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Liesegang Rings and a Theory of Fast Reaction and Slow Diffusion

Joseph B. Keller

1. INTRODUCTION.

A chemically reacting system usually proceeds mono-
tonically in time toward a state of completed reaction, a
state of equilibrium, or a steady state. The ultimate state
is generally homogeneous in space, as a consequence of diffu-
sion, *even though the initial state may be* inhomogeneous.
Therefore it is surprising that certain chemical systems do
not behave in this way. Instead some of them exhibit temporal
oscillations which do not decay in time, but which become
temporally periodic. Other systems, or the same systems with
different parameter values, perform oscillations which persist
but remain chaotic rather than becoming periodic. Still
others reach steady states which are not homogeneous in space,
but which are spatially periodic or quasi-periodic. There are
also systems which continue to oscillate in time and to be
quasi-periodic in space.

Examples of such unusual behavior of chemically reacting
systems have been known for a long time. One of them was dis-
covered by R. E. Liesegang [1] in 1896 and studied by him for
many years thereafter. The spatial pattern which occurs in
this system consists of a family of concentric circular rings
which are called Liesegang rings in his honor. It is this
system and similar ones which we shall examine. In order to
analyze them, first we shall have to describe in detail the
physical and chemical processes which occur. This we shall do
following the ideas of W. Ostwald [2] who, in 1897, outlined

an explanation involving <u>precipitation</u> and supersaturation.
We shall extend that outline to a complete description of the
various processes. Then we shall convert this verbal descrip-
tion into a mathematical theory, and formulate an appropriate
mathematical problem corresponding to it. All of these con-
siderations, and the subsequent analysis of the problem, are
the joint work of S. I. Rubinow and the author, and are pre-
sented in greater detail in their paper [3].

In order to treat the mathematical problem we shall make
certain appropriate simplifications. The most important of
these is a consequence of the fact that the reaction process
is fast compared to the diffusion process. This has led us to
formulate and develop a general theory of reaction and diffu-
sion processes involving <u>fast reaction and slow diffusion</u>. We
shall describe this theory, which should have many other
applications, and use it to simplify the present problem. It
was worked out by P. S. Hagan and the author, and is contained
in their paper [4] together with the related theory of some
fast and some slow reactions.

The subsequent analysis of the simplified problem is
carried out in [3] employing approximations like those in the
related work of C. Wagner [5], S. Prager [6] and Zeldovich,
Barenblatt and Salganik [7]. Some predictions of the theory
are compared in [3] with certain relevant experimental results.
More detailed comparison between theory and experiment will be
possible when the equations of the theory are solved
numerically. J.-M. Vanden-Broeck and the author [8] are
solving them by using the method of finite differences to re-
place the partial differential equations by a finite set of
algebraic equations. These equations, together with the
initial, boundary, and jump conditions, are being solved by an
appropriate iterative method.

Before turning to the study of Liesegang rings, we shall
comment on the periodic or non-monotonic behavior of chemical
systems in general. Let us first consider why chemical reac-
tions are expected to proceed monotonically in time. The
reason is that the rate of reaction is positive as long as the
concentrations of the reactants are positive and the concen-
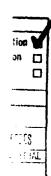tration of the reaction products is not too large. Furthermore

there is no mechanism, such as inertia, to cause the reaction
to overshoot the equilibrium state or the state of completed
reaction. Therefore it is to be expected that a reaction will
proceed in one direction at a decreasing rate. However when
several reactions occur, there is no longer any justification
for this belief. The presence of other processes, such as
flow and diffusion, can also lead to destabilization of uni-
form steady states and the occurrence of temporal or spatial
periodicity.

The interest in periodic and non-uniform states of reac-
ting systems has been generated by scientific curiosity, by
engineering necessity, and by potential applications in
biology. The engineering reason is the need to control and
eliminate oscillations which arise in stirred tank and other
continuous flow reactors. The biological reasons include the
possible explanation of biological clocks and a proposed ex-
planation of morphogenesis. The latter of these, made by
A. M. Turing in 1952, is that chemical reaction and diffusion
are responsible for the creation of form in biological systems
through the development of spatially nonuniform steady states.
All of these reasons have led to extensive experimental and
theoretical investigations of the non-monotonic and non-uniform
behavior of chemical systems.

2. THE MECHANISM OF LIESEGANG RING FORMATION.

Liesegang, while experimenting with photographic
materials, prepared a gel containing potassium dichromate.
This gel was in the form of a thin layer on a glass plate. He
placed a drop of silver nitrate solution on the gel and found,
after some time, that concentric circular rings of silver
chromate formed in the gel. The process whereby the silver
chromate formed into visible solid particles is precipitation,
so this phenomenon has sometimes been called "periodic"
precipitation. However it is not periodic in either space or
time, so that is a misnomer. We prefer to call it recurrent
precipitation, since the rings form one after another with in-
creasing spatial and temporal intervals between successive
rings.

To explain this phenomenon we suppose that silver ions from the drop go into solution in the gel and diffuse radially outward. Within the gel they encounter chromate ions and react with them to form silver dichromate. As the chromate ions near the drop are consumed by the reaction, more chromate ions diffuse radially inward. Thus the two reactants diffuse toward one another and, upon coming together, continually produce silver dichromate. In this way an expanding circular region of silver dichromate is produced. This is the monotonic behavior to be expected in chemically reacting systems. The question then arises of why only separated rings of silver dichromate are visible at the end of the experiment?

To answer this question we assume that the silver dichromate is not visible until it precipitates out of solution and forms solid particles. Now it is well known that a dissolved substance cannot precipitate until its concentration c exceeds the saturation concentration c^S. Then the rate of precipitation is proportional to $c - c^S$, provided that some of the solid is present upon which it can precipitate. However if none of the solid is present the concentration c may have to reach a higher value $c^* > c^S$ before precipitation can start. Therefore, following Ostwald, we assume that precipitation starts only when $c \geq c^*$. After it has started, we assume that it continues as long as $c \geq c^S$.

In view of these assumptions, precipitation will begin in the gel at the location of the drop of silver nitrate solution, provided that the reactants are present in sufficiently high concentrations to produce silver dichromate with a concentration $c \geq c^*$. Precipitation will continue within a circle of increasing radius surrounding the drop until c falls below c^* at the outer boundary of this circle. Then the precipitation zone will stop growing.

The reason why c may fall below c^* is that the chromate ions in the gel are depleted by reaction and by diffusion toward the silver nitrate drop, while the concentration of silver ions is low far from the drop. As time goes on this latter concentration increases everywhere. A time may be reached when this silver ion concentration and that of the

chromate ions is high enough somewhere in the gel to produce
silver dichromate with a concentration $c \geq c^*$. Then a second
zone of precipitation will start to form at that place. After
a while this process will be repeated, forming a third zone,
and so on. The resulting zones are the Liesegang rings.

Since Liesegang's first experiment, the same phenomenon
has been found to occur in many other reactions. Furthermore,
it has also been shown to occur in test tubes and capillary
tubes, with the formation of planar bands at various positions
along the tube instead of rings. It is this one dimensional
case which we shall treat.

3. A MATHEMATICAL FORMULATION OF THE THEORY.

To formulate the preceding theory mathematically, we
denote by a, b, c and d the molar concentrations of the four
substances A, B, C and D. Here A is analogous to the silver
ions, B to the chromate ions, C to the dissolved silver
dichromate and D to the solid or precipitated silver dichro-
mate. We assume that ν_A moles of A can combine with ν_B moles
of B to form ν_C moles of C, and that this reaction is rever-
sible. We also assume that C can precipitate to form D, but
that D cannot dissolve to form C. In addition we suppose that
A, B and C can diffuse with diffusion coefficients D_A, D_B and
D_C, but that D cannot diffuse. Then the four concentrations,
which depend upon the time t and the coordinate x along the
tube, satisfy the following reaction-diffusion-precipitation
equations within the tube $x > 0$:

$$a_t = D_A a_{xx} - \nu_A r ,\qquad (3.1)$$

$$b_t = D_B b_{xx} - \nu_B r ,\qquad (3.2)$$

$$c_t = D_C c_{xx} + \nu_C r - p(c,d) ,\qquad (3.3)$$

$$d_t = p(c,d) .\qquad (3.4)$$

Here r is the reaction rate and p is the precipitation rate.

We shall assume that r is given by the law of mass action
with rate constant k_+ for the forward reaction and k_- for the
backward reaction. Thus

$$r(a,b,c) = k_+ a^{\nu_A} b^{\nu_B} - k_- c^{\nu_C} \ . \tag{3.5}$$

The precipitation rate was described in Section 2, and that description leads to the following expression for p:

$$p(c,d) = 0 \ , \qquad \text{if } c < c^* \text{ and } d = 0 \ ,$$
$$= q(c-c^S)_+ \ , \qquad \text{if } c \geq c^* \text{ or } d > 0 \ . \tag{3.6}$$

Here q is a rate constant and the subscript + on $c - c^S$ denotes the positive part of $c - c^S$.

Since we have assumed that the reactions occur in the tube occupying the region $x > 0$, we must specify the initial conditions within the tube and boundary conditions at the end $x = 0$. To correspond with Liesegang's experiment, we assume that only B is present initially. Thus we require that

$$a = c = d = 0 \ , \quad b = b_0 \text{ at } t = 0 \ , \quad x > 0 \ . \tag{3.7}$$

At the endpoint $x = 0$ we assume that A is kept at the concentration a_0 while B and C cannot leave the tube there. Thus

$$a = a_0 \ , \qquad b_x = c_x = 0 \text{ at } x = 0 \ , \quad t > 0 \ . \tag{3.8}$$

In addition to these conditions, we require that a, b, c and their x derivatives be continuous throughout the tube for $t > 0$:

$$a, b, c, a_x, b_x, c_x \quad \text{continuous for } x > 0, t > 0 \ . \tag{3.9}$$

The preceding equations constitute our mathematical formulation of the theory of Liesegang band formation. The bands are the regions within which $d(x,t)$ is positive after a sufficiently long time. Thus the bands are defined by

$$d(x,\infty) > 0 \ . \tag{3.10}$$

This formulation is complete in the sense that the equations seem to have a unique solution. This solution determines exactly where the bands will be, the thickness of each band, the time when each band starts to form, the concentrations between the bands, etc. The solution will also show whether or not bands do form, and it will yield the sets of

parameter values for which they form. The solution can be
calculated numerically, but we have not calculated it.
Instead we have first simplified the equations and then
analyzed the resulting problem.

4. REDUCTION TO A SIMPLER PROBLEM.

    In order to simplify the theory presented in Section 3,
we shall proceed heuristically, making assumptions based upon
intuitive chemical considerations. Later on we shall indicate
how the same simplified theory results from a systematic
application of the theory of fast reactions and slow diffusion.
Our first assumption is that the reaction term $\nu_A r$ in (3.1) is
negligible compared to the other terms in that equation. This
assumption is valid when the concentration of B is sufficiently
small. It yields

$$a_t = D_A a_{xx} .\tag{4.1}$$

    Next we add ν_B times (3.3) to ν_C times (3.1) to eliminate
r and obtain

$$\nu_C b_t + \nu_B c_t = \nu_C D_B b_{xx} + \nu_B D_C c_{xx} - \nu_B p(c,d) .\tag{4.2}$$

Then we assume that the reaction proceeds so rapidly that it
is in equilibrium, which yields

$$r(a,b,c) = 0 .\tag{4.3}$$

We shall use (4.2) and (4.3) instead of (3.2) and (3.3). But
then we must replace the two boundary conditions $b_x = c_x = 0$
by the single condition that B, both free and contained in
C, cannot escape through the boundary:

$$\nu_C D_B b_x + \nu_B D_C c_x = 0 \quad \text{at } x = 0 , \quad t > 0 .\tag{4.4}$$

    With these modifications the problem can be dealt with by
first solving (4.1) for $a(x,t)$ with $a(x,0) = 0$ and $a(0,t) = a_o$.
The solution is

$$a(x,t) = a_o \, \text{erfc}[x/(4D_A t)^{1/2}] .\tag{4.5}$$

Then (4.3) can be solved for b in terms of a and c and the re-
sult can be used to eliminate b from (4.2) and (4.4). In this
way (4.2) becomes an equation for c in which d seems to occur.

However both in the region where p = 0 and in the region where
p > 0, d does not occur in the equation for c.   Thus the
entire problem can be reduced to one of determining c in each
of these regions.   The boundary between the two regions con-
sists of the curve on which c = c* together with the upper
halves of the tangents to this curve where it becomes
vertical.   Once c is found, d can be calculated from (3.4).

The reduction just described is carried out in detail in
the work of Keller and Rubinow [3], where the resulting equa-
tions are analyzed and solved approximately.   Therefore we
shall not describe that analysis and its results.   Instead we
shall present the theory of fast reaction and slow diffusion,
and show how it ~an be used to derive the reduced problem in a
systematic way.

5.   FAST REACTION AND SLOW DIFFUSION.

In many chemical systems, such as those considered in the
preceding sections, both chemical reactions and diffusion
occur, but at quite different rates.   Often the reactions pro-
ceed much more rapidly than the diffusion.   In such cases it
is possible to simplify the analysis of the process by taking
advantage of this difference in rates.   One way of doing so is
to assume that the reactions have reached equilibrium or
completion at each point of space.   Then the state at each
point is constrained to satisfy the condition that the reac-
tion rates vanish there.   When this constraint is adjoined to
the reaction-diffusion equations governing the system, an over-
determined set of equations results.   Chemical and physical
reasoning can then be used to delete some of the equations and
thereby make the system determined.

Instead of using this intuitive procedure, it is possible
to proceed in a systematic mathematical manner which we shall
now describe.   The basic idea is to introduce into the equa-
tions a small parameter ε which is the ratio of the time scale
for reaction to the time scale for diffusion.   Then the
asymptotic expansion of the solution is sought for ε near zero.
This expansion can be constructed either by the "two time"
method or by the method of matched asymptotic expansions.

Both of these methods are utilized in [4]. Here we shall indicate how to obtain the desired results via the latter method.

Let us begin by considering the following system of reaction-diffusion equations:

$$u_t(x,t,\epsilon) = f(u,x,t,\epsilon t,\epsilon) + \epsilon D \Delta u , \tag{5.1}$$

$$u(x,0,\epsilon) = g(x) . \tag{5.2}$$

Here $u(x,t,\epsilon)$ is a vector of concentrations, temperature and possibly other dependent variables, $f(u,x,t,\epsilon t,\epsilon)$ is a vector of reaction rates, heat production rate, etc. and D is the diffusion coefficient matrix. The essential feature of (5.1) is that the diffusion term contains the factor ϵ, which is small. We have also permitted f to depend upon all the variables, and even to have a slow time dependence via the argument ϵt. In the initial condition (5.2) the function $g(x)$ is given.

To obtain the asymptotic expansion of u for ϵ small, we assume first that it is of the form

$$u(x,t,\epsilon) \sim u^0(x,t) + \epsilon u^{(1)}(x,t) + \cdots . \tag{5.3}$$

We shall call this the initial layer expansion of u. We now substitute (5.3) into (5.1) and (5.2) and set $\epsilon = 0$ to obtain

$$u_t^0(x,t) = f(u^0,x,t,0,0) , \tag{5.4}$$

$$u^0(x,0) = g(x) . \tag{5.5}$$

These equations describe the evolution in time of u^0 at the point x. Further terms in (5.3) can be found by considering the coefficients of higher powers of ϵ in (5.1) and (5.2), but we shall not examine them. Instead we shall consider the behavior of $u^0(x,t)$ as $t \to \infty$.

To this end, we assume that $f(u^0,x,t,0,0)$ has a limit as $t \to \infty$, and then we may expect $u^0(x,t)$ to have a limit also. Let us denote it by $u^0(x,\infty)$. Thus we assume that

$$\lim_{t \to \infty} u^0(x,t) = u^0(x,\infty) . \tag{5.6}$$

Then $u^0(x,\infty)$ must satisfy the following equilibrium condition, which results from (5.4):

$$f[u^0(x,\infty),x,\infty,0,0] = 0 . \tag{5.7}$$

In general $u^0(x,\infty)$ is not uniquely determined by (5.7), but it is determined by the initial condition (5.5) and the reaction equation (5.4). It is this limit $u^0(x,\infty)$ of $u^0(x,t)$ which will be needed to determine the outer expansion of u, which we consider next.

The initial layer expansion (5.3) is presumably valid for fixed values of t as $\varepsilon \to 0$. In order to find the behavior of u for long times of order ε^{-1}, we must construct another expansion, called the outer expansion of u. In order to construct it we first introduce the new time variable $\tau = \varepsilon t$ and consider u to be a function of τ:

$$u(x,t,\varepsilon) = v(x,\tau,\varepsilon) , \qquad \tau = \varepsilon t . \tag{5.8}$$

Then (5.1) becomes the following equation for v:

$$\varepsilon v_\tau(x,\tau,\varepsilon) = f(v,x,\varepsilon^{-1}\tau,\tau,\varepsilon) + \varepsilon D\Delta v . \tag{5.9}$$

Next we assume that v has the expansion

$$v(x,\tau,\varepsilon) = v^0(x,\tau) + \varepsilon v^{(1)}(x,\tau) + \cdots . \tag{5.10}$$

This is the outer expansion of u.

Both the inner expansion (5.5) and the outer expansion (5.10) represent u. If there is a common region of validity of the two expansions, as we assume, then in it they must be asymptotic to one another. Thus we have in this region

$$u^0(x,t) + \varepsilon u^{(1)}(x,t) + \cdots \sim v^0(x,\varepsilon t) + \varepsilon v^{(1)}(x,\varepsilon t)$$

$$+ \cdots . \tag{5.11}$$

Suppose that (5.11) holds for $t = \varepsilon^{-1/2}$. Then by *choosing* this value for t and letting $\varepsilon \to 0$ in (5.11) we obtain

$$u^0(x,\infty) = v^0(x,0) . \tag{5.12}$$

This procedure for deriving (5.12) is called "matching" the initial layer expansion and the outer expansion. The result (5.12) yields the initial value of v^0 terms of the "final" value of u^0.

We next substitute (5.1) into (5.9) and set $\epsilon = 0$ to get

$$f[v^0(x,\tau),x,\infty,\tau,0] = 0 \ . \tag{5.13}$$

This equation shows that v^0 must lie on the equilibrium sur-
face (or manifold) defined by $f = 0$, but it does not determine
where v^0 is on this surface unless the surface is a single
point. From (5.12) we know that at $\tau = 0$, v^0 starts at the
point $u^0(x,\infty)$. To find how it evolves as τ increases, we
consider the terms of order ϵ in (5.9), which yield the
following linear equation for $v^{(1)}$:

$$f_v[v^0(x,\tau),x,\infty,\tau,0]v^{(1)} = v^0 - D\Delta v^0 - f_\epsilon \ . \tag{5.14}$$

The gradient matrix f_v in (5.14), which is the coefficient
of $v^{(1)}$, will generally be a singular matrix. Let us suppose
that it has k left null-vectors $\ell_1,\ldots,\ell_k$. Then (5.14) is
solvable for $v^{(1)}$ only if the right side satisfies k
solvability conditions. They can be obtained by scalar multi-
plication of (5.14) on the left by each left null-vector.
This yields the k conditions

$$\ell_j \cdot \{v^0_\tau(x,\tau) - D\Delta v^0(x,\tau) - f_\epsilon[v^0(x,\tau),x,\infty,\tau,0]\} = 0 \ ,$$
$$j = 1,\ldots,k \ . \tag{5.15}$$

This is a system of k equations that determine how $v^0(x,\tau)$
diffuses on the equilibrium surface upon which it is con-
strained to lie by (5.13). The initial value of v^0 is given
by (5.12). Thus (5.12), (5.13) and (5.15) determine the slow
diffusion of $v^0(x,\tau)$, the leading term in the outer expansion
of $u(x,\epsilon^{-1}\tau,\epsilon)$. The fast reaction is governed by (5.4) and
(5.5), which determine $u^0(x,t)$, the leading term in the initial
layer expansion of $u(x,t,\epsilon)$.

The system of equations (5.13) and (5.15), together with
the initial condition (5.12), are the results of our analysis.
We shall not examine them further here, nor shall we obtain
more terms in the outer expansion. These and other matters are
considered in [4].

6. APPLICATION TO THE THEORY OF LIESEGANG BANDS.

We shall now apply the theory of Section 5 to the equations of Section 3, which constitute our formulation of the theory of Liesegang bands. To do so we first rewrite these equations, introducing explicitly the small parameter ε to indicate the relative size of each term. In (3.1) for example, the diffusion term is small of order ε, say, but the reaction term is supposed to be smaller. Therefore we shall assume it to be of order ε^2. Thus we write (3.1) in the form

$$a_t = \varepsilon D_A a_{xx} - \varepsilon^2 \nu_A r \ . \tag{6.1}$$

Similarly we write (3.2) and (3.3) in the form

$$b_t = \varepsilon D_B b_{xx} - \nu_B r \ , \tag{6.2}$$

$$c_t = \varepsilon D_C c_{xx} + \nu_C r - \varepsilon p \ . \tag{6.3}$$

The system (6.1)-(6.3) is of the form (5.1) with the following identifications:

$$u = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \ , \qquad f = \begin{pmatrix} -\varepsilon^2 \nu_A r \\ -\nu_B r \\ \nu_C r - \varepsilon p \end{pmatrix} \ ,$$

$$\tag{6.4}$$

$$D = \begin{pmatrix} D_A & 0 & 0 \\ 0 & D_B & 0 \\ 0 & 0 & D_C \end{pmatrix} \ .$$

Thus the leading term of the initial layer expansion, $u^0 = (a^0, b^0, c^0)$, satisfies (5.4) which becomes

$$a_t^0 = 0 \ , \tag{6.5}$$

$$b_t^0 = -\nu_B r(a^0, b^0, c^0) \ , \tag{6.6}$$

$$c_t^0 = \nu_C r(a^0, b^0, c^0) \ . \tag{6.7}$$

From (3.7) the corresponding initial conditions are

$$a^0(x,0) = c^0(x,0) = 0 \ , \quad b^0(x,0) = b_0 \ , \quad x > 0 \ . \tag{6.8}$$

With these initial conditions and r given by (3.5), the solution of (6.5)-(6.7) is

$$a^0(x,t) = c^0(x,t) = 0 \ , \quad b^0(x,t) = b_0 \ , \quad x > 0 \ . \quad (6.9)$$

Thus nothing changes in the initial layer, to this order.

To treat the outer expansion we shall write

$$u(x,t,\varepsilon) = v(x,\tau,\varepsilon) = \begin{pmatrix} \alpha(x,\tau,\varepsilon) \\ \beta(x,\tau,\varepsilon) \\ \gamma(x,\tau,\varepsilon) \end{pmatrix} \ . \quad (6.10)$$

Then by using (6.4) for f in (5.13) we get the single equation

$$r[\alpha^0(x,\tau), \beta^0(x,\tau), \gamma^0(x,\tau)] = 0 \ . \quad (6.11)$$

Thus the leading term in the outer expansion satisfies (4.3), as we assumed in Section 4.

Next from (6.4) we compute f_v at $\varepsilon = 0$, and obtain

$$f_v = (f_a, f_b, f_c) = \begin{pmatrix} 0 & 0 & 0 \\ -\nu_B r_a & -\nu_B r_b & -\nu_B r_c \\ \nu_C r_a & \nu_C r_b & \nu_C r_c \end{pmatrix} \ . \quad (6.12)$$

This matrix has two linearly independent left null-vectors

$$\ell_1 = (1,0,0) \ , \qquad \ell_2 = (0, \nu_C, \nu_B) \ . \quad (6.13)$$

We also compute f_ε at $\varepsilon = 0$, which is given by

$$f_\varepsilon = \begin{pmatrix} 0 \\ 0 \\ -p \end{pmatrix} \ . \quad (6.14)$$

We can now use each of the vectors ℓ_1 and ℓ_2 given by (6.13) in (5.15), together with f_ε given by (6.14) and D given by (6.4). By using ℓ_1 in (5.15), we obtain

$$\alpha_\tau^0 = D_A \alpha_{xx}^0 \ . \quad (6.15)$$

This is just (4.1) of Section 4, with a replaced by α^0 and t replaced by τ. Then by using ℓ_2 in (5.15) we get

$$\nu_C \beta_\tau^0 + \nu_B \gamma_\tau^0 = \nu_C D_B \beta_{xx}^0 + \nu_B D_C \gamma_{xx}^0 - p(\gamma^0, d) \ . \quad (6.16)$$

This is exactly (4.2) with b, c and t replaced by β^0, γ^0 and τ respectively.

We have now shown how the simplified equations of Section 4 follow from the equations for the outer asymptotic expansion derived in Section 5. The corresponding initial conditions are obtained by using (6.9) in (5.12), and they are just those used in Section 4. An extension of the theory of Section 5 is needed to derive the boundary conditions (4.4) and $\alpha^0(0,\tau) = a_0$, but we shall not present it here.

This completes the application of the theory of fast reaction and slow diffusion to the equations of Section 3, when they are written in the form (6.1)-(6.3). Of course that way of writing the equations depends upon the parameters in the problem. They must be such that the terms in these equations have the indicated relative magnitudes.

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