AD-A076 176 ARMY ARMAMENT RESEARCH AND DEVELOPMENT COMMAND WATER--ETC F/G 12/1
A NUMERICAL COMPARISON BETWEEN TWO UNCONSTRAINED VARIATIONAL FO--ETC(U)
SEP 79 J J WU , T E SIMKINS
UNCLASSIFIED ARLCB-TR-79024 SBIE-AD-E440 050 NL

1 OF 1
AD
A076176

END
DATE
FILMED
11-79
DDC

TECHNICAL REPORT ARLCB-TR-79024

A NUMERICAL COMPARISON BETWEEN TWO UNCONSTRAINED VARIATIONAL FORMULATIONS

J.J. Wu
T.E. Simkins

September 1979

US ARMY ARMAMENT RESEARCH AND DEVELOPMENT COMMAND
LARGE CALIBER WEAPON SYSTEMS LABORATORY
BENÉT WEAPONS LABORATORY
WATERVLIET, N. Y. 12189

AMCMS No. 6111.0191A011

PRON No. 1A924225GGGG

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED

DISCLAIMER

The findings in this report are not to be construed as an official Department of the Army position unless so designated by other authorized documents.

The use of trade name(s) and/or manufacturer(s) does not constitute an official indorsement or approval.

DISPOSITION

Destroy this report when it is no longer needed. Do not return it to the originator.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: (14) ARLCB-TR-79024	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle): (6) A NUMERICAL COMPARISON BETWEEN TWO UNCONSTRAINED VARIATIONAL FORMULATIONS.		5. TYPE OF REPORT & PERIOD COVERED: (12) 35
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): (10) J. J. Wu, T. E. Simkins		8. CONTRACT OR GRANT NUMBER(s): (9) Technical rept.
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Benet Weapons Laboratory, Watervliet Arsenal, Watervliet, N.Y. 12189, DRDAR-LCB-TL		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: AMCMS No. 6111.0191A011, PRON No. 1A924225GGGG
11. CONTROLLING OFFICE NAME AND ADDRESS: US Army Armament Research and Development Command, Large Caliber Weapon System Laboratory, Dover, New Jersey 07801		12. REPORT DATE: (11) Sep[redacted]79
		13. NUMBER OF PAGES: 30
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report): UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

DDC RECEIVED NOV 6 1979 B

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

(18) SBIE (19) AD-E440 050

18. SUPPLEMENTARY NOTES

Presented at 25th (Silver Jubilee) Conference of Army Mathematicians, 6-8 June 79, Johns Hopkins Univ. Baltimore, Maryland.
To be published in Journal of Applied Mechanics.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Approximation Methods
Finite Element
Initial Boundary Problems
Variational Methods

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In an effort to relieve the often cumbersome burden of meeting the requirements on the end conditions and to unify the solution formulation for boundary- and initial-value problems, unconstrained variational statements have been introduced in conjunction with some approximate methods. In the case of a boundary value problem, it is shown in this paper that two different variational statements can be established: one is arrived at by the use of the (over)

DD FORM 1473 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE 410 224

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

Cont. from Block 20

LaGrange multipliers, the other by energy considerations. The numerical convergence of the solutions associated with finite element schemes using one of these two different variational statements is compared with that of the other. In the case of an initial value problem, both formulations can again be established when the adjoint field variable and the adjoint variational statement are introduced. The numerical data presented here indicate that while both methods generate excellent convergent results for the boundary value problem, the method of stiff springs yields results which show much better convergence for the initial value problem than those achieved by LaGrange multipliers.

ACCESSION for		
NTIS	White Section	☑
DDC	Buff Section	☐
UNANNOUNCED		☐
JUSTIFICATION		
BY		
DISTRIBUTION/AVAILABILITY CODES		
Dist.	AVAIL. and/or SPECIAL	
A		

CONTENTS

TABLES

TABLES CONT.

A NUMERICAL COMPARISON BETWEEN TWO UNCONSTRAINED VARIATIONAL FORMULATIONS

ABSTRACT. In an effort to relieve the often cumbersome burden of meeting the requirements on the end conditions and to unify the solution formulation for boundary- and initial-value problems, unconstrained variational statements have been introduced in conjunction with some approximate methods. In the case of a boundary value problem, it is shown in this paper that two different variational statements can be established: one is arrived at by the use of the Lagrange multipliers, the other by energy considerations. The numerical convergence of the solutions associated with finite element schemes using one of these two different variational statements is compared with that of the other. In the case of an initial value problem, both formulations can again be established when the adjoint field variable and the adjoint variational statement are introduced. The numerical data presented here indicate that while both methods generate excellent convergent results for the boundary value problem, the method of stiff springs yields results which show much better convergence for the initial value problem than those achieved by Lagrange multipliers.

I. INTRODUCTION. In conjunction with variational methods of mathematical physics, it is often burdensome to select trial functions which are required to satisfy some or all of the end conditions (see, for example, reference [1]). Efforts thus have been made to relieve such requirements on these trial functions. Courant and Hilbert have pointed out that in conjunction with boundary value problems, this can always be done by adding extra boundary terms in the variational statement [2]. Such a concept has been applied successfully by Wu in obtaining solutions to nonconservative stability problems [3]. Wu has further extended the application to the solutions of initial value problems [4]. Simkins also developed unconstrained variational statements for initial and boundary value problems [5]. The approaches used by Wu and Simkins are different in that while Wu, after Courant and Hilbert, employed the concept of a very large constant (very stiff spring constant), Simkins used the method of Lagrange multipliers. For any given problem, the variational statements arrived at by the two approaches are different in boundary terms. The purpose of this paper is to compare the numerical convergence of them in terms of some simple, but specific, examples. Both boundary and initial value problems are considered.

II. UNCONSTRAINED VARIATIONAL STATEMENTS FOR A BOUNDARY VALUE PROBLEM. Let us first consider the transverse vibrations of an Euler-Bernoulli beam under axial load. The differential equation in nondimensionalized form can be written as [1]:

$$y'''' + Qy'' + \lambda^2 y = 0 \tag{2-1}$$

where $y = y(x)$ is the transverse displacement of the beam, as a function of the variable x along the column's length ($0<x<1$). The axial force is denoted by Q; λ is the eigenvalue and a prime $(')$ denotes a differentiation with respect to x. The problem is not defined completely, of course, without appropriate boundary conditions. Consider the following given conditions:

$$y(0) = y'(0) = 0 \tag{2-2a,2b}$$

$$y''(1) = y'''(1) + Qy'(1) = 0 \tag{2-2c,2d}$$

Eqs. (2-1) and (2-2) define the familiar buckling problem of an Euler column. It can be solved by methods of approximation in conjunction with a variational statement.

$$\delta I_o = 0 \tag{2-3a}$$

where

$$I_o(y) = \frac{1}{2}\int_0^1 [(y'')^2 - Q(y')^2 + \lambda^2 y^2]dx \tag{2-3b}$$

Through integrations-by-parts, Eqs. (2-3) leads directly the following

$$\begin{aligned} \delta I_o &= 0 \\ &= \int_0^1 (y'''' + Qy'' + \lambda^2 y)\delta y dx \\ &+ y''(1)\delta y'(1) - y''(0)\delta y'(0) \\ &- [y'''(1) + Qy'(1)]\delta y(1) + [y'''(0) + Qy'(0)]\delta y(0) \end{aligned} \tag{2-4}$$

Eq. (2-4) indicates that $\delta I_o = 0$ is equivalent to the differential equation (2-1) and the last two of the b.c. Eq. (2-2c,2d) provided that the variations $\delta y(1)$ and $\delta \dot{y}(1)$ are chosen arbitrarily (thus causing their coefficients to vanish) and that $\delta y(0)$ and $\delta y'(0)$ vanish identically. Thus, $\delta I_o = 0$ can be used as a basis of approximate solution if trial functions are chosen which identically satisfy (2-2a) and (2-2b). Since (2-2a,2b) must be "imposed" they are called "imposed boundary conditions".

The choice of trial functions is otherwise arbitrary and convergence, when achieved, will tend 'naturally' toward a solution satisfying (2-2c) and (2-2d) which are called the 'natural boundary conditions' of the problem. The imposed conditions on the trial functions are often burdensome in the process of obtaining approximate solutions [1]. In this paper, two different methods are compared which remove these constraints on the trial functions.

The first approach is an extension of the method of the Lagrange multipliers in classical mechanics. Suppose one desires to unconstrain the boundary condition (2-2a) $y(0) = 0$. The modified variational statement shall take the form of

$$\delta I_1 = 0 \tag{2-5a}$$

where

$$I_1 = I_o + \alpha y(0) \tag{2-5b}$$

and I_o in (2-5b) is given by (2-3b). Eqs. (2-5) then become

$$\delta I_1 = 0 = \delta I_o + \alpha \delta y(0) = y(0)\delta\alpha \tag{2-6a}$$

$$\begin{aligned} &= \int_0^1 (y'''' + Qy'' + \lambda^2 y)\delta y dx \\ &+ y''(1)\delta y'(1) - y''(0)\delta y'(0) + y(0)\delta\alpha \\ &- [y'''(1) + Qy'(1)]\delta y(1) + [y'''(0) + Qy'(0) + \alpha]\delta y(0) \end{aligned} \tag{2-6b}$$

It is clear from Eq. (2-6b) that if one defines

$$\alpha = -[y'''(0) + Qy'(0)] \tag{2-7a}$$

thus

$$\delta\alpha = -[\delta y'''(0) + Q\delta y'(0)] \tag{2-7b}$$

equation (2-6b) becomes

$$\begin{aligned} \delta I_1 = 0 = &\int_0^1 (y'''' + Qy'' + \lambda^2 y)\delta y dx \\ &+ y''(1)\delta y(1) - [y''(0) + Qy(0)]\delta y'(0) - y(0)\delta y'''(0) \\ &- [y'''(1) + Qy'(1)]\delta y(1) \end{aligned} \tag{2-8}$$

Thus, with α given in (2-7a) and I_1 in (2-5b) the variational statement $\delta I = 0$ is equivalent to the given differential equation and the boundary conditions (2-2a), (2-2c) and (2-2d). Only (2-2b) is imposed on the set of trial functions. This last constraint condition can also be removed by the same process used above. The completely unconstrained variational statement through the means of the Lagrange multipliers is the following:

$$\delta I = 0 \tag{2-9a}$$

with

$$I = \frac{1}{2}\int_0^1 [(y'')^2 - Q(y')^2 + \lambda^2 y^2]dx - y(0)y'''(0) + y'(0)y''(0) \tag{2-9b}$$

Since then

$$\delta I = 0 = \int_0^1 (y'''' + Qy'' + \lambda^2 y)\delta y dx + y''(1)\delta y'(1) - [y'''(1) + Qy'(1)]\delta y(1) - y(0)\delta y'''(0) + y'(0)[\delta y''(0) + Q\delta y(0)] \tag{2-9c}$$

It is clear from Eq. (2-9c) that all the boundary conditions of Eq. (2-2) are natural if the variational statement of (2-9a) and (2-9b) is used.

The second approach to remove the imposed conditions may be referred to as "the method of infinitely stiff springs". The functional I_0 in Eqs. (2-3) can be identified with the nondimensionalized energy stored in the beam. If the beam is considered to be supported by two springs at $x = 0$, one reacting to deflection and the other to rotation, the energy stored in these springs can be included in the total energy of the system. Thus consider

$$\delta I = 0 \tag{2-10a}$$

where

$$I = I_0 + \frac{1}{2}k_1[y(0)]^2 + \frac{1}{2}k_2[y'(0)]^2 = \frac{1}{2}\int_0^1 [(y'')^2 - Q(y')^2 + \lambda^2 y^2]dx + \frac{1}{2}k_1[y(0)]^2 + \frac{1}{2}k_2[y'(0)]^2 \tag{2-10b}$$

where k_1 and k_2 are the nondimensionalized spring constants for deflection and rotation respectively at $x = 0$. Now since

$$\delta I = 0$$

$$= \int_0^1 (y'''' + Qy'' + \lambda^2 y)\delta y dx$$

$$+ y''(1)\delta y'(1) - [y''(0) - k_2 y'(0)]\delta y'(0)$$

$$- [y'''(1) + Qy'(1)]\delta y(1) + [y'''(0) + Qy'(0) + k_1 y(0)]\delta y(0) \quad (2\text{-}11)$$

the natural boundary conditions are

$$y'''(0) + Qy'(0) + k_1 y(0) = 0 , \quad y''(0) - k_2 y'(0) = 0 \quad (2\text{-}12a,12b)$$

$$y''(1) = 0 , \quad y'''(1) + Qy'(1) = 0 \quad (2\text{-}12c,12d)$$

It is clear that Eqs. (2-12) reduce to (2-2) if k_1 and k_2 become infinitely large. Hence, the variational statement (2-10) can serve as a basis of an approximate solution formulation for the problem defined by Eqs. (2-1) and (2-2) if k_1 and k_2 are taken to be very large compared with unity in actual computations.

III. UNCONSTRAINED VARIATIONAL STATEMENTS FOR AN INITIAL VALUE PROBLEM. In the case of initial value problems, similar procedures can be used to free the initial conditions imposed on the trial functions. Examples have been given in two previous papers [4,5]. Since initial value problems are nonself adjoint by nature, adjoint field variables must be introduced to form variational statements which provide the basis for approximate solutions. In this section Lagrange multiplier formulations will be compared with those using the method of infinitely stiff springs - each method being used to relax the requirement that trial functions satisfy identically the imposed conditions arising from an initial value problem. Forced motions of a spring-mass system is used for illustration. The differential equation for such a system can be written as

$$\ddot{y} + \omega^2 y = f(t) \quad (3\text{-}1)$$

where $y = y(t)$ is a function of the time t and a dot $(\dot{})$ denotes differentiation with respect to t. The constant $\omega^2 = k/m$ where k is the spring constant and m, the mass. The initial conditions are:

$$y(0) = a , \quad \dot{y}(0) = b \quad (3\text{-}2a,2b)$$

No generality is lost if, in establishing the corresponding variational statements, one considers only a homogeneous system. Hence we consider the differential equation:

$$\ddot{y} + \omega^2 y = 0 \tag{3-1'}$$

and initial condition

$$y(0) = 0 \quad , \quad \dot{y}(0) = 0 \tag{3-2'a,2'b}$$

The fact that the system of Eqs. (3-1') and (3-2') leads to a trivial solution only is not of concern here.

Let $z = z(t)$ be the adjoint field variable. First, the variational statement obtained by the use of Lagrange multipliers is verified to be:

$$\delta I_o = 0 \tag{3-3a}$$

where

$$I_o = -\int_0^1 \dot{y}\dot{z}dt + \omega^2\int_0^1 yzdt + \dot{y}(1)z(1) - y(0)\dot{z}(0) \tag{3-3b}$$

Eqs. (3-3) lead to

$$\begin{aligned} \delta I_o &= 0 \\ &= \int_0^1 (\ddot{y} + \omega^2 y)\delta z dt + \dot{y}(0)\delta z(0) - y(0)\delta\dot{z}(0) \\ &+ \int_0^1 (\ddot{z} + \omega^2 z)\delta y dt - \dot{z}(1)\delta y(1) + z(1)\delta\dot{y}(1) \end{aligned} \tag{3-4}$$

Eq. (3-4) states that $\delta I_o = 0$ is equivalent to the problem of Eqs. (3-1') and (3-2') and the adjoint problem defined by

$$\ddot{z} + \omega^2 z = 0 \tag{3-5}$$

and

$$z(1) = 0 \quad , \quad \dot{z}(1) = 0 \tag{3-6a,6b}$$

In as much as the variations of the field variable δy, δz, etc. are quite arbitrary and δy is quite independent of δz, one can take $\delta y = 0$, $\delta y(1) = 0$ and $\delta\dot{y}(1) = 0$. Hence the association of the problem of (3-1') and (3-2') with the variational statement Eqs. (3-3) is established.

Now for the inhomogeneous system of Eqs. (3-1) and (3-2), one may similarly verify the corresponding variations statement:

$$\delta I_1 = 0 \qquad (3\text{-}7a)$$

where

$$I_1(y,z) = -\int_0^1 \dot{y}\dot{z}dt + \int_0^1 [\omega^2 y - f(t)]zdt$$

$$+ \dot{y}(1)z(1) - [y(0) - a]\dot{z}(0) - bz(0) \qquad (3\text{-}7b)$$

On the other hand, when the "infinitely stiff spring" approach is used to treat the homogeneous case, the variational statement takes the following form [4]:

$$\delta I = 0 \qquad (3\text{-}8a)$$

where

$$I = -\int_0^1 \dot{y}\dot{z}dt + \omega^2\int_0^1 yzdt + ky(0)z(1) \qquad (3\text{-}8b)$$

Eqs. (3-8) result in

$$\delta I = 0$$

$$= \int_0^1 (\ddot{y} + \omega^2 y)\delta z dt + \dot{y}(0)\delta z(0) + [ky(0) - \dot{y}(1)]\delta z(1)$$

$$+ \int_0^1 (\ddot{z} + \omega^2 z)\delta y dt - \dot{z}(1)\delta y(1) + [kz(1) + \dot{z}(0)]\delta y(0) \qquad (3\text{-}9)$$

The differential equations for the problem and for the adjoint problem are unchanged. The end condition for the original and the adjoint problem are

$$\dot{y}(0) = 0 \quad , \quad ky(0) - \dot{y}(1) = 0 \qquad (3\text{-}10a,10b)$$

and

$$\dot{z}(1) = 0 \quad , \quad kz(1) + \dot{z}(0) = 0 \qquad (3\text{-}11a,11b)$$

respectively, Eqs. (3-10) and (3-11) reduce to (3-2') and (3-6) respectively as k becomes infinitely large.

From Eqs. (3-8), extension to a variational statement is easily made for the inhomogeneous case of Eqs. (3-1) and (3-2):

$$\delta I_1 = 0 \tag{3-12a}$$

where

$$I_1 = -\int_0^1 \dot{y}\dot{z}dt + \int_0^1 [\omega^2 y - f(t)]zdt$$

$$+ ky(0)z(1) - kaz(1) - bz(0) \tag{3-12b}$$

IV. NUMERICAL COMPARISONS. In this section, the two methods for the unconstraining of the coordinate (trial) functions described in the previous section will be compared numerically. The approximate solutions are formulated through the finite element discretizations.

IV.A. Boundary Value Problem. The example given in Section II shall be used. The set of Eqs. (2-1) and (2-2) constitute an eigenvalue problem. Using the method of Lagrange multipliers, the associated variational statement is given in Eqs. (2-9) which can also be written as

$$\delta I = 0 = \int_0^1 (y''\delta y'' - Qy'\delta y' + \lambda^2 y\delta y)dx$$

$$- y(0)\delta y'''(0) - y'''(0)\delta y(0)$$

$$+ y'(0)\delta y''(0) + y''(0)\delta y'(0) \tag{4-1}$$

In applying the standard finite element discretization the beam is divided into K equal elements. Denoting the local coordinate by ξ, one has, for the m-th element:

$$\xi = \xi^{(m)} = Kx - m + 1 \tag{4-2a}$$

$$d\xi = Kdx \tag{4-2b}$$

Thus, in terms of local variables, Eq. (4-1) becomes

$$\delta I = 0 = \sum_{m=1}^{K} \int_0^1 [K^3 y^{(m)''}\delta y^{(m)''} - QKy^{(m)'}\delta y^{(m)'} + \frac{\lambda^2}{K} y^{(m)}\delta y^{(m)}]d\xi$$

$$- K^3 y^{(1)}(0)\delta y^{(1)'''}(0) - K^3 y^{(1)'''}(0)\delta y(0)$$

$$+ K^3 y^{(1)'}(0)\delta y^{(1)''}(0) + K^3 y^{(1)''}(0)\delta y^{(1)'}(0) \tag{4-3}$$

Now, let

$$y^{(m)}(\xi) = \underset{\sim}{a}^T(\xi)\, \underset{\sim}{Y}^{(m)} \tag{4-4}$$

where

$$\underset{\sim}{a}(\xi) = \begin{Bmatrix} a_1(\xi) \\ a_2(\xi) \\ a_3(\xi) \\ a_4(\xi) \end{Bmatrix} = \begin{Bmatrix} 1 - 3\xi^2 + 2\xi^3 \\ \xi - 2\xi^2 + \xi^3 \\ 3\xi^2 - 2\xi^3 \\ -\xi^2 + \xi^3 \end{Bmatrix} \tag{4-5}$$

$$\underset{\sim}{Y}^{(m)} = \begin{Bmatrix} Y_1^{(m)} \\ Y_2^{(m)} \\ Y_3^{(m)} \\ Y_4^{(m)} \end{Bmatrix} \tag{4-6}$$

and a superscript T denotes the transpose of a matrix. Eq. (4-3) now can be written as

$$\delta I = 0 = \sum_{m=1}^{K} \delta\underset{\sim}{Y}^{(m)T}\Big[K^3\int_0^1 \underset{\sim}{a}''(\xi)\underset{\sim}{a}''^T(\xi)d\xi - QK\int_0^1 \underset{\sim}{a}'(\xi)\underset{\sim}{a}'^T(\xi)d\xi$$
$$+ \frac{\lambda^2}{K}\int_0^1 \underset{\sim}{a}(\xi)\underset{\sim}{a}^T(\xi)d\xi\Big]\underset{\sim}{Y}^{(m)}$$
$$- K^3\delta\underset{\sim}{Y}^{(1)T}\left[\underset{\sim}{a}'''(0)\underset{\sim}{a}^T(0) + \underset{\sim}{a}(0)\underset{\sim}{a}'''^T(0) - \underset{\sim}{a}''(0)\underset{\sim}{a}'^T(0) - \underset{\sim}{a}'(0)\underset{\sim}{a}''^T(0)\right]\underset{\sim}{Y}^{(1)} \tag{4-7a}$$

Or

$$\delta I = 0 = \sum_{m=1}^{K} \delta\underset{\sim}{Y}^{(m)T}\left[K^3\underset{\sim}{C} - QK\underset{\sim}{B} + \frac{\lambda^2}{K}\underset{\sim}{A}\right]\underset{\sim}{Y}^{(m)}$$
$$- K^3\delta\underset{\sim}{Y}^{(1)}\left[\underset{\sim}{B}_1 + \underset{\sim}{B}_1^T - (\underset{\sim}{B}_2 + \underset{\sim}{B}_2^T)\right]\underset{\sim}{Y}^{(1)} \tag{4-7b}$$

where

$$\underset{\sim}{A} = \int_0^1 \underset{\sim}{a}\underset{\sim}{a}^T d\xi, \quad \underset{\sim}{B} = \int_0^1 \underset{\sim}{a}'\underset{\sim}{a}'^T d\xi, \quad \underset{\sim}{C} = \int_0^1 \underset{\sim}{a}''\underset{\sim}{a}''^T d\xi \tag{4-8a}$$

$$\underset{\sim}{B}_1 = \underset{\sim}{a}'''(0)\underset{\sim}{a}(0) = \begin{bmatrix} 12 \\ 6 \\ -12 \\ 6 \end{bmatrix} [1\ 0\ 0\ 0] = \begin{bmatrix} 12 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ -12 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{bmatrix} \quad (4\text{-}8b)$$

$$\underset{\sim}{B}_2 = \underset{\sim}{a}''(0)\underset{\sim}{a}'(0) = \begin{bmatrix} -6 \\ -4 \\ 6 \\ -2 \end{bmatrix} [0\ 1\ 0\ 0] = \begin{bmatrix} 0 & -6 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix} \quad (4\text{-}8c)$$

Now, Eq. (4-7) can be assembled into a global matrix equation

$$\delta I - \delta \underset{\sim}{Y}^T[\underset{\sim}{K} + \lambda^2 \underset{\sim}{M}]\underset{\sim}{Y} = 0 \quad (4\text{-}9)$$

where

$$\underset{\sim}{Y}^T = [Y_1^{(1)}\ Y_2^{(1)}\ Y_3^{(1)}\ Y_4^{(1)}\ Y_3^{(2)}\ Y_4^{(2)} \ldots\ldots Y_3^{(K)}\ Y_4^{(K)}] \quad (4\text{-}10)$$

The details of obtaining the global matrices K and M have been given elsewhere [1] and will not be repeated here.

Since $\delta\underset{\sim}{Y}$ in (4-9) is unconstrained, the equation reduces to

$$(\underset{\sim}{K} + \lambda^2 \underset{\sim}{M})\underset{\sim}{Y} = 0 \quad (4\text{-}11)$$

which will be solved for the eigenvalues λ^2.

When the method of infinitely stiff springs is used, the variational statement is given by Eqs. (2-10), which can also be written as

$$\delta I = 0 = \int_0^1 (y''\delta y'' - Qy'\delta y' + \lambda^2 y \delta y)dx$$

$$+ k_1 y(0)\delta y(0) + k_2 y'(0)\delta y'(0) \quad (4\text{-}12a)$$

$$= \sum_{m=1}^{K} \int_0^1 (K^3 y^{(m)''}\delta y^{(m)''} - QK y^{(m)'}\delta y^{(m)'} + \frac{\lambda^2}{K} y^{(m)}\delta y^{(m)})d\xi$$

$$+ k_1 y^{(1)}(0)\delta y^{(1)}(0) + k_2 K^2 y^{(1)'}(0)\delta y^{(1)'}(0) \quad (4\text{-}12b)$$

Or,

$$\delta I = 0 = \sum_{m=1}^{K} \delta Y^{(m)T} [K^3 C - QKB + \frac{\lambda^2}{K} A] Y^{(m)}$$

$$+ \delta Y^{(1)T} [k_1 B_3 + k_2 K^2 B_4] Y^{(1)} \tag{4-13}$$

where

$$B_3 = a(0) a^T(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4-14a}$$

$$B_4 = a'(0) a'^T(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4-14b}$$

As before, Eq. (4-13) can be assembled into a global equation

$$\delta I = 0 = \delta Y^T (K + \lambda^2 M) Y \tag{4-15}$$

so that the eigenvalue λ^2 can be solved from

$$(K + \lambda^2 M) Y = 0 \tag{4-16}$$

Numerical data for the vibration frequencies of a cantilevered column are given in Tables I and II for both the method of Lagrange multipliers and the method of infinitely stiff springs. As shown in these Tables, both methods display excellent convergence.

In the case of the stiff spring method, Tables I and II also indicate that the greater values of k_1 and k_2 may not give more accurate results, although all the results are good when k_1 and k_2 are sufficiently large. This point is further demonstrated by the computations shown in Table III. Since greater values of k_1 and k_2 mean that the prescribed end conditions are more accurately satisfied, Table III suggests that forcing the solution to greater accuracy at one point may cause a decline in overall acceptability of the results as evidenced by the declining accuracy of the eigenvalue. This same conclusion was first presented in [1].

TABLE I. NUMERICAL COMPARISONS OF TWO UNCONSTRAINED VARIATIONAL FORMULATIONS

The First Eigenvalue of a Cantilevered Beam

No. of Elements	Method of Lagrange Multipliers	Stiff Spring Method $k_1 = k_2 = 10^8$	Stiff Spring Method $k_1 = k_2 = 10^{12}$
1	3.585387	3.532731	3.534027
3	3.516379	3.516371	3.515999
5	3.516063	3.516063	3.514741
7	3.516028	3.516027	3.516168
9	3.516020	3.516022	3.516549

From the exact solution: 3.516015.....

TABLE II. NUMERICAL COMPARISONS OF TWO UNCONSTRAINED VARIATIONAL FORMULATIONS

The Second Eigenvalue of a Cantilevered

No. of Elements	Method of Lagrange Multipliers	Stiff Spring Method $k_1 = k_2 = 10^8$	Stiff Spring Method $k_1 = k_2 = 10^{12}$
1	47.91346	34.80686	34.80688
3	22.13741	22.10685	22.10853
5	22.04607	22.04550	22.04783
7	22.03750	22.03746	22.05306
9	22.03560	22.03559	22.09871

From the exact solution: 22.03449.....

TABLE III. EFFECT OF THE MAGNITUDE OF THE "SPRING CONSTANTS" ON CONVERGENCE

The First Two Eigenvalues of a Cantilevered Beam

No. of Elements = 9

$k_1 = k_2 =$	10^4	10^6	10^8	10^{10}	10^{12}
λ_1	3.513985	3.516000	3.516022	3.516008	3.516549
λ_2	21.930259	22.034547	22.035588	22.035538	22.098706

Exact values: $\lambda_1 = 3.516015.....$

$\lambda_2 = 22.034491....$

IV.B. An Initial Value Problem. For our numerical comparisons in the case of an initial value problem, we shall consider the one defined by:

$$\text{D.E.:} \qquad m\ddot{y} + ky = f_o \cos\omega_f t, \quad 0 \le t \le T \tag{4-17}$$

$$\text{I.C.:} \qquad y(0) = a \;, \quad \dot{y}(0) = b \tag{4-18a,18b}$$

The specific values of the constants m, k, fo, ω_f, a, b and T will be given later. The upper limit of the time interval T can take any positive value other than infinity. Before one applies the variational formulation given in Section III, it will be convenient to normalize the time variable t with respect to T. Thus let

$$\tau = t/T \;, \quad t = T\tau, \; dt = Td\tau \tag{4-19}$$

$$y(t) = \bar{y}(\tau) \;, \quad \frac{dy}{dt} = \frac{1}{T}\frac{d\bar{y}}{d\tau} \;, \quad \frac{d^2y}{dt^2} = \frac{1}{T^2}\frac{d^2\bar{y}}{d\tau^2} \tag{4-20}$$

Also define

$$\bar{\omega} = \omega T \;, \quad \bar{f} = \frac{f_o T^2}{m}$$

$$\bar{\omega}_f = \omega_f T \;, \quad \bar{a} = a, \quad \bar{b} = bT \tag{4-21}$$

With these new parameters, Eqs. (4-17) and (4-18) become

$$\text{D.E.} \qquad \frac{d^2\bar{y}}{d\tau^2} + \bar{\omega}^2\bar{y} = \bar{f}\cos(\bar{\omega}_f\tau) \;, \quad 0 \le \tau \le 1 \tag{4-22}$$

$$\text{I.C.} \qquad \bar{y}(0) = \bar{a} \;, \quad \dot{\bar{y}}(0) = \bar{b} \tag{4-23a,23b}$$

Now we are ready to apply the formulations given in Section III. We shall first consider the solution formulation by the method of Lagrange multipliers. Comparing Eqs. (4-22) and (4-23) with (3-1) and (3-2), one observes that the variational statement follows that of Eqs. (3-7). Or,

$$\delta I = 0 \tag{4-24a}$$

where

$$I = -\int_0^1 \dot{\bar{y}}\,\dot{\bar{z}}\,d\tau + \int_0^1 [\bar{\omega}^2\bar{y} - \bar{f}\cos(\bar{\omega}_f\tau)]\bar{z}\,d\tau$$

$$+ \dot{y}(1)z(1) - \bar{y}(0)\dot{\bar{z}}(0) + \bar{a}\,\dot{\bar{z}}(0) - \bar{b}\,\bar{z}\,(0) \tag{4-24b}$$

Since δy and δz are quite independent of each other, one can set $\delta y = 0$ in Eqs. (4-24) and obtain

$$(\delta I)_{\delta y=0} = -\int_0^1 \dot{\bar{y}}\delta\dot{\bar{z}}d\tau + \int_0^1 \bar{\omega}^2\bar{y}\delta\bar{z}d\tau - \int_0^1 \bar{f}\cos(\bar{\omega}_f\tau)d\bar{z}d\tau$$

$$+ \dot{\bar{y}}(1)\delta\bar{z}(1) - \bar{y}(0)\delta\dot{\bar{z}}(0) + a\delta\dot{\bar{z}}(0) - b\delta\bar{z}(0) = 0 \qquad (4\text{-}25)$$

The same process of finite element discretization used for the boundary value problem in the previous subsection can be employed here. The same shape functions and generalized coordinates are also used. In terms of the element variables, ξ, defined before, except now that

$$\xi = k\tau - m + 1 \qquad (4\text{-}26)$$

etc., Eq. (4-25) becomes:

$$(\delta I)_{\delta y=0} = 0 = \sum_{m=1}^{K} \delta \underset{\sim}{z}^{(m)^T}[-K\int_0^1 \underset{\sim}{a}'\underset{\sim}{a}'^T d\xi + \frac{\bar{\omega}^2}{K}\int_0^1 \underset{\sim}{a}\underset{\sim}{a}^T d\xi]\underset{\sim}{Y}^{(m)}$$

$$+ \delta\underset{\sim}{z}^{(K)^T}K\underset{\sim}{a}(1)\underset{\sim}{a}'^T(1)\underset{\sim}{Y}^{(K)} - \delta\underset{\sim}{z}^{(1)^T}K\underset{\sim}{a}'(0)\underset{\sim}{a}^T(0)\underset{\sim}{Y}^{(1)}$$

$$- \sum_{m=1}^{K}\delta\underset{\sim}{z}^{(m)^T}\frac{\bar{f}}{K}\int_0^1 \cos[\frac{\bar{\omega}_f}{K}(\xi + m - 1)]\underset{\sim}{a}\,d\xi$$

$$+ \delta\underset{\sim}{z}^{(1)^T}\bar{a}K\underset{\sim}{a}'(0) - \delta\underset{\sim}{z}^{(1)^T}\bar{b}\underset{\sim}{a}(0) \qquad (4\text{-}27)$$

Or,

$$\sum_{m=1}^{K}\delta\underset{\sim}{z}^{(m)^T}[-K\underset{\sim}{B} + \frac{\bar{\omega}^2}{K}\underset{\sim}{A}]\underset{\sim}{Y}^{(m)} + \delta\underset{\sim}{z}^{(K)^T}K\underset{\sim}{B}_5\underset{\sim}{Y}^{(K)} - \delta\underset{\sim}{z}^{(1)^T}K\underset{\sim}{B}_6\underset{\sim}{Y}^{(1)}$$

$$- \sum_{m=1}^{K}\delta\underset{\sim}{z}^{(m)^T}\frac{\bar{f}}{K}\underset{\sim}{F}^{(m)} + \delta\underset{\sim}{z}^{(1)^T}[\bar{a}k\underset{\sim}{a}'(0) - \bar{b}\underset{\sim}{a}(0)] = 0 \qquad (4\text{-}28)$$

where $\underset{\sim}{A}$, $\underset{\sim}{B}$ have been defined in Eqs. (4-8a) and

$$B_5 = a(1)a'^T(1) = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}[0\ 0\ 0\ 1] = \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix} \qquad (4\text{-}29a)$$

$$B_6 = a'(0)a^T(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} [1\ 0\ 0\ 0] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (4\text{-}29b)$$

$$\underset{\sim}{F}^{(m)} = \int_0^1 \cos\left[\frac{\bar{\omega}_f}{K} (\xi + m - 1)\underset{\sim}{a}\ d\xi \right. \quad (4\text{-}29c)$$

In terms of global generalized coordinates $\underset{\sim}{Y}$ and $\underset{\sim}{Z}$ defined by

$$\underset{\sim}{Y}^T = [Y_1^{(1)}\ Y_2^{(1)}\ Y_3^{(1)}\ Y_4^{(1)}\ Y_3^{(2)}\ Y_4^{(2)} \ldots\ldots Y_3^{(K)}\ Y_4^{(K)}] \quad (4\text{-}30a)$$

and

$$\underset{\sim}{Z}^T = [Z_1^{(1)}\ Z_2^{(1)}\ Z_3^{(1)}\ Z_4^{(1)}\ Z_3^{(2)}\ Z_4^{(2)} \ldots\ldots Z_3^{(K)}\ Z_4^{(K)}] \quad (4\text{-}30b)$$

Eq. (4-28) can be assembled as before into the matrix equation

$$\delta \underset{\sim}{Z}^T [\underset{\sim\sim}{K}\underset{\sim}{Y} - \underset{\sim}{F}] = 0 \quad (4\text{-}31)$$

Or, since $\delta \underset{\sim}{Z}$ is not constrained in any way,

$$\underset{\sim\sim}{K}\underset{\sim}{Y} = \underset{\sim}{F} \quad (4\text{-}32)$$

which can be solved for $\underset{\sim}{Y}$.

When the method of infinitely stiff springs is used, the variational statement must be modified according to Eqs. (3-12). Thus, the finite element discretization begins with

$$(\delta I)_{\delta y=0} = 0$$

$$= - \int_0^1 \dot{\bar{y}}\delta\dot{\bar{z}}d\tau + \int_0^1 [\bar{\omega}^2\bar{y} - \bar{F}\cos(\bar{\omega}_f\tau)]\bar{z}\ d\tau$$

$$+ ky(0)z(1) - k\bar{a}z(1) - \bar{b}z(0) \quad (4\text{-}33)$$

Hence,

$$\sum_{m=1}^{K} \delta Z^{(m)^T} [-K\int_0^1 a'a'^T d\xi + \frac{\omega^2}{K} \int_0^1 aa^T d\xi] Y^{(m)}$$

$$+ k\delta Z^{(K)^T} a(1) a^T(0) Y^{(1)}$$

$$- \sum_{m=1}^{K} \delta Z^{(m)^T} \frac{\bar{f}}{K} \int_0^1 \cos[\frac{\omega_f}{K}(\xi + m - 1)] a \, d\xi$$

$$- k\delta Z^{(K)^T} \bar{a} a(1) - \delta Z^{(1)^T} \bar{b} a(0) \quad (4\text{-}34)$$

Or,

$$\sum_{m=1}^{K} \delta Z^{(m)^T} [-KB + \frac{\bar{\omega}^2}{K} A] Y^{(m)} + \delta Z^{(K)^T} k B_7 Y^{(1)}$$

$$- \sum_{m=1}^{K} \delta Z^{(m)^T} \frac{\bar{f}}{K} F^{(m)} - \delta Z^{(K)^T} k\bar{a} a(1) - \delta Z^{(1)^T} \bar{b} a(0) \quad (4\text{-}35)$$

where A, B, $F^{(m)}$ have all been defined before and

$$B_7 = a(1) a^T(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} [1\ 0\ 0\ 0] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (4\text{-}36)$$

Now, as with Eq. (4-28), here Eq. (4-35) can be assembled in a global equation in the form of Eqs. (4-31) and (4-32) and be solved.

The specific problem considered is as follows:

$$my + k\ddot{y} = fo \cos(\omega_f t) , \quad 0 \le t \le T$$

with

$$y(0) = y_o \text{ and } \dot{y}(0) = y_o$$

The numerical values of the parameters are:

$$m = 1.0, \quad k = 1.0, \quad f_o = 1.0, \quad \omega_f = 0.5$$

$$y_o = 1.0, \quad y_1 = 1.0$$

The plot for the forcing function $f_0 \cos(\omega_f t)$ and the exact solution $y(t)$ is shown in Figure 1. The numerical solutions of the problem using both the method of Lagrange multipliers and the method of stiff springs are given in Tables IV through IX.

Tables IV through VI show the stiff spring method generates excellent convergent results for various lengths of intervals of solution.

The results using the method of Lagrange multipliers are shown in Tables VII through IX. Table VII shows that for moderately long intervals, the convergence at the initial point is non-existent although it improves remarkably away from the initial point. This data may lead one to doubt whether the method of Lagrange multipliers works at all in treating i.v. problems. However, when the length of the interval of solution is reduced, as shown in Tables VIII and IX, it is clear that the results do converge. Hence, both methods generate convergent results. The length of interval used in the Lagrange multipliers approach is so small compared with the stiff spring method for comparable convergence that the practical value of the former is doubtful in treating initial value problems when finite element discretization is employed. Simkins [4] has shown, however, that when global approximating functions are employed, (consisting of higher ordered polynomials), very good results can be achieved over an acceptable interval of solution.

V. CONCLUSIONS. From the numerical data presented in this paper, the following conclusions are suggested:

1. Both the method of Lagrange multipliers and the method of stiff springs generate convergent results.

2. In the case of boundary value problems, both methods give excellent results and equally fast convergence. The method of stiff springs appears to be easier to use and more general in a practical sense.

3. For initial problems discretized by finite elements (piecewise continuous third order polynomials), convergence of the Lagrange multiplier method, as compared to the method of stiff springs, is so inferior as to be of dubious practical value. (This statement does not apply, however, where a global discretization is employed using higher ordered (e.g. 8th order [4]) polynomials continuous over the entire domain of integration.)

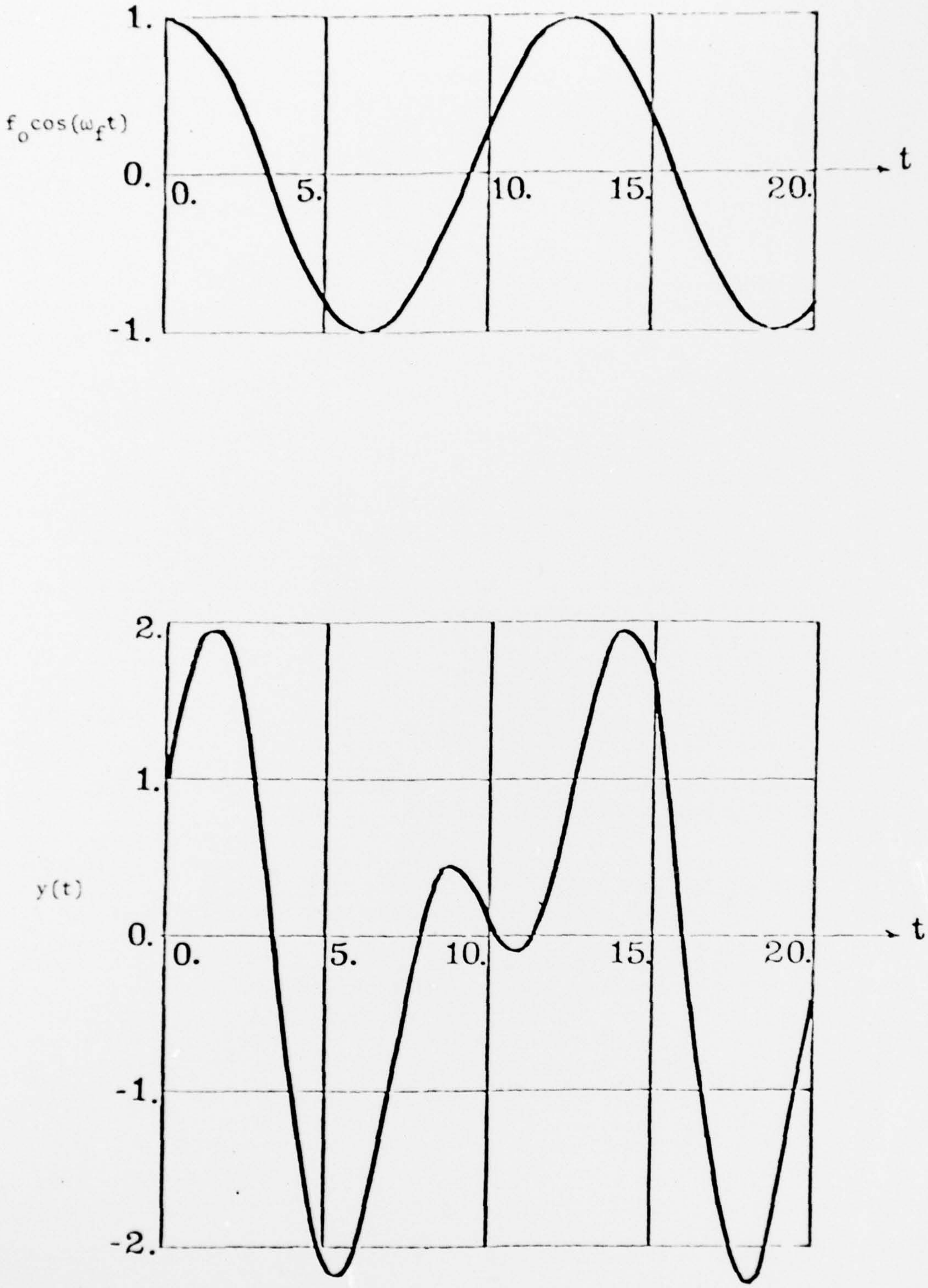

FIGURE 1. Plots for the forcing function $f_o \cos(\omega_f t)$ and the exact solution y(t) for a simple initial value problem.

TABLE IV. SOLUTIONS TO THE SIMPLE INITIAL-VALUE PROBLEM: BY METHOD OF STIFF SPRINGS

($0 \le t \le 2.0$, 10 Elements)

t	y(t)		$\dot{y}(t)$	
0.	1.0000000	(1.0000000)	1.00000	(1.00000)
0.4	1.3891537	(1.3891534)	0.91843	(0.91842)
0.8	1.7132029	(1.7132018)	0.67622	(0.67621)
1.2	1.9117024	(1.9117006)	0.29662	(0.29661)
1.6	1.9382512	(1.9382491)	-0.17424	(-0.17425)
2.0	1.7684161	(1.7684161)	-0.67413	(-0.67403)

TABLE V. SOLUTIONS TO THE SIMPLE INITIAL-VALUE PROBLEM: BY METHOD OF STIFF SPRINGS

($0 \leq t \leq 10.0$, 10 Elements)

t	y(t)		$\dot{y}(t)$	
0.	1.000	(1.000)	1.004	(1.000)
2.0	1.770	(1.768)	-0.675	(-0.674)
4.0	-1.094	(-1.094)	-1.518	(-1.512)
6.0	-1.920	(-1.919)	0.778	(0.773)
8.0	0.167	(0.166)	0.690	(0.689)
10.0	0.114	(0.114)	-0.385	(-0.381)

TABLE VI. SOLUTIONS TO THE SIMPLE INITIAL-VALUE PROBLEM: BY METHOD OF STIFF SPRINGS

($0 \le t \le 20.0$, 10 Elements)

t	y(t)	$\dot{y}(t)$
0.	1.000 (1.000)	1.05 (1.00)
4.	-1.097 (-1.094)	-1.57 (-1.51)
8.	0.173 (0.176)	0.71 (0.69)
12.	0.453 (0.462)	0.88 (0.85)
16.	-0.156 (-0.162)	-1.76 (-1.71)
20.	-0.348 (-0.342)	1.10 (-1.08)

TABLE VII. SOLUTIONS TO THE SIMPLE INITIAL-VALUE PROBLEM: BY METHOD OF LAGRANGE MULTIPLIERS

($0 \leq t \leq 2.0$, 10 Elements)

t	y(t)	$\dot{y}(t)$
0.	-54.1858 (1.0000)	2343. (1.000)
0.4	0.2348 (1.3892)	69.784 (0.918)
0.8	1.6623 (1.7132)	2.697 (0.676)
1.2	1.9058 (1.9117)	0.354 (0.297)
1.6	1.9330 (1.9382)	-0.173 (-0.174)
2.0	1.7635 (1.7684)	-0.673 (-0.674)

TABLE VIII. SOLUTIONS TO THE SIMPLE INITIAL-VALUE PROBLEM: BY METHOD OF LAGRANGE MULTIPLIERS

($0 \leq t \leq 0.5$, 10 Elements)

t	y(t)		$\dot{y}(t)$	
0.	0.95409	(1.00000)	8.79210	(1.00000)
0.1	1.09848	(1.09983)	1.22432	(0.99496)
0.2	1.19861	(1.19865)	0.98649	(0.97973)
0.3	1.29543	(1.29544)	0.95442	(0.95422)
0.4	1.38915	(1.38915)	0.91843	(0.91842)
0.5	1.47878	(1.47878)	0.87246	(0.87246)

TABLE IX. SOLUTIONS TO THE SIMPLE INITIAL-VALUE PROBLEM: BY METHOD OF LAGRANGE MULTIPLIERS

($0 \leq t \leq 0.1$, 10 Elements)

t	$y(t)$		$\dot{y}(t)$	
0	1.000049	(1.000000)	0.958591	(1.000000)
.02	1.020000	(1.019999)	0.998581	(0.999800)
.04	1.039989	(1.039989)	0.999162	(0.999197)
.06	1.059964	(1.059964)	0.998190	(0.998192)
.08	1.079914	(1.079914)	0.996780	(0.996780)
.10	1.099832	(1.099832)	0.994963	(0.994963)

REFERENCES

1. J. J. Wu, "On The Numerical Convergence of Matrix Eigenvalue Problems," Journal of Sound and Vibration (1974), 37, pp. 349-358.

2. R. Courant and D. Hilbert, Methods of Mathematical Physics, McGraw-Hill, New York (1953), see page 211.

3. J. J. Wu, "Effects of Support Flexibility on the Stability of a Beam Under a Follower Thrust and Inertia," Developments in Theoretical and Applied Mechanics (1976), Vol. 8, pp. 391-402.

4. J. J. Wu, "Solutions to Initial Value Problems by Use of Finite Elements - Unconstrained Variational Formulations," Journal of Sound and Vibration (1977), Vol. 53, pp. 341-356.

5. T. E. Simkins, "Unconstrained Variational Statements for Initial and Boundary Value Problems," American Institute fo Aeronautics and Astronautics Journal (1978), Vol. 16, pp. 559-563.

WATERVLIET ARSENAL INTERNAL DISTRIBUTION LIST

		NO. OF COPIES
COMMANDER		1
DIRECTOR, BENET WEAPONS LABORATORY		1
CHIEF, DEVELOPMENT ENGINEERING BRANCH		1
	ATTN: DRDAR-LCB-DA	1
	-DM	1
	-DP	1
	-DR	1
	-DS	1
	-DC	1
CHIEF, ENGINEERING SUPPORT BRANCH		1
CHIEF, RESEARCH BRANCH		2
	ATTN: DRDAR-LCB-RA	1
	-RC	1
	-RM	1
	-RP	1
TECHNICAL LIBRARY		5
TECHNICAL PUBLICATIONS & EDITING UNIT		2
DIRECTOR, OPERATIONS DIRECTORATE		1
DIRECTOR, PROCUREMENT DIRECTORATE		1
DIRECTOR, PRODUCT ASSURANCE DIRECTORATE		1

NOTE: PLEASE NOTIFY DIRECTOR, BENET WEAPONS LABORATORY, ATTN: DRDAR-LCB-TL, OF ANY REQUIRED CHANGES.

EXTERNAL DISTRIBUTION LIST (CONT)

	NO. OF COPIES
COMMANDER US ARMY RESEARCH OFFICE P.O. BOX 1211 RESEARCH TRIANGLE PARK, NC 27709	1
COMMANDER US ARMY HARRY DIAMOND LAB ATTN: TECH LIB 2800 POWDER MILL ROAD ADELPHIA, MD 20783	1
DIRECTOR US ARMY INDUSTRIAL BASE ENG ACT ATTN: DRXPE-MT ROCK ISLAND, IL 61201	1
CHIEF, MATERIALS BRANCH US ARMY R&S GROUP, EUR BOX 65, FPO N.Y. 09510	1
COMMANDER NAVAL SURFACE WEAPONS CEN ATTN: CHIEF, MAT SCIENCE DIV DAHLGREN, VA 22448	1
DIRECTOR US NAVAL RESEARCH LAB ATTN: DIR, MECH DIV CODE 26-27 (DOC LIB) WASHINGTON, D.C. 20375	1 1
NASA SCIENTIFIC & TECH INFO FAC P.O. BOX 8757, ATTN: ACQ BR BALTIMORE/WASHINGTON INTL AIRPORT MARYLAND 21240	1
COMMANDER DEFENSE DOCU CEN ATTN: DDC-TCA CAMERON STATION ALEXANDRIA, VA 22314	12
METALS & CERAMICS INFO CEN BATTELLE COLUMBUS LAB 505 KING AVE COLUMBUS, OHIO 43201	1
MPDC 13919 W. BAY SHORE DR. TRAVERSE CITY, MI 49684	1
MATERIEL SYSTEMS ANALYSIS ACTV ATTN: DRXSY-MP ABERDEEN PROVING GROUND MARYLAND 21005	1

NOTE: PLEASE NOTIFY COMMANDER, ARRADCOM, ATTN: BENET WEAPONS LABORATORY, DRDAR-LCB-TL, WATERVLIET ARSENAL, WATERVLIET, N.Y. 12189, OF ANY REQUIRED CHANGES.

EXTERNAL DISTRIBUTION LIST

	NO. OF COPIES
ASST SEC OF THE ARMY RESEARCH & DEVELOPMENT ATTN: DEP FOR SCI & TECH THE PENTAGON WASHINGTON, D.C. 20315	1
COMMANDER US ARMY MAT DEV & READ. COMD ATTN: DRCDE 5001 EISENHOWER AVE ALEXANDRIA, VA 22333	1
COMMANDER US ARMY ARRADCOM ATTN: DRDAR-TSS DRDAR-LCA (PLASTICS TECH EVAL CEN) DOVER, NJ 07801	 2 1
COMMANDER US ARMY ARRCOM ATTN: DRSAR-LEP-L ROCK ISLAND ARSENAL ROCK ISLAND, IL 61299	1
DIRECTOR US ARMY BALLISTIC RESEARCH LABORATORY ATTN: DRDAR-TSB-S (STINFO) ABERDEEN PROVING GROUND, MD 21005	1
COMMANDER US ARMY ELECTRONICS COMD ATTN: TECH LIB FT MONMOUTH, NJ 07703	1
COMMANDER US ARMY MOBILITY EQUIP R&D COMD ATTN: TECH LIB FT BELVOIR, VA 22060	1
COMMANDER US ARMY TANK-AUTMV R&D COMD ATTN: TECH LIB - DRDTA-UL MAT LAB - DRDTA-RK WARREN, MICHIGAN 48090	 1 1
COMMANDER US MILITARY ACADEMY ATTN: CHMN, MECH ENGR DEPT WEST POINT, NY 10996	1
COMMANDER REDSTONE ARSENAL ATTN: DRSMI-RB DRSMI-RRS DRSMI-RSM ALABAMA 35809	 2 1 1
COMMANDER ROCK ISLAND ARSENAL ATTN: SARRI-ENM (MAT SCI DIV) ROCK ISLAND, IL 61202	1
COMMANDER HQ, US ARMY AVN SCH ATTN: OFC OF THE LIBRARIAN FT RUCKER, ALABAMA 36362	1
COMMANDER US ARMY FGN SCIENCE & TECH CEN ATTN: DRXST-SD 220 7TH STREET, N.E. CHARLOTTESVILLE, VA 22901	1
COMMANDER US ARMY MATERIALS & MECHANICS RESEARCH CENTER ATTN: TECH LIB -DRXMR-PL WATERTOWN, MASS 02172	2

NOTE: PLEASE NOTIFY COMMANDER, ARRADCOM, ATTN: BENET WEAPONS LABORATORY, DRDAR-LCB-TL, WATERVLIET ARSENAL, WATERVLIET, N.Y. 12189, OF ANY REQUIRED CHANGES.