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A CONSTRUCTIVE PROOF OF THE BORSUK-ULAM ANTIPODAL POINT THEOREM--ETC(U)
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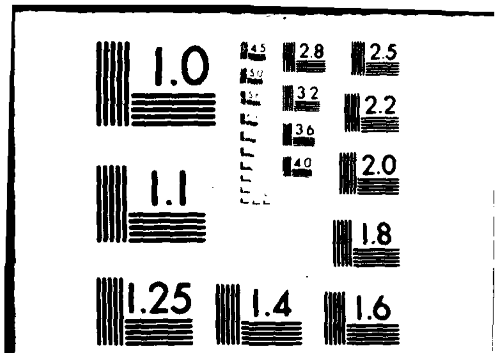

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A CONSTRUCTIVE PROOF OF THE BORSUK-ULAM ANTIPODAL POINT THEOREM

by

R.M. Freund

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I. Introduction

In this paper, *a proof of the Borsuk-Ulam Antipodal Point* Theorem is presented by means of a constructive algorithm that computes an approximate solution by means of a simplicial subdivision and integer labels.

II. Main Results

For $x \in \mathbb{R}^n$, let $\|x\|$ denote the L^∞ norm. Let $S^n = \{x \in \mathbb{R}^n : \|x\| = 1\}$. By an odd function, we mean a function such that $f(-x) = -f(x)$. The Borsuk-Ulam Antipodal Point Theorem [1] can be stated as follows:

Theorem: Let $f : S^n \to \mathbb{R}^{n-1}$ be an odd continuous function. Then there exists a point $x^* \in S^n$ such that $f(x^*) = 0$. ⊠

Let T be any symmetric triangulation of S^n such that its restriction to $S^n \cap \{x | x_i = 0 \ i \in U\}$ for any U is also a triangulation, with grid size δ. For example, consider a scaling of J^1 [2] restricted to S^n. Let T^0 denote the vertices of the triangulation. Consider the following labeling function on T^0:

$$\ell(x) = \begin{cases} i & \text{if } i \text{ is the smallest index such that } \|f(x)\| = f_i(x) \text{ and } f_i(x) > 0 \\ -i & \text{if } i \text{ is the smallest index such that } \|f(x)\| = f_i(x) \text{ and } f_i(x) \le 0 \end{cases}$$

Note $\ell(x) = -\ell(-x)$.

Fix $\varepsilon > 0$ and choose δ such that $\|x - y\| < \delta$ implies $\|f(x) - f(y)\| < \varepsilon$.

<u>Lemma 1</u>: Suppose $\ell(x) = i > 0$ and $\ell(y) = -i$ and $\|x - y\| < \delta$. Then $\|f(x)\| < 3\varepsilon$.

<u>Proof</u>:

$$f_i(x) > 0$$

$$f_i(y) \le 0$$

$$f_i(x) - f_i(y) < \varepsilon$$

$$f_i(x) \le \varepsilon + f_i(y) < \varepsilon$$

$$f_i(y) > f_i(x) - \varepsilon \ge -\varepsilon \quad .$$

Therefore

$$|f_i(x)| < \varepsilon \quad \text{and} \quad |f_i(y)| < \varepsilon \ ,$$

also

$$f_j(x) \le f_i(x) \quad \text{for any } j = 1, \ldots, n$$

$$f_j(y) \ge f_i(y) \quad \text{for any } j = 1, \ldots, n \quad .$$

Therefore

$$f_i(x) - 2\varepsilon < f_i(y) - \varepsilon \le f_j(y) - \varepsilon < f_j(x) \le f_i(x) \ .$$

Therefore

$$|f_j(x) - f_i(x)| < 2\varepsilon \ ,$$

hence

$$\|f(x)\| < 3\varepsilon \ . \qquad \boxtimes$$

In the next section, we will prove constructively:

<u>Lemma 2</u>: For a given T and induced labeling $\ell(\cdot)$ as above, there exists a pair of adjacent vertices x and $y \in S^n$ such that $\ell(x) = i$ and $\ell(y) = -i$. ⊠

Combining Lemmas 1 and 2 and taking a limiting subsequence of x's as $\varepsilon \to 0$, we obtain the main theorem.

3. An Algorithm for Computing Oppositely Labeled Adjacent Vertices

The algorithm of this section is a modification of that of Reiser [3]. Let T^i denote the collection of i-dimensional simplices of T. We shall define a simplex $\sigma \in T$ to be oppositely labeled if there are two vertices x, y of σ such that $\ell(x) = i$ and $\ell(y) = -i$. The algorithm will terminate with an oppositely labeled simplex. Let $R \subset \{1, \ldots, n-1, -1, \ldots, -n+1\}$ such that $i \in R$ implies $-i \notin R$. Define

$$A(R) = \{x \in S^n : \begin{array}{ll} x_i \geq 0 & \text{for } 0 < i \in R, \\ x_i \leq 0 & \text{for } 0 < -i \in R, \\ x_i = 0 & \text{otherwise} \end{array} \} .$$

The following algorithm, analogous to that of Reiser, will produce an oppositely labeled simplex. d is the dimension of the simplex under question. q is the index of the newly added vertex. R is the index set of the orthant of $\mathbb{R}^{n-1}$ under consideration, and X is the set of vertices of the simplex under question.

<u>Step 0</u>: $R \leftarrow \emptyset$. $v' \leftarrow e^n$. $X \leftarrow \{v'\}$. $d \leftarrow 0$. $q \leftarrow 1$.

<u>Step 1</u>: Let $\ell = \ell(v^q)$. If there is a vertex $v \in X$ with $\ell(v) = -\ell$, stop. If there is a vertex $v^k \in X$, $k \neq q$ with $\ell(v^k) = \ell$, go to Step 2, otherwise go to Step 3.

<u>Step 2</u>: v^k is replaced by the unique vertex $\bar{v}^k$ in $A(R)$ for which we have a d-dimensional simplex of T in $A(R)$, if such a $\bar{v}^k$ exists. In this case set $v^k \leftarrow \bar{v}^k$, set $q \leftarrow k$ and go to Step 1. Otherwise, go to Step 4.

<u>Step 3</u>: $R \leftarrow R \cup \{\ell\}$, $d \leftarrow d + 1$. Define v^{d+2} to be the unique vertex $v \in A(R)$ such that $\langle v^1, \ldots, v^{d+1}, v\rangle \in T^d$. $X \leftarrow X \cup \{v^{d+2}\}$, $q \leftarrow d + 2$. Go to Step 1.

<u>Step 4</u>: $X \leftarrow X \setminus \{v^k\}$. There now is a unique index $i \in R$ such that $v_i = 0$ for all $v \in X$. Set $R = R \setminus \{i\}$. $d \leftarrow d - 1$. $q \leftarrow$ that index s.t. $v^q \in X$ and $\ell(v^q) = i$. Set $k \leftarrow q$ and go to Step 2.

Note that upon returning to Step 1, we have $X = \{v^1, \ldots, v^{d+1}\}$, each $v^i \in A(R)$, and R has d elements. The algorithm cannot cycle, and must terminate with either an oppositely-labeled subsimplex σ, or the simplex $\{-e^n\}$. The fact that $\ell(x) = -\ell(-x)$ quarantees that $\{-e^n\}$ cannot be the terminal simplex.

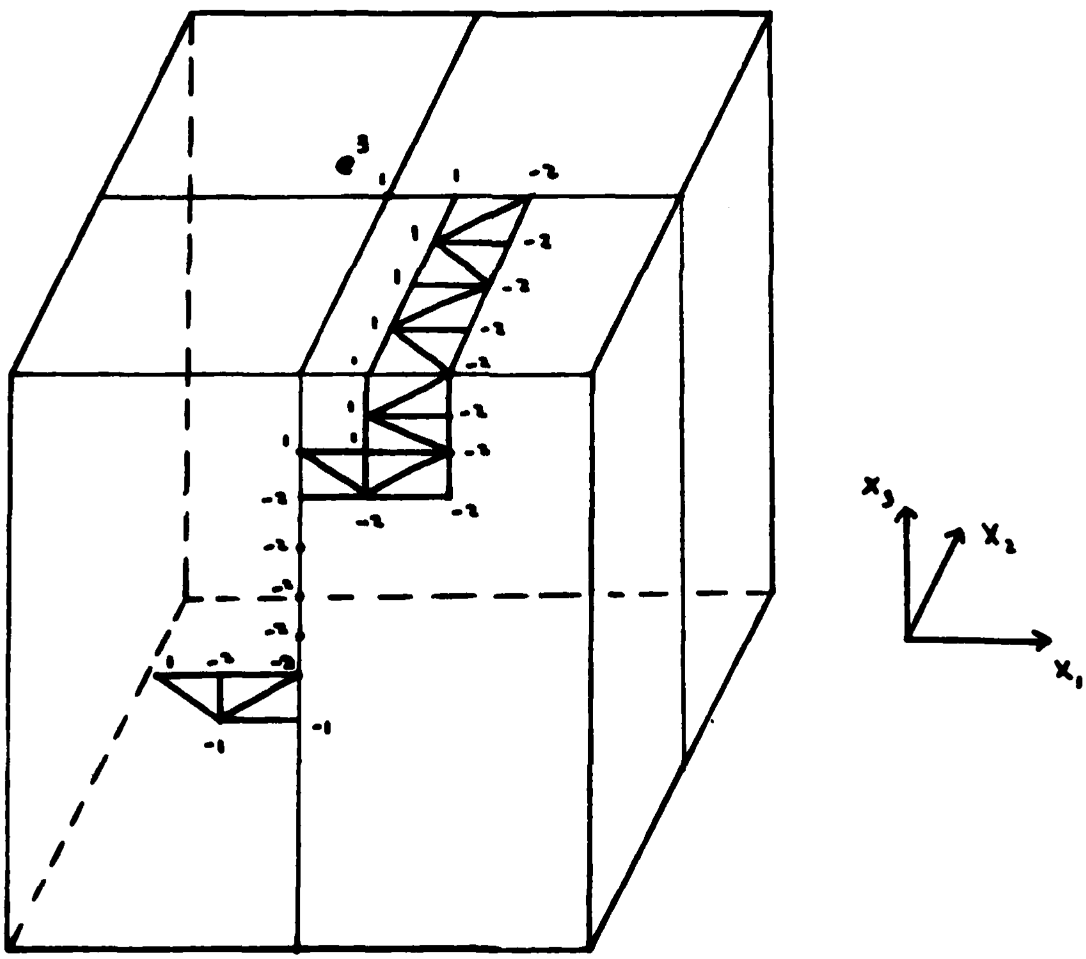

Figure 1. Sample path of algorithm

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[1] Lefschetz, S., Introduction to Topology, Princeton University Press, 1949.

[2] Todd, M. J., "Union Jack Triangulations," in Fixed Points; Algorithms and Applications, Ed: Stepan Karamardian, Academic Press, Inc., New York, 1977.

[3] Reiser, P. M., "A Modified Integer Labelling for Complementarity Algorithms," Institut für Operations Research, Universität Zürich, June 1978.

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