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GAUGE THEORIES AND SPONTANEOUS SYMMETRY BREAKING.(U)

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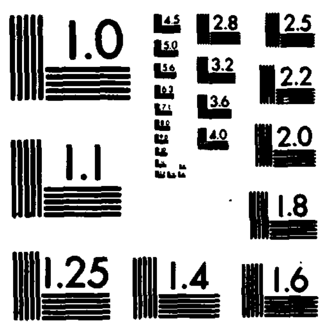

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This report ~~is a summary of~~ attempts to understand in what way spontaneous symmetry breaking arose in the context of gauge field theories of elementary particles. The main interest was in knowing whether techniques of bifurcation theory could be applied to the problem of spontaneous symmetry breaking in gauge field theories. It was felt that the symmetry breaking used by the physicists (a procedure known as the Higgs mechanism) is not precisely a bifurcation problem in the usual sense of the term, but more a matter of fixing a gauge and thereby reducing the amount of symmetry of the problem. In other words, it is not really a matter of "spontaneous" symmetry breaking.

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GAUGE THEORIES AND SPONTANEOUS SYMMETRY BREAKING

FINAL TECHNICAL REPORT

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Preface

This report is a summary of discussions and reading undertaken by O.L. Weaver and D.H. Sattinger during October-November, 1979. During that period Professors Weaver and Sattinger attempted to understand in what way spontaneous symmetry breaking arose in the context of gauge field theories of elementary particles. They were interested in knowing whether techniques of bifurcation theory could be applied to the problem of spontaneous symmetry breaking in gauge field theories. It was their feeling, after some discussions, that the symmetry breaking used by the physicists (a procedure known as the Higgs mechanism) is not precisely a bifurcation problem in the usual sense of the term, but more a matter of fixing a gauge and thereby reducing the amount of symmetry of the problem. In other words, it is not really a matter of "spontaneous" symmetry breaking. Sattinger and Weaver felt that it would be useful to compile the results of their discussions in the present form for possible future reference.

They thank the U.S. Army research office for their support in these studies.

Minneapolis, Minnesota
November 1980

Electrodynamics and Abelian Gauge Field Theories.

0. Units of measurement.

In mechanics there are three fundamental quantities, mass (m), length (ℓ), and time (τ), in terms of which all other quantities may be measured. For example, velocity is $\ell\tau^{-1}$, energy is $m\ell^2\tau^{-2}$, action is $m\ell^2\tau^{-1}$.

In quantum electrodynamics there are two characteristic quantities, namely c, the velocity of light, and h, Plancks constant, which has the units of action (energy × time). There is no third characteristic quantity, as is often the case in fluid dynamics, so the equations of electrodynamics cannot be written in a completely non-dimensional form. But if we choose c as a characteristic velocity and h as a characteristic action, then h and c disappear from the equations of quantum field theory, and all quantities can be measured in terms of one unit, for example time. Since h has dimensions of energy × time, setting $h = 1$ in effect makes the dimensions of energy τ^{-1}. Similarly setting $c = 1$ gives length and time equivalent dimensions. From $E = mc^2$ (or $E = \frac{1}{2} mv^2$) we see that, with $c = 1$, mass has the dimensions of energy. Recalling that e^2/hc is a pure number (the fine structure constant when e is the charge of an electron) we see that charge is a pure number in these units.

Finally, in this choice of units the energy and momentum operators are

$$E = -\frac{1}{c}\frac{\partial}{\partial t} \quad , \qquad P^j = \frac{1}{c}\frac{\partial}{\partial x_j} \quad .$$

The 4-vector x^μ is $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$.

1. Euler-Lagrange equations

The equations of motion of field theory are derived from a principle of least action

$$\delta L = 0 \qquad (1)$$

where $$L = \iiiint \mathcal{L}(\phi^a, \partial\mu\phi^a)\, dt\, dx^1 dx^2 dx^3 \quad ,$$

$a = 1,\ldots,N$, $\mu = 0,1,2,3$, and $\partial\mu = \dfrac{\partial}{\partial x^\mu}$. The Lagranian density $\mathcal{L}$ has dimensions $E\ell^{-3}$, so that L has the dimensions of action. ∂_μ transforms as a covariant 4-vector under a Lorentz transformation. The Euler-Lagrange equations $\delta L = 0$ are

$$\frac{\partial\mathcal{L}}{\partial\phi^a} - \partial\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial\mu\phi^a)}\right) = 0 \qquad (2)$$

2. Currents and Conservation Laws

Consider a variation in the Lagrangian due to a continuous transformation group acting on the fields ϕ^a . Differentiating at the identity we get

$$\delta\mathcal{L} = \sum_a \frac{\partial\mathcal{L}}{\partial\phi^a}\,\delta\phi^a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)}\,\delta\,\partial_\mu\phi^a \qquad (3)$$

If the transformation group is spatially independent (i.e. a gauge transformation of the first kind), then $\delta\partial_\mu\phi^a = \partial_\mu\delta\phi^a$. Furthermore, if ϕ^a is a solution of the Euler-Lagrange equations, then (2) holds, and

$$\delta\mathcal{L} = \sum_a \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)}\right)\delta\phi^a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)}\partial_\mu\delta\phi^a$$

$$= \partial_\mu\left(\sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)}\delta\phi^a\right) . \qquad (4)$$

Defining $J^\mu = \sum_a \frac{\partial\mathcal{L}}{\partial(\partial\mu\phi^a)}\delta\phi^a$, we get

$$\partial_\mu J^\mu = \delta\mathcal{L} \qquad (5)$$

If $\mathcal{L}$ is invariant under the given transformation group then $\delta\mathcal{L} = 0$ and we obtain the conservation law

$$\partial_\mu J^\mu = 0 .$$

This is a special case of Noether's theorem.

3. Mass terms

Quadratic terms in the Lagrangian density $\mathcal{L}$ correspond physically to mass terms. To see this let

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi) . \qquad (6)$$

$\mathcal{L}$ has the form kinetic minus potential energy. (Actually, energy/unit volume). The Euler-Lagrange equations for $\mathcal{L}$ are

$$\Box^2\phi + \frac{\partial U}{\partial \phi} = 0 \tag{7}$$

where

$$\Box^2 = \partial_\mu \partial^\mu = \partial_o^2 - \Delta \quad .$$

Now look at a plane wave $\phi = Ae^{ip_\mu x^\mu}$,

$$p_\mu x^\mu = p_o x^o + p_i x^i = Et - \vec{p} \cdot \vec{x} \quad .$$

$(p^\mu = (E,\vec{p})$, $p_\mu = (E,-\vec{p}))$.

We find

$$\Box^2 \phi = (-E^2 + p^2)\phi \quad . \tag{8}$$

The equations (7) describe a free, relativistic particle. Comparing (7) and (8) we see

$$\frac{\partial U}{\partial \phi} = \left(E^2 - p^2\right)\phi \quad .$$

For a free relativistic particle $E^2 = p^2 + m^2$, and therefore

$$\frac{\partial U}{\partial \phi} = m^2 \phi \ , \qquad U = m^2 \frac{\phi^2}{2}$$

where m is the rest mass of the particle.

In this heuristic argument we have tacitly assumed that a free particle may be formed as a superposition of plane waves; hence the equation (7) is linear and U is quadratic.

4. Charge

To describe charged particles we use a complex scalar field. The Lagrangian density is

$$\mathcal{L} = \partial_\mu \phi^* \, \partial^\mu \phi - U(\phi^*, \phi) \ .$$

The equations of motion are then

$$\Box^2 \phi + \frac{\partial U}{\partial \phi^*} = 0$$

and for a free particle $U = m^2 \phi^* \phi$.

This Lagrangian density is invariant under the gauge group $U(1)$: $\phi \to e^{i\omega}\phi$, $\phi^* \to e^{-i\omega}\phi^*$ where ω is real. Let's compute the conserved current J^μ . Taking $\varphi^1 = \phi$ and $\varphi^2 = \phi^*$ we obtain

$$J^{\mu} = \{(\partial^{\mu}\phi)(-i\phi^{*}) + (\partial_{\mu}\phi^{*})(i\phi)\}$$
$$= i\{(\partial_{\mu}\phi^{*})\phi - (\partial^{\mu}\phi)\phi^{*}\} \quad .$$

In particular $J^{0} = 2\,\mathrm{Im}\,\phi_{t}\phi^{*}$ is the charge density. Note that if $\mathcal{L}$ is to have dimensions $E/\ell^{3} \sim \tau^{-4}$ ϕ must have dimensions τ^{-1} ; then J^{0} has dimensions $\tau^{-3} \sim \ell^{-3}$.

5. Hamiltonian formulation

The Hamiltonian density is obtained from $\mathcal{L}$ as follows:

Define $\pi^{a} = \dfrac{\partial\mathcal{L}}{\partial(\partial_{o}\psi^{a})}$. Then

$$\mathcal{H} = \sum_{n} \pi^{a}(\partial_{o}\psi^{a}) - \mathcal{L} \quad .$$

In our case, $\pi = \dfrac{\partial\mathcal{L}}{\partial(\partial_{o}\phi)} = \partial^{o}\phi^{*}$, $\pi^{*} = \partial^{o}\phi$. Therefore

$$\mathcal{H} = \partial^{o}\phi^{*}\,\partial_{o}\phi + \partial^{o}\phi\,\partial_{o}\phi - \mathcal{L}$$
$$= (\partial_{o}\phi^{*})(\partial_{o}\phi) + \nabla\phi^{*}\cdot\nabla\phi + u(\phi^{*},\phi)$$
$$= \pi^{*}\pi + \vec{\nabla}\phi^{*}\cdot\vec{\nabla}\phi + u(\phi^{*},\phi) \quad .$$

"Quantizition" means interpreting π,ϕ as self-adjoint operators that obey

$$[\pi(x'),\phi(x)] = -2\delta(x'-x) \quad .$$

The space on which they operate is left for later. For now we look at the current, specifically, at J^o .

$$J^o(x) \sim -i(\phi^* \pi^* - \pi\phi) \quad .$$

Because $\frac{d}{dt} \int J^o(x) d^3x = 0$, $\int J^o(x) d^3x$ is a constant operator.

6. Gauge transformations and electromagnetism.

Classical electrodynamics is invariant under the gauge transformation

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \Lambda(x) \quad ,$$

for the only observable quantities are the field strengths $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$. In quantum mechanics we must determine the corresponding gauge transformation of the wave function. Consider the case where the scalar (electric) potential is shifted by a constant: $A_o \rightarrow A_o + \lambda$. Then the energy of a particle of charge e is increased by $e\lambda$. If that particle is described by a wave function ψ , what operation must we perform on ψ to increase the energy by $e\lambda$? Recalling that the energy operator in quantum mechanics is $-\frac{1}{i} \frac{\partial}{\partial t}$ we see that the transformation $\psi \rightarrow e^{-i e\lambda t} \psi$ is the required gauge transformation. In general, then, we should like our theory to be invariant under gauge transformations of the second kind

$$\psi^a(x) \to e^{-i\,e\,\Lambda(x)}\psi^a(x)$$
$$A_\mu(x) \to A_\mu + \partial_\mu \Lambda \qquad (9)$$

Note that $\overline{\psi^a}$ then transforms as

$$\overline{\psi^a} \to e^{+ie\,\Lambda(x)}\,\overline{\psi^a} \quad .$$

In our example above this suggests that a particle described by the field $\overline{\psi^a}$ might be one of opposite charge, since its energy is decreased by $e\lambda$. However, it is not quite correct to interpret ψ and $\overline{\psi}$ as representing oppositely charged particles when viewed as quantum fields.

7. Structure of gauge invariant Lagrangians.

The variation $\delta\mathcal{L}$ of $\mathcal{L}$ under a gauge transformation (9) is

$$\delta\mathcal{L} = \sum_a \frac{\partial\mathcal{L}}{\partial\psi^a}(-i\,e_a\,\Lambda\psi^a) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^a)}(-ie_a\partial_\mu(\Lambda\,\psi^a))$$
$$+ \frac{\partial\mathcal{L}}{\partial A_\mu}\partial_\mu\Lambda + \frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}\partial_\nu\partial_\mu\Lambda \quad .$$

Here $\delta A_\mu = \partial_\mu\Lambda$; $\delta\partial_\nu A_\mu = \partial_\nu\delta A_\mu = \partial_\nu\partial_\mu\Lambda$; $\delta\psi^a = -ie\,\Lambda\psi^a$; and $\delta\partial_\mu\psi^a = \partial_\mu\delta\psi^a = -\partial_\mu(ie\,\Lambda\psi^a)$. So

$$\delta\mathcal{L} = \sum_a \left[\frac{\partial\mathcal{L}}{\partial\psi^a} (-iea\,\Lambda\psi^a) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^a)} (-iea\,\Lambda\partial_\mu\psi^a) \right.$$

$$+ \frac{\partial\mathcal{L}}{\partial(\partial\mu\psi^a)} (-iea\psi^a)\partial\mu\,\Lambda + \frac{\partial\mathcal{L}}{\partial A\mu}\partial\mu\,\Lambda$$

$$+ \frac{\partial\mathcal{L}}{\partial(\partial_\nu A\mu)} \partial_\nu\partial\mu\,\Lambda \quad .$$

Now assume ψ^a is an extremal; then

$$\frac{\partial\mathcal{L}}{\partial\psi^a} - \partial\mu \frac{\partial\mathcal{L}}{\partial(\partial\mu\psi^a)} = 0$$

and

$$\delta\mathcal{L} = \Lambda\,\partial_\mu \sum_a -iea \frac{\partial\mathcal{L}}{\partial(\partial\mu\psi^a)}\psi^a$$

$$+(\partial_\mu\Lambda)\left[\sum_a -iea \frac{\partial\mathcal{L}}{\partial(\partial\mu\psi^a)}\psi^a + \frac{\partial\mathcal{L}}{\partial A\mu} \right]$$

$$+(\partial_\mu\partial_\nu\Lambda)\frac{\partial\mathcal{L}}{\partial(\partial_\nu A\mu)} = 0 \quad .$$

Since Λ is an arbitrary function of space time we may draw the following conclusions:

1) From the choice Λ = const. we derive the conservation law

$$\partial\mu \sum_a iea \frac{\partial\mathcal{L}}{\partial(\partial\mu\psi^a)} \psi^a = 0 \quad ,$$

or (10)

$$\partial\mu \, J^\mu = 0$$

2) The coefficient of $\partial_\mu \Lambda$ must also vanish, which leads to the relationship

$$\frac{\partial\mathcal{L}}{\partial A\mu} = J^\mu \qquad (11)$$

where $J^\mu = i \sum_a ea \frac{\partial\mathcal{L}}{\partial(\partial\mu\psi^a)} \psi^a$.

3) The term $(\partial_\mu \partial_\nu \Lambda) \frac{\partial\mathcal{L}}{\partial(\partial\nu A\mu)}$ must vanish, so

$$\frac{\partial\mathcal{L}}{\partial(\partial_\nu A\mu)} \qquad (12)$$

must be anti-symmetric in μ and ν. This means that the dependence of $\mathcal{L}$ on $\partial_\nu A_\mu$ is of the form $\mathcal{L} = \mathcal{L}(\partial_\nu A\mu - \partial\mu A\nu)$.

From

$$\frac{\partial\mathcal{L}}{\partial A\mu} - \sum_a iea \frac{\partial\mathcal{L}}{\partial(\partial\mu\psi^a)} \psi^a = 0$$

we get that $\mathcal{L}$ depends on $A\mu$; ψ^a and $\partial\mu\psi^a$ only through the quantities

$$D_\mu \psi^a = \partial_\mu \psi^a + iea \, A\mu\psi^a \quad .$$

These are called the gauge covariant derivatives of ϕ. The $A\mu$'s are then called the gauge potentials, and the four-curl $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is called the field strength. Under gauge transformations of the second kind these quantities transform as follows

$$\phi'^{a} = e^{-ie_a\Lambda}\phi^{a}$$

$$A'_\mu = A_\mu + \partial_\mu\Lambda$$

$$(D_\mu\phi^{a})' = e^{-ie_a\Lambda} D_\mu\phi^{a}$$

$$F'_{\mu\nu} = F_{\mu\nu} \quad .$$

The gauge covariance of D_μ means $(D_\mu\phi^{a})' = D'_\mu\phi'^{a}$.

8. <u>Charged scalar field coupled to the electromagnetic potential.</u>

The Lagrangian $\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - U(\phi,\phi^*)$ is now replaced by the gauge covariant Lagrangian

$$D\mu\phi^*D^\mu\phi - U(\phi,\phi^*) + \mathcal{E}(F_{\mu\nu})$$

where $D^\mu = \partial^\mu + ie\, A^\mu$. The equations of motion are derived by taking variations of $\mathcal{L}$ relative to ϕ,ϕ^* and A_μ . To determine $\mathcal{E}(F_{\mu\nu})$ note that our equations must read

$$\partial\sigma \frac{\partial\mathcal{L}}{\partial(\partial\sigma A\mu)} = \frac{\partial\mathcal{L}}{\partial A\mu} = J^\mu$$

The left hand side of this equation should be $\partial_\sigma F^{\sigma\mu}$. $\mathcal{E}(F_{\mu\nu})$ is the Lagrangian of the field equations in a vacuum. In the electromagnetic case $\mathcal{E}(F_{\mu\nu}) = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$.

II - Non-Abelian Gauge Theories.

Suppose we wish to describe a system of spinless particle fields which we denote by a vector field $\underset{\sim}{\varphi}$, $\underset{\sim}{\varphi} : V_4 \to V$ where V is an n-dimensional vector space. We assume the Lagrangian is invariant under a gauge group $\mathcal{G}$, so that $\mathcal{L}(\varphi, \partial_\mu \varphi) = \mathcal{L}(\varphi', \partial_\mu \varphi')$ where $\varphi' = g\varphi, g \in \mathcal{G}$. By analogy with the electromagnetic case we suppose that these particles $\underset{\sim}{\varphi}$ interact with a force field in a gauge covariant way, that is, so that the complete Lagrangian is invariant with respect to gauge transformations of the second kind. In order to achieve this we must replace the partial derivatives ∂_μ by gauge covariant derivatives D_μ given by

$$D_\mu = \partial_\mu + Q_\mu$$

where the Q_μ are matrices. In fact, if the infinitesimal generators of the gauge group are τ_k, $k = 1, \ldots, m$ we may take Q_μ in the form

$$Q_\mu = Q^\alpha_\mu \tau_\alpha \quad .$$

Gauge covariance then requires the relationship

$$(D'_\mu \underset{\sim}{\varphi}') = (D_\mu \underset{\sim}{\varphi})'$$

or

$$(\partial_\mu + Q'^\alpha_\mu \tau_\alpha) g\varphi = g(\partial_\mu + Q^\alpha_\mu \tau_\alpha)\underset{\sim}{\varphi} \quad .$$

This leads to the transformation law

$$Q'^{\alpha}_{\mu}\tau_{\alpha} = gQ^{\alpha}_{\mu}\tau_{\alpha}g^{-1} + g\,\partial_{\mu}g^{-1} \quad .$$

Remark: Let us verify that $g\,\partial_{\mu}\,g^{-1}$ is an element of the Lie algebra of $\mathcal{G}$. Writing

$$g(x) = e^{\sigma^1(x)\tau_1 + \sigma^2(x)\tau_2 + \ldots + \sigma^m(x)\tau_m}$$

we have

$$g\,\partial_{\mu}\,g^{-1} = g\,\frac{\partial g^{-1}}{\partial\sigma^i}\,\partial_{\mu}\sigma^i \quad .$$

Now for any matrix $A(t)$, $e^{A(t)}\frac{d}{dt}e^{-A(t)} = -f(\mathrm{Ad}\,A)\dot{A}$, where

$f(z) = \frac{e^z-1}{z} = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \ldots$ Thus $f(\mathrm{Ad}A)\dot{A} = \dot{A} + \frac{1}{2!}[A,\dot{A}] + \frac{1}{3!}[A,[A,\dot{A}]] + \ldots$

Applying this relationship to the case at hand we have

$$g\,\frac{\partial g^{-1}}{\partial\sigma^i} = -f(\mathrm{Ad}\,A)\,\tau_i$$

where $A = \sigma^1\tau_1 + \ldots + \sigma^m\tau_m$. Since A belongs to the Lie algebra,

$g\,\frac{\partial g^{-1}}{\partial\sigma^i}$ also belongs to the Lie algebra.

The 4-potentials Q^{α}_{μ} are the analogs of the electromagnetic 4-potential A_{μ}. The analogs of the electromagnetic field strengths are obtained by considering the commutation $[D_{\mu}, D_{\nu}]$ operating on $\underset{\sim}{\varphi}$. In the electromagnetic case $D_\mu = \partial_\mu + eA_\mu$ and

$$[D_\mu, D_\nu]\varphi = eF_{\mu\nu}\ \varphi = e(\partial_\mu A_\nu - \partial_\nu A_\mu)\varphi \quad .$$

The left side is clearly gauge covariant so therefore $(F_{\mu\nu}\varphi)' = e^{i\theta}\ F_{\mu\nu}\ \varphi = F_{\mu\nu}'\ e^{i\theta}\varphi$ and $F_{\mu\nu}' = e^{i\theta}\ F_{\mu\nu}\ e^{-i\theta} = F_{\mu\nu}$. That is, the $F_{\mu\nu}$ are gauge invariant in the Abelian case.

In the non-Abelian case, however, the commutator is

$$\begin{aligned}[D_\mu, D_\nu] &= e(\partial_\mu Q_\nu^\alpha - \partial_\nu Q_\mu^\alpha)\tau_\alpha \\ &\quad + e^2\ Q_\mu^\alpha\ Q_\nu^\beta\ [\tau_\alpha, \tau_\beta] \\ &= e[\partial_\mu\ Q_\nu^\alpha - \partial_\nu\ Q_\mu^\alpha + e\ Q_\mu^\beta\ Q_\nu^\gamma\ C^{\alpha\beta\gamma}]\ \tau_\alpha\end{aligned}$$

where $C^{\alpha\beta\gamma}$ are the structure constants of the Lie algebra. The field strengths are given by

$$F_{\mu\nu}^\alpha = \partial_\mu\ Q_\nu^\alpha - \partial_\nu\ Q_\mu^\alpha + e\ Q_\mu^\beta\ Q_\nu^\gamma\ C^{\alpha\beta\gamma} \quad .$$

These field strengths transform according to the law

$$F_{\mu\nu}^{'\alpha}\ \tau_\alpha = F_{\mu\nu}^\alpha\ g\tau_\alpha\ g^{-1} \quad .$$

In the electromagnetic case the quantity $F_{\mu\nu}F^{\mu\nu}$ is quadratic in the derivatives $\partial_\mu A_\nu$ and invariant under gauge transformations. In the non-Abelian case the quantity

$$\mathrm{Tr}(F^{\alpha}_{\mu\nu}\,\tau_\alpha)(F^{\alpha,\mu\nu}\,\tau_\alpha)^+$$

is also quadratic in the derivatives. (Of course, there are other invariants as well; if we write

$$F_{\mu\nu} = F^{\alpha}_{\mu\nu}\,\tau_\alpha$$

then the matrix transforms as $F'_{\mu\nu} = gF_{\mu\nu}\,g^{-1}$ and det $F_{\mu\nu}$ is equally an invariant under gauge transformations.)

Equations of Motion

The equations of motion for the free Yang-Mills fields are derived in this section. We first summarize the notation we have already introduced.

The gauge potentials $Q_\mu(x)$ take values in the Lie algebra of the structural group G, and we write them

$$Q_\mu(x) = Q^a_\mu(x)T_a\ .$$

The field strengths $F_{\mu\nu}(x)$ are obtained from the potentials:

$$F_{\mu\nu}(x) = F^a_{\mu\nu}(x)T_a$$

$$F^a_{\mu\nu}(x) = \partial_\mu Q^a_\nu(x) - \partial_\nu Q^a_\mu(x) + e[Q_\mu(x),Q_\nu(x)]^a \ .$$

It is often convenient to write instead

$$[Q_\mu(x),Q_\nu(x)]^a = Q^b_\mu(x)Q^c_\nu(x)\ C^{abc}$$

where C^{abc} are the structure constants of the Lie algebra of G. Remember that while $Q_\mu(x)$ and $Q_\nu(x)$ do not commute, Q^a_μ and $Q^b_\nu(x)$ do: they are ordinary (real) functions.

The Lagrangian density for the free fields is

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu}(x)F^{a\mu\nu}(x)$$

$$= -\frac{1}{4} \mathrm{Tr}\ F_{\mu\nu}(x)F^{\mu\nu}(x) \quad .$$

From it we obtain the field equations in the standard way:

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu Q^a_\sigma)} - \frac{\partial\mathcal{L}}{\partial Q^a_\sigma} = 0 \quad .$$

Now

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu Q^a_\sigma)} = -F^{a\mu\sigma} \quad .$$

The calculation of $\frac{\partial\mathcal{L}}{\partial Q^a_\sigma}$ is a bit longer, giving

$$\frac{\partial\mathcal{L}}{\partial Q^a_\sigma} = -e\, Q^r_\nu\, F^{s\sigma\nu}\, C^{ars} \quad .$$

Thus the equations of motion are

$$-\partial_\mu F^{a\mu\sigma} + e\, Q^r_\nu\, F^{s\sigma\nu}\, C^{ars} = 0$$

or, rearranging indices,

$$\partial_\mu F^{a\mu\sigma} + e\, Q^r_\mu\, F^{s\mu\sigma}\, C^{ars} = 0 \quad .$$

These are the equations we sought. In matrix form they are

$$\partial_\mu F^{\mu\sigma} + e[Q_\mu, F^{\mu\sigma}] = 0$$

or

$$D_\mu F^{\mu\sigma} = 0 \quad .$$

III. Massless particles in gauge invariant theories.

We have seen that quadratic terms in the Lagrangian are interpreted as mass terms for a relatistic particle. If φ^a are a set of particle fields and $V(\varphi^a)$ is the potential, then, shifting the φ^a to a critical point of V and recombining the φ^a's so that the Hessian $\frac{\partial^2 V}{\partial\varphi^a \partial\varphi^b}$ is diagonal, the eigenvalues of this Hessian act as the squares of the masses in this theory:

$$\frac{\partial^2 V}{\partial\varphi^a \partial\varphi^b} = m_1^2(\varphi^1)^2 + m_2^2(\varphi^2)^2 + \ldots$$

In order that all the masses be real, we must operate at a local minimum of V .

Goldstone Bosons. If the Lagrangian $\mathcal{L}(\varphi^a, \partial_\mu \varphi^a)$ is invariant under an N-parameter gauge group $\mathcal{G}$ - that is $\varphi_a' = T(g)\varphi_a$ for elements g of a Lie group g , then $V(Tg\ \varphi^a) = V(\varphi^a)$ is an invariant function under the group action. Consequently we must expect that in general some of the eigenvalues of the Hessian of V at the critical point are going to vanish. This can be seen as follows.

Let the group parameters be $g_1, \ldots, g_N$ and suppose

$$\varphi_i' = T_{ij}(g_1, \ldots, g_N)\varphi_j$$

is the group action. Then

$$V(\varphi_1',\ldots,\varphi_n') = V(\varphi_1,\ldots,\varphi_n) .$$

Differentiating once with respect to the variables φ^j we get

$$\frac{\partial V}{\partial \varphi_i'} \frac{\partial \varphi_i'}{\partial \varphi_j} = \frac{\partial V}{\partial \varphi^j} ,$$

or

$$\frac{\partial V}{\partial \varphi_i} T_{ij} = \frac{\partial V}{\partial \varphi^j} (\varphi^1,\ldots,\varphi^n)$$

where $T_{ij} = T_{ij}(g_1,\ldots,g_N)$. Now differentiate with respect to the group parameters $g_1,\ldots,g_N$. We get

$$\frac{\partial^2 V}{\partial \varphi_i \partial \varphi_r} \frac{\partial \varphi_r'}{\partial g_\ell} T_{ij} + \frac{\partial V}{\partial \varphi_i} \frac{\partial T_{ij}}{\partial g_\ell} = 0 .$$

At a critical orbit $\nabla V = 0$ so this reduces to

$$\frac{\partial^2 V}{\partial \varphi_i \partial \varphi_r} \frac{\partial \varphi_r'}{\partial g_\ell} = \frac{\partial^2 V}{\partial \varphi_i \partial \varphi_r} L^\ell_{rs} \varphi_s = 0$$

at the identity, where L^ℓ_{rs} is the Lie derivative

$$L^\ell_{rs} = \left.\frac{\partial T_{rs}}{\partial g\ell}\right|_{g^1=\ldots=g^N=0} .$$

Therefore the null vectors of the Hessian $\frac{\partial^2 V}{\partial\varphi_i \partial\varphi_r}$ are the vectors $L^{\ell}_{rs}\,\varphi_s$; but these vectors span the tangent space to the orbit of critical points of V under the group action. The dimension of this tangent space is $\nu = \dim \mathcal{G} - \dim \mathcal{J}$, where $\mathcal{J}$ is the isotropy subgroup of the critical point, and $\dim \mathcal{J}$ is counted as zero if the isotropy subgroup is discrete.

If we choose as a new set of basis vectors in φ-space the eigenvectors of $\frac{\partial^2 V}{\partial\varphi_i \partial\varphi_j}$, the normal form of V is

$$V(\varphi_1, \ldots \varphi_n) = m^2_{\nu+1}\varphi^2_{\nu+1} + \ldots + m^2_n\, \varphi^2_n \;,$$

and the fields $\varphi_1, \ldots, \varphi_\nu$ have no mass terms. They therefore describe massless particles, called Goldstone Bosons. They are extraneous because they do not really occur in nature.

Massless Vector Mesons

Massless particles also arise in quite another context when one tries to couple vector force fields to scalar fields in a gauge-invariant way.

Suppose we are trying to construct a theory for a set of particles $\varphi^1, \ldots, \varphi^n$ which is invariant under some gauge group $\mathcal{G}$ of dimension N. By analogy with the electromagnetic field, we couple force fields Q^a_μ, $a = 1, \ldots N$ to the $\varphi^1, \ldots \varphi^n$ in a gauge-invariant way - that is, so that the total Lagrangian

$$\mathcal{L}(F^a_{\mu\nu}, \varphi^i, D_\mu \varphi^i)$$

is invariant under gauge transformations of the second kind

$$e^{\omega^\alpha \tau_\alpha} \varphi$$

where the ω^α are functions of the space-time variables x and the τ_α are the generators of the gauge group.

As in §II this is accomplished by introducing the gauge covariant derivatives

$$D_\mu = \partial_\mu + e\, Q^\alpha_\mu \tau_\alpha$$

where the Q^α_μ are the gauge potentials. The Lagrangian $\mathcal{L} = \mathcal{L}(F^\alpha_{\mu\nu}, \varphi^a, D_\mu \varphi^a)$, is then invariant under gauge transformations of the second kind provided the Q^α_μ transform according to the rule

$$Q'_\mu = g\, Q_\mu g^{-1} + g\, \partial_\mu g^{-1} \quad (Q_\mu = Q^\alpha_\mu \tau_a) \ .$$

If the dependence of $\mathcal{L}$ on $D_\mu \varphi^a$ is of the form $D_\mu \varphi^a D^\mu \varphi^a$ then, expanding out, one sees that there are no quadratic terms in the Q^α_μ . The particles associated with these fields are thus massless - called massless vector mesons.

These massless vector mesons are equally undesirable, for they signify long-range forces (forces that decay like $\frac{1}{r}$) rather than the short range forces that decay like $\frac{e^{-mr}}{r}$ which are typical of nuclear forces. That is, we expect that the force fields which describe the interaction of the φ-particles to be transmitted by massive particles.

We thus see that massless particles are inherent in any gauge-invariant theory. Massless particles occur both as vector mesons of the fields and as Goldstone bosons of the particles. These massless particles can be eleminated (or at least reduced in number) by a procedure due to Higgs which is usually called "spontaneous symmetry breakdown". What is involved is to break the gauge-invariance of the theory by fixing the gauge in an appropriate way so as to eliminate the massless terms. In fact, it should really come as no surprise that in the end we do not want that arbitrary choice of gauge. If the gauge were in fact arbitrary, then the distinction between the particles described by the φ's would be lost: Max would see a proton where Sam sees a neutron. The nature of the particles of the theory would simply be an artifact of the choice of gauge - that is, of the way in which they were measured.

So the building of a gauge invariant Lagrangian is only a preliminary first step, not an ultimate goal. The next step is to fix the gauge in a way that eliminates the unwanted massless particles of the theory. If the field theory is to include the electromagnetic interaction then at the end we still want the theory to be invariant under a one-parameter gauge group. On the other hand, if no massless particles are to occur, the final gauge group should be trivial (or at least discrete).

IV

Symmetry breaking and Elimination of Massless Particles

In this section we discuss a method, generally ascribed to Higgs for eliminating some or all of the massless Goldstone bosons and giving mass to the massless vector mesons. The procedure may be outlined as follows. If $U(\varphi)$ is the potential for the boson fields φ, let us first minimize $U(\varphi)$. Suppose U takes its minimum at a point $\vec{a} \in \mathbb{R}^n$ (φ now is regarded as a vector in $\mathbb{R}^n$). Due to the gauge invariance of the theory, $U(Tg\,\varphi) = U(\varphi)$ for any φ, so the action of the gauge group on $\vec{a}$ generates an orbit $\{Tg\ a\} = \Theta a$. As we observed in the previous sections, the tangent directions to Θa give the Goldstone bosons.

Let us first suppose the gauge group acts transitively on the φ-space, so that $\Theta a = \mathbb{R}^n$. Then we can fix the gauge so that the minimum φ' has the form

$$\varphi = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ a \end{pmatrix}.$$

For example, suppose the gauge group is $SU(2)$ and that it acts on $\mathbb{R}^3$ via the representation D^1. Then we choose the gauge (i.e. we fix a particular gauge) so that the particle field has the form

$$\varphi = \begin{pmatrix} 0 \\ 0 \\ \varphi^3 \end{pmatrix}$$

everywhere in space-time.

Then

$$D_\mu \varphi = \begin{pmatrix} 0 \\ 0 \\ \partial_\mu \varphi^3 \end{pmatrix} + e\, Q_\mu^a \, \tau_a \begin{pmatrix} 0 \\ 0 \\ \varphi^3 \end{pmatrix}$$

$$\tau_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad \tau_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \tau_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$D_\mu \varphi = \begin{pmatrix} -e\, A_\mu^2 \, \varphi^3 \\ e\, A_\mu^1 \, \varphi^3 \\ \partial_\mu \, \varphi^3 \end{pmatrix}$$

$$(D_\mu \varphi)^+ \, (D^\mu \varphi) = (\partial_\mu \varphi^3 \, \partial^\mu \varphi^3) + e^2 (Q_\mu^2 \, Q^{2\mu} + Q_\mu^1 \, Q^{\mu 1}) (\varphi^3)^2$$

Now the Lagrangian takes the form

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} \, F^{\mu\nu} + \frac{1}{2} (\partial_\mu \varphi^3) \, (\partial^\mu \varphi^3)$$
$$+ e^2 ((Q_\mu^2)^2 + (Q_\mu^1)^2)(\varphi^3)^2 + U(\varphi^2)$$

where $(Q_\mu^2)^2$ means $Q_\mu^2 \, Q^{2\mu}$; etc. Now in this form, where the gauge has been fixed, we minimize $U(\varphi)$ where $\varphi = \begin{pmatrix} 0 \\ 0 \\ \varphi^3 \end{pmatrix}$. By assumption this occured at $\varphi^3 = a$. So write $\varphi^3 = a + \rho$ where ρ is a function of x .

Then $\mathcal{L}$ has the form

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu \rho)^2$$

$$+ e^2 a^2((Q^2_\mu)^2 + (Q^1_\mu)^2) + \frac{U''(a)\rho^2}{2} + \ldots$$

$$+ (\rho^2 + 2a\rho)e^2((Q^2_\mu)^2 + (Q^1_\mu)^2)$$

From this form of the Lagrangian we see that there are no massless bosons and that two of the three vector mesons have acquired a mass $2e^2a^2$.

The final Lagrangian still possesses a gauge invariance of the second kind, but where now the gauge group is $SO(2)$ - the rotations about the 3-axis. This group is in fact the isotropy subgroup of the minimum $\begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix}$ of U . For these transformations $\rho' = \rho$ and $\partial_\mu \rho' = \partial_\mu \rho$ is therefore gauge invariant. Let us see how the Q^a_μ transform under this restricted gauge invariance. The Q's originally transformed according to the rule

$$Q'_\mu = g\, Q_\mu g^{-1} - (\partial_\mu g)g^{-1} + g\, \partial_\mu g^{-1}$$

or

$$Q'^a_\mu \tau_a = g\, Q^a_\mu \tau_a\, g^{-1} - (\partial_\mu g)g^{-1} \; .$$

Now restrict g to a gauge transformation of rotations about the 3-axis:

$$g(x) = e^{\sigma(x)\tau_3} .$$

Then $\partial_\mu g = e^{\sigma(x)\tau_3}(\partial_\mu\sigma)\tau_3;\ (\partial_\mu g)g^{-1} = e^{\sigma(x)\tau_3}(\partial_\mu\sigma)\tau_3\, e^{-\sigma(x)\tau_3} = (\partial_\mu\sigma)\tau_3$.

Now note that

$$e^{\sigma\tau_3}\tau_1 e^{-\sigma\tau_3} = \cos\sigma\,\tau_1 + \sin\sigma\,\tau_2$$

$$e^{\sigma\tau_3}\tau_2 e^{-\sigma\tau_3} = -\sin\sigma\,\tau_1 + \cos\sigma\,\tau_2$$

$$e^{\sigma\tau_3}\tau_3 e^{-\sigma\tau_3} = \tau_3 .$$

Therefore

$$Q^{1\ a}_\mu \tau_a = Q^1_\mu(\cos\sigma\,\tau_1 + \sin\sigma\,\tau_2)$$

$$Q^2_\mu(-\sin\sigma\,\tau_1 + \cos\sigma\,\tau_2)$$

$$Q^3_\mu\tau_3 + (\partial_\mu\sigma)\tau_3 ,$$

and

$$Q^{1'}_\mu = \cos\sigma\ Q^1_\mu - \sin\sigma\,Q^2_\mu$$

$$Q^{2'}_\mu = \sin\sigma\ Q^1_\mu + \cos\sigma\,Q^2_\mu$$

$$Q^{3'}_\mu = Q^3_\mu + \partial_\mu\sigma .$$

Therefore under the restricted gauge transformations the ρ-field is invariant, Q_μ^1 and Q_μ^2 transform as a rotation, and Q_μ^3 transforms as the electromagnetic potential. Note that Q_μ^3 is the field component associated with the massless particle.

In summary, we have eliminated the massless Goldstone bosons and given a mass to two of the vector mesons by breaking the gauge invariance of the theory - that is, by fixing a gauge. The resulting theory then possesses the gauge invariance of the isotropy subgroup of the minimum of the potential U .

V

Spinor Analysis and Covariant Wave Equations

Notation.

Points in space-time are denoted by x, coordinates $x^{\mu}(\mu = 0,1,2,3)$. The vector between two such points, $s = x - y$, defines a relativistically invariant metric denoted by (s,s) :

$$(s,s) = g_{\mu\nu}\, s^{\mu} s^{\nu} = (s^{0})^{2} - \vec{s}\cdot\vec{s} \quad .$$

The metric $g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$. There are four classes of intervals:

(a) $(s,s) > 0$ Time-like

(b) $(s,s) < 0$ Spacelike

(c) $(s,s) = 0$ Lightlike $(s \neq 0)$

(d) $s = 0$ Null vector.

Classes (a) and (c) have two pieces: $s^{0} > 0$ and $s^{0} < 0$. The set of points equipped with the above metric is called Minkowski space, or space-time.

§1. The Poincare group $\mathcal{P}$ is the set of linear transformations of Minkowski space into itself such that the distance between any two points is preserved. Thus

$$\mathcal{P}: M \to M$$
$$x,y \to x',y'$$

and $$(x'-y',x'-y') = (x-y),x-y) \quad .$$

An element of $\mathcal{P}$ is written (a,Λ) and its action on a point $x \in M$ is

$$(a,\Lambda): x \to x' = a + \Lambda x$$

$$x'^{\mu} = a^{\mu} + \Lambda^{\mu}{}_{\nu} x^{\nu} .$$

Thus 'a' is a translation in space-time and Λ is a 4×4 real matrix. Exercise: Show $(a,\Lambda)(a',\Lambda') = (a + \Lambda a'), \Lambda\Lambda')$. Important subgroups of $\mathcal{P}$ are the translations, $T_4 = \{a, I)\}$ with I the unit matrix, and the Lorentz group, $L = \{(0,\Lambda)\}$. Exercise: Show that $T_4 \, G \mathcal{P}$.

<u>§2.</u> The Lorenz group is also the subgroup of $GL(4,R)$ whose elements Λ obey

$$\Lambda^{T} g \Lambda = g \; .$$

Exercise: Show this. Written out explicitly it is

$$\Lambda^{\mu}{}_{\nu} \Lambda^{\sigma}{}_{\tau} g_{\mu\sigma} = (\Lambda^{T})_{\nu}{}^{\mu} g_{\mu\sigma} \Lambda^{\sigma}{}_{\tau} = g_{\nu\tau}$$

Exercise: Show that $(\Lambda^{0}{}_{0})^{2} = 1 + \sum_{i=1}^{3} (\Lambda^{i}{}_{0})^{2} \geq 1$

Exercise: Show that $\det \Lambda = \pm 1$.

The last two exercises and the observation that $\Lambda^0{}_0$ and $\det \Lambda$ are continuous functions of Λ show that L is split into four disconnected parts:

$$L_+^{\uparrow} - \Lambda^0{}_0 \geq 1 \, , \ \det \Lambda = +1$$

$$L_+^{\downarrow} - \Lambda^0{}_0 \leq -1 \, , \ \det \Lambda = +1$$

$$L_-^{\uparrow} - \Lambda^0{}_0 \geq 1 \, , \ \det \Lambda = -1$$

$$L_-^{\downarrow} - \Lambda^0{}_0 \leq -1 \, , \ \det \Lambda = -1$$

If we put $P = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$, $T = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$, then we can write

$$L = L_+^{\uparrow} + \underbrace{TL_+^{\uparrow}}_{L_+^{\downarrow}} + \underbrace{PL_+^{\uparrow}}_{L_-^{\uparrow}} + \underbrace{PTL_+^{\uparrow}}_{L_-^{\downarrow}}$$

. The component $L_+^{\uparrow}$ is continuously connected to the identity and is called the proper $(\det \Lambda = +1)$ orthochronous $(\Lambda^0{}_0 \geq 1)$ Lorentz group. It is connected but it is not simply connected, as we will see below.

§3. The Lorentz group $L_+^{\uparrow}$ is a homomorphic image of $SL(2,\mathbb{C})$.

We establish this result in this section. We first map Minkowski space into the set of Hamiltian 2×2 matrices:

$$X = \begin{pmatrix} x_0 + z & x - iy \\ x + iy & x_0 - z \end{pmatrix}$$

(*) $$x \to \underset{\sim}{x} = x^{\mu}\sigma_{\mu}$$

where $\sigma_{\mu} = \left(\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&-i\\i&0\end{pmatrix}, \begin{pmatrix}1&0\\0&-1\end{pmatrix}\right) = (I, \vec{\sigma})$. Using the metric $g^{\mu\nu}$ we can raise the index on σ_{μ}: $\sigma^{\mu} = (I, -\vec{\sigma})$. We can find the set $(I, -\vec{\sigma})$ another way which will prove useful later:

let $\epsilon = \begin{pmatrix}0&1\\-1&0\end{pmatrix} = \epsilon^{\alpha\beta}$. Define $\hat{\sigma}_{\mu} = \epsilon^{-1}\sigma_{\mu}\epsilon$, and $\hat{\sigma}_{\mu} = (I, -\vec{\sigma})$, $\hat{\sigma}^{\mu} = (I, \vec{\sigma})$. So we have

$$\sigma_{\mu} = (I, \vec{\sigma}) \qquad \sigma^{\mu} = (I, -\vec{\sigma})$$

$$\hat{\sigma}_{\mu} = (I, -\vec{\sigma}) \qquad \hat{\sigma}^{\mu} = (I, \vec{\sigma}) \ .$$

Now $\mathrm{Tr}\,\sigma_{\mu}\sigma_{\nu} = \mathrm{Tr}\,\sigma_{\mu}\hat{\sigma}^{\nu} = 2\,\delta_{\mu\nu}$, $\mathrm{Tr}\,\sigma_{\mu}\hat{\sigma}_{\nu} = 2g_{\mu\nu}$. Thus the mapping *) can be inverted:

*2) $$x^{\mu} = \frac{1}{2}\,\mathrm{Tr}\,\underset{\sim}{x}\,\hat{\sigma}^{\mu}$$

The inner product (x,x) is captured easily:

*3) $$(x,x) = \det \underset{\sim}{x} \ .$$

Finally, we associate $\Lambda \in L_{+}^{\uparrow}$ with $A \in SL(2,C)$ as promised in the opening sentence:

$$*^4) \qquad \underset{\sim}{x}' = A \underset{\sim}{x} A^* \quad , \quad A \in SL(2,C)$$

is linear and preserves the inner product because det $A = +1$. [We could allow det $A = e^{i\omega}$, but this gains us nothing, for $e^{i\omega/2} I$ maps $\underset{\sim}{x} \to \underset{\sim}{x}$.] Thus, if

$$\Lambda: x \to x' = \Lambda x$$

we have

$$\underset{\sim}{x}'^{\mu} = \frac{1}{2} \operatorname{Tr} A \underset{\sim}{x} A^* \hat{\sigma}^{\mu}$$

$$= x^{\nu} \left(\frac{1}{2} \operatorname{Tr} A \sigma_{\nu} A^* \hat{\sigma}^{\mu}\right)$$

$$= x^{\nu} \Lambda_{\nu}^{\mu} .$$

Exercise: Find A for $\Lambda \in SO(3)$.

Exercise: Find A for Λ = boost along z-axis.

Exercise: Show that the mapping $A \to \Lambda_A$ is a homomorphism with kernel $\{\pm I\}$.

Exercise: Prove that $\Lambda_A \in L_+^{\uparrow}$ (and not, say, $L_-^{\downarrow}$) .

§4. Spinors I. The group $SL(2,C)$ acts naturally on a two dimensional complex vector space $V = \{\binom{v_1}{v_2} : v_i \in C\}$ via

$$A: v \to v' = Av \;.$$

$$v'_\alpha = A_\alpha^\beta v_\beta \,, \quad \alpha \text{ and } \beta = 1,2.$$

If $A \in SU(2)$ then v transforms according to the two-dimensional representation $D^{1/2}$ of $SU(2)$, so we say v carries spin $1/2$. It is a two-valued representation of $L_+^\dagger$, for $\Lambda_A = \Lambda_{-A}$. Thus $\Lambda: v \to v' = \pm Av$, although $A: v \to Av$.

Consider the product $v_{\alpha_1} v_{\alpha_2} = \omega_{\alpha_1\alpha_2}$. This transforms like

$$A: \omega \to AA\omega = \omega'$$

$$\omega'_{\alpha_1\alpha_2} = A_{\alpha_1\beta_1} A_{\alpha_2\beta_2} \omega_{\beta_1\beta_2} \;.$$

Notice that ω is symmetric in its indices. On restricting A to $SU(2)$ we see that ω is the symmetric product of two spin $\frac{1}{2}$ objects, so ω carries spin 1. Further, this is a single-valued representation of $L_+^\dagger$. This object ω belongs to $V \otimes V$. More generally we construct objects belonging to $V\, V \;\ldots\; V$, i.e. $\omega_{\alpha_1\alpha_2\ldots\alpha_k}$, totally symmetric in the k-indices α_i, $\alpha_i = 1,2$. These are called spinors with k undotted indices. (We will get to dotted indices below.) Using the matrix ϵ introduced in §3 we can raise and lower indices

$$\epsilon^{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$v^\alpha = \epsilon^{\alpha\beta} v_\beta \,, \quad v_2 = \epsilon_{\alpha\beta} v^\beta \;;\; \epsilon^{12} = 1, \; \epsilon_{12} = -1 \;.$$

(To keep signs straight we need a convention for ϵ with upper and lower indices.)

§5. Spinors II. There is another "natural" action of $SL(2,C)$ on a two dimensional complex vector space $\dot{V} = \{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} : v_{\dot\alpha} \in C\}$:

$$A: v \to \bar{A}v$$

$$v_{\dot\alpha} \to v'_{\dot\alpha} = \bar{A}_{\dot\alpha\dot\beta}\, v_{\dot\beta} \quad .$$

When restricted to $SU(2)$ this is equivalent (though not equal) to the action discussed in the preceding section.

$$A \in SU(2) \qquad A = \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix} \qquad A \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{A}$$

so $D_{1/2} \cong \bar{D}_{1/2}$. In fact $A\epsilon = \epsilon \bar{A}$ for $A \in SU(2)$

$\epsilon A \epsilon^{-1} = (A^t)^{-1}$ when $\det A = 1$. As before we can construct spinors with ℓ dotted indices belonging to $\underbrace{\dot{V} \otimes \dot{V} \ \ldots \otimes \dot{V}}_{\ell\text{-factors}}$ that satisfy

$$A: w \to w'_{\dot\alpha_1 \ldots \dot\alpha_\ell} = \bar{A}_{\dot\alpha_\ell \dot\beta_\ell}\, w_{\dot\beta_1 \ldots \dot\beta_\ell} \quad .$$

Again, we may use ϵ to raise indices

$$v^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\, v_{\dot\beta} \quad , \quad \epsilon^{\dot1\dot2} = +1 \ .$$

We stress: as far as SU(2) is concerned, spinors with ℓ dotted or ℓ undotted indices carry spin $\ell/2$.

§6. Spinors III. The general case is a spinor with k undotted and ℓ dotted indices, symmetric in each class separately. An example is $\underset{\sim}{x}$ itself:

$$A: \underset{\sim}{x} \to \underset{\sim}{x}' = A \underset{\sim}{x} A^*$$

$$\underset{\sim}{x}'_{\alpha\dot\beta} = A_\alpha^{\alpha'} \bar{A}_{\dot\beta}^{\dot\beta'} \underset{\sim}{x}_{\alpha'\dot\beta'} = A_\alpha^{\alpha'} \underset{\sim}{x}_{\alpha'\dot\beta'} (A^*)^{\dot\beta'}{}_{\dot\beta} \quad .$$

Here $(k,\ell) = (1,1)$. Under SU(2) it splits into a spin 1 $(\vec{x})$ and spin $0(x^0)$ part. That is,

$$(k,\ell)\big|_{SU(2)} = (k) \oplus (\ell) \text{ , (k) is the k+1 dimensional}$$

unitary rep of SU(2) . Thus

$$(k,\ell)\big|_{SU(2)} = (k+\ell) \oplus (k+\ell-2) \oplus \ldots \oplus (|k-\ell|) \quad .$$

Again we may raise indices with ϵ . It is often convenient to use spinors with all undotted indices lowered, all dotted ones raised:

$$w_{\alpha_1 \ldots \alpha_k}^{\dot\beta_1 \ldots \dot\beta_\ell} \quad .$$

The transformation properties are

$$A: w \to w'$$

$$w'^{\dot\alpha_1 \ldots \dot\alpha_\ell}_{\alpha_1 \ldots \alpha_k} = A_{\alpha_1\beta_1} \ldots A_{\alpha_n\beta_k} \hat{A}^{\dot\alpha_1\dot\beta_1} \ldots \hat{A}^{\dot\alpha_\ell\dot\beta_\ell} w^{\dot\beta_1 \ldots \dot\beta_\ell}_{\beta_1 \ldots \beta_k}$$

where $\hat{A} = \epsilon \bar{A} \epsilon^{-1}$, $(\hat{A})^{\dot\alpha\dot\beta} = \epsilon^{\dot\alpha\dot\alpha'} \bar{A}_{\dot\alpha'\dot\beta'} \epsilon^{\dot\beta'\dot\beta}$.

Exercise: If $A^* = A^{-1}$, how is $\hat{A}$ related to A ? Hint: A is unitary, so work this out for $A = \begin{pmatrix} e^{i\,\varphi/2} & 0 \\ 0 & e^{-i\,\varphi/2} \end{pmatrix}$ and

$A = \begin{pmatrix} c\ \theta/2 & s\ \theta/2 \\ -s\ \theta/2 & c\ \theta/2 \end{pmatrix}$. Then all unitary matrices may be found by composition of these. In fact, ϵ is precisely the matrix P such that $AP = \bar{P}A$ for $A \in SU(2)$.

VI. The Relativistic Gauge Group and its Lie Algebra.

We begin by reviewing the results of §2, this time with somewhat greater precision. The gauge group $\mathcal{G}$ consists of finite transformations $g = e^{\omega(x)}$ where $\omega(x) = \omega^a(x)\tau_a$ is a function on space-time with values in the Lie algebra $\underset{\sim}{g}$ of the structural group. The τ_a are a basis for the Lie algebra of the structural group.

The gauge group acts on fields ϕ through a representation D of $\mathcal{G}$ as follows:

$$g: \phi \rightarrow U_g\phi = \phi' = D(g)\phi$$

$$(\phi')^a = D^{ab}(g)\phi \quad . \qquad (6\text{-}1)$$

The matrix $D(g) = \exp \omega^a(x)T_a$, where T_a are generators of the representation of $\underset{\sim}{g}$ associated with the rep D of $\mathcal{G}$. We take T_a to be skew-Hermitian and $\omega^a(x)$ real, so that D is a unitary rep.

The elements of $\underset{\sim}{\mathcal{G}}$ are therefore the $\underset{\sim}{g}$ valued functions on V_4. The commutator of two elements $\omega,\sigma \in \underset{\sim}{\mathcal{G}}$ is thus

$$[\omega,\sigma] = \rho = \omega^a(x)\sigma^b(x)c^{abc}\tau_c$$

where c^{abc} are the structure constants of $\underset{\sim}{g}$. We call $\underset{\sim}{\mathcal{G}}$ the gauge algebra.

In general the ψ's may also transform non-trivially under the Lorentz group, as discussed in §5. For example, in §2,3,4 we dealt with fields ψ that transformed merely as scalars under the Poincaré group $\mathcal{P}$:

$$(U_\Lambda \psi^a)(\underline{x}) = \psi^a(\Lambda^{-1}\underline{x}) \quad .$$

More generally the fields ψ^a_σ may transform as a spinor under the Poincaré transformations

$$(U_\Lambda \psi)^a_\sigma\,(\underline{x}) = S_{\sigma\sigma'}(\Lambda)\psi^a_{\sigma'}(\Lambda^{-1}\underline{x}) \tag{6-2}$$

where $S(\Lambda)$ is a representation of $L^\uparrow_+$ if Λ is a Lorentz transformation, and $S(\Lambda) = I$ if Λ is a translation. $\Lambda^{-1}x$ is $x - a$ if Λ is a translation.

In the notation ψ^a_σ the Greek indices are the spinor indices and the Roman indices are the gauge indices. Lorentz transformations act only on the σ-index and gauge transformations act only on the a-index.

<u>Definition</u>: The full group of gauge and Poincaré transformations is called the relativistic gauge group. We denote it by $\mathcal{P}\mathcal{J}$.

Below we use the notation g_Λ to mean $g_\Lambda(x) = g(\Lambda^{-1}x)$.

<u>Proposition 6.1</u>. i) $U_{g^{-1}} U_{\Lambda^{-1}} U_g U_\Lambda = U_{g^{-1} g_{\Lambda^{-1}}}$.

ii) The gauge group $\mathcal{G}$ is a normal subgroup of $\mathcal{P}\mathcal{G}$.

iii) $\mathcal{P}\mathcal{G}$ is a semi-direct product of $\mathcal{G}$ and $\mathcal{P}$.

Proof. To prove (i) we proceed directly:

$$(U_{g^{-1}} U_{\Lambda^{-1}} U_g U_\Lambda \psi)^a_\sigma (\underline{x})$$

$$= (U_{g^{-1}}(U_{\Lambda^{-1}} U_g U_\Lambda \psi))_\sigma(\underline{x})$$

$$= D^{ab}(g^{-1}(\underline{x}))(U_{\Lambda^{-1}} U_g U_\Lambda \psi)^b_\sigma(\underline{x})$$

$$= D^{ab}(g^{-1}(\underline{x}))S_{\sigma\sigma'}(\Lambda^{-1})(U_g U_\Lambda \psi)^b_{\sigma'} (\Lambda\underline{x})$$

$$= D^{ab}(g^{-1}(\underline{x}))S_{\sigma\sigma'}(\Lambda^{-1})D^{bc}(g(\Lambda x))S_{\sigma'\sigma''}(\Lambda)\psi^c_{\sigma''}(\underline{x})$$

$$= D^{ac}(g^{-1}(\underline{x})g(\Lambda\underline{x}))\psi^c_\sigma(\underline{x})$$

$$= (U_{g^{-1} g_{\Lambda^{-1}}} \psi)^a_\sigma (\underline{x})$$

From (i) we have $U_{\Lambda^{-1}} U_g U_\Lambda = U_{g_{\Lambda^{-1}}}$ and therefore the gauge group is invariant under inner automorphisms of $\mathcal{P}\mathcal{G}$ (invariance under inner automorphisms by $\mathcal{G}$ itself is trivial). Consequently $\mathcal{G}$ is a normal subgroup of $\mathcal{P}\mathcal{G}$. To show that it is a semi-direct product let (g,Λ) represent the transformation $U_g U_\Lambda$. The product of two such transformations is then

$$(g,\Lambda)\ (g',\Lambda') = U_g U_\Lambda U_{g'} U_{\Lambda'}$$

$$= U_g U_\Lambda U_{g'} U_{\Lambda^{-1}} U_\Lambda U_{\Lambda'}$$

$$= U_g U_{g'_\Lambda}\ U_{\Lambda\Lambda'} = U_{gg'_\Lambda}\ U_{\Lambda\Lambda'}$$

$$= (gg'_\Lambda\ ,\ \Lambda\Lambda')\ .$$

<u>Proposition 6.2</u>. <u>Let $\omega,\sigma \in \underset{\sim}{\mathcal{J}}$, let m belong to the Lie algebra of the Lorentz group $L_+^\uparrow$, and M the infinitesimal generator of the action (6-2) corresponding to m</u>. (We will compute M below). Let P_μ generate the translations. Then

(i) $[\omega,\sigma] = \rho\ ,\ \ \rho(x) = \omega^a(x)\sigma^b(x)c^{abd}\ \tau_d$

(ii) $[P_\mu,\omega] = i\ \partial_\mu\omega$

(iii) $[M,\omega] = -\partial_\mu\omega\ m^\mu{}_\nu x^\nu$.

Equations (ii) and (iii) show that the commutators of infinitesimal gauge and Poincaré transformations are in the gauge algebra. This means that $\underset{\sim}{\mathcal{J}}$ is an ideal in $\underset{\sim}{\mathcal{PJ}}$, which we should expect, since $\mathcal{J}$ is a normal subgroup of $\mathcal{PJ}$.

We do not need M but for completeness we compute it. Let

$$(U_\Lambda\phi)\sigma(\underline{x}) = S_{\sigma\sigma'}(\Lambda)\phi_{\sigma'}(\Lambda^{-1}\underline{x})$$

$$\Lambda = I + tm \qquad S = I + t\Gamma$$

$$U_\Lambda = I + tM\ .$$

We have

$$((I + tM)\phi)_\sigma(\underline{x}) = (I + t\Sigma)_{\sigma\sigma'}\, \phi_{\sigma'}(x - tmx) \quad .$$

Comparing coefficients of t we find

$$(M\phi)_\sigma(\underline{x}) = \Sigma_{\sigma\sigma'}\phi_{\sigma'} - m^\mu{}_\nu\, x^\nu \partial_\mu \phi_\sigma \quad .$$

<u>Proof of (6.2).</u> (i) has already been done. From (6.1)(i) we have

$$(e^{-\omega} U_{\Lambda^{-1}} e^{\omega} U_\Lambda \phi)(x) = e^{-\omega} e^{\omega_\Lambda} \phi$$

where $e^{\omega(x)} = \exp \omega^a(x) T_a$. Now replace e^ω by $e^{s\omega}$, U_Λ by e^{tM} , and Λ by e^{tm} ; expand both sices in power series; and collect the coefficients of st . Keeping only lowest terms, we have

$$(I - s\omega)(I - tM)(I + s\omega)(I + tM)\phi$$
$$= (I - s\omega)(I + s\omega_\Lambda)\phi$$
$$(I - st[M,\omega])\phi = (I - s\omega(x))(I + s\omega(\underline{x} + tm\underline{x}))\phi$$
$$= (I + st(\partial_\mu \omega) m^\mu{}_\nu x^\nu)\phi \quad .$$

Therefore $[M,\omega]$ is an infinitesimal gauge transformation, namely $-(\partial_\mu \omega)(m^\mu{}_\nu x^\nu)$. Part (ii) is even easier to prove.

Action of the relativistic gauge group on the gauge potentials Q_μ and the covariant derivatives

As we saw in §2 the gauge potentials Q_μ transform according to the law

$$Q'_\mu = T^\omega Q_\mu = e^\omega Q_\mu e^{-\omega} + e^\omega \partial_\mu e^{-\omega}$$

This action is not linear, though it is a group action. In fact

$$\begin{aligned}
\sigma \circ \omega : Q_\mu \to & \; e^\sigma e^\omega Q_\mu e^{-\omega} e^{-\sigma} + e^\sigma e^\omega \partial_\mu (e^{-\omega} e^{-\sigma}) \\
= & \; e^\sigma e^\omega Q_\mu e^{-\omega} e^{-\sigma} + e^\sigma \partial_\mu e^{-\sigma} \\
& \qquad\qquad + e^\sigma (e^\omega \partial_\mu e^{-\omega}) e^{-\sigma} \\
= & \; e^\sigma (e^\omega Q_\mu e^{-\omega} \quad + e^\omega \partial_\mu e^{-\omega}) e^{-\sigma} \\
& \qquad\qquad + e^\sigma \partial_\mu e^{-\sigma} \\
= & \; T^\sigma (T^\omega Q_\mu) \; .
\end{aligned}$$

The infinitesimal generator of this (nonlinear) action is found by replacing ω by $t\omega$ and expanding in powers of t. The coefficient of t is then the infinitesimal generator of the action and is

$$X_\omega (Q_\mu) = [\omega, Q_\mu] - \partial_\mu \omega \; .$$

We may formally think of the X_ω as vector fields. In that case their commutator is formally obtained by the methods of proof of proposition 6.2(ii) and (iii). That is we form the product

$$T^{-t\omega}\, T^{-s\sigma}\, T^{t\omega}\, T^{s\sigma}\, Q_\mu$$

and compute the coefficient of st . The result is that

$$[X_\omega, X_\sigma] = X_{[\omega,\sigma]} \quad .$$

The covariant derivatives D_μ however, transform under a linear action of the gauge group; namely

$$D'_\mu \rightarrow e^{\omega} D_\mu\, e^{-\omega}$$

and the infinitesimal generator of this action is

$$[\omega, D_\mu] = [\omega, \partial_\mu] + [\omega, Q_\mu]$$

$$= [\omega, Q_\mu] - \partial_\mu \omega = X\omega(Q_\mu) \; .$$

Note that this action is linear on the D_μ but nonlinear on the Q_μ .

Under the action of the Poincaré group, the D_μ transform as a 4-vector (also the Q_μ) .