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EXACT SOLUTIONS FOR CONVECTIVE HEAT TRANSFER. (U)
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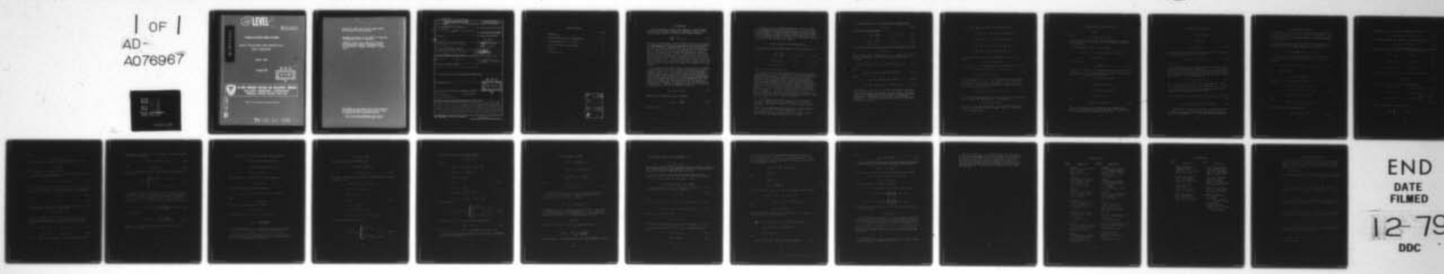

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TECHNICAL REPORT ARBRL-TR-02186

EXACT SOLUTIONS FOR CONVECTIVE HEAT TRANSFER

John F. Polk

August 1979

US ARMY ARMAMENT RESEARCH AND DEVELOPMENT COMMAND
BALLISTIC RESEARCH LABORATORY
ABERDEEN PROVING GROUND, MARYLAND

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REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: Technical Report ARBRL-TR-02186	2. GOVT ACCESSION NO.
	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle): EXACT SOLUTIONS FOR CONVECTIVE HEAT TRANSFER	5. TYPE OF REPORT & PERIOD COVERED: Final rept.
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): John F. Polk	8. CONTRACT OR GRANT NUMBER(s): 12 25
9. PERFORMING ORGANIZATION NAME AND ADDRESS: US Army Ballistic Research Laboratory (ATTN: DRDAR-BLT) Aberdeen Proving Ground, MD 21005	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: RDT&E 1L161102AH43
11. CONTROLLING OFFICE NAME AND ADDRESS: US Army Armament Research and Development Command US Army Ballistic Research Laboratory (ATTN: DRDAR-BL) Aberdeen Proving Ground, MD 21005	12. REPORT DATE: AUG[redacted]79
	13. NUMBER OF PAGES: 24
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): SBIE AD-E430 318	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release, distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Heat Equation
Diffusion Equation
Convective Heat Transfer
Radiation Boundary Condition
Heat Polynomials
Exact Solutions

20. ABSTRACT (Continue on reverse side if necessary and identify by block number) (mba)

The convective heat transfer at a fluid/solid interface is investigated from a mathematical point of view. Through the use of a transformation, T_h, a number of exact solutions of the heat equation satisfying convection or radiation type boundary conditions are developed from previously known solutions satisfying temperature-specified type boundary conditions. A general solution is obtained in the form of an integral representation and specific solutions are determined for some important particular choices of the boundary and initial data.

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I. INTRODUCTION

The heat transfer occurring at the surface of a solid in contact with a fluid at different temperature is frequently modeled by the radiation, Newton cooling or convective heat transfer condition

$$K \frac{\partial u_s}{\partial \vec{n}} = - H\,(u_f - u_s); \tag{1.1}$$

here $\vec{n}$ is the inward normal from the surface of the solid, u_f and u_s are the temperatures of the fluid and solid respectively at their interface, K is the coefficient of thermal conductivity for the metal and H is the effective heat transfer coefficient (which might include radiation terms). In most cases H, K and u_f are not known *a priori* since they may vary with existing conditions and their determination becomes a major portion of the problem. However, to make progress and to understand the role of these parameters more completely we may idealize the problem by assuming that $h = H/K$ is a known positive constant and u_f is a given function of time. Ultimately these are not serious restrictions when incorporated into a larger scale fluid flow calculation since h and u_f can be reset over each sufficiently small time increment of the computation.

The objective of the present report is the construction of a number of exact solutions for the heat or diffusion equation which satisfy boundary condition (1.1). These functions will be used subsequently in a separate report to develop an effective numerical algorithm which models the heat transfer occurring at the bore surface of gun barrels. In this report however we shall not be concerned with engineering applications but formulate the problem as a purely mathematical one. Our analysis will suppose that the temperature of the solid depends on a single spatial coordinate x and on time t and that the solid exists in the region $x > 0$. Denoting the temperature of the solid by $u(x,t)$ the boundary condition (1.1) may be rewritten in the form

$$L_h\, u\,(0,t) = u_f\,(t) \tag{1.2}$$

where L_h is the differential operator defined by

$$L_h\, u \equiv u - \frac{1}{h}\frac{\partial u}{\partial x} \tag{1.3}$$

with $h = H/K > 0$.

In Section II we shall define our problem precisely and introduce the transformation T_h which is the inverse of L_h. In Section III we find transformations for the functions H_γ and H_γ^* which were introduced in previous work[1,2]. Finally in Section IV explicit solutions of the problem are obtained for specific choices of the boundary and initial data. We shall make frequent use of the results in Reference 2 and shall assume that this is available to the reader.

II. PROBLEM STATEMENT AND THE TRANSFORMATION T_h

Consider the following problem defined in the quarter plane $x \geq 0$, $t \geq 0$.

$$\frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2} \qquad x > 0,\ t > 0 \qquad (2.1)$$

$$u\ (x,0) = f\ (x) \qquad x > 0 \qquad (2.2)$$

$$L_h\ u\ (0,t) = g\ (t) \qquad t > 0 \qquad (2.3)$$

where $a > 0$ and L_h is defined by (1.3).

To obtain solutions for this problem we shall proceed using purely formal derivations. That is, we shall not be concerned with mathematical legitimacy but presume that all functions in question exist and are sufficiently smooth to justify whatever operations are performed. Vindication of this procedure follows *a posteriori* by noting that the functions thereby obtained do in fact satisfy the desired equations. On the other hand, the important questions of uniqueness and continuous dependence on the data will not be considered; in fact, the solution to problem (2.1) - (2.3) is not unique unless an additional exponential growth restriction is stipulated. (Consider in this regard the example given by Friedman[3]). Such restrictions serve mainly to rule out physically meaningless solutions and will certainly be satisfied by the functions constructed below which are either bounded or grow more slowly than a polynomial in x (as $x \to \infty$ with t held finite).

---

[1] *J. F. Polk, "Asymptotic Expansions for the Solutions of Parabolic Differential Equations with a Small Parameter," Ph.D dissertation, Department of Mathematics, University of Delaware, Newark, Delaware, June 1979.*

[2] *John F. Polk, "Special Function Solutions of The Diffusion Equation," Technical Report, Ballistic Research Laboratory, TR 02182*

[3] *Abner Friedman, "Partial Differential Equation of Parabolic Type," Prentice-Hall Corp, Englewood Press, N.J., p. 31, 1964.*

To solve problem (2.1) - (2.3) note that the related function

$$v = L_h u \tag{2.4}$$

satisfies the following equations

$$\frac{\partial v}{\partial t} = a \frac{\partial^2 v}{\partial x^2} \qquad x > 0,\ t > 0 \tag{2.5}$$

$$v(x,0) = L_h\ f\ (x) \qquad x > 0 \tag{2.6}$$

$$v(0,t) = g\ (t) \qquad t > 0\ . \tag{2.7}$$

Using linearity we may express the solution to this problem in the form

$$v = v_f + v_g$$

where v_f and v_g are the solutions to (2.5) - (2.7) with $g\ (t) = 0$ and $f\ (x) = 0$ respectively. But these problems have the well known solutions

$$v_f(x,t) = \int_0^\infty [H_{-1}\ (x-s,\ at) - H_{-1}\ (x+s,\ at)][L_h\ f]\ (s)\ ds \tag{2.8}$$

and

$$v_g(x,t) = -2a \int_0^t H_{-2}\ (x,a\ (t-r))\ g(r)\ dr \tag{2.9}$$

where

$$H_{-1}\ (x,t) = (4\pi t)^{-1/2} \exp\ (-x^2/4t) \tag{2.10}$$

$$H_{-2}\ (x,t) = \frac{\partial}{\partial x} H_{-1}\ (x,t) = (-x/2t)\ H_{-1}\ (x,t). \tag{2.11}$$

The functions $H_\gamma\ (x,t)$, for all $\gamma \in R$, were investigated in References 1 and 2 and will be briefly reviewed in the next section. The expression for v_f is somewhat unsatisfactory in that it requires $f(x)$ to be differentiable, although this was not needed in the formulation of problem (2.1) - (2.3). We can eliminate this difficulty using integration by parts, as follows

$$v_f(x,t) = \int_0^\infty [H_{-1}(x-s, at) - H_{-1}(x+s, at)]\, f(s)\, ds$$

$$- \frac{1}{h}\int_0^\infty [H_{-1}(x-s, at) - H_{-1}(x+s, at)]\, f'(s)\, ds$$

$$= \int_0^\infty [H_{-1}(x-s, at) - H_{-1}(x+s, at)]\, f(s)\, ds$$

$$- \frac{1}{h}\int_0^\infty [H_{-2}(x-s, at) + H_{-2}(x+s, at)]\, f(s)\, ds$$

$$+ [H_{-1}(x-s, at) - H_{-1}(x+s, at)]\, f(s)\, \Big|_{s=0}^{\infty} .$$

The last term vanishes because of the exponential decay of $H_{-1}(x,t)$ provided

$$|f(x)| \leq \text{const. } \exp[\bar{a}\, x^2]$$

for some $\bar{a} < 1/4at$. The remaining terms can be recombined in the form

$$v_f(x,t) = \int_0^\infty \{[L_h H_{-1}](x-s, at) - [L_h H_{-1}](x+s, at) - \frac{2}{h} H_{-2}(x+s, at)\}\, f(s)\, ds. \tag{2.12}$$

The solution to problem (2.1) - (2.3) can now be obtained by inverting the transformation $u \to v = L_h u$. This amounts to solving an ordinary differential equation and can be accomplished using the integrating factor method. The appropriate transformation is denoted by T_h and defined for a function $w = w(x)$ by

$$[T_h w](x) \equiv h \int_x^\infty \exp[h(x-s)]\, w(s)\, ds. \tag{2.13}$$

This is well defined and finite for all $w(x)$ in the class S_h of functions which are defined and continuous for $x > 0$ and bounded by

$$|w(x)| \leq \text{const. } \exp(\bar{h}\, x)$$

for some $\bar{h} < h$ and sufficiently large x. The transformed function is differentiable and can be similarly bounded. Thus $T_h w \in S_h$ whenever $w \in S_h$. Moreover we have

$$|T_h w(x)| \leq h M(x) \int_x^\infty \exp [h (x-s)] ds$$

$$\leq M(x)$$

where

$$M(x) = \sup \{|w(s)| : s \geq x\}.$$

It follows that if w is bounded or vanishes asymptotically as some negative power of x as $x \to \infty$ then $T_h w$ is also bounded or vanishes at least as rapidly.

Applying Leibnitz's rule to (2.13) we have

$$[T_h w]'(x) = h^2 \int_x^\infty \exp [h(x-s)] w(s) ds - h w(x)$$

$$= h [T_h w - w] (x)$$

and consequently

$$L_h [T_h w] = w \tag{2.14}$$

whenever w is of class S_h. The transformations L_h and T_h are thus seen to be inverses. If $w(x)$ has a derivative $w'(x)$ which is also of class S_h then using intergration by parts we have

$$[T_h w'] (x) = h \int_x^\infty \exp [h(x-s)] w'(s) ds$$

$$= h \exp [h(x-s)] w(s) \Big|_{s=x}^\infty$$

$$+ h^2 \int_x^\infty \exp [h(x-s)] w(s) ds$$

$$= h [T_h w - w] (x).$$

Thus

$$[T_h w'] (x) = [T_h w]'(x). \tag{2.15}$$

That is, the derivative of the transform is the transform of the derivative; this permits us to reverse the order of the operations in (2.14) for functions with derivatives of class S_h:

$$T_h\,[L_h\,w] = T_h\,w - \frac{1}{h}\,T_h\,[w']$$

$$= T_h\,w - \frac{1}{h}\,[T_h\,w]'$$

$$= L_h\,[T_h\,w]$$

or

$$T_h\,[L_h\,w] = w\,. \qquad (2.14)'$$

In general if the n-th derivative $w^{(n)}(x)$ exists and is of class S_h then by repeated applications of (2.14) we obtain

$$[T_h\,w^{(n)}]\,(x) = [T_h\,w]^{(n)}\,(x) \qquad (2.16)$$

for $n = 0,1,2,\ldots.$

We can now return to the original problem. The functions v_1 and v_2 are twice differentiable with respect to x, for $t > 0$, since they are solutions of the heat equation. Furthermore they can easily be shown to be of class S_h (for each fixed $t > 0$) provided f is of class S_h and g is bounded. Thus their transforms are well defined and the solution to problem (2.1) - (2.3) can be written as

$$u = u_f + u_g = T_h\,v_f + T_h\,v_g \qquad (2.17)$$

where u_f and u_g are the solutions of the problem when $g(t) = 0$ and $f(x) = 0$ respectively. But applying T_h to the expressions (2.9) and (2.12) and using Fubini's Theorem to interchange the order of the integration and transformation operators yields

$$u_f(x,t) = \int_0^\infty [H_{-1}(x-s,a\,t) - H_{-1}(x+s,a\,t) - \frac{2}{h}\,T_h\,H_{-2}(x+s,a\,t)]\,f(s)\,ds \qquad (2.18)$$

$$u_g(x,t) = -2a \int_0^t T_h\,H_{-2}(x,a\,(t-r))\,g(r)\,dr\,. \qquad (2.19)$$

Transformation of the functions H_{-2} and H_{-1} is permissible since they are not only continuous but entire with respect to x for $t > 0$ and vanish exponentially as $|x| \to \infty$. We also note that the identity

$$T_h\,[w\,(x+y)] = [T_h\,w](x+y) \tag{2.20}$$

for any function w of class S_h follows directly from the definition of T_h. Thus the application of T_h to H_{-2} in (2.18) is unambiguous in that the shifting (by s) can be performed before or after the transformation (with respect to x).

In the next two sections we shall obtain explicit expressions for the transformations of the functions $H_\gamma\,(x,y)$, $\gamma \in R$ and solutions to problem (2.1) - (2.3) for specific choices of f(x) and g(t).

III. T_h TRANSFORMATIONS OF H_γ, H_γ^* and v_n

Let us briefly review the salient properties of the functions H_γ, H_γ^* and v_n which were discussed in References 1 and 2. These were defined by

$$H_\gamma\,(x,t) = (4\pi t)^{-1/2} \int_0^\infty \frac{x^\gamma}{\gamma!} \exp\,(-\,(x-s)^2/4t)\,ds \tag{3.1}$$

for $\gamma > -1$ and recursively

$$H_\gamma\,(x,t) = \frac{\partial}{\partial x} H_{\gamma+1}\,(x,t) \tag{3.2}$$

for $\gamma \leq -1$. (The latter equation also holds for $\gamma > -1$). In particular

$$\begin{aligned} H_o\,(x,t) &= (1/2)\,\mathrm{erfc}\,\,(-\,x/2\,\sqrt{t}) \\ H_{-1}\,(x,t) &= (4\pi t)^{-1/2} \exp\,(-\,x^2/4t) \\ H_{-2}\,(x,t) &= (-\,x/2t)\,H_{-1}\,(x,t) \end{aligned} \tag{3.3}$$

where

$$\mathrm{erfc}\,(z) = (2/\sqrt{\pi}\,) \int_z^\infty \exp\,(-\,s^2)\,ds\;.$$

The associated functions H_γ^* were given by

$$H_\gamma^*\,(x,t) = H_\gamma\,(-\,x,t) \tag{3.4}$$

and for any integer n the heat polynomials were defined by

$$v_n (x,t) = H_n (x,t) + (-1)^n H_n (x,t); \qquad (3.5)$$

for negative integers these vanished in view of the identity

$$H_n^* (x,t) = (-1)^{n+1} H_n (x,t) \qquad (3.6)$$

for $n = -1, -2, \ldots$. All of these functions were seen to be infinitely differentiable in x and t for $t > 0$, and satisfy the formulas

$$\frac{\partial}{\partial x} H_\gamma = H_{\gamma-1} \qquad \frac{\partial}{\partial t} H_\gamma = H_{\gamma-2} \qquad (3.7)$$

$$\frac{\partial}{\partial x} H_\gamma^* = - H_{\gamma-1}^* \qquad \frac{\partial}{\partial t} H_\gamma^* = H_{\gamma-2}^* \qquad (3.8)$$

$$\frac{\partial}{\partial x} v_n = v_{n-1} \qquad \frac{\partial}{\partial t} v_n = v_{n-2} . \qquad (3.9)$$

Thus they are all solutions of the heat or diffusion equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad (3.10)$$

for $t > 0$ and satisfy the initial and boundary values

$$H_\gamma (x,0) = h_\gamma (x) \qquad H_\gamma (0,t) = \frac{1}{2} h_{\gamma/2}(t) \qquad (3.11)$$

$$H_\gamma^* (x,0) = h_\gamma^* (x) \qquad H_\gamma^* (0,t) = \frac{1}{2} h_{\gamma/2} (t) \qquad (3.12)$$

$$v_n (x,0) = x^n/n! \qquad v_n (0,t) = \begin{cases} h_{n/2} (t) & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \qquad (3.13)$$

where h_γ and h_γ^* denote the jump functions

$$h_\gamma(x) = \begin{cases} x^\gamma/\gamma! & x > 0 \\ 0 & x < 0 \end{cases}$$

and

$$h^*_\gamma(x) = h_\gamma(-x).$$

(The reciprocal of the factorial function is defined for all $\gamma \in R$ by

$$\frac{1}{\gamma!} = \frac{1}{\Gamma(\gamma+1)}$$

where Γ denotes the usual Gamma function.)

From the discussion in Reference 1 it is known that for each $\gamma \in R$ there exist a constant $K_\gamma > 0$ such that

$$|H_\gamma(x,t)| \le K_\gamma\,[h_\gamma(x) + \sqrt{t}^{\,\gamma}]. \tag{3.14}$$

Thus it is clear that the functions $H_\gamma(x,t)$, $H^*_\gamma(x,t)$ and $v_n(x,t)$ are of class S_h (with respect to x), for each fixed $t > 0$, and their transformations are well defined. For convenience we shall use the notation

$$Z_\gamma = T_h\, H_\gamma \tag{3.15}$$

$$Z^\#_\gamma = T_h\, H^*_\gamma \tag{3.16}$$

(Note here that in contrast to the simple formulas (2.16) and (2.20) the definition of T_h (2.13) implies in general that

$$T_h\,[w(-x)] \neq [T_h\, w](-x).$$

Thus the transformation of H^*_γ is <u>not</u> given by $Z^*_\gamma(x,t) = Z_\gamma(-x,t)$.) Using differentiation formulas (3.7) - (3.9) and the bound (3.14) to show uniform convergence of the defining integral we arrive at the formulas

$$\frac{\partial}{\partial x} Z_\gamma = Z_{\gamma-1} \qquad \frac{\partial}{\partial t} Z_\gamma = Z_{\gamma-2} \tag{3.17}$$

$$\frac{\partial}{\partial x} Z^\#_\gamma = -\,Z^\#_{\gamma-1} \qquad \frac{\partial}{\partial t} Z_\gamma = Z^\#_{\gamma-2} \tag{3.18}$$

for $t > 0$. These, in turn, imply that Z_γ and $Z^\#_\gamma$ are solutions of the heat equation (3.10) for any $\gamma \in R$. Applying T_h to the initial values (3.11) and (3.12) we have

$$Z_\gamma(x,0) = h \int_x^\infty e^{h(x,s)} \frac{s^\gamma}{\gamma!} ds \tag{3.19}$$

and

$$Z^*_\gamma(x,0) = 0 \tag{3.20}$$

for $x > 0$. The former expression can be explicitly evaluated when γ is an integer by

$$Z_n(x,0) = \begin{cases} h^n \sum_{k=0}^{n} (hx)^k/k! & n = 0,1,2,\ldots \\ 0 & n = -1,-2,\ldots \end{cases} \tag{3.21}$$

It would be useful to obtain explicit functional forms for these transformations and for integer values of γ this is possible, as we now show. First consider $\gamma = -1$, which may be regarded as a fundamental case, since H_{-1} is the fundamental solution of the heat equation. The functions H_{-1} and H^*_{-1} are identical and their transformation is given by

$$Z_{-1}(x,t) = Z^*_{-1}(x,t) = \frac{h}{2} \operatorname{erfc}\left(\frac{x+2ht}{\sqrt{4t}}\right) \exp(hx + h^2 t) \tag{3.22}$$

or, alternatively,

$$Z_{-1}(x,t) = h H_{-1}(x,t) \frac{H^*_0(x+2ht,\ t)}{H_{-1}(x+2ht,\ t)} \tag{3.23}$$

for $t > 0$. To verify this note that the former expression may be differentiated to give

$$\frac{\partial}{\partial x} Z_{-1}(x,t) = (-h/\sqrt{4\pi t}) \exp[-(x+2ht)^2/4t] \exp[hx+h^2 t]$$

$$+ \frac{h^2}{2} \operatorname{erfc}[(x+2ht)/\sqrt{4t}] \exp[hx+h^2 t]$$

$$= (-h/\sqrt{4\pi t}) \exp[-(x^2/4t)] + h\, Z_{-1}(x,t)$$

$$= -h\,[H_{-1} - Z_{-1}]\,(x,t).$$

Thus the function Z_{-1} defined by (3.22) clearly satisfies

$$L_h Z_{-1} = H_{-1}.$$

Note furthermore that (3.22) may be written as

$$Z_{-1}(x,t) = \frac{h}{2} \operatorname{erfc}(z) \exp(z^2) \exp(-x^2/4t)/\sqrt{4\pi t}$$

where

$$z = (x+2ht)/\sqrt{4t}.$$

For large z the standard asymptotic formula

$$\operatorname{erfc}(z) \exp(z^2) \sim (\sqrt{\pi}\, z)^{-1}$$

then implies that as $x \to \infty$ with $t > 0$ fixed we have

$$Z_{-1}(x,t) \sim \frac{\exp[-x^2/4t]}{\pi(x+2ht)}$$

which vanishes exponentially. It follows that the expression defined by (3.22) is of class S_h and must indeed be the transform of H_{-1}.

We are now able to determine functional forms for Z_n and $Z_n^{\#}$ for other integer values of n with the help of a useful recurrence relation. Using (3.7) and the definition of L_h we have

$$L_h H_n = H_n - \frac{1}{h} H_{n-1}.$$

But then, applying T_h to both sides yields

$$H_n = T_h H_n - \frac{1}{h} T_h H_{n-1}$$

or

$$Z_n = \frac{1}{h} Z_{n-1} + H_n \ . \tag{3.24}$$

This permits us to express Z_n in terms of Z_{-1} and the functions H_k with k between -1 and n:

$$\begin{aligned}
Z_2 &= h^{-3} Z_{-1} + h^{-2} H_o + h^{-1} H_1 + H_2 \\
Z_1 &= h^{-2} Z_{-1} + h^{-1} H_o + H_1 \\
Z_o &= h^{-1} Z_{-1} + H_o \\
Z_{-1} &= Z_{-1} \qquad\qquad (3.25) \\
Z_{-2} &= h Z_{-1} - h H_{-1} \\
Z_{-3} &= h^2 Z_{-1} - h^2 H_{-1} - h H_{-2} \\
Z_{-4} &= h^3 Z_{-1} - h^3 H_{-1} - h^2 H_{-2} - h^3 H_{-3}.
\end{aligned}$$

In general, using induction, we have

$$Z_n = h^{-n-1} Z_{-1} + \begin{cases} \sum_{k=o}^{n} h^{k-n} H_k & n > 0 \\ 0 & n = -1 \\ -\sum_{k=1}^{-n-1} h^{-n-k} H_{-k} & n < -2 \end{cases} \tag{3.26}$$

Similar for $Z_n^{\#}$ we obtain the induction formula

$$Z_n^{\#} = H_n^{\bullet} - \frac{1}{h} Z_{n-1}^{\#}$$

whence

$$\begin{aligned}
Z_2^{\#} &= -h^{-3} Z_{-1} + h^{-2} H_o^{\bullet} - h^{-1} H_1^{\bullet} + H_2^{\bullet} \\
Z_1^{\#} &= h^{-2} Z_{-1} - h^{-1} H_o^{\bullet} + H_1^{\bullet} \\
Z_o^{\#} &= -h^{-1} Z_{-1} + H_o^{\bullet} \\
Z_{-1}^{\#} &= Z_{-1} \\
Z_{-2}^{\#} &= -h Z_{-1} + h H_{-1}^{\bullet} \\
Z_{-3}^{\#} &= h^2 Z_{-1} - h^3 H_{-1}^{\bullet} + h H_{-2}^{\bullet} \\
Z_{-4}^{\#} &= -h^3 Z_{-1} + h^3 H_{-1}^{\bullet} - h^2 H_{-2}^{\bullet} + h H_{-3}^{\bullet}
\end{aligned} \tag{3.27}$$

and, in general,

$$Z_n^{\#} = (-h)^{-n-1} Z_{-1} + \begin{cases} \sum_{k=0}^{n} (-h)^{k-n} H_k^{\bullet} & n > 0 \\ 0 & n = -1 \\ -\sum_{k=1}^{-n-1} (-h)^{-n-k} H_{-k}^{\bullet} & n < -2 \end{cases} \tag{3.28}$$

Comparing (3.26) and (3.28) for negative integers we have

$$Z_n^{\#} = (-1)^{n+1} Z_n \tag{3.29}$$

for $n = -1, -2, -3, \ldots$. This is to be expected in view of (3.6).

Transformation of the heat polynomials can now be easily accomplished. From (3.5), (3.26) and (3.28)

$$T_h\, v_n = T_h\,[H_n + (-1)^n H_n^*]$$

$$= h^{-n-1} Z_{-1} + \sum_{k=o}^{n} h^{k-n} H_k$$

$$- h^{-n-1} Z_{-1} + (-1)^k h^{k-n} H_k^*$$

$$= \sum_{k=o}^{n} h^{k-n} [H_k + (-1)^k H_k^*].$$

Thus,

$$T_h\, v_n = \sum_{k=o}^{n} h^{k-n} v_k\ . \tag{3.30}$$

A closed form expression such as (3.22) has not been obtained for Z_γ or $Z_\gamma^\#$ when γ is not an integer. However an alternative series representation for $Z_\gamma^\#$ can be used in such cases:

$$Z_\gamma^\# = - \sum_{k=1}^{\infty} (-h)^k H_{\gamma+k}^* (k,t).$$

Using inequality (3.14) it is easy to show that for any $\gamma \in R$ this series is uniformly convergent in compact subsets of $x > 0$, $t \geq 0$. Thus from (3.7) we see that term by term differentiation is possible and that the series satisfies formulas (3.17), the heat equation (3.10) and

$$L_h\, Z_\gamma^\# = H_\gamma^*$$

for each $\gamma \in R$. Along $x = 0$ this series takes on the values

$$Z_\gamma^\# (0,t) = - \frac{\sqrt{t}^{\,\gamma}}{2} \sum_{k=1}^{\infty} \frac{(-h\,\sqrt{t})^k}{((\gamma+k)/2)!}$$

whereas along $t = 0$ it vanishes for $x > 0$, as it must according to (3.20).

The analogous series for Z_γ would seem to be

$$\sum_{k=1}^{\infty} h^k H_{\gamma+k}(x,t)$$

which converges and satisfies the appropriate equations. However, using (3.11) this series can be shown to grow exponentially as $x \to \infty$ and thus cannot be of class S_h, even though the individual terms $H_{\gamma+k}$ belong.

In conclusion we note that although our interest has been in the region $x \geq 0$, $t \geq 0$ the functions Z_γ and Z_γ^* can be defined in the domain $x < 0$, $t > 0$ as well, and in some cases along $t = 0$ also.

IV. SOLUTION OF BOUNDARY VALUE PROBLEMS

Let us now apply the results of Section III to problem (2.1) - (2.3). Substituting (3.15) into (2.17) and (2.18) we have

$$u = u_f + u_g \tag{4.1}$$

with

$$u_f(x,t) = \int_0^\infty [H_{-1}(x-s,at) - H_{-1}(x+s,at) - \frac{2}{h} Z_{-2}(x+s,at)]\, f(s)\, ds \tag{4.2}$$

$$u_g(x,t) = -2a \int_0^t Z_{-2}(x,a(t-r))\, g(r)\, dr \tag{4.3}$$

where Z_{-1} is given by (3.22) and

$$Z_{-2} = h\,[Z_{-1} - H_{-1}].$$

These formulas represent the general solution for problem (2.1) - (2.3).

Next we may consider the specific choices of boundary and initial data

$$f(x) = h_\gamma(x-x_0) \tag{4.4}$$

$$g(t) = h_{\gamma/2}(t) \ . \tag{4.5}$$

The solutions u_f and u_g for these cases can be obtained more easily by first solving for $v_f = L_h u_f$ ang $v_g = L_h u_g$ and transforming by T_h than by using formulas (4.2) and (4.3) directly. These functions are required to satisfy Equation (2.5) with

$$v_f(x,0) = h_\gamma(x-x_o) - \frac{1}{h} h_{\gamma-1}(x-x_o)$$

$$v_f(0,t) = 0$$

and

$$v_g(x,0) = 0$$

$$v_g(0,t) = \frac{\sqrt{t}^\gamma}{(\gamma/2)!} .$$

But in previous work[2] we have seen that these problems have the solutions

$$v_f(x,t) = H_\gamma(x-x_o, at) - H^\bullet_\gamma(x+x_o, at)$$

$$- \frac{1}{h} H_{\gamma-1}(x-x_o, at) + \frac{1}{h} H^\bullet_{\gamma-1}(x+x_o, at)$$

and

$$v_g(x,t) = 2H^\bullet_\gamma(x, at).$$

(These can be directly verified using (3.11) and (3.12).) By applying the transform T_h to v_f and v_g we obtain

$$u_f(x,t) = [Z_\gamma - \frac{1}{h} Z_{\gamma-1}](x-x_o,at) - [Z^\#_\gamma - \frac{1}{h} Z^\#_{\gamma-1}](x+x_o,at)$$

$$= [L_h Z_\gamma](x-x_o,at) - [L_h Z^\#_\gamma](x+x_o,at)$$

$$+ \frac{2}{h} Z^\#_{\gamma-1}(x+x_o,at)$$

or

$$u_f(x,t) = H_\gamma(x-x_o,at) - H^\bullet_\gamma(x+x_o,at) + \frac{2}{h} Z^\#_{\gamma-1}(x+x_o,at) \qquad (4.6)$$

and

$$u_g(x,t) = 2\, Z_\gamma^{\#}(x,at). \tag{4.7}$$

We can illustrate our procedures by giving as examples the solutions to the following two important particular choices for f(x) and g(t):

$$\begin{matrix} f(x) = 1 \\ g(t) = 0 \end{matrix} \quad \text{and} \quad \begin{matrix} f(x) = 0 \\ g(t) = 1. \end{matrix}$$

From (4.6) and (4.7) these are seen to have the respective solutions

$$u_f(x,t) = H_0(x,at) - H_0^{*}(x,at) + \frac{2}{h} Z_{-1}(x,at)$$

$$u_g(x,t) = 2\, Z_0^{\#}(x,at).$$

Using (3.22) and (3.27) to expand these yields

$$u_f(x,t) = \frac{1}{2}\,\mathrm{erfc}(-x/\sqrt{4at}) - \frac{1}{2}\,\mathrm{erfc}(x/\sqrt{4at}) + \frac{2}{h} Z_{-1}(x,at)$$

$$= \mathrm{erf}\,(x/\sqrt{4at}) + \mathrm{erfc}\left(\frac{x+2hat}{\sqrt{4at}}\right) \exp(hx + h^2at)$$

$$u_g(x,t) = \mathrm{erfc}(x/\sqrt{4at}) - \mathrm{erfc}\left(\frac{x+2hat}{\sqrt{4at}}\right) \exp(hx + h^2at).$$

In this form these solutions are seen to be identical with the results of Carslaw and Jaeger[4].

V. CONCLUSION

The transformation T_h defined by (2.13) was introduced in order to convert solutions of the heat equation with known boundary values into solutions satisfying the convective heat transfer condition (1.2) along $x = 0$. In particular the functions H_γ and H_γ^{*} were transformed into functions Z_γ and $Z_\gamma^{\#}$. A general solution (4.1) - (4.3) was found for the convective heat transfer problem (2.1) - (2.3) and specific solutions were found for particular choices (4.4) and (4.5) of the boundary and initial data.

[4] *H. S. Carslaw and J. C. Jaeger, "Conduction of Heat in Solids," Oxford Clarendon Press, 1959, 2nd Addition (pp 70-72).*

The basic problem (2.1) - (2.3) studied in this report was posed for the semi-infinite domain $x > 0$, $t > 0$. However all of the functions encountered were extendable into $x < 0$, $t > 0$ and thus can be used in a more general context. They can also be used to develop approximate solutions for more general equations in bounded spatial domains. In a subsequent report we shall use these functions to construct a numerical algorithm which models the heat transfer occuring at the bore surface of a gun barrel.

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nd boundary values

$$H_\gamma\ (0,t) = \frac{1}{2}\, h_{\gamma/2}(t) \qquad (3.11)$$

$$H^*_\gamma\ (0,t) = \frac{1}{2}\, h_{\gamma/2}\ (t) \qquad (3.12)$$

$$v_n\ (0,t) = \begin{cases} h_{n/2}\ (t) & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \qquad (3.13)$$

ctions

$x > 0$

$x < 0$

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