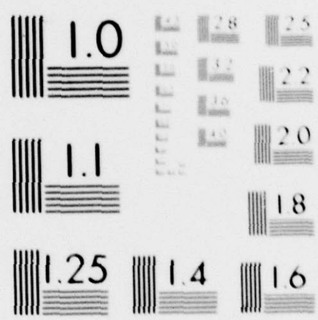
AD-A078 130 SOUTH CAROLINA UNIV COLUMBIA DEPT OF MATHEMATICS COM--ETC F/G 12/1
ON THE WEAK LAW OF LARGE NUMBERS FOR D(0,1).(U)
JUN 79 R L TAYLOR F49620-79-C-0140
UNCLASSIFIED TR-38 AFOSR-TR-79-1038 NL

| OF |

AD
A078130

END
DATE
FILMED
1-80
DDC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS 1963-A

AFOSR-TR- 79-1038

Department of Mathematics,
Computer Science and Statistics
The University of South Carolina
Columbia, South Carolina 29208

(2) LEVEL II

AD A078130

DDC FILE COPY

79 11 27 045

DISTRIBUTION STATEMENT A

Approved for public release;
Distribution Unlimited

LEVEL II

ON THE WEAK LAW OF LARGE NUMBERS FOR D[0,1]

by

Robert Lee Taylor[1]
University of South Carolina
Statistics Technical Report No. 38
60B05

June, 1979

University of South Carolina
Department of Mathematics, Computer Science
and Statistics
Columbia, South Carolina 29208

1. Research supported in part by the Air Force Office of Scientific Research under Contract No. F49620-79-C-0140.

Abstract

Weak laws of large numbers are obtained for random elements in D[0,1] where the convergence is in the sup-norm topology. For identically distributed random elements satisfying a compact integral condition, the weak law of large numbers holding pointwise is shown to be necessary and sufficient for the weak law of large numbers. In addition to a discussion of the compact integral condition, a weak law of large numbers is obtained for monotone increasing random elements, and convergence of weighted sums of independent, identically distributed random elements is obtained.

AMS Classification Number: Primary 60B05, 60F15; Secondary 60G99.

Key words and phrases: Skorohod topology, laws of large numbers, convergence in probability, partitions, and identical distributions.

1. Introduction and Preliminaries. Let D=D[0,1] denote the space of real-valued functions on [0,1] which are right-continuous and possess left-hand limits at each $t \in [0,1]$. Let the linear space D be equipped with the topology generated by the Skorohod metric d and let $|| \ ||$ denote the uniform norm, $||x|| = \sup_{0 \le t \le 1} |x(t)|$ for $x \in D$ (see Billingsley (1968), pages 109-153 for detailed topological and probabilistic properties of D). With the Skorohod metric topology, the linear space D is separable but not a linear metric space (addition is not continuous). However, the major obstacle in developing laws of large numbers is the absence of local convexity for D with the Skorohod topology.

Ranga Rao (1963) proved a strong law of large numbers for independent, identically distributed random elements in D while Sethuraman (1965) proved a large deviation result for independent, identically distributed random elements in D. Their methods of proof used truncation to a compact set and the following lemma.

Lemma 1 (Ranga Rao (1963)): Let X be a random element in D with $E||X|| < \infty$. For each $\varepsilon > 0$ there exists a partition $0 = t_0 < t_1 < \ldots < t_m = 1$ of [0,1] such that

$$\max_{1 \le i \le m} \ \sup_{t_{i-1} \le s,t < t_i} E|X(t) - X(s)| \le \varepsilon.$$

To compensate for the absence of local convexity, Taylor and Daffer (1978) & (1979) required the random elements to be convex tight and obtained laws of large numbers in the Skorohod topology. In this paper a weak law of large numbers is obtained in the uniform norm topology for identically distributed random elements satisfying the following compact integral condition.

Property MT: A random element X in D[0,1] has property MT if for each $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ there exists a compact set K and a partition $0 = t_0 < \ldots < t_m = 1$ such that

$$E[\max_{0 \le i \le m-1} \sup_{t_i \le t < t_{i+1}} |X(t) - X(t_i)| I_{[X \in K]}] < \varepsilon_1$$

and where $P[X \in K] > 1 - \varepsilon_2$.

Since convex tightness and a finite first moment is sufficient for property MT, the convergence in the $\|\ \|$ - topology significantly improves the laws of large numbers for convex tight random elements. A brief discussion of property MT will be included, and a weak law for the convergence of weighted sums of random elements will also be obtained in Section 3.

2. A Weak Law of Large Numbers. Given a partition $0 = t_0 < t_1 < \ldots < t_m = 1$, for notational convenience let

$$Tx = \sum_{i=0}^{m} x(t_i) I_{[t_i, t_{i+1})}(t) \tag{2.1}$$

where $x = x(t) \in D[0,1]$ and $[t_m, t_{m+1}) = \{1\}$. It is easy to see that T is a linear, Borel measurable (with respect to the Skorohod topology) function from D into D and that $\|Tx\| \le \|x\|$ for all $x \in D$.

Theorem 1: Let $\{X_n\}$ be identically distributed random elements in D such that $E\|X_1\| < \infty$ and let X_1 have property MT. Then

$$\frac{1}{n} \sum_{k=1}^{n} X_k(t) \to EX_1(t) \text{ in probability}$$

for each $t \in [0,1]$ if and only if

$$\left\|\frac{1}{n} \sum_{k=1}^{n} X_k - EX_1\right\| \to 0 \text{ in probability.}$$

<u>Proof</u>: The "if" part is immediate. For the "only if" part let $\varepsilon > 0$ and $0 < \eta \leq 1$ be given. Since $E||X_1|| < \infty$ and X_1 has property MT, there exists a compact set K such that

$$E(||X_1||I_{[X_1 \notin K]}) < \frac{\varepsilon\eta}{12} \tag{2.2}$$

and a partition $0 = t_0 < t_1 < \ldots < t_m = 1$ such that

$$E[\max_{0 \leq i \leq m-1} \sup_{t_i \leq t < t_{i+1}} |X_1(t) - X_1(t_i)| I_{[X_1 \in K]}] < \frac{\varepsilon\eta}{12}. \tag{2.3}$$

For each n

$$\begin{aligned} P[||\tfrac{1}{n} \textstyle\sum_{k=1}^n X_k - EX_1|| > \varepsilon] \\ \leq P[||\tfrac{1}{n} \textstyle\sum_{k=1}^n TX_k - E(TX_1)|| > \tfrac{\varepsilon}{4}] \\ + P[||\tfrac{1}{n} \textstyle\sum_{k=1}^n X_k - \tfrac{1}{n}\sum_{k=1}^n TX_k|| > \tfrac{\varepsilon}{2}] \\ + P[||EX_1 - E(TX_1)|| > \tfrac{\varepsilon}{4}]. \end{aligned} \tag{2.4}$$

The third term of (2.4) is zero since

$$\begin{aligned} ||EX_1 - E(TX_1)|| \\ \leq E(||X_1 - TX_1||I_{[X_1 \in K]}) + E(||X_1 - TX_1||I_{[X_1 \notin K]}) \\ < E[\max_i \sup_{t_i \leq t < t_{i+1}} |X_1(t) - X_1(t_i)| I_{[X_1 \in K]}] + \frac{2\varepsilon\eta}{12} \\ < \frac{\varepsilon}{12} + \frac{2\varepsilon}{12} = \frac{\varepsilon}{4} \end{aligned} \tag{2.5}$$

by (2.2) and (2.3). Using Markov's inequality and the identical distributions for the second term of (2.4),

$$P[\|\frac{1}{n}\sum_{k=1}^{n} X_k - \frac{1}{n}\sum_{k=1}^{n} TX_k\| > \frac{\varepsilon}{2}]$$

$$\le \frac{2}{\varepsilon n}\sum_{k=1}^{n} E\|X_k - TX_k\|$$

$$\le \frac{2}{\varepsilon}(E[\max_{i} \sup_{t_i \le t < t_{i+1}} |X_1(t) - X_1(t_i)| I_{[X_1 \in K]}]$$

$$+ 2E(\|X_1\| I_{[X_1 \notin K]}))$$

$$< \frac{2}{\varepsilon}(\frac{\varepsilon\eta}{12} + \frac{2\varepsilon\eta}{12}) = \frac{\eta}{2}. \qquad (2.6)$$

The weak law of large numbers holds for $\{X_k(t_i): k \ge 1\}$ with $i = 0,1,\ldots,m$ by hypothesis. Thus, for the first term of (2.4),

$$P[\|\frac{1}{n}\sum_{k=1}^{n} TX_k - E(TX_1)\| > \frac{\varepsilon}{4}]$$

$$\le P[\max_{0\le i\le m} |\frac{1}{n}\sum_{k=1}^{n} X_k(t_i) - E(X_1(t_i))| > \frac{\varepsilon}{4}]$$

$$< \frac{\eta}{2} \qquad (2.7)$$

for all $n \ge N(\varepsilon,\eta)$. Combining (2.4), (2.5), (2.6), and (2.7) yields

$$P[\|\frac{1}{n}\sum_{k=1}^{n} X_k - EX_1\| > \varepsilon] < \frac{\eta}{2} + \frac{\eta}{2} = \eta$$

for all $n \ge N(\varepsilon,\eta)$. ///

A random element X in D is said to be <u>convex tight</u> if for each $\varepsilon > 0$ there exists a compact, convex set K such that $P[X \in K] > 1-\varepsilon$. Characterizations of convex tightness were given by Daffer and Taylor (1979).

The following lemma shows that convex tightness will imply the property MT.

<u>Lemma 2</u>: If a random element X in D is convex tight and $E\|X\| < \infty$, then X has property MT.

Proof: Given $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, choose K convex and compact such that $P[X \in K] > 1 - \varepsilon_2$. By Theorem 6 of Daffer and Taylor (1979),

$$\{t \in [0,1]: \sup_{x \in K} |x(t)-x(t-0)| > \frac{\varepsilon_1}{2}\}$$

is finite ($=\{s_1, \ldots, s_r\}$). Let

$$\{t_0, t_1, \ldots, t_m\} = \{s_1, \ldots, s_r\} \cup \{0, \frac{1}{k}, \ldots, \frac{k}{k}\}.$$

Then, $k \le m \le k + r$, and

$$\limsup_{m \to \infty} \{\max_i \sup_{t_i \le t < t_{i+1}} |x(t)-x(t_i)|\} \le \frac{\varepsilon_1}{2}.$$

By the bounded convergence theorem

$$\limsup_{m \to \infty} E[\max_i \sup_{t_i \le t < t_{i+1}} |X(t)-X(t_i)| I_{[X \in K]}] \le \frac{\varepsilon_1}{2}$$

$$\le E[\limsup_{m \to \infty} \max_i \sup_{t_i \le t < t_{i+1}} |X(t)-X(t_i)| I_{[X \in K]}] \le \frac{\varepsilon_1}{2}.$$

Thus, there exists m such that

$$E[\max_{0 \le i \le m-1} \sup_{t_i \le t < t_{i+1}} |X(t)-X(t_i)| I_{[X \in K]}] < \varepsilon_1. \qquad ///$$

From Lemma 2 it follows that the weak law of large numbers with convergence in the $\| \; \|$-topology holds for many situations. The most standard condition may be pointwise uncorrelated identically distributed random elements which satisfy a convexity condition and hence yields an appropriate partition of [0,1].

3. Related Results. An example of a random element which does not have property MT is somewhat (by Lemma 2) related to nonseparable support with respect to the $\|\ \|$-topology. Let $\Omega = [0,1]$, $\mathcal{A}$ be the Borel subsets, and $X(t) = I_{[\omega,1]}(t)$. The uniform probability measure on [0,1] is assumed, that is, $P[X \in \{I_{[s,1]}: 0 \le s_1 \le s \le s_2 \le 1\}] = s_2 - s_1$. Let K be any compact set such that $P[X \notin K] < \varepsilon$. For any partition $\{t_0, \dots, t_m\}$ of [0,1]

$$\sup_{t_i \le t < t_{i+1}} |X(t) - X(t_i)| = 1 \quad \text{for } \omega \in (t_i, t_{i+1}).$$

Thus,

$$\max_{0 \le i \le m-1} \sup_{t_i \le t < t_{i+1}} |X(t) - X(t_i)| I_{[X \in K]} = 1$$

if and only if $\omega \in [X \in K] \cap (\bigcup_{i=0}^{m-1} (t_i, t_{i+1}))$. Hence,

$$\begin{aligned} E[\max_{0 \le i \le m-1} &\sup_{t_i \le t < t_{i+1}} |X(t) - X(t_i)| I_{[X \in K]}] \\ &= P[[X \in K] \cap (\bigcup_{i=0}^{m-1} (t_i, t_{i+1}))] \\ &= P[X \in K] > 1 - \varepsilon. \end{aligned} \tag{3.1}$$

By (3.1) X does not have property MT.

The property MT is certainly not necessary for the weak law of large numbers in D[0,1]. Note that X is monotone increasing with probability one. The following weak law of large numbers for monotone increasing random elements can be proved in the same manner as Theorem 2 of Daffer and Taylor (1979).

Theorem 2: Let $D^{\uparrow}$ denote the cone of nondecreasing elements of D. Let $\{X_n\}$ be a sequence of random elements in D such that $X_n \in D^{\uparrow}$ with probability one and $EX_n = EX_1$ for each n. Then,

$$\left|\frac{1}{n}\sum_{k=1}^{n} X_k(t) - EX_1(t)\right| \to 0 \text{ in probability}$$

for each $t \in [0,1]$ if and only if

$$\left\|\frac{1}{n}\sum_{k=1}^{n} X_k - EX_1\right\| \to 0 \text{ in probability.}$$

The major use of property MT in obtaining the weak law of large numbers for D is in developing Inequality (2.6). It is instructive (for applications) to consider

$$P\left[\left\|\frac{1}{n}\sum_{k=1}^{n} X_k - \frac{1}{n}\sum_{k=1}^{n} TX_k\right\| > \frac{\varepsilon}{2}\right]$$

$$= P\left[\max_{0 \le i \le m-1} \sup_{t_i \le t < t_{i+1}} \left|\frac{1}{n}\sum_{k=1}^{n} (X_k(t) - X_k(t_i))\right| > \frac{\varepsilon}{2}\right]$$

when the $\{X_k(t)\}$ are stochastic processes with independent increments. For these applications, Levy's inequalities may yield Inequality (2.6).

The generality of the pointwise condition is balanced by a compact, smooth type condition (property (MT)). Additional probability structure on the random elements can eliminate the need for property MT as is illustrated in Theorem 3 for the convergence of weighted sums. Only the essential steps of the proof for Theorem 3 will be listed for comparison with the proof of Theorem 1.

Theorem 3: Let $\{X_n\}$ be independent, identically distributed random elements in D such that $E\|X_1\| < \infty$. Let $\{a_{nk}\}$ be a double array of constants such that (i) $\sum_{k=1}^{\infty}|a_{nk}| \leq 1$ for each n and (ii) $\lim_{n\to\infty} \sum_{k=1}^{n} a_{nk} = 1$. A necessary and sufficient condition that

$$||\sum_{k=1}^{n} a_{nk}X_k - EX_1|| \to 0 \text{ in probability}$$

is that $\max_{1\leq k\leq n} |a_{nk}| \to 0$.

Outline of Proof: Since real numbers can be identified with constant functions in D, $\max_{1\leq k\leq n} |a_{nk}| \to 0$ is necessary. For sufficiency pick K compact such that (2.2) holds. Choose δ so that $\alpha\leq t<\beta<\alpha+\delta$ implies that $|x(t) - x(\alpha)| \leq |x(\beta-0)-x(\alpha)| + \frac{\varepsilon\eta}{8}$ uniformly for $x \in K$. In place of (2.3), Lemma 1 provides a partition such that

$$\max_{0\leq i\leq m-1} \sup_{t_i\leq t,s<t_{i+1}} E|X(s)-X(t)| \leq \frac{\varepsilon\eta}{12},$$

and it can be assumed that $t_{i+1}-t_i < \delta$ for all $0\leq i\leq m-1$. For (2.5)

$$||EX_1 - ETX_1||$$

$$= \max_{0\leq i\leq m-1} \sup_{t_i\leq t<t_{i+1}} E|X(t)-X(t_i)| \leq \frac{\varepsilon\eta}{12} < \frac{\eta}{6}. \quad (3.2)$$

Using the independence of $\{X_k(t_{i+1}-0) - X_k(t_i): k = 1,2,3,\ldots\}$ for each i, (2.6) becomes

$$P[||\sum_{k=1}^{n} a_{nk}X_k - \sum_{k=1}^{n} a_{nk}TX_k|| > \frac{\varepsilon}{2}]$$

$$\leq P[\max_{0\leq i\leq m} \sup_{t_i\leq t<t_{i+1}} |\sum_{k=1}^{n} a_{nk}(X_k(t)-X_k(t_i))I_{[X_k \in K]}| > \frac{\varepsilon}{4}]$$

$$+ P[\sum_{k=1}^{n}|a_{nk}| \; ||X_k-TX_k||I_{[X_k \notin K]} > \frac{\varepsilon}{4}]$$

$$\le P[\max_{0\le i\le m-1} \sum_{k=1}^{n} |a_{nk}| |X_k(t_{i+1}-0) - X_k(t_i)| I_{[X_k \in K]} > \frac{\varepsilon}{8}]$$

$$+ \frac{4}{\varepsilon} \sum_{k=1}^{n} |a_{nk}| 2E(\|X_k\| I_{[X_k \notin K]})$$

$$\le \sum_{i=0}^{m-1} P[\sum_{k=1}^{n} |a_{nk}| (|X_k(t_{i+1}-0) - X_k(t_i)| I_{[X_k \in K]}$$

$$- E(|X_k(t_{i+1}-0) - X_k(t_i)| I_{[X_k \in K]}) > \frac{\varepsilon}{8} - \frac{\varepsilon}{12}]$$

$$+ \frac{8}{\varepsilon} \cdot \frac{\varepsilon\eta}{12}. \tag{3.3}$$

Since the random variables $\{|X_k(t_{i+1}-0) - X_k(t_i)| I_{[X_k \in K]}: k = 1,2,\ldots\}$ are independent, identically distributed random elements, the first term can be made less than $\frac{\eta}{6}$. ///

REFERENCES

Billingsley, P. (1968). Convergence of Probability Measures. Wiley, New York.

Daffer, P. and Taylor, R. L. (1979). Laws of large numbers for D[0,1]. Annals, Probability, 7, 85-95.

Rao, R. Ranga (1963). The law of large numbers for D[0,1]-valued random variables. Theor. Probability Appl., 8, 70-74.

Sethuraman, J. (1965). On the probability of large deviations of the mean for random variables in D[0,1]. Annals, Math. Statist., 36, 280-285.

Taylor, R. L. and Daffer, P. (1978). Some weak laws of large numbers for probability measures on vector spaces. Proceedings of the Conference on Vector Space Measures and Applications. Springer-Verlag Lecture Notes, Vol. 644, 411-417.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) UNCLASSIFIED

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: AFOSR-TR- 79-1038 2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle): ON THE WEAK LAW OF LARGE NUMBERS FOR D[0,1]	5. TYPE OF REPORT & PERIOD COVERED: Interim
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Robert Lee Taylor	8. CONTRACT OR GRANT NUMBER(s): F49620-79-C-0140
9. PERFORMING ORGANIZATION NAME AND ADDRESS: University of South Carolina, Dept. of Mathematics, Computer Science, & Statistics, Columbia, SC 29208	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F 2304/A5
11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Office of Scientific Research/NM, Bolling AFB, Washington, DC 20332	12. REPORT DATE: June 1979
	13. NUMBER OF PAGES: 12
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION ST. MENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Skorohod topology, laws of large numbers, convergence in probability, partitions, and identical distributions.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Weak laws of large numbers are obtained for random elements in D[0,1] where the convergence is in the sup-norm topology. For identically distributed random elements satisfying a compact integral condition, the weak law of large numbers holding pointwise is shown to be necessary and sufficient for the weak law of large numbers. In addition to a discussion of the compact integral condition, a weak law of large numbers is obtained for monotone increasing random elements, and convergence of weighted sums of independent, identically distributed random elements is obtained.

DD FORM 1 JAN 73 1473

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)