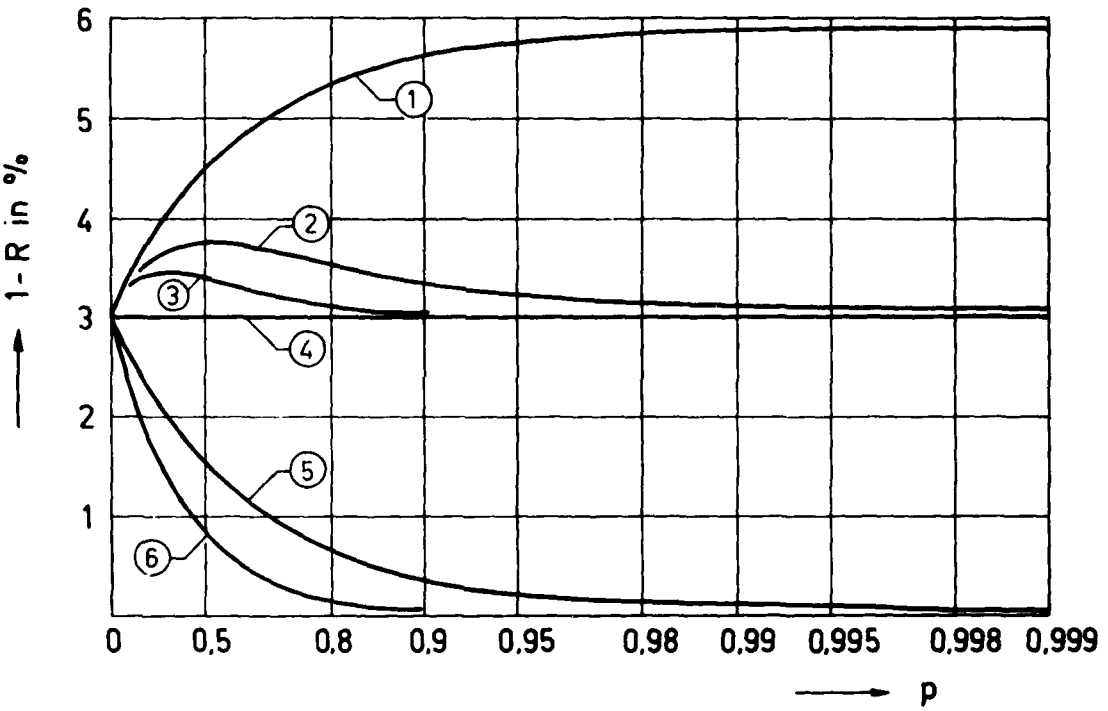

FIGURE 1. $1 - R_p$ and $1 - R_{ap}$ as functions of p.

TABLE 1. *Probabilities of Survival in the Uncorrelated Case*

	$1 - R_p$ in %	$1 - R_{ap}$ in %
$n = 1$	3.00	5.91
$n = 2$	0.0900	3.09
$n = 3$	0.00270	3.00
$n \to \infty$	0	3.00

4. THE MISSION RELIABILITY OF A SYSTEM COMPOSED OF TWO SUBSYSTEMS

We consider a system S composed of two subsystems S_1 and S_2. For the operation/maintenance of S, crewmembers are on board in two categories, 1 and 2. Crewmen in category 1 are competent to operate/maintain S_1 (but not S_2), while crewmen in category 2 are competent to operate/maintain S_2 (but not S_1). S cannot be in operation unless z_1 crewmen in category 1 and z_2 in category 2 are on hand. At the beginning of the mission there are n_1 crewmen on board in category 1 and n_2 in category 2. The ship incurs t hits, where t is an integer-valued Poisson-distributed random variable with mean g. At each hit, the probability of survival of an individual is p, while the probability of survival of the materiel of S_1 and S_2 is a_1 and a_2, respectively.

The probability that after t hits the materiel of subsystem S_1 survives, along with a sufficient number of crewmen to operate/maintain it, is

$$\binom{n_1}{z_1}\sum_{u_1=0}^{n_1-z_1}(-1)^{u_1}\frac{z_1}{z_1+u_1}\binom{n_1-z_1}{u_1}(a_1p^{z_1+u_1})^t \quad \text{(see appendix)}. \tag{4.1}$$

For S_2 a similar expression can be written. After multiplying the two expressions we obtain the following expression for the probability that the materiel of both subsystems, along with enough crew, survives t hits

$$\binom{n_1}{z_1}\binom{n_2}{z_2}\sum_{u_1=0}^{n_1-z_1}\frac{z_1}{z_1+u_1}\binom{n_1-z_1}{u_1}\sum_{u_2=0}^{n_2-z_2}(-1)^{u_1+u_2}\frac{z_2}{z_2+u_2}\binom{n_2-z_2}{u_2}(a_1a_2p^{z_1+z_2+u_1+u_2})^t. \tag{4.2}$$

By multiplying this by $P(t)$ and summing with respect to t, we obtain the mission reliability

$$R_{ap}=\binom{n_1}{z_1}\binom{n_2}{z_2}\sum_{u_1=0}^{n_1-z_1}\frac{z_1}{z_1+u_1}\binom{n_1-z_1}{u_1}\sum_{u_2=0}^{n_2-z_2}(-1)^{u_1+u_2}\frac{z_2}{z_2+u_2}\binom{n_2-z_2}{u_2}\exp(a_1a_2p^{z_1+z_2+u_1+u_2}-1)g. \tag{4.3}$$

In the uncorrelated case this changes into

$$R_{ap}=\binom{n_1}{z_1}\binom{n_2}{z_2}\sum_{u_1=0}^{n_1-z_1}\frac{z_1}{z_1+u_1}\binom{n_1-z_1}{u_1}\sum_{u_2=0}^{n_2-z_2}(-1)^{u_1+u_2}\frac{z_2}{z_2+u_2}\binom{n_2-z_2}{u_2}r_{a_1}r_{a_2}r_p^{z_1+z_2+u_1+u_2} \tag{4.4}$$

where r_{a_1} and r_{a_2} are the probabilities that the materiel of S_1 and S_2, respectively, survives the mission. The probability, R_p, that enough crew survives the mission to operate/maintain the two subsystems, irrespective of materiel survival, can be derived from R_{ap} by putting in (4.3): $a_1 = a_2 = 1$ and in (4.4): $r_{a_1} = r_{a_2} = 1$.

For the probability R_a that the materiel of both subsystems survives the mission, we have in the correlated case

$$R_a = \exp(a_1 a_2 - 1)g \tag{4.5}$$

which can be derived in the same fashion as (2.3), and which in the uncorrelated case changes into

$$R_a = r_{a_1} r_{a_2}. \tag{4.6}$$

5. SOME NUMERICAL RESULTS

The quantities R_p, R_{ap} and R_a as dealt with in the preceding section were calculated for the case where

$$z_1 = 8 \text{ and } z_2 = 1$$

while g, p, n_1 and n_2 were varied; a_1 and a_2 were varied along with p such that always

$$a_1 = a_2 = p.$$

As p was varied, g was varied at the same time, keeping

$$(p - 1)g = \text{constant}.$$

Three runs were made, with

$$(p - 1)g = \ln 0.990, \ln 0.970 \text{ and } \ln 0.900,$$

respectively.

Samples of the numerical results are given in Tables 2, 3, 4 and 5, and in Figures 2 and 3. In Tables 2 and 3 we list $1 - R_p$ and $1 - R_{ap}$, respectively, as functions of (n_1, n_2), as obtained at the second run, in the uncorrelated case. In Tables 4 and 5 we do the same for the case where $p = 0.700$. In Figure 2 we plot $1 - R_p$ as a function of p for several pairs of values of (n_1, n_2), again as obtained at the second run. The corresponding plots of $1 - R_{ap}$ and $1 - R_a$ are given in Figure 3.

Tables like 2, 3, 4 and 5 can be used for the determination of optimal sequences of allocation. Let us suppose that the cost per crewman per mission is k_1 for a man in category 1 and k_2 for a man in category 2. Let us further suppose that we can afford to spend an amount K on the crew in the two categories. This means that we have to choose n_1 and n_2 under the restriction that

$$n_1 k_1 + n_2 k_2 \leqslant K. \tag{5.1}$$

For every value of K, it is easy to find a pair of values (n_1, n_2) such that under the restriction (5.1), the lowest value of $1 - R_p$ or $1 - R_{ap}$ is obtained. By doing this for increasing values of K, the optimal sequence of allocation is established.

TABLE 2. $1 - R_p$ *in % as a function of* (n_1, n_2), *for* $(p-1)g = \ln 0.970$ and $p \to 1$.

n_1 \ n_2	1	2	3	4
8	24.0	21.7	21.6	21.6
9	5.73	2.90	2.82	2.82
10	3.27	0.366	0.279	0.277
11	3.02	0.113	0.0253	0.0226
12	3.00	0.0916	0.00432	0.00169
13	3.00	0.0900	0.00280	0.185×10^{-3}

TABLE 3. $1 - R_{ap}$ *in % as a function of* (n_1, n_2) for $(p-1)g = \ln 0.970$ and $p \to 1$.

n_1 \ n_2	1	2	3	4
8	28.5	26.3	26.3	26.3
9	11.3	8.64	8.56	8.56
10	8.99	6.25	6.17	6.17
11	8.75	6.02	5.93	5.93
12	8.73	6.00	5.91	5.91
13	8.73	5.99	5.91	5.91
14	8.73	5.99	5.91	5.91

TABLE 4. $1 - R_p$ *in % as a function of* (n_1, n_2), *for* $(p-1)g = \ln 0.970$ *and* $p = 0.7$.

n_1 \ n_2	1	2	3	4
8	9.28	9.17	9.14	9.13
9	8.39	8.01	7.90	7.86
10	7.19	6.44	6.22	6.15
11	5.97	4.87	4.53	4.43
12	4.97	3.55	3.12	2.99
13	4.23	2.60	2.10	1.95
14	3.75	1.96	1.41	1.24
15	3.45	1.56	0.985	0.806
16	3.27	1.33	0.728	0.542
17	3.16	1.19	0.576	0.385
18	3.10	1.10	0.485	0.290
19	3.05	1.02	0.394	0.195

TABLE 5. $1 - R_p$ *in % as a function of* (n_1, n_2), *for* $(p-1)g = \ln 0.970$ *and* $p = 0.7$.

n_1 \ n_2	1	2	3	4
8	9.47	9.42	9.40	9.40
9	9.04	8.85	8.80	8.78
10	8.45	8.09	7.98	7.94
11	7.86	7.32	7.15	7.10
12	7.37	6.68	6.47	6.41
13	7.01	6.22	5.98	5.91
14	6.78	5.92	5.65	5.57
15	6.64	5.73	5.45	5.37
16	6.56	5.62	5.34	5.25
17	6.51	5.56	5.27	5.18

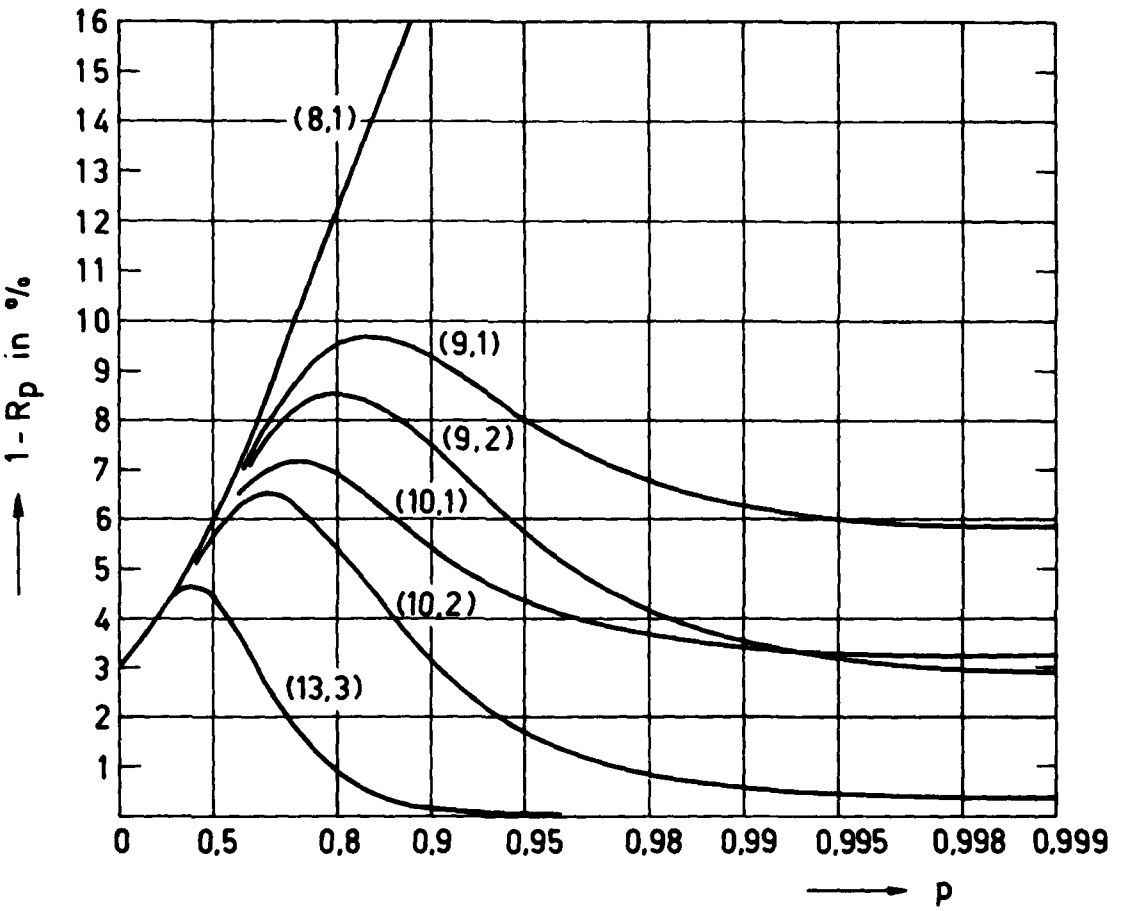

FIGURE 2. $1 - R_p$ vs. p for several values of (n_1, n_2), for $(p-1)\,g = \ln 0.970$

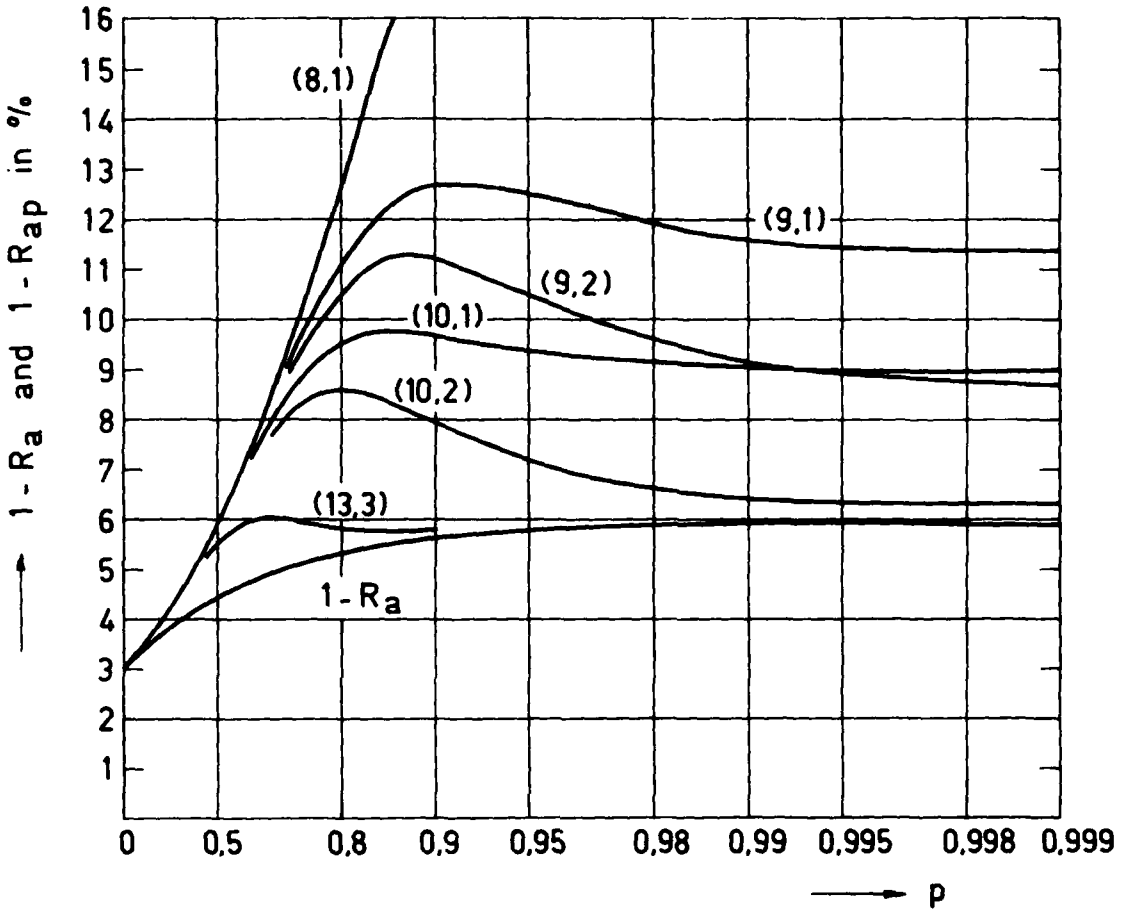

FIGURE 3. $1 - R_a$ and $1 - R_{ap}$ vs. p for several values of (n_1, n_2), for $(p - 1)\,g = \ln 0.970$

Results are depicted in Figures 4 and 5, which show allocation sequences optimizing $1 - R_p$ and $1 - R_{ap}$, respectively. In both of these figures, the black dots refer to the situation where

$$k_1 = k_2$$

whereas the open dots apply to the situation where

$$k_1 : k_2 = 1 : 2.$$

In the former case the sequence is discontinued after the first step where $1 - R_p$ or $1 - R_{ap}$ is reduced by less than 0.5%; in the latter case the sequence is continued up to the same final value.

The effect of the first four optimal allocation steps (that is to say, optimal for the case $k_1 = k_2$) on the numerical values of $1 - R_p$ and $1 - R_{ap}$ is shown in Tables 6 and 7, respectively. The parameters in these tables are the same as in Figures 4 and 5. In Table 7 we also give the limit values $1 - R_a$ of $1 - R_{ap}$ as n_1 and n_2 approach infinity.

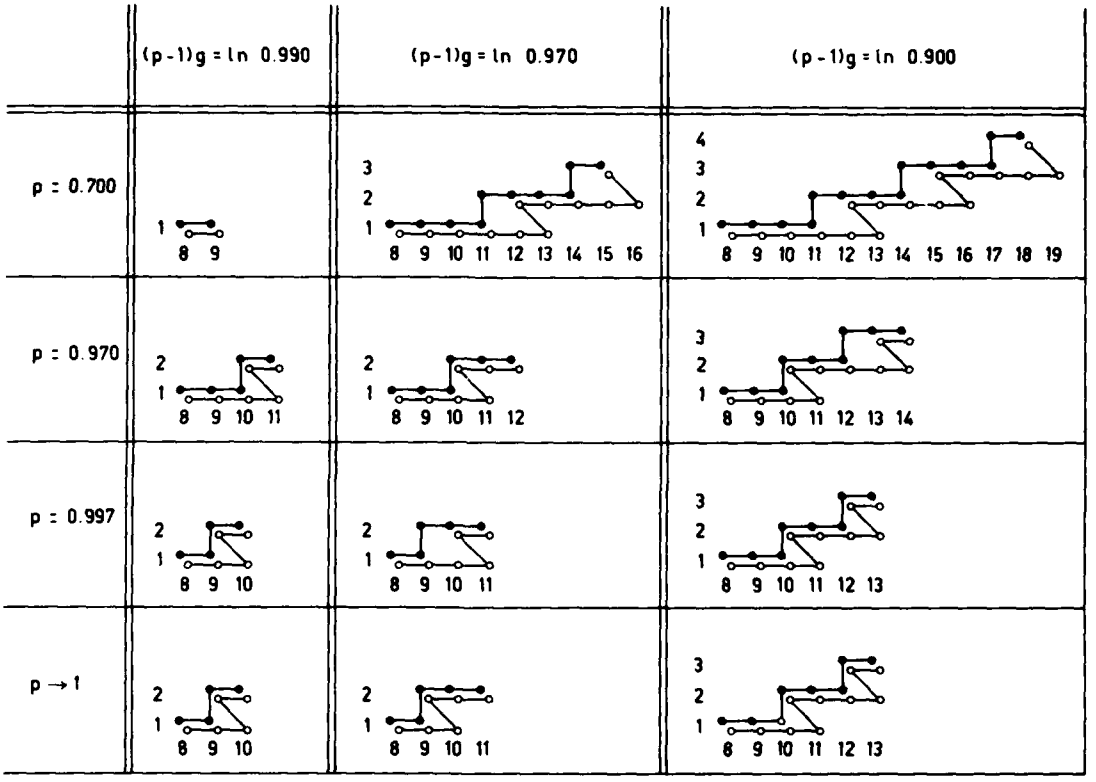

FIGURE 4. Allocation sequences optimizing R_p

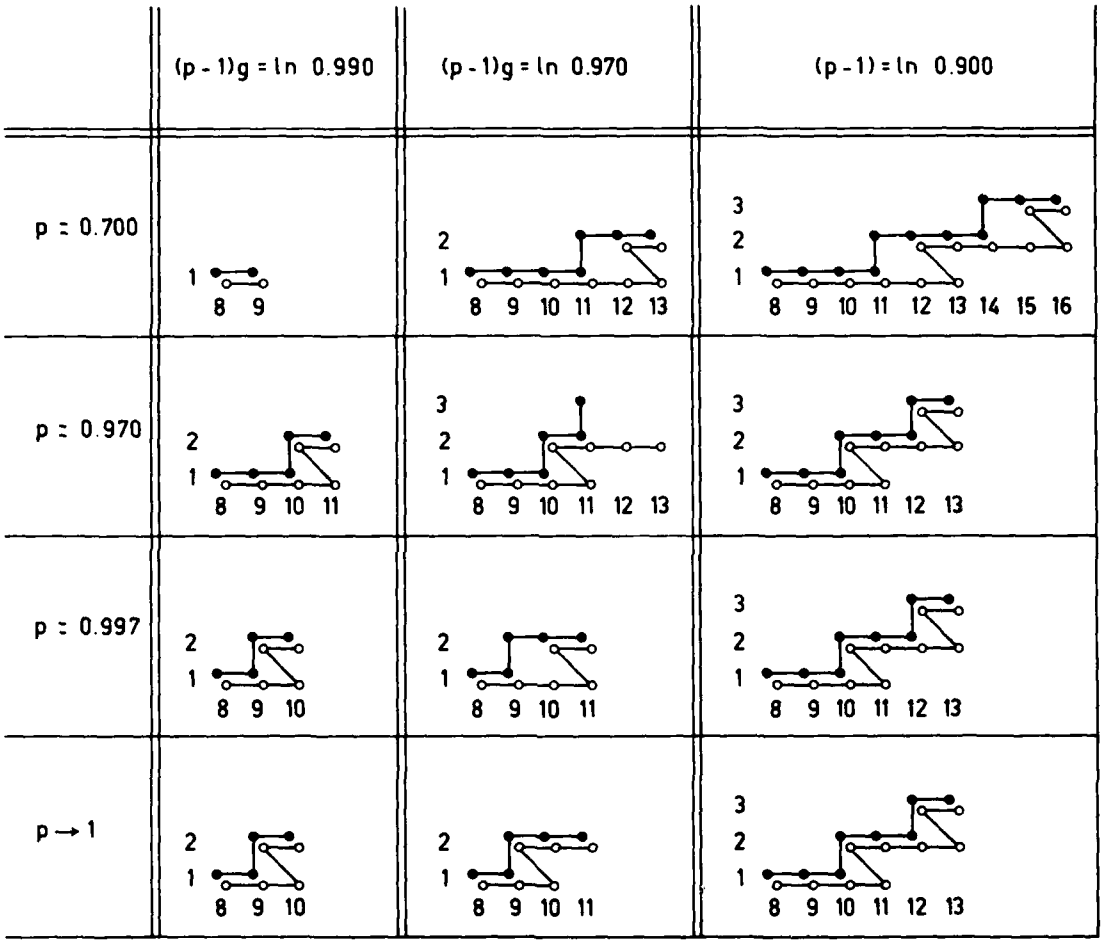

FIGURE 5. Allocation sequences optimizing R_{ap}

TABLE 6. $1 - R_p$ *in ‰ as a Function of* (n_1, n_2); Allocation Steps Optimal With Respect to $n_1 + n_2$.

	$(p-1)g = \ln 0.990$		$(p-1)g = \ln 0.970$		$(p-1)g = \ln 0.900$	
	(n_1, n_2)	$1 - R_p$	(n_1, n_2)	$1 - R_p$	(n_1, n_2)	$1 - R_p$
$p = 0.700$	(8,1)	3.16	(8,1)	9.28	(8,1)	28.6
	(9,1)	2.85	(9,1)	8.39	(9,1)	26.2
	(10,1)	2.43	(10,1)	7.19	(10,1)	22.9
	(11,1)	2.00	(11,1)	5.97	(11,1)	19.5
	(11,2)	1.61	(11,2)	4.87	(11,2)	1[illegible]
$p = 0.970$	(8,1)	7.72	(8,1)	21.6	(8,1)	56.9
	(9,1)	2.10	(9,1)	7.25	(9,1)	30.4
	(10,1)	1.16	(10,1)	3.90	(10,1)	18.0
	(10,2)	0.204	(10,2)	1.14	(10,2)	10.4
	(11,2)	0.0629	(11,2)	0.382	(11,2)	4.72
$p = 0.997$	(8,1)	8.55	(8,1)	23.7	(8,1)	60.8
	(9,1)	1.43	(9,1)	5.90	(9,1)	30.3
	(9,2)	0.447	(9,2)	3.10	(10,1)	16.5
	(10,2)	0.0346	(10,2)	0.442	(10,2)	8.23
	(11,2)	0.0140	(11,2)	0.133	(11,2)	3.03
$p \to 1$	(8,1)	8.65	(8,1)	24.0	(8,1)	61.3
	(9,1)	1.34	(9,1)	5.73	(9,1)	30.3
	(9,2)	0.354	(9,2)	2.90	(10,1)	16.3
	(10,2)	0.0214	(10,2)	0.366	(10,2)	7.95
	(11,2)	0.0103	(11,2)	0.113	(11,2)	2.83

TABLE 7. $1 - R_{ap}$ *in ‰ as a Function of* (n_1, n_2); Allocation Steps Optimal With Respect to $n_1 + n_2$.

	$(p-1)g = \ln 0.990$		$(p-1)g = \ln 0.970$		$(p-1)g = \ln 0.900$	
	(n_1, n_2)	$1 - R_{ap}$	(n_1, n_2)	$1 - R_{ap}$	(n_1, n_2)	$1 - R_{ap}$
$p = 0.700$	(8,1)	3.23	(8,1)	9.47	(8,1)	29.1
	(9,1)	3.08	(9,1)	9.04	(9,1)	27.9
	(10,1)	2.87	(10,1)	8.45	(10,1)	26.3
	(11,1)	2.66	(11,1)	7.86	(11,1)	24.7
	(11,2)	2.47	(11,2)	7.32	(11,2)	23.3
	(∞, ∞)	1.69	(∞, ∞)	5.05	(∞, ∞)	16.4
$p = 0.970$	(8,1)	9.10	(8,1)	25.1	(8,1)	63.2
	(9,1)	3.89	(9,1)	12.2	(9,1)	41.8
	(10,1)	3.02	(10,1)	9.25	(10,1)	32.2
	(10,2)	2.14	(10,2)	6.80	(10,2)	26.4
	(11,2)	2.02	(11,2)	6.15	(11,2)	22.1
	(∞, ∞)	1.96	(∞, ∞)	5.82	(∞, ∞)	18.7
$p = 0.997$	(8,1)	10.3	(8,1)	28.1	(8,1)	68.1
	(9,1)	3.38	(9,1)	11.4	(9,1)	43.3
	(9,2)	2.42	(9,2)	8.79	(10,1)	32.2
	(10,2)	2.02	(10,2)	6.31	(10,2)	25.6
	(11,2)	2.00	(11,2)	6.03	(11,2)	21.4
	(∞, ∞)	1.99	(∞, ∞)	5.90	(∞, ∞)	19.0
$p \to 1$	(8,1)	10.5	(8,1)	28.5	(8,1)	68.6
	(9,1)	3.30	(9,1)	11.3	(9,1)	43.5
	(9,2)	2.34	(9,2)	8.64	(10,1)	32.2
	(10,2)	2.01	(10,2)	6.25	(10,2)	25.4
	(11,2)	2.00	(11,2)	6.02	(11,2)	21.3
	(∞, ∞)	1.99	(∞, ∞)	5.91	(∞, ∞)	19.0

SOME OBSERVATIONS ON THE RESULTS:

1. Comparing situations where p is equal to or close to 1 (so $C_{corr} \approx 0$) with situations where p is smaller (so C_{corr} is larger), we find that in the former case the mission reliability with $(n_1, n_2) = (8, 1)$ is smaller, but the addition of spare crewmen is more effectual.

2. Comparing the optimal allocation sequences in the various cases, we find that these sequences are not too strongly dependent on p, g, a_1a_2 or k_2/k_1. In all cases $n1/n_2$ tends to become smaller than z_1/z_2. In other words we recognize a tendency to allocate a relatively large portion of the spares to the smaller number.

3. Comparing the allocation sequences for a particular value of p, while ln $(p-1)g$ is varied, we recognize the tendency of n_1/n_2 to approach z_1/z_2 as exp $(p-1)g$ decreases: if the risk of the mission increases, the extra allocations go primarily to the larger number.

4. There is a significant difference between spare parts in a spare part kit and "spare" crewmen aboard a warship. Only parts in active state can fail, whereas parts in storage ("cold standby") will not get out of order. A "spare" crewman, on the other hand, is vulnerable as soon as the mission is begun (in terms of reliability theory, he is in "hot standby"). Despite this difference, however, the results obtained here are in accordance with the rules which were found to apply to optimal spare part kits [4].

5. Comparing R_p with R_{ap} under the same circumstances, we find that the addition of spare crewmen has less effect on R_{ap} than on R_p. Since the difference between R_{ap} and R_p is caused by the vulnerability of the materiel, this illustrates that the addition of spare crewmen in less effectual as the materiel is more vulnerable. Results like the ones we have obtained here could be useful for making the choice between spending available funds on spare crew or on improving the mission reliability of the equipment.

6. CONCLUDING REMARKS

In this paper, mathematical models for the mission reliabilities of personnel, and personnel and materiel jointly, were presented. An example was worked out, where the optimal allocation of crewmen in different categories of specialization was calculated. Despite its simplicity, this example is indicative of the results that can be obtained by application of the same techniques to more complex configurations.

From our calculations we have learned that the mission reliability of a manned system not only depends on the probabilities of survival of individual crewmen and separate pieces of equipment, but equally on the correlations between survivals. This implies that efforts must be made to gain insight into the statistical characteristics of risks that personnel and equipment are exposed to during a mission.

APPENDIX

Consider n possible events $e_1, e_2, \ldots, e_n$ which are symmetrical in the sense that

$$P(e_1) = P(e_2) = \ldots = P(e_n) \tag{A.1a}$$

$$P(e_1e_2) = P(e_1e_3) = \ldots = P(e_{n-1}e_n) \tag{A.1b}$$

$$P(e_1e_2e_3) = P(e_1e_2e_4) = \ldots = P(e_{n-2}e_{n-1}e_n) \tag{A.1c}$$

Then the following property is known [2]. The probability $P(\geqslant z, n)$ that at least z events happen is

$$P(\geqslant z, n) = \binom{n}{z} \sum_{u=0}^{n-z} (-1)^u \binom{n-z}{u} \frac{z}{z+u} \pi(z+u) \tag{A.2}$$

where $\pi(z+u)$ is the probability that $e_1, e_2, \ldots, e_{z+u}$ all happen, irrespective of whether the others happen or not.

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STOCHASTIC DUELS WITH DISPLACEMENTS (SUPPRESSION)*

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ABSTRACT

The stochastic duel is extended to include the possibility of a near-miss on each round fired, which causes the opponent to displace. During displacement, the displacing contestant cannot return the fire but is still a target for his opponent. An alternative interpretation of this model is to consider the displacement time as the time a contestant's fire is suppressed by his opponent's fire and that he does not move, but merely ceases fire temporarily. All times are exponentially distributed.

1. INTRODUCTION

This paper continues the development of the Theory of Stochastic Duels. Although combat is a very complex process, microscopic models of this type help to develop important insights into the more complex processes and are also useful in weapons systems analyses, among other things.

Some earlier work has considered mobility by incorporating varying projectile time-of-flight, see Ancker [2] and Jaiswal and Bhashyam [5], or by varying hit probability, see Williams [8] and Bhashyam and Singh [4] and Ancker [1]. Some very simple models, where the contestants displace, are provided in Ancker and Williams [3] and Schoderbeck [6] and [7]. In this paper we consider a more general situation of the latter type.

In general, most earlier studies of stochastic duels have not considered an interaction between the two contestants, A and B. That is to say, the same results would have been effected if each fired separately at his own target in two different localities and they subsequently compared their times to score a kill, having previously agreed that the quicker of the two would be the winner. We shall now consider a duel where A's behavior is contingent upon B's, and vice versa.

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2. THE MODEL

We assume, as usual, that both duelists begin to load and fire simultaneously, but we now add the possibility that one or the other may score a near-miss, the effect of which on a duelist is to make him move to a new firing position. We may imagine that if he did not move, he would immediately be killed, since his opponent has now gotten his range. During his displacement time, he is subject to fire from his opponent but cannot return it. We assume that the probabilities of a near-miss and a kill are the same from round to round, and that once a duelist has displaced because of a near-miss, he proceeds to load and fire as before. His displacement and firing times are random variables whose probability density functions are known and are not necessarily the same.

An alternate interpretation that may be useful is as follows: Upon receiving a near-miss, the duelist merely seeks cover in his present position and ceases fire for a period of time equal to the corresponding displacement time, i.e., merely interpret displacement time as fire-suppression time, and the model is a fire-suppression model as it stands.

3. ANALYSIS

Our first step is to eliminate from consideration all complete misses on both sides. We are thus left with a series of near-misses that form, so to speak, a succession of turning points on which the duel hinges. It does not matter which of the duelists scores a near-miss, since it still interrupts the course of the duel. We thus make a list of when the near-misses occurred and who scored them. Ultimately, the duel ends on a near-miss which was actually a kill.

We introduce the following notation. Let

(1)
$$\begin{cases} p_{AA} = \text{the conditional probability that } A \text{ scores the next near-miss, given that } A \text{ scored the last one,} \\ p_{BA} = \text{the conditional probability that } B \text{ scores the next near-miss, given that } A \text{ scored the last one,} \end{cases}$$

and similarly for p_{AB} and p_{BB}. Then, we plainly have

(2) $$p_{AA} + p_{BA} = p_{AB} + p_{BB} = 1.$$

Next, let

(3) p_A = the unconditional probability that A scores a near-miss on any round,

and

(4) k_A = the conditional probability that A scores a kill, given a near-miss,

that is, on any near-miss, a kill may or may not have occurred, and

(5) $k_A p_A$ = the joint probability of a near-miss and a kill, or just simply a kill.

Also, let

$$q_A = 1 - p_A = \text{the probability that } A \text{ misses entirely on any round} \tag{6}$$

and

$$l_A = 1 - k_A = \text{the conditional probability that } A \text{ does not score a kill, given a near-miss.} \tag{7}$$

Of course, there is a similar notation for B. We may remark here, that it will be important in some future research to allow these probabilities, i.e., (3), (4), (6) and (7), to be different in the suppressed and nonsuppressed states.

We further let

$$\begin{cases} p_{OA} = \text{the probability that } A \text{ scores the first near-miss} \\ p_{OB} = \text{the probability that } B \text{ scores the first near-miss} \end{cases} \tag{8}$$

so that

$$p_{OA} + p_{OB} = 1 \tag{9}$$

and we call

$$\begin{cases} P_n(A) = \text{the probability that } A \text{ kills, for the first time, on near-miss number } n \\ P_n(B) = \text{the probability that } B \text{ kills, for the first time, on near-miss number } n \end{cases} \tag{10}$$

Then, the probability, $P(A)$, that A wins the duel is

$$\begin{cases} P(A) = \sum_{n=1}^{\infty} P_n(A) \ , \\ \text{and, similarly for } B \\ P(B) = \sum_{n=1}^{\infty} P_n(B) \ . \end{cases} \tag{11}$$

Also, $P(A) + P(B) = 1$. Clearly, when $n = 1$, we have

$$P_1(A) = p_{OA}k_A \text{ and } P_1(B) = p_{OB}k_B. \tag{12}$$

Now, the probability that A scored a near miss on round $n - 1$, but did not kill, is given by

$$\frac{l_A}{k_A} P_{n-1}(A)$$

since $P_{n-1}(A)$ includes a factor k_A for the kill on round $n - 1$, which factor must be replaced by l_A if there was no kill. Thus,

$$P_n(A) = k_A \left[p_{AA} \frac{l_A}{k_A} P_{n-1}(A) + p_{AB} \frac{l_B}{k_B} P_{n-1}(B) \right] \tag{13}$$

provides the two near-miss situations on a given round that can lead to a kill by A on the next round. Hence,

$$P_n(A) = l_A p_{AA} P_{n-1}(A) + l_B p_{AB} \frac{k_A}{k_B} P_{n-1}(B), \tag{14}$$

and similarly,

$$P_n(B) = l_B p_{BB} P_{n-1}(B) + l_A p_{BA} \frac{k_B}{k_A} P_{n-1}(A). \tag{15}$$

Summing both sides of these simultaneous difference equations, from 2 to ∞, and making use of Equations (11) and (12), we find that

$$P(A) - p_{OA} k_A = l_A p_{AA} P(A) + l_B p_{AB} \frac{k_A}{k_B} P(B), \tag{16}$$

and

$$P(B) - p_{OB} k_B = l_B p_{BB} p(B) + l_A p_{BA} \frac{k_B}{k_A} P(A). \tag{17}$$

When we solve these two equations simultaneously for $P(A)$, we obtain

$$P(A) = \frac{k_A (p_{OA} k_B + l_B p_{AB})}{k_B - l_A p_{AA} k_B + l_B p_{AB} k_A}, \tag{18}$$

with a corresponding expression for $P(B)$. By using Notations (1), (2), and (7), it is easily shown that the denominators of $P(A)$ and $P(B)$ are equal.

Now, adding the numerator of $P(A)$ to that of $P(B)$, we find

$$k_A k_B p_{OA} + k_A l_B p_{AB} + k_A k_B p_{OB} + k_B l_A p_{BA},$$

which, by using Equations (2) and (9), is equal to

$$k_A k_B + k_A l_B p_{AB} + k_B l_A - k_B l_A p_{AA}.$$

And now, using Equation (7), this is equal to

$$k_B + k_A l_B p_{AB} - k_B l_A p_{AA},$$

which is equal to the common denominator of $P(A)$ and $P(B)$, thus checking that $P(A) + P(B) = 1$.

Now, assume that A's firing time is exponentially distributed with mean $1/r_A$ (r_A is A's rate of fire), and his displacement time is also an exponential with mean δ_A, and similarly for B. Then, by exactly the same reasoning as used in the fundamental duel, [9], to arrive at the first kill, t he probability that A makes the first near-miss, p_{OA}, is

$$p_{OA} = \frac{p_A r_A}{p_A r_A + p_B r_B}. \tag{19}$$

Also, from Notation (1),

$$p_{AA} = P[T_A \leqslant T_B + D_B] \tag{20}$$

where

T_A = random variable time between A's near-misses,

T_B = time between B's near-misses, and

D_B = B's time to displace.

This expression accounts for the fact that after A scores a near-miss, he proceeds to fire again, whereas B must first displace and then fire. And, as we have already seen, when the probability of the event (near-miss) is p_A and the firing time is exponential with mean $1/r_A$, then the time to score a near-miss is exponentially distributed with mean $1/p_A r_A$. Again, we may use the solution of the fundamental duel to obtain this probability, p_{AA}, by noting that the characteristic function of $T_B + D_B$ is simply the product of their individual characteristic functions. Hence,

$$p_{AA} = \frac{1}{2\pi i} \int_{-\infty - i\epsilon}^{\infty - i\epsilon} \frac{p_A r_A p_B r_B \, du}{(p_A r_A + iu)(p_B r_B - iu)(1 - i\delta_B u)} \tag{21}$$

and, using residue theory, we have

$$p_{AA} = \frac{p_A r_A (1 + p_A r_A \delta_B + p_B r_B \delta_B)}{(p_A r_A + p_B r_B)(1 + p_A r_A \delta_B)}, \tag{22}$$

and similarly,

$$p_{AB} = \frac{p_A r_A}{(p_A r_A + p_B r_B)(1 + p_B r_B \delta_A)}. \tag{23}$$

When we substitute these results into Equation (18), we obtain, after a little algebra,

$$P(A) = \frac{k_A p_A r_A (1 + p_A r_A \delta_B)(1 + k_B p_B r_B \delta_A)}{\left[\begin{array}{l} k_A p_A r_A (1 + p_A r_A \delta_B)(1 + k_B p_B r_B \delta_A) \\ + k_B p_B r_B (1 + p_B r_B \delta_A)(1 + k_A p_A r_A \delta_B) \end{array}\right]}. \tag{24}$$

4. CONCLUSION

It is interesting to note that if $k_A = k_B = 1$, then all near-misses are really kills, and Equation (24) reduces to the solution for the fundamental duel with exponential firing times, [9], and the same result occurs if $\delta_A = \delta_B = 0$ (zero displacement), as it should.

The expression in Equation (24) is unwieldy, since it contains eight parameters. If we define

$P_B = p_B r_B \delta_B$,

$P_A = p_A r_A \delta_A$, and

$D = \delta_B / \delta_A$

then Equation (24) becomes

$$P(A) = \frac{D k_A P_A (D + k_B P_B)(1 + D P_A)}{k_A D P_A (D + k_B P_B)(1 + D P_A) + k_B P_B (D + P_B)(1 + k_A D P_A)} \tag{25}$$

which reduces the parameters to only five.

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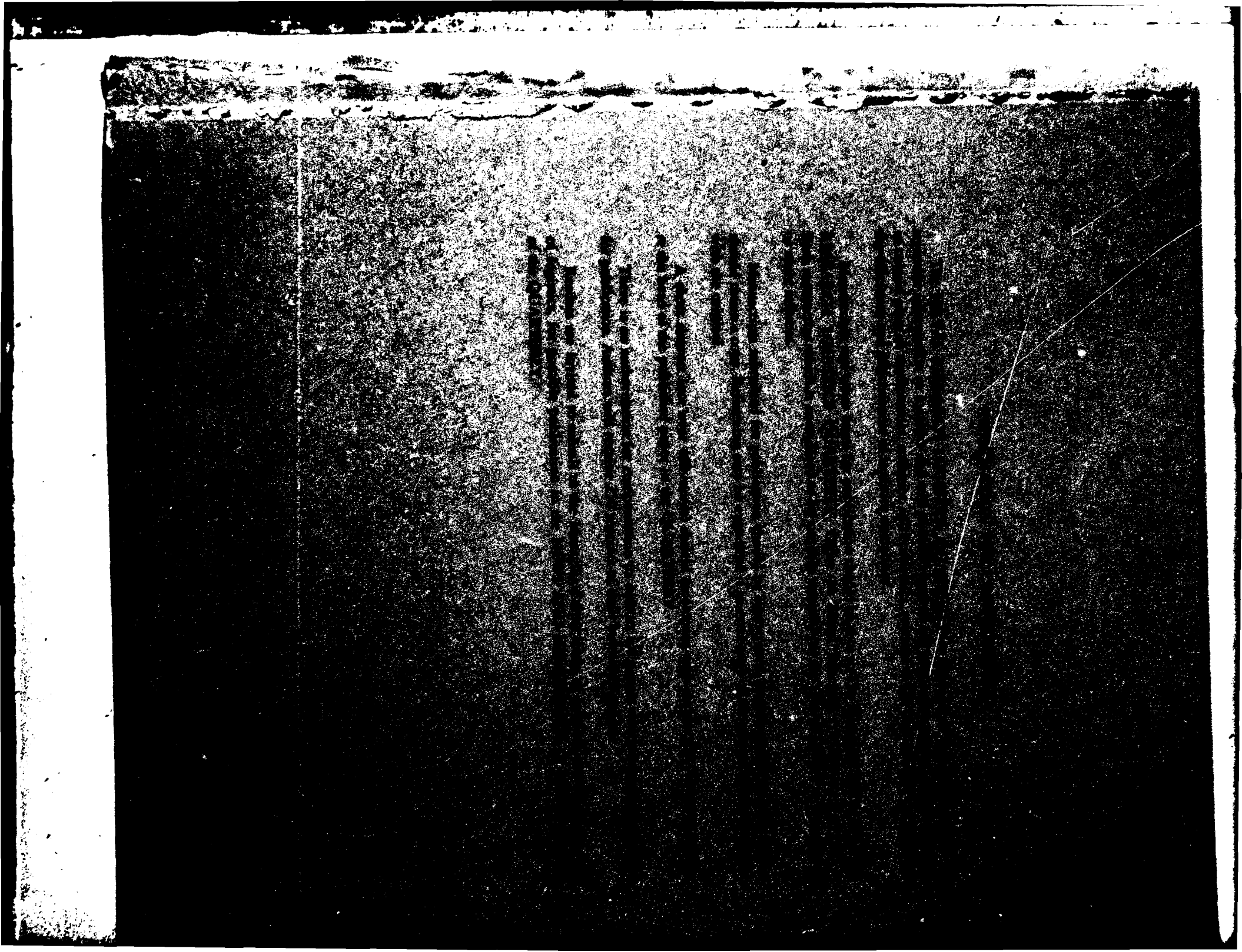

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