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THE S-MATRIX AND SONAR ECHO STRUCTURE.(U)  
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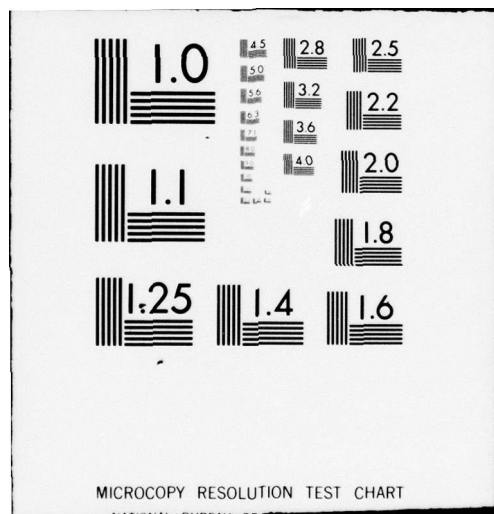
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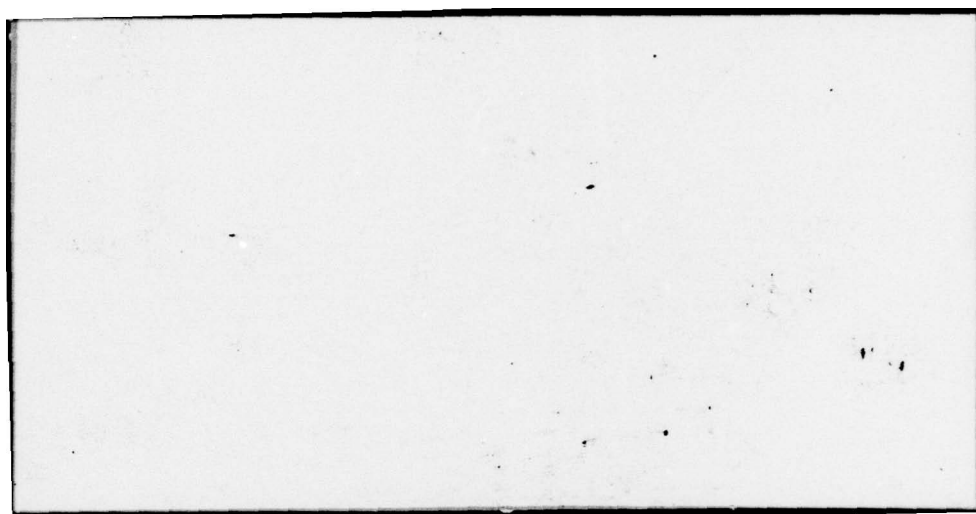
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THE S-MATRIX  
AND SONAR ECHO STRUCTURE

Calvin H. Wilcox  
Technical Summary Report #35  
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## ABSTRACT

Pulse mode sonar operation is analyzed under the physical hypotheses that

- . The medium is a stationary homogeneous fluid.
- . Both the sonar system and the scattering objects are stationary.
- . The scattering objects are rigid bodies.
- . The scattering objects lie in the far fields of the transmitter and receiver.

It is shown that if the sonar signal waveform in the far field is

$$\frac{s(|x| - t, \theta)}{|x|}, \quad x = |x|\theta$$

then the sonar echo waveform in the far field is

$$\frac{e(|x| - t, \theta)}{|x|}, \quad x = |x|\theta$$

where

$$e(\tau, \theta) = \operatorname{Re} \left\{ \frac{i}{4} \int_0^\infty \omega \exp(i\tau\omega) \int_{|\theta'|=1} T_+(\omega\theta, \omega\theta') \hat{s}(\omega, \theta') d\theta' d\omega \right\}.$$

Here  $\hat{s}(\omega, \theta)$  is the Fourier transform of  $s(\tau, \theta)$  with respect to  $\tau$  and  $T_+(\omega\theta, \omega\theta')$  is the differential scattering cross section of the scattering objects.

§1. INTRODUCTION. This paper deals with pulse mode sonar echo prediction; that is, the calculation of sonar echoes when the characteristics of the transmitter, scattering objects and ambient medium are known. The physical hypotheses are

- The medium is a stationary homogeneous unlimited fluid.
- Both the sonar system and the scattering objects are stationary.
- The scattering objects are rigid bodies.

The analysis is based on the theory of scattering for the wave equation developed in the author's monograph [5]. The principal result of this paper is an asymptotic calculation of the echo waveform, valid when both transmitter and receiver are in the far-field of the scatterers. The results show that, in this approximation, the dependence of the echo waveform on the scatterers is determined by the S-matrix of scattering theory. The work is a sequel to the author's article [6] where similar results are derived for plane-wave signals.

THE BOUNDARY VALUE PROBLEM. In what follows  $x = (x_1, x_2, x_3) \in \mathbb{R}^3$  denotes the coordinates of a Cartesian system fixed in the medium and  $t \in \mathbb{R}$  denotes a time coordinate. The acoustic field is characterized by a real-valued acoustic potential  $u(t, x)$  which is a solution of the wave equation

$$(1.1) \quad \frac{\partial^2 u}{\partial t^2} - \nabla^2 u = f(t, x)$$

The function  $f$ , which characterizes the transmitter, will be called the source function. The scattering of a single pulse of duration  $T$ , emitted by a transmitter localized near a point  $x_0$ , will be analyzed. Hence, the space-time support of  $f$  is assumed to satisfy

$$(1.2) \quad \text{supp } f \subset \{(t, x) \mid t_0 \leq t \leq t_0 + T \text{ and } |x - x_0| \leq \delta_0\}$$

where  $\delta_0$  and  $t_0$  are constants. The scatterers are represented by a closed



bounded set  $\Gamma \subset \mathbb{R}^3$  with complement  $\Omega = \mathbb{R}^3 - \Gamma$ . The common boundary  $\partial\Gamma = \partial\Omega$  is assumed to be a smooth surface. It will be convenient to let the origin of coordinates lie in  $\Gamma$  and

$$(1.3) \quad \Gamma \subset \{x \mid |x| \leq \delta\}$$

It is also assumed that

$$(1.4) \quad \delta + \delta_0 < |x_0|$$

(the transmitter and scatterers are disjoint) and

$$(1.5) \quad T < |x_0| - \delta - \delta_0$$

(the sources cease acting before the signal reaches the scatterers).

The total acoustic field produced by the transmitter in the presence of the scatterers  $\Gamma$  is the solution  $u(t, x)$  of (1.1) in  $\mathbb{R} \times \Omega$  that satisfies the boundary condition

$$(1.6) \quad \frac{\partial u}{\partial \nu} = \vec{\nu} \cdot \nabla u = 0$$

for  $(t, x) \in \mathbb{R} \times \partial\Omega$ , where  $\vec{\nu}$  is a normal to  $\partial\Omega$ , and the initial condition

$$(1.7) \quad u(t, x) = 0 \quad \text{for } t < t_0 \quad \text{and } x \in \Omega$$

The corresponding signal field  $u_0(t, x)$ , generated by  $f(t, x)$  when no scatterers are present, is given by the retarded potential

$$(1.8) \quad u_0(t, x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(t - |x - x'|, x')}{|x - x'|} dx'$$

where  $dx' = dx'_1 dx'_2 dx'_3$ . The sonar echo  $u_s(t, x)$  produced by the source function

$f$  and the scatterers  $\Gamma$  is defined by

$$(1.9) \quad u_s(t, x) = u(t, x) - u_0(t, x), \quad t \in \mathbb{R}, \quad x \in \Omega$$

Both  $\text{supp } u_0(t, \cdot)$  and  $\text{supp } u(t, \cdot)$  are contained in  $\{x \mid |x - x_0| \leq t - t_0 + \delta_0\}$  which is disjoint from  $\Gamma$  for  $t - t_0 + \delta_0 \leq |x_0| - \delta$ . It follows that

$$(1.10) \quad u_s(t, x) = 0 \quad \text{for } t \leq t_0 + |x_0| - \delta - \delta_0 \quad \text{and } x \in \Omega$$

The goal of this paper is to calculate  $u_s(t, x)$ , especially in the far field ( $|x| \gg 1$ ) and to analyze its dependence on the source function  $f$  and the scatterers  $\Gamma$ .

§2. THE WAVE OPERATORS AND PULSE MODE SONAR ECHO STRUCTURE. The starting point for the calculation of  $u_s(t, x)$  below is a construction of  $u(t, x)$  in the Hilbert space  $L_2(\Omega)$ . To describe it let

$$(2.1) \quad L_2^m(\Omega) = \{u(x) \mid D^\alpha u(x) \in L_2(\Omega) \quad \text{for } 0 \leq |\alpha| \leq m\}$$

where  $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ ,  $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$  and  $D^\alpha = \partial^{|\alpha|} / \partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}$ . Then the operator  $A : L_2(\Omega) \rightarrow L_2(\Omega)$  defined by

$$(2.2) \quad D(A) = L_2^2(\Omega) \cap \{u \mid \partial u / \partial \nu = 0 \quad \text{on } \partial\Omega\},$$

$$(2.3) \quad Au = -\nabla^2 u \quad \text{for all } u \in D(A)$$

is selfadjoint and non-negative; see [5] for details. The solution of the initial-boundary value problem (1.1), (1.6), (1.7) is given by Duhamel's integral [6]:

$$(2.4) \quad u(t, \cdot) = \int_{t_0}^t \{A^{-1/2} \sin(t-\tau) A^{1/2}\} f(\tau, \cdot) d\tau, \quad t \geq t_0$$



In particular, for  $t \geq t_0 + T$

$$(2.5) \quad u(t, \cdot) = \int_{t_0}^{t_0 + T} \{A^{-1/2} \sin(t-\tau)A^{1/2}\} f(\tau, \cdot) d\tau = \operatorname{Re}\{v(t, \cdot)\}$$

where

$$(2.6) \quad v(t, \cdot) = i \int_{t_0}^{t_0 + T} A^{-1/2} \exp\{-i(t-\tau)A^{1/2}\} f(\tau, \cdot) d\tau = \exp\{-itA^{1/2}\} h$$

and

$$(2.7) \quad h = i \int_{t_0}^{t_0 + T} A^{-1/2} \exp\{i\tau A^{1/2}\} f(\tau, \cdot) d\tau$$

In the special case where  $\Omega = \mathbb{R}^3$  (no scatterer) the operator  $A$  will be denoted by  $A_0$ . Thus  $A_0 : L_2(\mathbb{R}^3) \rightarrow L_2(\mathbb{R}^3)$ , defined by  $D(A_0) = L_2^2(\mathbb{R}^3)$  and  $A_0 u = -\Delta u$ , is selfadjoint in  $L_2(\mathbb{R}^3)$  and the signal  $u_0(t, x)$  is given by

$$(2.8) \quad u_0(t, \cdot) = \operatorname{Re}\{v_0(t, \cdot)\}, \quad t \geq t_0 + T$$

where

$$(2.9) \quad v_0(t, \cdot) = \exp\{-itA_0^{1/2}\} h_0$$

and

$$(2.10) \quad h_0 = i \int_{t_0}^{t_0 + T} A_0^{-1/2} \exp\{i\tau A_0^{1/2}\} f(\tau, \cdot) d\tau$$

To compare  $u(t, x)$  and  $u_0(t, x)$  introduce the operator  $J : L_2(\Omega) \rightarrow L_2(\mathbb{R}^3)$  defined by

$$(2.11) \quad Jh(x) = \begin{cases} j(x)h(x) & \text{for } x \in \Omega \\ 0 & \text{for } x \in \mathbb{R}^3 - \Omega \end{cases}$$

where  $j \in C^\infty(\mathbb{R}^3)$ ,  $0 \leq j(x) \leq 1$ ,  $j(x) = 1$  for  $|x| \geq \delta$  and  $j(x) = 0$  in a neighborhood of  $\Gamma$ .  $J$  is a bounded operator with bound  $\|J\| = 1$ . It will be convenient to extend the definition (1.9) of  $u_s$  by defining  $u_s(t, x) = \text{Re}\{v_s(t, x)\}$  where

$$(2.12) \quad v_s(t, x) = Jv(t, x) - v_0(t, x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^3$$

The calculation of the far field form of  $u_s(t, x)$  will be based on the theory of wave operators as developed in [5]. The wave operators  $W_+$  and  $W_-$  are defined by the strong limits

$$(2.13) \quad W_\pm = s - \lim_{t \rightarrow \pm \infty} \exp\{itA_0^{1/2}\} J \exp\{-itA^{1/2}\}$$

It is shown in [5] that these limits exist and define unitary operators  $W_\pm : L_2(\Omega) \rightarrow L_2(\mathbb{R}^3)$ . It follows that for each  $h \in L_2(\Omega)$

$$(2.14) \quad Jv(t, \cdot) = J \exp\{-itA^{1/2}\} h = \exp\{-itA_0^{1/2}\} W_+ h + o_t(1), \quad t \rightarrow +\infty$$

where  $o_t(1)$  denotes an  $L_2(\mathbb{R}^3)$ -valued function of  $t$  that tends to zero in  $L_2(\mathbb{R}^3)$  when  $t \rightarrow \infty$ . (2.14) and (2.9) imply that

$$(2.15) \quad v_s(t, \cdot) = \exp\{-itA_0^{1/2}\} (W_+ h - h_0) + o_t(1), \quad t \rightarrow \infty$$

This result is used below to calculate the far field form of  $u_s(t, x) = \text{Re}\{v_s(t, x)\}$ .

§3. THE FAR FIELD APPROXIMATION AND THE SCATTERING OPERATOR. The scattering operator for the scatterer  $\Gamma$  is the unitary operator  $S$  in  $L_2(\mathbb{R}^3)$  defined by

$$(3.1) \quad S = W_+ W_-^*$$

where  $W_-^*$  denotes the adjoint of  $W_-$ . A connection between  $S$  and the approximation (2.15) will be derived by calculating the relationship between  $h$  and  $h_0$ . Equation (1.10) implies that  $v_s(t, x) = 0$  for  $t_0 + T \leq t \leq t_0 + |x_0| - \delta - \delta_0$  and  $x \in \mathbb{R}^3$ . It will be convenient to choose

$$(3.2) \quad t_0 = -|x_0| + \delta_0 + \delta$$

so that the arrival time of the signal at  $\Gamma$  is non-negative; see (1.10). With this convention

$$(3.3) \quad J \exp\{-itA^{1/2}\}h = \exp\{-itA_0^{1/2}\}h_0 \quad \text{for } t_1 \leq t \leq 0$$

where

$$(3.4) \quad t_1 = t_0 + T = -|x_0| + \delta_0 + \delta + T$$

Taking  $t = 0$  in (3.3) gives  $Jh = h_0$  while taking  $t = t_1$  gives

$$(3.5) \quad \exp\{it_1 A_0^{1/2}\}J \exp\{-it_1 A^{1/2}\}h = h_0$$

The scatterer  $\Gamma$  is in the far field of the transmitter if  $|x_0| \gg 1$  or, by (3.4),  $t_1 \ll -1$ . Combining this with (3.5) and the definition (2.13) gives

$$(3.6) \quad h_0 = W_- h + o_{x_0}(1), \quad |x_0| \rightarrow \infty$$

where  $o_{x_0}(1)$  is an  $L_2(\mathbb{R}^3)$ -valued function of  $x_0$  that tends to zero in  $L_2(\mathbb{R}^3)$  when  $|x_0| \rightarrow \infty$ . Multiplying (3.6) by  $S$  gives

$$(3.7) \quad W_+ h = S h_0 + o_{x_0}(1), \quad |x_0| \rightarrow \infty$$

because  $W_-^* W_- = 1$  ( $W_-$  is unitary). Combining (3.7) and (2.15) gives

$$(3.8) \quad v_s(t, \cdot) = \exp\{-itA_0^{1/2}\}(S-1)h_0 + o_t(1) + o_{x_0}(1)$$

Note that the term  $o_{x_0}(1)$  tends to zero in  $L_2(\mathbb{R}^3)$  when  $|x_0| \rightarrow \infty$  uniformly in  $t$  because  $\exp\{-itA_0^{1/2}\}$  is unitary. Equation (3.8) shows that, in the far field approximation, the dependence of the echo waveform on the scatterer is determined by the scattering operator.

The approximation (3.8) is used in §6 to derive an explicit integral formula for the far field echo waveform. The derivation is based on a known integral representation for  $S$ , formulated in §4, and the theory of asymptotic wave functions of [5] which is summarized in §5.

§4. THE STRUCTURE OF THE SCATTERING OPERATOR. The steady-state theory of scattering and associated eigenfunction expansions for  $A$  are reviewed briefly in this section and applied to the construction of the scattering operator for  $\Gamma$ .

$A_0$  is a selfadjoint operator in  $L_2(\mathbb{R}^3)$  with a purely continuous spectrum and the plane waves



$$(4.1) \quad w_0(x,p) = (2\pi)^{-3/2} \exp\{ix \cdot p\}, \quad p \in \mathbb{R}^3$$

are a complete family of generalized eigenfunctions. The corresponding eigenfunction expansion is the well-known Plancherel theory of the Fourier transform (see [5], Ch. 6).

Generalizations of the Plancherel theory to acoustic scattering by bounded objects were first given by N. A. Shenk [2] and Y. Shizuta [4]. In this work the generalized eigenfunctions are the distorted plane waves

$$(4.2) \quad w_{\pm}(x,p) = w_0(x,p) + w_{\pm}^S(x,p), \quad x \in \Omega, \quad p \in \mathbb{R}^3$$

which are characterized by the properties that  $w_{\pm}(x,p)$  is locally in  $D(A)$  (i.e.,  $\phi w_{\pm}(\cdot, p) \in D(A)$  for all  $\phi \in C_0^{\infty}(\mathbb{R}^3)$ ),

$$(4.3) \quad (\nabla^2 + |p|^2)w_{\pm}(x,p) = 0 \quad \text{for } x \in \Omega$$

$$(4.4) \quad \frac{\partial w_{\pm}^S}{\partial |x|} \mp i |p| w_{\pm}^S = O\left(\frac{1}{|x|^2}\right), \quad |x| \rightarrow \infty$$

For the existence, uniqueness and construction of  $w_{\pm}(x,p)$  see [2,3,4,5,6].

Physically,  $w_{\pm}^S(x,p)$  is the steady-state scattered field produced when the plane wave (4.1) is scattered by  $\Gamma$ . It has the far field form [5,6]

$$(4.5) \quad w_{\pm}^S(x,p) = \frac{e^{\pm i |p| |x|}}{4\pi |x|} T_{\pm}(|p|\theta, p) + O\left(\frac{1}{|x|^2}\right), \quad |x| \rightarrow \infty$$

where  $\theta = x/|x|$ .  $T_{\pm}(p, p')$ , the *scattering amplitude* or *differential scattering cross section* of  $\Gamma$ , is defined for all  $p$  and  $p'$  in  $\mathbb{R}^3$  such that  $|p| = |p'|$  and has the symmetry properties

$$(4.6) \quad T_{\pm}(p, p') = T_{\pm}(-p', -p) = \overline{T_{\pm}(p, -p')} = \overline{T_{\pm}(-p, -p')}$$

where the bar denotes the complex conjugate.

The connection between  $S$  and  $T_{\pm}(p, p')$  is based on the eigenfunction expansion theorem for  $A$ . The latter states that for all  $h \in L_2(\Omega)$  the limits

$$(4.7) \quad \hat{h}_{\pm}(p) = (\Phi_{\pm} h)(p) = L_2(\mathbb{R}^3)\text{-}\lim_{M \rightarrow \infty} \int_{\Omega_M} \overline{w_{\pm}(x, p)} h(x) dx$$

exist, where  $\Omega_M = \Omega \cap \{x \mid |x| < M\}$ , and

$$(4.8) \quad h(x) = L_2(\Omega)\text{-}\lim_{M \rightarrow \infty} \int_{|p| < M} w_{\pm}(x, p) \hat{h}_{\pm}(p) dp$$

Moreover, the operators  $\Phi_{\pm} : L_2(\Omega) \rightarrow L_2(\mathbb{R}^3)$  are unitary and for each bounded measurable function  $\Psi(\lambda)$  on  $\lambda \geq 0$

$$(4.9) \quad (\Phi_{\pm} \Psi(A) h)(p) = \Psi(|p|^2) \Phi_{\pm} h(p)$$

The Fourier transform will be denoted by

$$(4.10) \quad \hat{h}(p) = (\Phi h)(p) = L_2(\mathbb{R}^3)\text{-}\lim_{M \rightarrow \infty} \int_{|x| < M} \overline{w_0(x, p)} h(x) dx$$

An exposition of the eigenfunction expansion theory is given in [5]. In what follows the relations (4.7), (4.8) are written

$$(4.11) \quad \hat{h}_{\pm}(p) = \int_{\Omega} \overline{w_{\pm}(x, p)} h(x) dx$$



$$(4.12) \quad h(x) = \int_{\mathbb{R}^3} w_{\pm}(x,p) \hat{h}_{\pm}(p) dp$$

for brevity. However, the integrals are not convergent, in general, and (4.11), (4.12) must be interpreted in the sense of (4.7), (4.8).

The wave operators  $W_{\pm}$  defined by (2.3) are known to have the representations [5]

$$(4.13) \quad W_{+} = \Phi^{*} \Phi_{-}, \quad W_{-} = \Phi^{*} \Phi_{+}$$

Combining (4.13) and (3.1) gives the representation

$$(4.14) \quad S = W_{+} W_{-}^{*} = \Phi^{*} \hat{S} \Phi$$

where the operator

$$(4.15) \quad \hat{S} = \Phi_{-} \Phi_{+}^{*}$$

is called the S-matrix for the scatterer  $\Gamma$ . The operator  $\hat{S}^{-1}$  has the integral representation

$$(4.16) \quad (\hat{S}^{-1}) \hat{h}(p) = \frac{i|p|}{2(2\pi)^{1/2}} \int_{S^2} T_{+}(p, |p|\theta') \hat{h}(|p|\theta') d\theta'$$

whose kernel is the differential scattering cross section of  $\Gamma$ . The integration in (4.16) is over the points  $\theta'$  of the unit sphere  $S^2$  in  $\mathbb{R}^3$ . The first proof of (4.16) for acoustic scattering is due to Shenk [2].

§5. PULSE MODE SONAR SIGNALS IN THE FAR FIELD. The signals  $u_0(t, x)$  originate in the region  $|x - x_0| \leq \delta_0$  and reach points  $x$  in the far field, charac-

terized by  $|x-x_0| \gg 1$ , after a time interval of magnitude comparable with  $|x-x_0|$ . Hence the far field form of  $u_0(t,x)$  coincides with its asymptotic form for large  $t$ . The latter is provided by the theory of asymptotic wave functions developed in [5]. The theory is applied here to determine the far field form of  $u_0(t,x)$ .

The complex wave function  $v_0(t,x)$  defined by (2.9), (2.10) has the Fourier representation

$$(5.1) \quad v_0(t,x) = (2\pi)^{-3/2} \int_{\mathbf{R}^3} \exp\{i(x \cdot p - t|p|)\} \hat{h}_0(p) dp$$

Equations (2.10) and (4.9) imply that

$$(5.2) \quad \hat{h}_0(p) = (2\pi)^{1/2} i|p|^{-1} \hat{f}(-|p|, p)$$

where

$$(5.3) \quad \hat{f}(\omega, p) = (2\pi)^{-2} \int_{\mathbf{R}^4} \exp\{-i(t\omega + x \cdot p)\} f(t,x) dt dx$$

is the 4-dimensional Fourier transform of  $f$ . Note that (5.2) suggests the concept of a *non-radiating source function*.  $f$  is said to be non-radiating if

$$(5.4) \quad f(t,x) = \partial^2 u_0 / \partial t^2 - \nabla^2 u_0, \text{ supp } u_0 \text{ bounded in } \mathbf{R}^4$$

In this case  $\hat{u}_0(\omega, p)$  exists and (5.4) is equivalent to

$$(5.5) \quad \hat{f}(\omega, p) = (|p|^2 - \omega^2) \hat{u}_0(\omega, p)$$

Equations (5.2) and (5.5) imply  $\hat{h}_0(p) = 0$  and hence  $u_0(t,x) = 0$  for  $t \geq t_0 + T$ .

The asymptotic wave function associated with  $u_0(t, x) = \operatorname{Re}\{v_0(t, x)\}$  is defined by

$$(5.6) \quad u_0^\infty(t, x) = s(|x| - t, \theta) / |x|, \quad x = |x|\theta$$

where  $s \in L_2(\mathbb{R} \times S^2)$  is defined by (see [5], Ch. 2)

$$(5.7) \quad \begin{aligned} s(\tau, \theta) &= \operatorname{Re} \left\{ (2\pi)^{-1/2} \int_0^\infty \exp\{i\tau\omega\} (-i\omega) \hat{h}_0(\omega\theta) d\omega \right\} \\ &= \operatorname{Re} \left\{ \int_0^\infty \exp\{i\tau\omega\} \hat{f}(-\omega, \omega\theta) d\omega \right\} \end{aligned}$$

Direct calculation of  $s(\tau, \theta)$  from (1.8) yields the alternative representation

$$(5.8) \quad s(\tau, \theta) = \frac{1}{4\pi} \int_{\mathbb{R}^3} f(-\tau + \theta \cdot x, x) dx$$

It was shown in [5] that  $u_0^\infty$  describes the asymptotic behavior of  $u_0$  in  $L_2(\mathbb{R}^3)$  for  $t \rightarrow \infty$ :

$$(5.9) \quad u_0(t, \cdot) = u_0^\infty(t, \cdot) + o_t(1), \quad t \rightarrow \infty$$

The integral

$$(5.10) \quad E(u, K, t) = \frac{1}{2} \int_K \{ |\nabla u(t, x)|^2 + (\partial u(t, x) / \partial t)^2 \} dx$$

may be interpreted as the acoustic energy in the set  $K \subset \mathbb{R}^3$  at time  $t$ .

It was shown in [5] that if  $h_0 \in L_2^1(\mathbb{R}^3)$  then  $u_0^\infty(t, x)$  converges to  $u_0(t, x)$  in energy when  $t \rightarrow \infty$ . More precisely,

$$(5.11) \quad \partial u_0(t, x) / \partial x_j = u_{0,j}^\infty(t, x) + o_t(1), \quad t \rightarrow \infty, \quad j = 0, 1, 2, 3$$

where  $x_0 = t$ ,

$$(5.12) \quad u_{0,j}^\infty(t, x) = s_j(|x| - t, \theta) / |x|$$

$$(5.13) \quad s_0(\tau, \theta) = -\partial s(\tau, \theta) / \partial \tau$$

and

$$(5.14) \quad s_j(\tau, \theta) = -\theta_j s_0(\tau, \theta) \quad \text{for } j = 1, 2, 3$$

The result (5.11) was used in [5] to calculate the asymptotic distribution of energy in cones

$$(5.15) \quad C = \{x = r\theta \mid r > 0, \quad \theta \in C_0 \subset S^2\}$$

Applying the results to the signal  $u_0(t, x)$  generated by  $f(t, x)$  gives

$$(5.16) \quad E(u_0, C, \infty) = \lim_{t \rightarrow \infty} E(u_0, C, t) = \pi \int_C |\hat{f}(-|p|, p)|^2 dp$$

In particular, the total signal energy introduced by the source function  $f$  is

$$(5.17) \quad E(u_0, \mathbb{R}^3, \infty) = \pi \int_{\mathbb{R}^3} |\hat{f}(-|p|, p)|^2 dp$$

§6. PULSE MODE SONAR ECHOES IN THE FAR FIELD. Equation (3.8) implies that the echo  $u_s(t, x)$  satisfies



$$(6.1) \quad u_s(t, \cdot) = \operatorname{Re}\{\exp(-itA_0^{1/2})(S-1)h_0\} + o_t(1) + o_{x_0}(1)$$

The first term on the right has the same form as the signal but with  $h_0$  replaced by  $(S-1)h_0$ . It follows from the results of §5 that

$$(6.2) \quad u_s(t, x) = u_s^\infty(t, x) + o_t(1) + o_{x_0}(1)$$

where

$$(6.3) \quad u_s^\infty(t, x) = e(|x| - t, \theta) / |x|, \quad x = |x|\theta$$

and

$$(6.4) \quad e(\tau, \theta) = \operatorname{Re}\{(2\pi)^{-1/2} \int_0^\infty \exp(i\tau\omega) (-i\omega) ((S-1)h_0)^\wedge(\omega\theta) d\omega\}$$

Now by (4.14)

$$(6.5) \quad ((S-1)h_0)^\wedge = \phi(S-1)\phi^*\phi h_0 = (\hat{S}-1)\hat{h}_0$$

and hence by (4.16)

$$(6.6) \quad ((S-1)h_0)^\wedge(\omega\theta) = \frac{i\omega}{2(2\pi)^{1/2}} \int_{S^2} T_+(\omega\theta, \omega\theta') \hat{h}_0(\omega\theta') d\theta'$$

Combining (6.4) and (6.6) gives

$$(6.7) \quad e(\tau, \theta) = \frac{1}{4\pi} \operatorname{Re} \left\{ \int_0^\infty \exp(i\tau\omega) \omega^2 \int_{S^2} T_+(\omega\theta, \omega\theta') \hat{h}_0(\omega\theta') d\theta' d\omega \right\}$$

Finally, by (5.2)

$$(6.8) \quad e(\tau, \theta) = \frac{1}{2(2\pi)^{1/2}} \operatorname{Re} \left\{ i \int_0^\infty \exp(i\tau\omega) \omega \int_{S^2} T_+(\omega\theta, \omega\theta') \hat{f}(-\omega, \omega\theta') d\theta' d\omega \right\}$$



Equation (6.2) implies that  $u_s^\infty$  gives the far field form of  $u_s$ . For  $|x_0| \gg 1$ ,  $\Gamma$  is in the far field of the transmitter and the term  $o_{x_0}(1)$  is small, uniformly for all  $t$ . For receivers in the far field region  $|x| \gg 1$  for  $\Gamma$ , the echo  $u_s(t, x)$  arrives at times  $t \gg 1$  and hence the term  $o_t(1)$  is small.

§7. CONCLUDING REMARKS. Actual sonar transmitters do not, of course, generate signals by means of a source function  $f(t, x)$ . However, the purpose of a well designed transmitter is to generate a signal with a prescribed waveform  $s(\tau, \theta)$ . Now

$$(7.1) \quad \hat{s}(\omega, \theta) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \exp(-i\omega\tau) s(\tau, \theta) d\tau = \left(\frac{\pi}{2}\right)^{1/2} \hat{f}(-\omega, \omega\theta)$$

(see (5.7)) and hence

$$(7.2) \quad e(\tau, \theta) = \operatorname{Re} \left\{ \frac{i}{4} \int_0^\infty \exp(i\tau\omega) \omega \int_{S^2} T_+(\omega\theta, \omega\theta') \hat{s}(\omega, \theta') d\theta' d\omega \right\}$$

In particular, the transmitter characteristics influence the echo waveform only through  $s(\tau, \theta)$ . Hence, (7.2) is applicable to real transmitters with known waveforms  $s(\tau, \theta)$ .

It is known that  $T_+(\omega\theta, \omega\theta')$  is a meromorphic function of  $\omega$  with poles in the lower half plane [3]. The other functions in the integrand of (7.2) are entire holomorphic functions. Hence, the integral in (7.2) can be transformed by deforming the contour of integration in the complex  $\omega$ -plane. This leads to an expansion of the echo waveform of the type occurring in the singularity expansion method [1].

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| 13. ABSTRACT<br>Pulse mode sonar operation is analyzed under the physical hypotheses that<br><div style="margin-left: 40px;">       The medium is a stationary homogeneous fluid.<br/>       Both the sonar system and the scattering objects are stationary,<br/>       The scattering objects are rigid bodies, and<br/>       The scattering objects lie in the far fields of the transmitter and receiver.     </div> It is shown that if the sonar signal waveform in the far field is<br>$\frac{s( x  - t, \theta)}{ x }, \quad x =  x \theta$ then the sonar echo waveform in the far field is<br>$\frac{e( x  - t, \theta)}{ x }, \quad x =  x \theta$ where<br>$e(\tau, \theta) = \operatorname{Re} \left\{ \frac{i}{4} \int_0^\infty \omega \exp(i\tau\omega) \int_{ \theta' =1} T_+(\omega\theta, \omega\theta') \hat{S}(\omega, \theta') d\theta' d\omega \right\}.$ Here $\hat{S}(\omega, \theta)$ is the Fourier transform of $s(\tau, \theta)$ with respect to $\tau$ and $T_+(\omega\theta, \omega\theta')$ is the differential scattering cross section of the scattering objects. |                                                                                     |                                                                                                     |

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## S-matrix

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