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INVESTIGATION OF INVERSE VANDERMONDE MATRIX CALCULATION FOR LIN--ETC(U)
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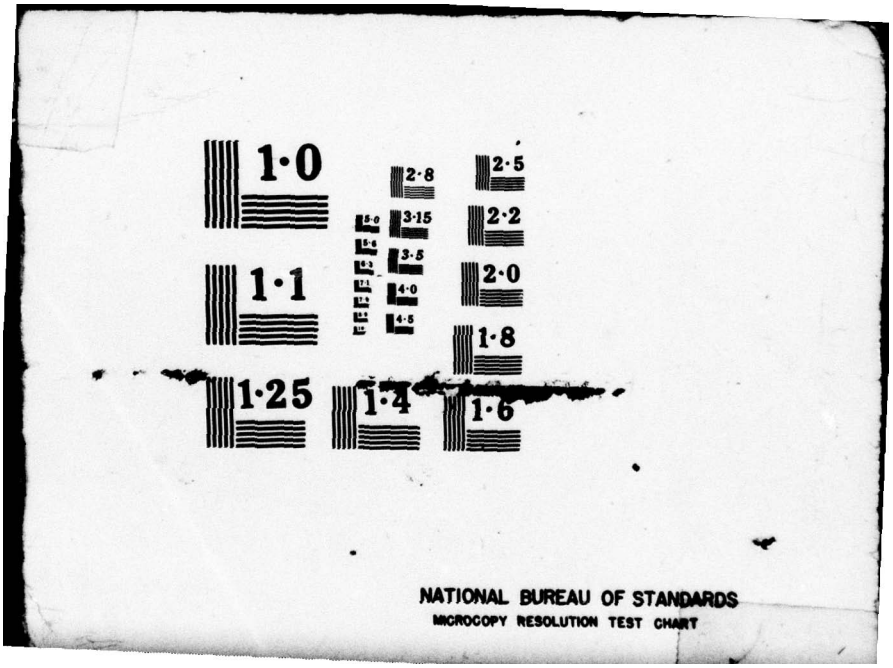
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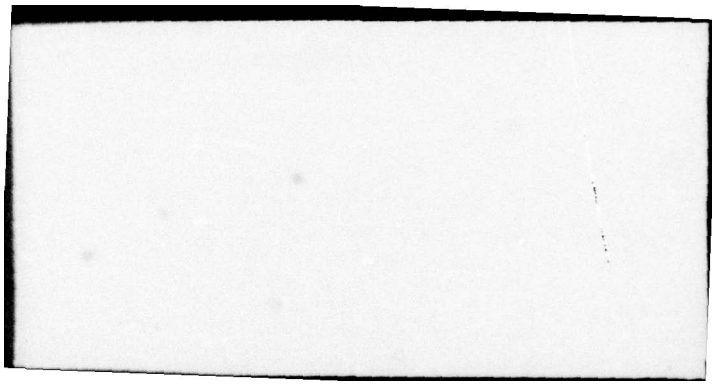

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(6) INVESTIGATION OF INVERSE VANDERMONDE MATRIX CALCULATION FOR LINEAR SYSTEM APPLICATIONS.

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INVESTIGATION OF INVERSE VANDERMONDE MATRIX CALCULATION FOR LINEAR SYSTEM APPLICATIONS

THESIS

Presented to the Faculty of the School of Engineering
of the Air Force Institute of Technology
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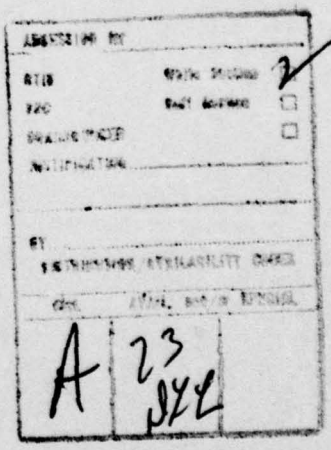

by

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March 1979

Preface

This thesis is intended as a source document for further research in system identification processes. Specifically, this research concentrated on efficiently finding the inverse of confluent (i.e., repeated eigenvalue) Vandermonde matrices, which is an important part of the component matrix approach to system identification. However, in order to provide a broader understanding of system identification in general, and the component matrix approach in particular, the background and theory sections are more comprehensive than might otherwise be needed. Several supplementary mathematical developments and some example problems are included to aid this broad understanding, but they are placed in appendices to preserve the continuity of the text, and referenced as appropriate. A list of special symbols and abbreviations is also included, to give the reader a ready reference for those that may be unfamiliar.

I wish to give special thanks to Maj. J. Gary Reid, my thesis advisor, for his support and encouragement in this endeavor. Professor Charles W. Richard and Dr. Peter S. Maybeck, who read the manuscript, gave many helpful suggestions. For their patient consultation regarding the computer programming and example problems used in the course of the research, I wish to thank Maj. Edward Reeves and Lt. Paul Vergez. I am also deeply indebted to a close personal friend of mine, Rev. Roy Dorsett, for his understanding support

throughout the entire project. There are many others like him who "cheered from the sidelines," but are too numerous to mention. Finally I want to thank Mrs. Evelyn Shaw for her perseverance in the final days while she was typing the final draft.

Donald P. Seyler

Contents

Contents

List of Figures

List of Tables

Symbols and Abbreviations

CLDE = Coupled Linear Differential Equation

CM = Component Matrix

n = System order

p = Number of system parameters

P/O = "Part Of"

q = Number of system outputs

r = Number of system control inputs

STM = State Transition Matrix

w.r.t. = "With Respect To"

$\underline{\bar{x}}$)
$\underline{\bar{A}}$) = Augmented vector and augmented matrix

1. Lower case denotes a vector
2. Upper case denotes a matrix
3. Underscore denotes "vector" or "matrix" as appropriate; used in addition to 1 and 2 if confusion with scalars is likely
4. Overscore denotes "augmented" in either case

$\underline{\bar{x}}_{(i)}(\cdot)$
$\underline{\bar{A}}_{(i)}(\cdot)$ = Partial derivatives of the system state and system matrix w.r.t. the i'th system parameter, θ_i

θ_i = i'th parameter component of a state space system

Abstract

Efficient inversion of a $2n \times 2n$ Vandermonde matrix is a key requirement of a new algorithm developed by J. Gary Reid (Ref 10, 11) for parameter identification in linear time-invariant systems. It has features very desirable for application to large systems with many unknown parameters, such as adaptive flight control systems.

A generalized algorithm for inverting the Vandermonde matrix was proposed by F. G. Csáki (Ref 1). It was chosen for study because its structure parallels that of Reid's algorithm. Csáki's algorithm was coded into a computer subroutine called "VANINV," and 43 eigensystems were used to test its computational accuracy and efficiency. Four potential problem areas tested were: (1) large system orders, (2) eigenvalues near each other, (3) eigenvalues near zero, and (4) eigenvalues with large magnitude differences. For a comparison, the Vandermonde matrix for each system was also inverted using routines from the International Mathematical & Statistical Library (IMSL).

Test results indicated that VANINV is not quite as fast or as accurate as the IMSL routines. However, the tests were limited to systems with real distinct eigenvalues because the IMSL routines cannot handle other types. VANINV in its present form can handle systems having any combination of complex and/or repeated eigenvalues. Therefore, recommendations for further research and several possible means of improving VANINV are outlined.

INVESTIGATION OF INVERSE VANDERMONDE MATRIX CALCULATION FOR LINEAR SYSTEM APPLICATIONS

I Introduction

Fundamental quantities needed for iterative computation of a quasi-linear estimate of unknown parameters in a linear dynamic system are the "parameter sensitivity variables." Finding these variables is a basic part of a "system identification process," as is shown in the background subsection on system identification.

Section II develops the process for finding the parameter sensitivity variables and shows that a basic part of the process is the inversion of a $2n \times 2n$ Vandermonde matrix. The development of an efficient computer program to invert the Vandermonde matrix, the major emphasis of this thesis effort, is presented in Section III. Section IV explains some test procedures used for determining the accuracy and efficiency of the program. The concluding two sections give the results of the research and recommendations for further research.

As an aid in understanding the concepts and processes presented in the theory sections of the thesis, several example problems and supplementary mathematical developments are worked out. However, because of their length they are placed in appendices and referenced as appropriate. The computer program developed to invert the Vandermonde

matrix and the programs used to test its computational speed and accuracy are also placed in appendices.

Motivational Background

If all systems were linear and time-invariant, the linear system model (noise-free):

$$\dot{x} = Ax + Bu \tag{1a}$$

$$y = Cx + Du \tag{1b}$$

would suffice. However, control systems for other than very simple applications are not linear or time-invariant: depending on operating conditions,[1] the control laws must change to maintain adequate system control. For example, the Wright Brothers didn't have to concern themselves with large changes in operating conditions. Their first powered flight, 120 feet in 12 seconds (Ref 2:Vol 29:557; 14:Vol 7: 388), was not even as fast as a man can run, and didn't last long enough for the environment to change much.

On the other hand, engineers designing flight control systems for modern high performance aircraft do need to concern themselves with large changes in operating conditions. As aircraft flight envelopes are extended over broader operating ranges (for example, a high performance

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[1] In this study, "operating conditions" is used to encompass both system configuration and environment. Each has definite effects on system response, and may also affect the other. For convenience however, the term "operating conditions" is used to refer to both when it is not necessary to distinguish between them.

aircraft may be required to have a Vertical or Short Take-off and Landing (V/STOL) capability), more and alternate loops must be added to the flight control system to handle all the various operating conditions. The increased requirement for control system flexibility often results in an overall system that is nonlinear and/or time-varying, and to maintain adequate control, some means of adjusting the control system parameters is necessary. This capacity to change the modes of the control system according to the aircraft flight operating conditions is the essence of an "adaptive" flight control system. Of course every time the operating conditions change, the control system variables must change accordingly. Thus the control problem becomes one of identifying the system parameters (the process is called "system identification") in order to determine the proper system model and match an appropriate control law to it.

The flight control system design engineer could try to think of a collection of representative types of operating conditions which would, hopefully, cover the continuum of possible operating conditions. A control loop could then be designed to handle each different representative operating condition. That would require an a priori identification of the system model and parameter values and a corresponding collection of hardware to implement the design. Then some type of master controller would be needed to choose which model and parameter values to use

for the given operating conditions.

In terms of an entirely analog flight control system, the increased flexibility required to handle the adaptive part of the control function results in a confusing mass of hardware and interconnecting links. Also, each new piece of hardware poses a possible reliability problem. These are some of the reasons for the trend toward digital flight control systems. Instead of having a discrete piece of hardware for each required control function, a digital controller need be simply reprogrammed to accommodate the new control system parameters. These parameters provide the relation between the system inputs, states, and outputs for the control variable calculations.

The overall control system can be visualized as shown in Figure 1. Upon starting the system, some nominal system model and control are used. However, as shown, the model must change to account for significant variations in the system operating conditions. Referring back to the high performance aircraft mentioned before, an example of a change in the system configuration is the change from a vertical takeoff to "conventional" horizontal flight. An example of an environmental change affecting the system model is the change in air density with altitude (eg: the air at 40,000 feet is much less dense than at sea level, so the control surface actuator gains must be changed accordingly to maintain the same performance relative to pilot inputs). The digital control system can be given

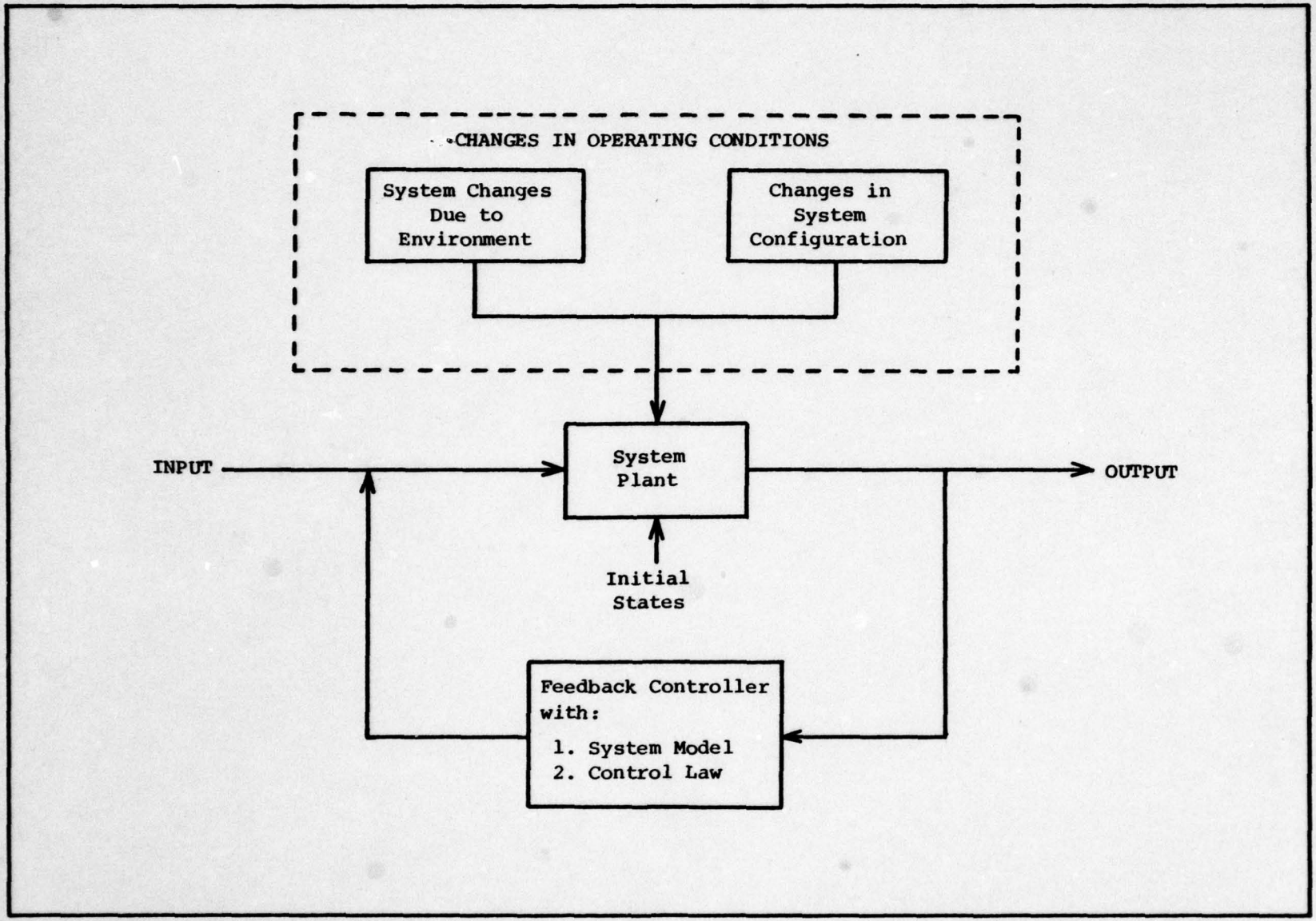

Figure 1. Conceptual model of an adaptive feedback control system.

the task of identifying the present system parameters, updating the system model with them, and then determining what adaptations must be made in the control to achieve the desired overall performance. The first two steps are what distinguish an adaptive control system from a fixed control system.

System Identification Process Background

A conceptually simple method for the system identification process is the use of a first order Taylor series approximation to update the system model according to:

$$y(t) \simeq \hat{y}(t)\Big|_{\substack{\text{a priori}\\ \text{model}}} + \frac{\partial y(t)}{\partial \theta(t)}\,\Delta\theta \qquad (2)$$

where:

$y(t)$ = physically measured system output

$\hat{y}(t)$ = calculated output based on an a priori model (or in an iterative process, the last updated model)

$\frac{\partial y(t)}{\partial \theta(t)}$ = system output "sensitivity" with respect to the system parameters

$\Delta\theta$ = change in the system parameters from those of the a priori model (or the change since the last model update, for an iterative process)

Notice that since $\Delta y = \frac{\partial y}{\partial \theta}\,\Delta\theta$, Eq (2) is equivalent to

$$y(t) = \hat{y}(t) + \Delta y(t) \qquad (3)$$

which appeals to the intuitive relation "new output equals old output plus the change in the output."

It is seen, then, that finding these sensitivity variables is a significant part of the system identification process. Therefore, real-time adaptive control applications, such as the adaptive flight control system mentioned earlier, require an efficient means of computing the sensitivity variables.

However, even when using minimal order sensitivity models, the computational requirements can be quite large for even modest size systems. For example, the complete solution of a sensitivity system of order n, with p unknown parameters, may require the solution of n(p+1) coupled linear differential equations (Ref 8:124). The basis for this number is presented in Section II.

An alternate method has been proposed by J. G. Reid in his dissertation (Ref 11:Section III) which may have significant computational advantages because it avoids the need to solve a collection of coupled linear differential equations (CLDE). This method is explained in more detail in Section II.

However, this alternate method also involves the inversion of a $2n \times 2n$ Vandermonde matrix, as is shown in Section II. (Vandermonde matrices are defined in Appendix C, Eqs (C2-2 through C2-6)). This process can be computationally quite costly (Ref 5:817). Therefore, a critical part of the proposed alternate method is the efficient inversion of the Vandermonde matrix. Thus the effort of this investigation

concentrates on implementing and evaluating a proposed method (Ref 1:154-157) of efficiently computing the inverse Vandermonde matrix. The inversion method used forms polynomials from the system eigenvalues and uses the polynomials to determine the elements of the inverse Vandermonde matrix. Section III amplifies this method and shows its parallel computation structure. The significance of this parallel structure is explained in terms of its potential application to on-line flight controllers. Section IV explains the tests used to evaluate the computation time and accuracy of the computer subroutine written to implement the method. The subroutine was called VANINV. Sections V and VI give the test results and resulting conclusions and resulting recommendations for further research.

II Parameter Sensitivity Calculations

Section I showed that calculation of the parameter sensitivities is an essential part of the system identification process. In this section, a theoretical comparison is made between the coupled linear differential equations (CLDE) approach to finding parameter sensitivities and the alternate component method (CM) approach proposed by Reid. Since the CLDE approach is already developed and documented (eg: Ref 13), it is merely outlined here to serve as a reference base for the CM approach.

Although the CLDE method is more straightforward, it has several drawbacks (Ref 8:124). Perhaps the most obvious difficulty is the computational burden. As is illustrated later in this section, the solution of the sensitivity system requires solving as many as $n(p+1)$ coupled linear differential equations, where n is the system dimension and p is the number of parameters whose sensitivities are to be found.[1] Attempts to reduce this computational burden with low-order sensitivity models have resulted in some other problems, such as: (1) requirement for special forms of the plant matrix, (2) numerical sensitivity of calculations performed with the reduced

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[1] This task grows increasingly formidable as n and p increase. Each new variable not only adds another CLDE, it may contribute to the complexity of any or all of the other equations.

order models, and (3) difficulty of obtaining the original state variable sensitivities from the calculated low-order sensitivities. In addition, physical insight is impaired when using the transformed coordinate systems resulting from the low-order models.

Among the advantages of the CM method are: (1) potential reduction in computational burden, (2) retention of physical system insight (the modes are visible throughout the calculation procedure), and essentially "free" availability of the eigenvalue sensitivities themselves (Ref 11:79-80). The entire set of parameter sensitivities may be found from, at most, 2nr "quadrature integrals" (where n is the system dimension and r is the number of control inputs), plus some matrix operations (Ref 8:133). The last part of this section shows that the CM method requires the inversion of a $2n \times 2n$ Vandermonde matrix. Since this inversion is potentially very costly, Section III is devoted to investigating the possible method that was referenced in the introduction for efficiently computing the inverse. At this point, some system definitions are needed.

Most control systems other than very simple ones are not time-invariant. However, assuming that the variations are slow w.r.t. the identification process (as would be the case for flight control systems), the system may be considered to be pseudo time-invariant (or "quasi-static"). Thus, for this discussion, the linear time-invariant state system is used:

$$\dot{x}(t) = A(\theta) \times (t) + B(\theta)u(t) \qquad (4a)$$

$$y(t) = C(\theta) \times (t) + D(\theta)u(t) \qquad (4b)$$

The relation of the initial conditions to the parameter sensitivities is defined by the vector

$$x(O) = \phi(\theta) \qquad (5)$$

This shows that the system matrices and the initial states are a function of the "parameter vector," θ, of which there are p elements, designated θ_i, i = 1, 2, ... p. The nominal value, θ_o, of the parameter vector is used for all evaluations. Furthermore, it is assumed that A, B, C, D and ϕ are real, bounded and continuously differentiable w.r.t. the parameter component θ_i at its nominal value.

The "state sensitivities" of this system are defined as the partial derivative of the system state fector w.r.t. the parameter components θ_i, and are written as

$$\underline{x}_{(i)}(t) = \frac{\partial x(t;\theta)}{\partial \underline{\theta}_i}\bigg|_{\theta=\theta_0} \qquad (6)$$

Coupled Linear Differential Equations Method

With the assumptions made in the introduction for this section, the system parameter sensitivities may be found by augmenting the original system with the desired state sensitivities (from Eq (6)) as follows (Ref 8:124):

$$\bar{x} \equiv \begin{bmatrix} x \\ x_{(1)} \\ \cdot \\ \cdot \\ \cdot \\ x_{(p)} \end{bmatrix}_{n(p+1)\times 1} \qquad \bar{y} \equiv \begin{bmatrix} y \\ y_{(1)} \\ \cdot \\ \cdot \\ \cdot \\ y_{(p)} \end{bmatrix}_{q(p+1)\times 1} \qquad \bar{\phi} \equiv \begin{bmatrix} \phi \\ \phi_{(1)} \\ \cdot \\ \cdot \\ \cdot \\ \phi_{(p)} \end{bmatrix}_{n(p+1)\times 1} \tag{7}$$

$$\bar{A} \equiv \begin{bmatrix} A & 0 & \cdots & 0 \\ A_{(i)} & A & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ A_{(p)} & 0 & \cdots & A \end{bmatrix}_{n(p+1)\times n(p+1)} \qquad \bar{B} \equiv \begin{bmatrix} B \\ B_{(1)} \\ \cdot \\ \cdot \\ \cdot \\ B_{(p)} \end{bmatrix}_{n(p+1)\times r} \tag{8}$$

$$\bar{C} \equiv \begin{bmatrix} C \\ C_{(1)} \\ \cdot \\ \cdot \\ \cdot \\ C_{(p)} \end{bmatrix}_{n(p+1)\times q} \qquad \bar{D} \equiv \begin{bmatrix} D \\ D_{(1)} \\ \cdot \\ \cdot \\ \cdot \\ D_{(p)} \end{bmatrix}_{n(p+1)\times r} \tag{9}$$

where q is the system output dimension and r is the system input (control) dimension. This augmentation gives a "sensitivity system" which contains all the original states plus the desired parameter sensitivities expressed as states. From Eq (7) it is evident that the complete solution of this sensitivity system involves solving up to n(p+1) coupled linear differential equations.

Component Matrix Method

The CM method of finding the parameter sensitivities seems at first to be much more indirect than the CLDE method. However, after all the development of the theory is complete, it turns out that some steps can be eliminated because they serve only as convenient dividing points for the problem when solved manually. In a computer solution process, once the inverse Vandermonde matrix is found, all the sensitivities of the parameters and the original system can be determined using the elements of the matrix.

Eq (4b), the system output equation, has a time domain solution (Ref 12:373)

$$y(t)=Ce^{A(t-t_o)}x(t_o)+\int_{to}^{t}e^{A(t-\tau)}B\,u(\tau)d\tau+D\,u(t) \quad (10)$$

For simplicity of discussion, the initial time, t_o, and the feed forward matrix, D, may be set to zero. Since the system is time-invariant, the output matrix, C, can be taken outside of the integral. Thus Eq (1) becomes

$$y(t) = Ce^{At}x(o) + C\int_{o}^{t}e^{A(t-\tau)}B\,u(\tau)d\tau \quad (11)$$

The state transition matrix (STM), e^{At}, is not a function of the variable of integration, so it may be removed from the integral. Therefore

$$y(t) = Ce^{At}x(o) + Ce^{At}\int_{o}^{t}e^{-A\tau}B\,u(\tau)d\tau \quad (12)$$

Notice that the integral is now only a function of one variable and is therefore simple to compute. Such single

variable integrals are called "quadrature integrals." Notice that there are a total of nr quadrature integrals since $e^{-A\tau}$ is an n × n matrix and B is an n × r matrix.

Eq (2) established the need to find the system output parameter sensitivities, so differentiating Eq (12) w.r.t. the parameter vector, θ, yields

$$\frac{\partial y}{\partial \theta_i} = C\, e^{At}\, \frac{\partial x(o)}{\partial \theta_i} + C\, \frac{\partial}{\partial \theta_i} e^{At} x(o) + \frac{\partial}{\partial \theta_i}\, C\, e^{At} x(o)$$

$$+ C \int_o^t e^{A\tau}\, \frac{\partial}{\partial \theta_i}\, B\, u\,(\tau) d\tau$$

$$+ C \int_o^t \frac{\partial}{\partial \theta_i}\, e^{A\tau}\, B\, u(\tau) d\tau$$

$$+ \frac{\partial}{\partial \theta_i}\, C \int_o^t e^{A\tau}\, B\, u(\tau) d\tau \qquad (13)$$

Note that none of the parameter sensitivity derivatives in Eq (13) are functions of time except that of the state transition matrix:

$$e^{At}_{(i)} \equiv \frac{\partial}{\partial \theta_i}\, e^{At} \qquad (14)$$

The sensitivity derivative of the state transition matrix is potentially very costly to obtain because, in the form shown in Eq (14), it must be reevaluated for each desired point in time.

It can be shown (see Appendix C, Section 1), that the sensitivity derivative of the original state transition matrix, $e^{At}_{(i)}$, is contained in the evaluation of the state transition matrix of the augmented sensitivity system, i.e:

$$\begin{bmatrix} e^{At} & 0 \\ e_{(i)}^{At} & e^{At} \end{bmatrix}_{2n\times 2n} = e^{\begin{bmatrix} A & 0 \\ A_{(i)} & A \end{bmatrix} t} \tag{15}$$

By an application of the Cayley-Hamilton theorem (see Appendix C, section 2) the state transition matrix of the augmented sensitivity system is found to be a summation of products of the remainder polynomial coefficients and powers of the sensitivity system matrix:

$$e^{\begin{bmatrix} A & 0 \\ A_{(j)} & A \end{bmatrix} t} = \sum_{j=1}^{2n} \begin{bmatrix} A & 0 \\ A_{(i)} & A \end{bmatrix}^{j-1} a_j(t) \tag{16}$$

However

$$\begin{bmatrix} A & 0 \\ A_{(i)} & A \end{bmatrix}^j = \begin{bmatrix} A^j & 0 \\ (A^j)_{(i)} & A^j \end{bmatrix} \tag{17}$$

that is: the power of a matrix can be brought inside and applied to each element (Ref 11:64; 3:241-248). Therefore applying Eqs (15) and (17) to Eq (16) results in

$$\begin{bmatrix} e^{At} & 0 \\ e_{(i)}^{At} & e^{At} \end{bmatrix} = e^{\begin{bmatrix} A & 0 \\ A_{(i)} & A \end{bmatrix} t} = \sum_{j=1}^{2n} \begin{bmatrix} A^{j-1} & 0 \\ A_{(i)}^{j-1} & A^{j-1} \end{bmatrix} a_j(t) \tag{18}$$

Eq (18) shows that the scalar multiplication and summation operations can be applied separately to each section

of the partitioned state transition matrix of the augmented sensitivity system. So the partial derivative of the original state transition matrix, which is needed to find the parameter sensitivities according to Eq (13), can be evaluated simply as

$$e^{At}_{(i)} = \frac{\partial}{\partial \theta_i} e^{At} = \sum_{j=1}^{2n} A^{j-1}_{(i)} \, a_j(t) \qquad (19)$$

At this point, it is still necessary to find the partial derivative of 2n powers of the original system matrix, A, and solve 2n differential equations to find the remainder polynomial coefficients, $a_j(t)$. The sensitivity problem appears to have become even more complicated.

However, Reid shows (Ref 11:70) that the partial derivative of the system state transition matrix can be found from a summation of products of partial derivatives of the system "component matrices" and the system modes:

$$e^{At}_{(i)} = \sum_{j=1}^{n} (Z_{j,o})_{(i)} e^{\lambda_j t} + \sum_{j=1}^{n} (Z_{j,o}) t (\lambda_j)_{(i)} e^{\lambda_j t} \qquad (20)$$

Eq (20) is a simplification of Reid's formulation and applies for the case of distinct eigenvalues. It is used here for clarity.

Eq (20) can be extended to the augmented system:

$$e^{\left[\begin{array}{c|c} A & O \\ \hline A_{(i)} & A \end{array}\right]t} = \sum_{j=1}^{n} \left[\begin{array}{c|c} Z_{j,o} & O \\ \hline (Z_{j,o})_{(i)} & Z_{j,o} \end{array}\right] e^{\lambda_j t}$$

$$+ \sum_{j=1}^{n} \left[\begin{array}{c|c} O & O \\ \hline Z_{j,o} & O \end{array}\right] t(\lambda_j)_{(i)} e^{\lambda_j t} \quad (21)$$

It is evident from Eq (21) that for every mode, $e^{\lambda_j t}$, in the original system, there is an additional mode, $te^{\lambda_j t}$, in the augmented system. However, the additional modes occur only in the sensitivity derivative of the STM. Thus the STM of the original matrix is preserved by the CM method. Only the original modes appear in the STM for the original system, e.g.:

$$e^{At} = \sum_{j=1}^{n} (Z_{j,o}) e^{\lambda_j t} \quad (22)$$

Thus the CM method is seen to have a physical insight advantage in comparison to the CLDE method: the modes of the system are clearly visible throughout the CM solution process, whereas usually they are not in the CLDE process. But there is now a total of 2n modes, so the Vandermonde matrix which must be inverted to find the component matrices for the sensitivity system is 2n × 2n. (See

Appendix C, Section 3 for a discussion of how the component matrices are formed.) And, a total of 2nr quadrature integrals must be found.

This section theoretically compared the CLDE method of computing system parameter sensitivities to the alternate CM method proposed by Reid. It showed that when the state system is augmented to include the parameter sensitivities, the CLDE method requires the solution of up to n(p+1) coupled linear differential equations. To reduce this computational burden, reduced order models are sometimes used, but the required coordinate transformations often impair physical insight into the system and make it difficult to obtain the original system sensitivities from the reduced order sensitivities. The alternate procedure, the CM method, was shown to retain the physical significance of the system variables throughout the calculation process, and to exchange the formidable task of solving n(p+1) coupled linear differential equations for 2nr much simpler quadrature integrals, plus some matrix operations. However, the potentially costly process of inverting a $2n \times 2n$ Vandermonde matrix was also shown to be part of the CM method. Thus Section III deals with the subject of efficiently inverting the Vandermonde matrix.

III Vandermonde Matrix Inversion

Preceding sections established the need to calculate parameter sensitivities as part of a system identification process. Two methods for finding those parameter sensitivities were compared: the CLDE method and the CM method. The latter was chosen for further study based on theoretical comparisons in Section II. However, a key issue associated with this method is the need to invert a $2n \times 2n$ Vandermonde matrix. This section addresses a number of problems associated with inverting the Vandermonde matrix, and investigates a generalized method of Vandermonde inversion proposed by F. G. Csáki (Ref 1:154-157). A special feature of Csáki's generalized algorithm is also discussed. That feature is that the algorithm can be implemented on a parallel architecture processor, which for a real-time flight control system in particular, is significant.

Potential Problem Sources

There are basically five sources of problems associated with computing the inverse of the Vandermonde matrix, all of them related to the characteristics of the eigenvalues. They are:

(1) Complex eigenvalues

(2) Repeated eigenvalues

(3) Eigenvalues close to each other

(4) Eigenvalues close to the origin

(5) Combinations of very large and very small eigenvalues

Each of these potential problems is addressed in turn.

Complex Eigenvalues. Vandermonde matrices formed from complex variables may be handled three ways: (1) conversion of the complex matrix to a real one (Ref 9: Problem 6.8), (2) use of two separate sets of real variables for all calculations--one set for the real parts and the other for the imaginary parts of the complex variables, and (3) use of complex variables directly. Investigation of Csáki's generalized algorithm was desirable because of its parallel implementation capability, as is discussed later in this section. The algorithm has an inherent capability for handling complex eigenvalues, so conversion of the complex Vandermonde to a real one was unnecessary. Few control systems have entirely complex eigenvalues, so with this in mind the second method of handling complex variables was chosen. In this way, tests for complexity can be performed, and for calculations involving only real quantities, the complex part of the calculations can be omitted to save computation time.

Repeated Eigenvalues. For systems with repeated eigenvalues, the derivative equations (Eqs (C2-4, C2-5)) must be used in the formation of the Vandermonde matrix. If the Vandermonde matrix was not defined this way, any system having repeated eigenvalue would have a singular, and thus

uninvertable, Vandermonde matrix. Csáki's generalized algorithm never forms the Vandermonde matrix, but incorporates the derivative feature directly in the formation of the inverse Vandermonde matrix.

Nearly Identical or Nearly Zero Eigenvalues. Eigenvalues that are close to each other or close to the origin may cause numerical instability because of the finite word length of digital computers. Algorithms using differences of eigenvalues, which Caśki's generalized algorithm does, are especially susceptible to this problem. The accuracy tests of this thesis therefore include some systems with close eigenvalues and some nearly zero eigenvalues.

Eigenvalues With Large Magnitude Differences. Combinations of eigenvalues with large differences in magnitude may cause scaling problems in the calculations. Some tests to investigate this problem are also included in the thesis.

Csáki's Generalized Algorithm

F. G. Csáki proposed (Ref 1:154-157) an algorithm for finding the inverse of a Vandermonde matrix directly from a collection of eigenvalues and their multiplicities. The Vandermonde matrix itself is never formed by the algorithm. Since the algorithm finds each element of the inverse Vandermonde matrix independently, it may be implemented on a parallel architecture computer to increase the real-time computation speed. This is discussed in more detail in the

next part of this section. Many inversion techniques cannot be implemented in parallel fashion because they do simple row or volumn operations rather than operations involving individual elements.

Csáki's generalized algorithm partitions the inverse Vandermonde matrix into m blocks, W_i, (i = 1, 2, ... m), with one block for each distinct eigenvalue. Each block is dimensioned k_i rows by n columns, where k_i is the eigenvalue multiplicity and n is the system order. The Vandermonde inverse may thus be represented by

$$V^{-1} = W = \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ \cdot \\ \cdot \\ \cdot \\ W_m \end{bmatrix}_{n \times n} \tag{23}$$

where

$$W_i = \begin{bmatrix} W_1^{T(i)} \\ W_2^{T(i)} \\ \cdot \\ \cdot \\ \cdot \\ W_{k_i}^{T(i)} \end{bmatrix}_{k_i \times n} \tag{24}$$

in which $W_j^{T(i)}$ (j = 1, 2, . . ., k_i) are the row vectors of the block W_i.

With this formulation and nomenclature, each element of the inverse Vandermonde matrix may be determined from

$$w_{j\ell}^{(i)} = \frac{1}{(k_i-j)!} \frac{d^{(k_i-j)}}{ds^{(k_i-j)}} \frac{N_\ell(s)}{D_i(s)} \Bigg|_{s=\lambda_i} \tag{25}$$

in which $w_{j\ell}^{(i)}$ represents the element in the j'th row and ℓ'th column of the i'th block, W_i. The fraction terms are defined as follows:

$$D_i(s) = \frac{D(s)}{(s-\lambda_i)^{k_i}} ;(i = 1, 2, \ldots m) \tag{26}$$

and

$$N_\ell(s) = s^{n-1} + a_{n-1}s^{n-\ell-1} + \ldots + a_{\ell+1}s + a_\ell ; \tag{27}$$

$$(\ell = 1, 2, \ldots n)$$

D(s) is the system characteristic equation, which may be represented as

$$D(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_o = \prod_{i-1}^{m} (s-\lambda_i)^{k_i} \tag{28}$$

Eq (26), the i'th "denominator polynomial," is therefore the characteristic equation with the i'th eigenvalue divided out completely. The terms defined by Eq (27) are called "truncated polynomials" and are formed by iteratively reducing by one the power of each term of the characteristic equation, Eq (28), and deleting the right-

most term until a total of n truncated polynomials are found. For example, suppose a third order system has the characteristic polynomial

$$D(s) = s^3 + 4s^2 - s + 3 \tag{29}$$

The n = 3 truncated polynomials are

$$N_1(s) = s^2 + 4s - 1 \tag{30a}$$

$$N_2(s) = s + 4 \tag{30b}$$

$$N_3(s) = 1 \tag{30c}$$

Note that the final truncated polynomial is always the integer one because of the form of D(s) used. Also notice from Eq (25) that the derivative procedure which is part of the definition of the Vandermonde matrix (see Appendix C, Section 2) is included in the formula for the elements of the inverse matrix.

In order to draw the foregoing together, suppose a sixth order system has three eigenvalues, having multiplicities of three, two, and one, respectively. Substituting Eq (25) into Eq (24), and the result into Eq (23) yields the following structure for the inverse Vandermonde matrix:

$$V^{-1} = W = \begin{bmatrix} \\ W_1 \\ \\ - - \\ \\ W_2 \\ - - \\ W_3 \end{bmatrix} = \left[\begin{array}{cccccc} W_{11}^{(1)} & W_{12}^{(1)} & W_{13}^{(1)} & W_{14}^{(1)} & W_{15}^{(1)} & W_{16}^{(1)} \\ W_{21}^{(1)} & W_{22}^{(1)} & W_{23}^{(1)} & W_{24}^{(1)} & W_{25}^{(1)} & W_{26}^{(1)} \\ W_{31}^{(1)} & W_{32}^{(1)} & W_{33}^{(1)} & W_{34}^{(1)} & W_{35}^{(1)} & W_{36}^{(1)} \\ \hline W_{11}^{(2)} & W_{12}^{(2)} & W_{13}^{(2)} & W_{14}^{(2)} & W_{15}^{(2)} & W_{16}^{(2)} \\ W_{21}^{(2)} & W_{22}^{(2)} & W_{23}^{(2)} & W_{24}^{(2)} & W_{25}^{(2)} & W_{26}^{(2)} \\ \hline W_{11}^{(3)} & W_{12}^{(3)} & W_{13}^{(3)} & W_{14}^{(3)} & W_{15}^{(3)} & W_{16}^{(3)} \end{array}\right] \quad (31)$$

A simple numerical example of the application of this algorithm is in Appendix D, Example 3.

Parallel Architecture Processors and Csáki's Generalized Algorithm

If a large problem can be broken up into a number of smaller independent ones, parallel processing can be used to reduce the real process time. No computational advantage is realized, but since several calculations can be done simultaneously, the clock time from start to finish of the large problem is less than if each calculation were done serially. This is particularly important for flight control applications where real processing time may be critical.

For a comparison between serial and parallel process times, suppose a third order system has two eigenvalues, one with a multiplicity of two. A serial processor would perform the Vandermonde inversion with Csáki's algorithm

as shown in Figure 2a. If each calculation takes p seconds of processing time, the minimum total time is 15p because a calculation cannot begin until the preceding one is finished. Figure 2b shows the same calculations as a parallel processor could perform them. Notice that the total time from start to finish of the group of calculations is a third of the amount of time the serial processing took because the parallel processing system has three times as many processors to do the job. More information on parallel processing is found in Ref 4.

Not only does Csáki's method for finding the inverse of the Vandermonde matrix lend itself to parallel processing, so does the component matrix approach itself, as Figure 3 shows. At the top of the figure is $\underline{\hat{\theta}}$, the current estimate of the system parameter vector. From that estimate, the system plant model, $\sigma(A)$, can be found, and from it, the current eigenvalues, λ_1, λ_2, ..., λ_n, can be determined. As shown, each eigenvalue can be used in parallel to compute the $1 \times 2nr$ vector, F, of quadrature integrals, while at the same time computing the elements of the inverse Vandermonde matrix which correspond to that eigenvalue. At this point, the value of the parallel computation structure of Csáki's algorithm is readily visible. If the algorithm to find the inverse Vandermonde matrix did not have a parallel structure, none of the calculations using the elements of the inverse Vandermonde matrix could proceed until the entire matrix was computed in serial form.

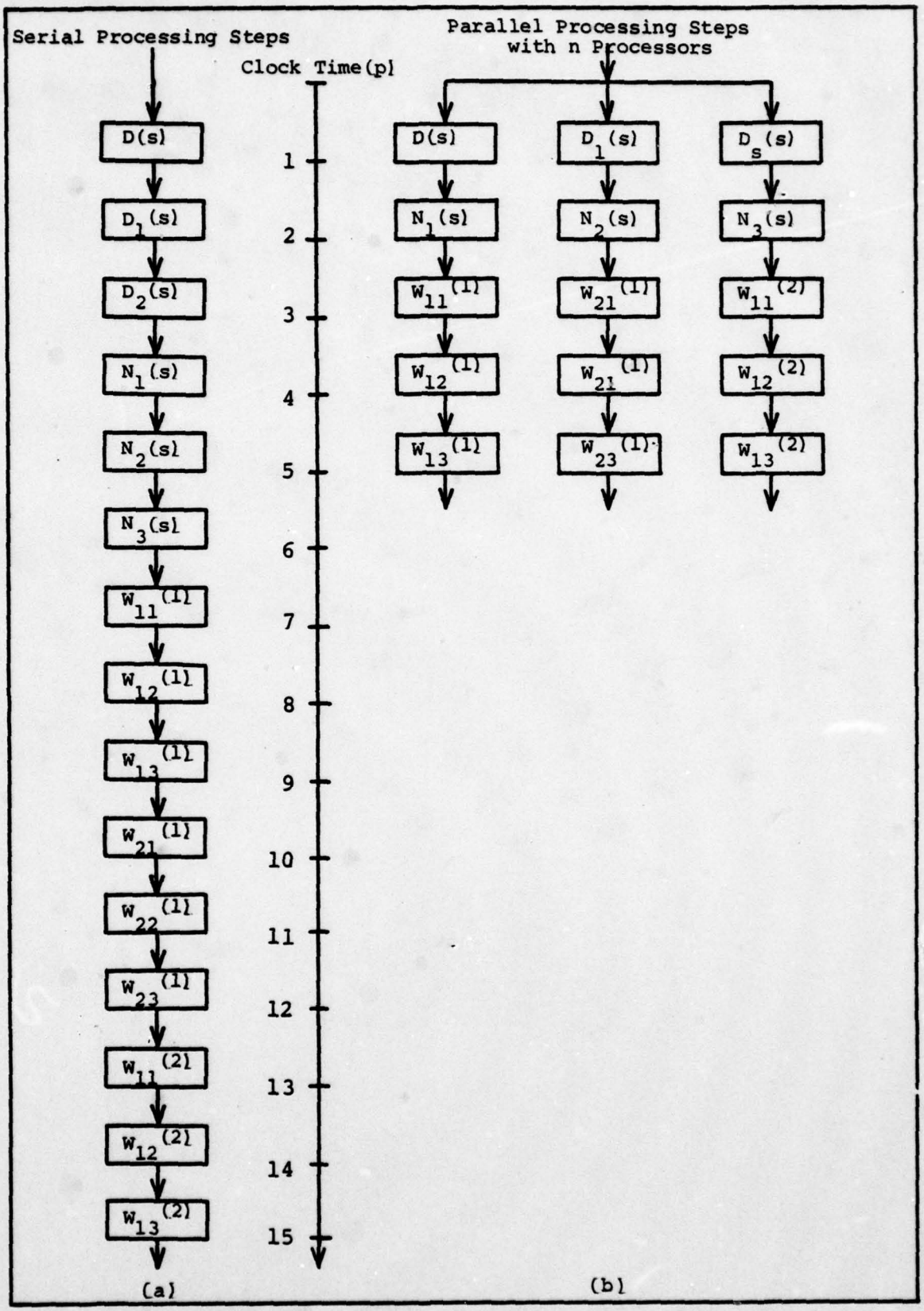

Figure 2. Processing time comparison:
a) Serial Processor,
b) Parallel Processor.

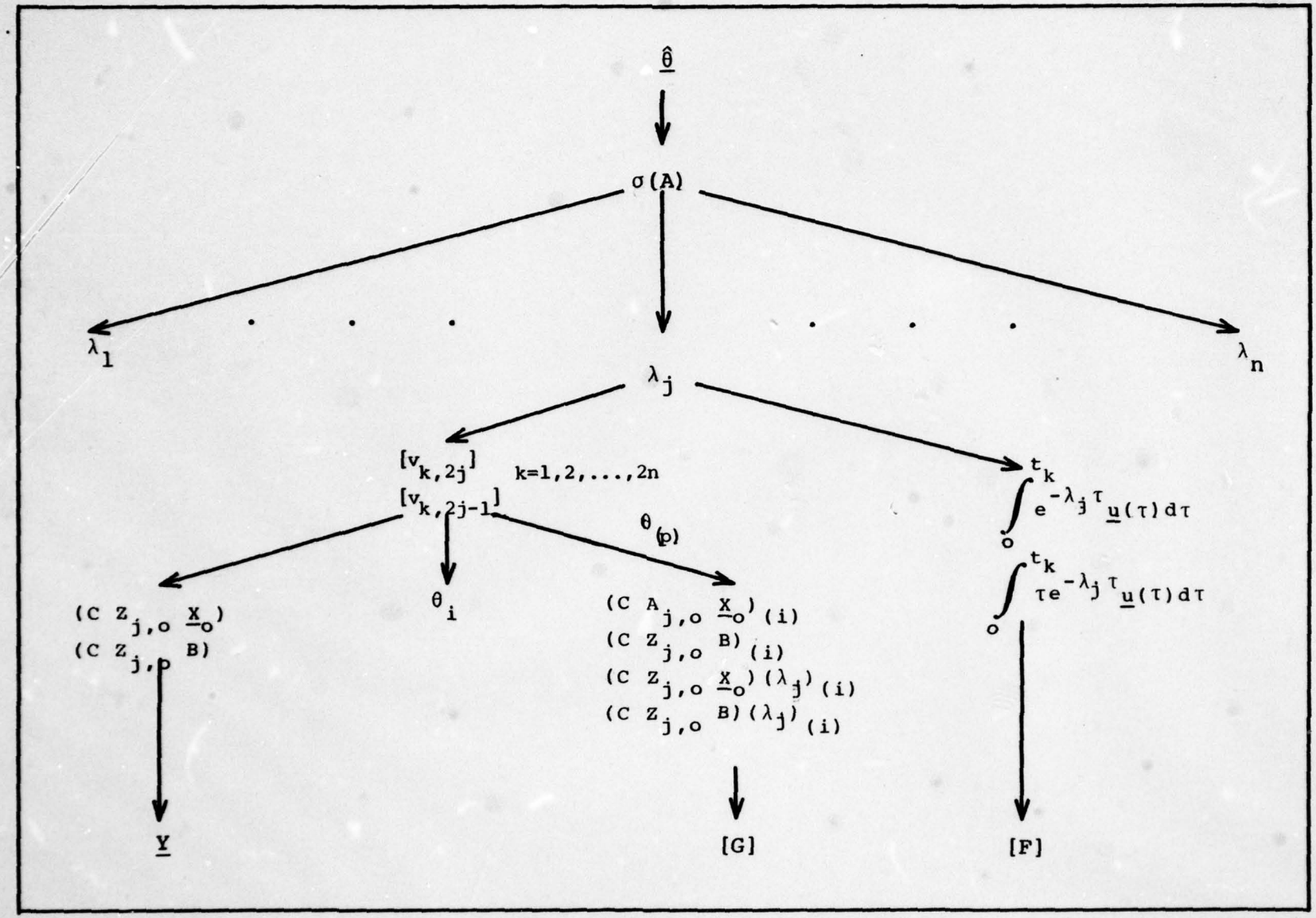

Figure 3. Parallel computation structure of the component matrix method of system identification. (Adapted from slides used to present Ref 10.)

That could stretch out the overall computation cycle time considerably, as Figure 2 showed. However, with the parallel structure of Csáki's algorithm, each eigenvalue branch of Figure 2 may find the elements of the inverse Vandermonde matrix pertaining to that eigenvalue and continue with the next calculations immediately. From the calculated inverse Vandermonde matrix elements, the vector of component matrices, $\underline{Y}$, and matrix of component sensitivities, G, can be computed. A change in the system parameters, $\Delta\theta$, can then be computed from (Ref 10)

$$\underline{Y} = [F] \cdot [G]^T \Delta\underline{\theta} \qquad (32)$$

where Y, [F], and [G] are defined as in Figure 3. It is now possible to use $\Delta\underline{\theta}$ to update the estimate of the parameter vector, $\hat{\underline{\theta}}$. This closes the computational loop, making the system identification process iterative, and thereby capable of applications in on-line flight controllers.

All of this sounds nice in theory. The immediate question that arises concerns whether or not the method actually works when implemented on a computer with a practical problem like limited word length which causes truncation and round off errors in the calculations performed. Therefore Csáki's algorithm was encoded in Fortram and some tests were run to evaluate its accuracy and efficiency. Those tests are the subject of the next section.

IV Tests Run on the Coded Version of Csáki's Algorithm

Previous sections of this thesis dealt with the theory behind a system identification process. The CM method for system identifications was shown to have theoretical computational advantages over the more straightforward CLDE method. However, part of the CM method is the inversion of a $2n \times 2n$ Vandermonde matrix, which can be computationally costly. Csáki's generalized algorithm for finding the inverse Vandermonde matrix was chosen for its parallel computation structure. This structure gives the algorithm the capability of operating much faster in real time than algorithms which must perform all the calculations in serial form. Also, the particular parallel structure of Csáki's algorithm is very compatible with the parallel structure of the CM method of system identification.

Csáki's generalized algorithm was coded in Fortran in a subroutine called "VANINV", and is hereafter referred to as such. Appendix A contains descriptions and flowcharts of VANINV and the subroutines associated with it. Part I of Appendix B contains source listings of all the subroutines in Appendix A. This section describes tests done subroutines. Source listings of the test programs, though not flowcharted, are found in Appendix B, Part II.

Types of Tests

Two types of tests were performed on VANINV. The first was actually a combination of four separate tests to determine VANINV's computational accuracy. The second type of test measured how long it took VANINV to find the total inverse Vandermonde matrix for a given system. These two types of tests were run on a number of different systems to determine what happens to the computation time and accuracy when: (1) the system order increases, (2) two eigenvalues become very close, causing the Vandermonde matrix to become more nearly singular and therefore harder to invert, (3) one eigenvalue approaches zero, which also makes the Vandermonde matrix more nearly singular, and (4) the difference between the magnitudes of the largest and smallest eigenvalue increases. The ratio, $C = \left|\frac{\lambda_{max}}{\lambda_{min}}\right|$, in (4) is called the "condition index."

In order to obtain a relative measure of VANINV's performance, the same tests with the same systems were run using a standard matrix inversion routine contained in the International Mathematical & Statistical Library (IMSL). Early tests were done with IMSL subroutine LINV1F, but it did not have sufficient accuracy, so the final tests were done using IMSL subroutine LINV2F, which has a capability for iterative improvement of the solution.

Accuracy Tests

The first accuracy test is actually only a test of the IMSL routine. It simply involves feeding the inverse

Vandermonde matrix back into the routine, and comparing the reinverted matrix to the original Vandermonde matrix. It is unlikely that the accuracy of the reinverted matrix is better than that of the first inversion. Thus a lower accuracy bound on the inversion is the minimum number of decimal places in the reinverted matrix that are identical to the original matrix.

Originally a second accuracy test was set up to multiply the Vandermonde matrix by the calculated inverse and subtract the identity matrix:

$$V V_c^{-1} - I = E \tag{33}$$

in which the subscript "c" means "calculated." Theoretically, E should always be a null matrix. The extent to which it is not zero gives an indication of the accuracy of the calculation of the inverse, V_c^{-1}. However, E is typically composed of numbers so small that the errors caused by the computational characteristics of the algorithm cannot be distinguished from the truncation and round off errors resulting from the finite word length of the computer.

The following analysis may be made of the problem (Ref 6:61-64). What is really desired is the error between the true inverse and the actual inverse, defined as:

$$A_c^{-1} - A^{-1} = \text{Error} \tag{34}$$

Now

$$A(A_c^{-1}-A^{-1}) = AA_c^{-1}-AA^{-1}=AA_c^{-1}-I = E \tag{35}$$

from Eq (33). Then Eq (34) can be expressed

$$(A_c^{-1}-A^{-1}) = \text{Error} = A^{-1}(AA_c^{-1}-I) = A^{-1}E \tag{36}$$

Tests with matrices having known inverses show that the absolute value of the smallest negative exponent in the elements of $A^{-1}E$ approximates the minimum number of decimal places in A_c^{-1} that are accurate. For example, if the elements in $A^{-1}E$ are .xxxxxE-10, the elements of A_c^{-1} may be considered accurate to at least ten decimal places. This may be expressed as follows:

let α_{ij} be the elements of A^{-1}

e_{ij} be the elements of E

β_{ij} be the elements of $A^{-1}E$

Therefore

$$\beta_{ij} = \sum_{k=1}^{n} \alpha_{ik}\, e_{kj} \tag{37}$$

and a bound on the error of α_{ij} is given by

$$| \beta_{ij} | \leq \sum_{k=1}^{n} | \alpha_{ik} | \, | e_{kj} | \tag{38}$$

However, $\alpha_{ik} \in A^{-1}$ is not available; so use $\gamma_{ik} \in A_c^{-1}$ and use the following approximation for the error bound:

$$|\rho_{ij}| \leq \sum_{k=1}^{n} |\gamma_{ik}||e_{kj}| \qquad (39)$$

hoping that the actual error of the element is bounded by ρ_{ij}:

$$|\alpha_{ik} - \gamma_{ik}| < |\rho_{ij}| \qquad (40)$$

When this test was applied to systems with known solutions, Eq (40) predicted inversion accuracy to within three decimal places for systems whose condition index was less than about 100, unless there were two eigenvalues closer than .2 units to each other.

The third test used was simply a summation of the elements, β_{ij}, of the check matrix, $A^{-1}E$, from test two. It was an attempt to quantify in a single number the accuracy of the inversion method being tested. In this thesis it is referred to as an "Accuracy Merit Figure" (AMF). The problem with it is that it is very dependent on how many elements of A_c^{-1} are nonzero. It tends, therefore, to be a cumulative, rather than an absolute (i.e.: "decimal place") accuracy test. Note, however, that the AMF can be used in conjunction with the largest element of the check matrix to carry the information contained in the matrix in two numbers rather than the n^2 elements of the matrix. Tests showed that if the AMF is much larger than the largest element of the check matrix (i.e.: AMF $>>(\beta_{ij})_{max}$), it indicates that many elements of the matrix are nearly the same size as $(\beta_{ij})_{max}$ and probably all the elements of the inverse matrix, V^{-1}, are accurate to the same number of decimal places.

However, if AMF $\simeq (\beta_{ij})_{max}$, one element of the check matrix is dominant, and most of the other elements of V^{-1} are more accurate than $(\beta_{ij})_{max}$ indicates. In this situation, $(\beta_{ij})_{max}$ is definitely a worst case error indicator.

Fourth and final of the accuracy tests used was a subjective comparison of the inverse Vandermonde matrix calculated by VANINV and the IMSL routine. A number of the systems used in the tests had known solutions, or had repeating decimals in the solution. The decimal place accuracy of each inversion was therefore estimated on the basis of these characteristics of the solutions.

A total of 43 different systems were designed in order to test VANINV and the IMSL routines. The tests were run on a CDC 6600 computer using single precision arithmetic. Results of the tests are given in the next section.

V Results of Tests

The total of 43 test systems (see Table I) which were run were divided among the following five application categories:

1. Computation time as a function of system order (Tests 1 through 43).
2. Accuracy as a function of system order (Tests 1 through 11).
3. Accuracy as a function of the distance between the closest two eigenvalues of the system, $|\lambda_x - \lambda_y|_{min}$ (Tests 12 through 20).
4. Accuracy as a function of the condition index, $C = \frac{|\lambda_{max}|}{|\lambda_{min}|}$ (Tests 21 through 27).
5. Accuracy as a function of increasing condition index, C, as the smallest eigenvalue, $|\lambda_{min}|$, of the system approaches zero (Tests 28 through 43).

Table II contains all the data from the tests, and the results are explained according to the categories above.

Computation Time

Tests 1 through 11 indicate that the computation time for finding the inverse Vandermonde matrix is proportional to a number between n^2 and n^3. This is to be expected since the number of elements to be computed is n^2 and some elements take more calculations than others. Figure 4 shows this relation (computation time between n = 8 and

Table I

Test Systems

Test No.	System Order	Eigenvalues			
1	2	λ_1= .5+j0.			
		λ_2= -.5+j0.			
2	3	λ_1= .5+j0.			
		λ_2= .5+j0.			
		λ_3= 0.			
3	4	λ_1= 1. +j0.	λ_3 = .5+j0.		
		λ_2= 1. +j0.	λ_4 = .5+j0.		
4	6	λ_1= 1.5+j0.	λ_3 = 1. +j0.	λ_5 = .5+j0.	
		λ_2=-1.5+j0.	λ_4 =-1. +j0.	λ_6 = -.5+j0.	
5	8	λ_1= 2. +j0.	λ_3 = 1.5+j0.	λ_5 = 1. +j0.	λ_7 = .5+j0.
		λ_2=-2. +j0.	λ_4 =-1.5+j0.	λ_6 =-1. +j0.	λ_8 = -.5+j0.
6	10	λ_1= 2.5+j0.	λ_4 =-2. +j0.	λ_6 = 1. +j0.	λ_9 = -.5+j0.
		λ_2=-2.5+j0.	λ_5 = 1.5+j0.	λ_7 =-1. +j0.	
		λ_3= 2. +j0.	λ_6 =-1.5+j0.	λ_8 = .5+j0.	
7	12	λ_1= 3. +j0.	λ_4 = 2.5+j0.	λ_7 = 1.5+j0.	λ_{10}=-1. +j0.
		λ_2=-3. +j0.	λ_5 = 2. +j0.	λ_8 =-1.5+j0.	λ_{11}= .5+j0.
		λ_3= 2.5+j0.	λ_6 =-2. +j0.	λ_9 = 1. +j0.	λ_{12}= -.5+j0.
8	14	λ_1= 3.5+j0.	λ_5 = 2.5+j0.	λ_9 = 1.5+j0.	λ_{13}= .5+j0.
		λ_2=-3.5+j0.	λ_6 =-2.5+j0.	λ_{10}=-1.5+j0.	λ_{14}= -.5+j0.
		λ_3= 3. +j0.	λ_7 = 2. +j0.	λ_{11}= 1. +j0.	
		λ_4=-3. +j0.	λ_8 =-2. +j0.	λ_{12}=-1. +j0.	
9	16	λ_1= 4. +j0.	λ_5 = 3. +j0.	λ_9 = 2. +j0.	λ_{13}= 1. +j0.
		λ_2=-4. +j0.	λ_6 =-3. +j0.	λ_{10}=-2. +j0.	λ_{14}=-1. +j0.
		λ_3= 3.5+j0.	λ_7 = 2.5+j0.	λ_{11}= 1.5+j0.	λ_{15}= .5+j0.
		λ_4=-3.5+j0.	λ_8 =-2.5+j0.	λ_{12}=-1.5+j0.	λ_{16}= -.5+j0.
10	18	λ_1= 4.5+j0.	λ_6 =-3.5+j0.	λ_{11}= 2. +j0.	λ_{16}=-1. +j0.
		λ_2=-4.5+j0.	λ_7 = 3. +j0.	λ_{12}=-2. +j0.	λ_{17}= .5+j0.
		λ_3= 4. +j0.	λ_8 =-3. +j0.	λ_{13}= 1.5+j0.	λ_{18}= -.5+j0.
		λ_4=-4. +j0.	λ_9 = 2.5+j0.	λ_{14}=-1.5+j0.	
		λ_5= 3.5+j0.	λ_{10}= 2.5+j0.	λ_{15}= 1. +j0.	
11	20	λ_1= 5. +j0.	λ_6 = 4. +j0.	λ_{11}= 2.5+j0.	λ_{16}=-1.5+j0.
		λ_2=-5. +j0.	λ_7 = 3.5+j0.	λ_{12}=-2.5+j0.	λ_{17}= 1. +j0.
		λ_3= 4.5+j0.	λ_8 =-3.5+j0.	λ_{13}= 2. +j0.	λ_{18}=-1. +j0.
		λ_4=-4.5+j0.	λ_9 = 3. +j0.	λ_{14}=-2. +j0.	λ_{19}= .5+j0.
		λ_5= 4. +j0.	λ_{10}=-3. +j0.	λ_{15}= 1.5+j0.	λ_{16}= -.5+j0.

Table I--continued

Test No.	System Order	Eigenvalues	
12	4	λ_1=-10.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-4.+j0.
13	4	λ_1=-10.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-4.2+j0.
14	4	λ_1=-10.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-4.4+j0.
15	4	λ_1=-10.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-4.6+j0.
16	4	λ_1=-10.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-4.8+j0.
17	4	λ_1=-10.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-4.9+j0.
18	4	λ_1=-10.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-4.99+j0.
19	4	λ_1=-10.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-4.999+j0.
20	4	λ_1=-10.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-4.9999+j0.
21	4	λ_1=-100.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-4.+j0.
22	4	λ_1=-1000.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-4.+j0.
23	4	λ_1=-10000.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-4.+j0.
24	4	λ_1=-100000.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-4.+j0.
25	4	λ_1=-1000000.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-4.+j0.
26	4	λ_1=-10000000.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-4.+j0.
27	4	λ_1=-100000000.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-4.+j0.
28	4	λ_1=-10.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-1.+j0.
29	4	λ_1=-10.+j0.	λ_3=-5.+j0.
		λ_3= -9.+j0.	λ_4=-.8+j0.

Table I--continued

Test No.	System Order	Eigenvalues	
30	4	λ_1=-10.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-.6+j0.
31	4	λ_1=-10.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-.4+j0.
32	4	λ_1=-10.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-.2+j0.
33	4	λ_1=-10.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-.1+j0.
34	4	λ_1=-10.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-.01+j0.
35	4	λ_1=-10.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-.001+j0.
36	4	λ_1=-10.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-.0001+j0.
37	4	λ_1=-10.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4= 0.+j0.
38	4	λ_1=-1000.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-1.+j0.
39	4	λ_1=-1000.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-.1+j0.
40	4	λ_1=-1000.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-.01+j0.
41	4	λ_1=-1000.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-.001+j0.
42	4	λ_1=-1000.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4=-.0001+j0.
43	4	λ_1=-1000.+j0.	λ_3=-5.+j0.
		λ_2= -9.+j0.	λ_4= 0.+j0.

Table II

Test Results

Test No.	System Order	No. of Iterations	Computation Time (Sec)		Accuracy											
					Accuracy Merit Figure		No. of Significant Digits in V^{-1} †		No. of Significant Digits in (V^{-1})†	Largest Non-Zero Element in CHECK		Bound on No. of Significant Digits in V^{-1} ‡		Condition No.	$\lvert\lambda_x-\lambda_y\rvert$ minimum	$\lvert\lambda_x\rvert$ minimum
			VANINV	IMSL	VANINV	IMSL	VANINV	IMSL	(IMSL ONLY)	VANINV	IMSL	VANINV	IMSL	$\frac{\lvert\lambda\rvert_{max}}{\lvert\lambda\rvert_{min}}$		
1	2	30	.07	.04	0.	0.	14	14	14	0.	0.	N/A	N/A	1	.5	.5
2	3	30	.14	.06	0.	0.	14	14	14	0.	0.	N/A	N/A	∞	.5	.0
3	4	30	.26	.18	0.	0.	14	14	13	0.	0.	N/A	N/A	2	.5	.5
4	6	30	.60	.53	8.21×10^{-14}	1.10×10^{-13}	14	14	13	1.42×10^{-14}	1.65×10^{-14}	13	13	3	.5	.5
5	8	30	1.14	.99	6.35×10^{-14}	1.35×10^{-14}	14	14	11	1.03×10^{-14}	1.03×10^{-14}	13	13	4	.5	.5
6	10	30	1.88	1.68	2.54×10^{-12}	2.54×10^{-12}	14	14	12	2.16×10^{-13}	2.16×10^{-13}	12	12	5	.5	.5
7	12	30	2.87	2.65	8.96×10^{-12}	8.96×10^{-12}	14	14	0*	6.48×10^{-13}	6.48×10^{-13}	12	12	6	.5	.5
8	14	30	4.13	3.87	2.86×10^{-11}	2.87×10^{-11}	14	14	10	1.32×10^{-12}	1.37×10^{-12}	11	11	7	.5	.5
9	16	30	5.56	5.65	8.62×10^{-11}	8.92×10^{-11}	13	13	9	3.74×10^{-12}	4.33×10^{-12}	11	11	8	.5	.5
10	18	30	7.43	7.45	9.72×10^{-6}	2.35×10^{-10}	13	13	8	7.14×10^{-7}	7.51×10^{-12}	6	11	9	.5	.5
11	20	30	9.68	11.12	5.17×10^{-5}	5.79×10^{-10}	10	12	6	3.90×10^{-6}	2.02×10^{-11}	5	10	10	.5	.5
12	4	100	.86	.82	1.68×10^{-12}	1.68×10^{-12}	14	14	12	5.37×10^{-12}	5.37×10^{-12}	11	11	2.5	1.	4.0
13	4	100	.87	.79	4.21×10^{-9}	2.89×10^{-11}	12	14	12	7.45×10^{-10}	9.96×10^{-13}	9	11	2.4	.8	4.2
14	4	100	.86	.77	7.92×10^{-10}	2.60×10^{-11}	14	13	11	2.05×10^{-10}	7.43×10^{-12}	9	11	2.3	.6	4.4
15	4	100	.86	.80	9.77×10^{-9}	6.84×10^{-11}	12	12	11	2.75×10^{-9}	2.15×10^{-11}	8	10	2.2	.4	4.6
16	4	100	.86	.80	7.00×10^{-8}	5.02×10^{-10}	14	13	11	2.14×10^{-8}	1.84×10^{-10}	7	9	2.1	.2	4.8
17	4	100	.90	.31	6.10×10^{-7}	8.37×10^{-9}	13	11	10	1.99×10^{-7}	7.58×10^{-9}	6	8	2.0	.1	4.9
18	4	100	.87	.32	1.55×10^{-5}	5.72×10^{-7}	10	10	9	5.24×10^{-4}	1.54×10^{-7}	3	6	2.0	.01	4.99
21	4	100	.86	.83	1.07×10^{-12}	3.03×10^{-12}	14	14	11	2.82×10^{-13}	1.32×10^{-12}	12	11	25	1.	4.
22	4	100	.87	.77	1.21×10^{-12}	1.21×10^{-12}	same		9	5.67×10^{-13}	5.67×10^{-13}	12	12	2.5×10^{2}	1.	4.
23	4	100	.87	.88	1.19×10^{-12}	1.19×10^{-12}	same		7	5.17×10^{-13}	5.17×10^{-13}	12	12	2.5×10^{3}	1.	4.
24	4	100	.86	1.12	1.82×10^{-12}	1.82×10^{-12}	same		4	6.96×10^{-13}	6.96×10^{-13}	12	12	2.5×10^{4}	1.	4.
25	4	100	.88	3.46	1.80×10^{-4}	118.	7	0	0*	9.00×10^{-5}	88.4	4	0	2.5×10^{5}	1.	4.
26	4	100	.87	3.90	.419	118.	7	0	0*	.210	88.4	0	0	2.5×10^{6}	1.	4.
27	4	100	.88	.336	2.07	10^{24}	7	0	0*	9.00×10^{-7}	10^{24}	6	0	2.5×10^{7}	1.	4.

† By subjective observation.

‡ By second accuracy test.

* IMSL had a terminal error.

Table II

Test Results

Table II--continued

28	4	100	.89	.92	7.49×10^{-13}	2.62×10^{-12}	14	14	10	1.85×10^{-13}	9.81×10^{-13}	12	12	10	1.	1.
29	4	100	.87	.77	2.52×10^{-13}	2.08×10^{-12}	12	14	11	5.66×10^{-11}	7.26×10^{-13}	10	12	12	1.	.8
30	4	100	.86	.78	2.91×10^{-10}	1.01×10^{-12}	12	14	11	5.62×10^{-11}	2.94×10^{-13}	10	12	16	1.	.6
31	4	100	.86	.78	1.97×10^{-10}	1.72×10^{-12}	12	14	10	3.99×10^{-11}	6.35×10^{-13}	10	12	25	1.	.4
32	4	100	.88	.79	1.36×10^{-10}	4.76×10^{-13}	13	14	9	2.99×10^{-11}	9.11×10^{-14}	10	13	50	1.	.2
33	4	100	.88	.79	1.79×10^{-10}	5.02×10^{-13}	10	14	8	3.76×10^{-11}	1.38×10^{-13}	10	12	10^2	1.	.1
34	4	100	.84	.79	9.38×10^{-11}	7.34×10^{-13}	9	12	6	2.97×10^{-11}	2.06×10^{-13}	10	12	10^3	1.	.01
35	4	100	.87	.77	1.25×10^{-10}	3.54×10^{-13}	8	12	5	3.53×10^{-11}	1.44×10^{-13}	10	12	10^4	1.	.001
36	4	100	.86	.80	6.35×10^{-11}	6.37×10^{-13}	5	10	3	1.91×10^{-11}	1.74×10^{-13}	10	12	10^5	1.	.0001
37	4	100	.85	.30	2.04×10^{-13}	1.54×10^{-12}	13	13	12	6.20×10^{-14}	5.31×10^{-13}	13	12	∞	1.	0.
38	4	100	.86	.79	3.66×10^{-14}	3.67×10^{-14}	14	14	9	1.16×10^{-14}	1.16×10^{-14}	13	13	10^3	4.	1.
39	4	100	.86	.78	8.02×10^{-10}	1.88×10^{-15}	?	?	11	4.00×10^{-10}	4.42×10^{-16}	9	15	10^4	4.	1.
40	4	100	.88	.80	1.42×10^{-9}	9.79×10^{-15}	?	?	9	7.08×10^{-10}	4.47×10^{-15}	9	14	10^5	4.	.01
41	4	100	.87	.80	1.16×10^{-9}	4.43×10^{-15}	?	?	6	5.81×10^{-10}	1.65×10^{-15}	9	14	10^6	4.	.001
42	4	100	.87	.80	1.30×10^{-9}	4.12×10^{-15}	?	?	4	6.47×10^{-10}	1.63×10^{-15}	9	14	10^7	4.	.0001
43	4	100	.87	.72	1.13×10^{-15}	1.15×10^{-15}	14	14	10	1.03×10^{-15}	1.02×10^{-15}	14	14	∞	4.	0.

+ By subjective observation.

± By second accuracy test.

* IMSL had a terminal error.

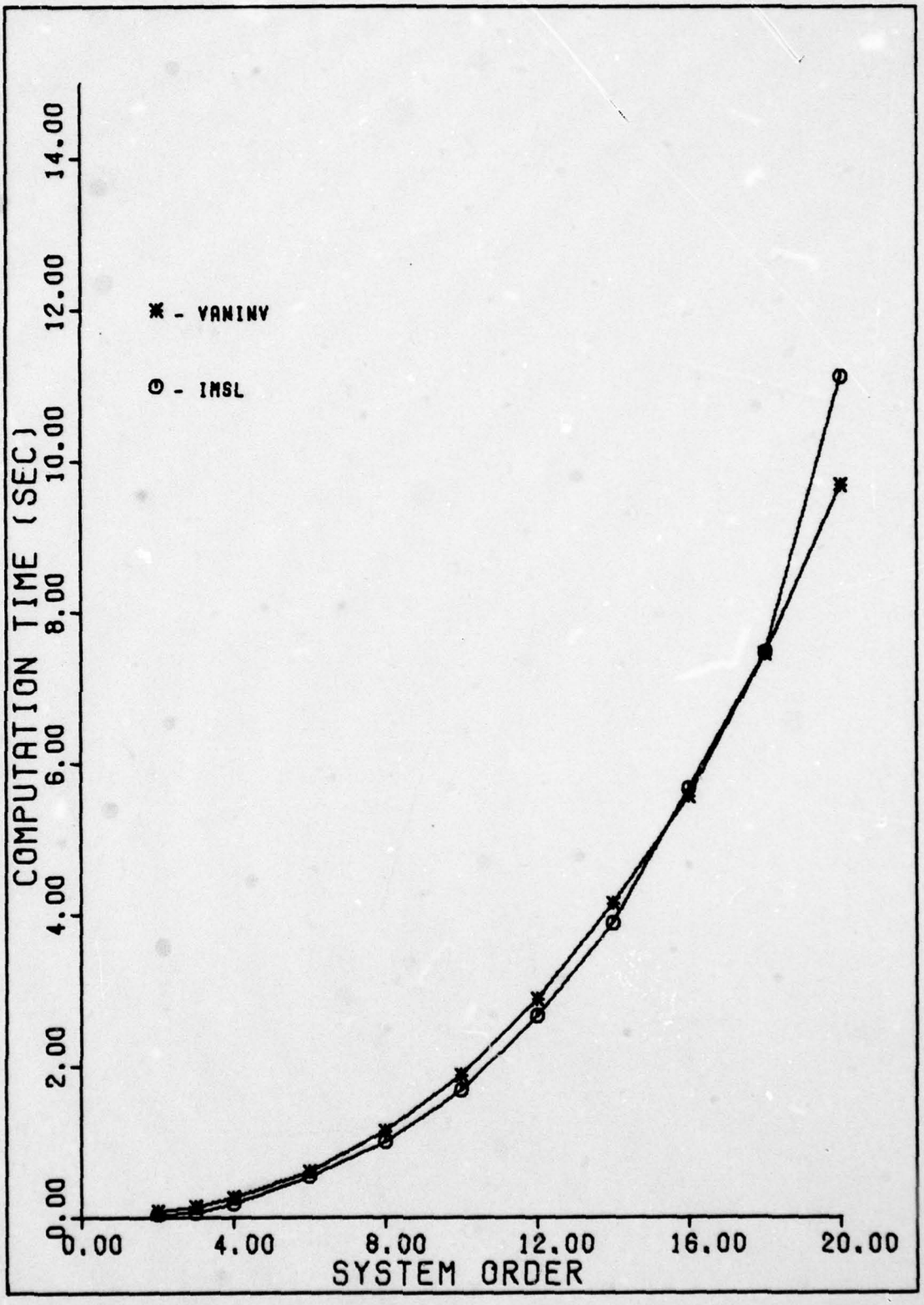

Figure 4. Computation time as a function. of system order (30 iterations with real eigenvalues).

n = 16 is proportional to about $n^{2.5}$) and also shows that the IMSL routines tended to run only slightly faster than VANINV. The remaining tests were all done with fourth order systems, and computation times remained nearly constant in every case for VANINV.

Accuracy as a Function of System Order

Figure 5 shows graphically what the results of Tests 1 through 11 indicate--accuracy is essentially constant at the maximum machine capability (14 decimal places for a CDC 6600 machine) over the range of system orders tested.

The second accuracy bound test indicates that although the full 14 decimal place accuracy is present up through the 14th order system, the probability of this being so is diminished as the system order increases. This may be explained as follows. The number of calculations done to find each element of the inverse Vandermonde matrix increases approximately to the n^2 to n^3 power with system order, as was shown in the preceding result. With the increase in the number of calculations, truncation and round off errors from the calculations may accumulate sufficiently to appear to propagate into the more significant decimal places. Since the second accuracy test uses absolute values of error quantities in determining a lower found on the accuracy, the test is actually a calculation of the worst possible accuracy, not the most probable accuracy. This hypothesis would be supported if these tests were run in double precision arithmetic and the accuracy tracked along

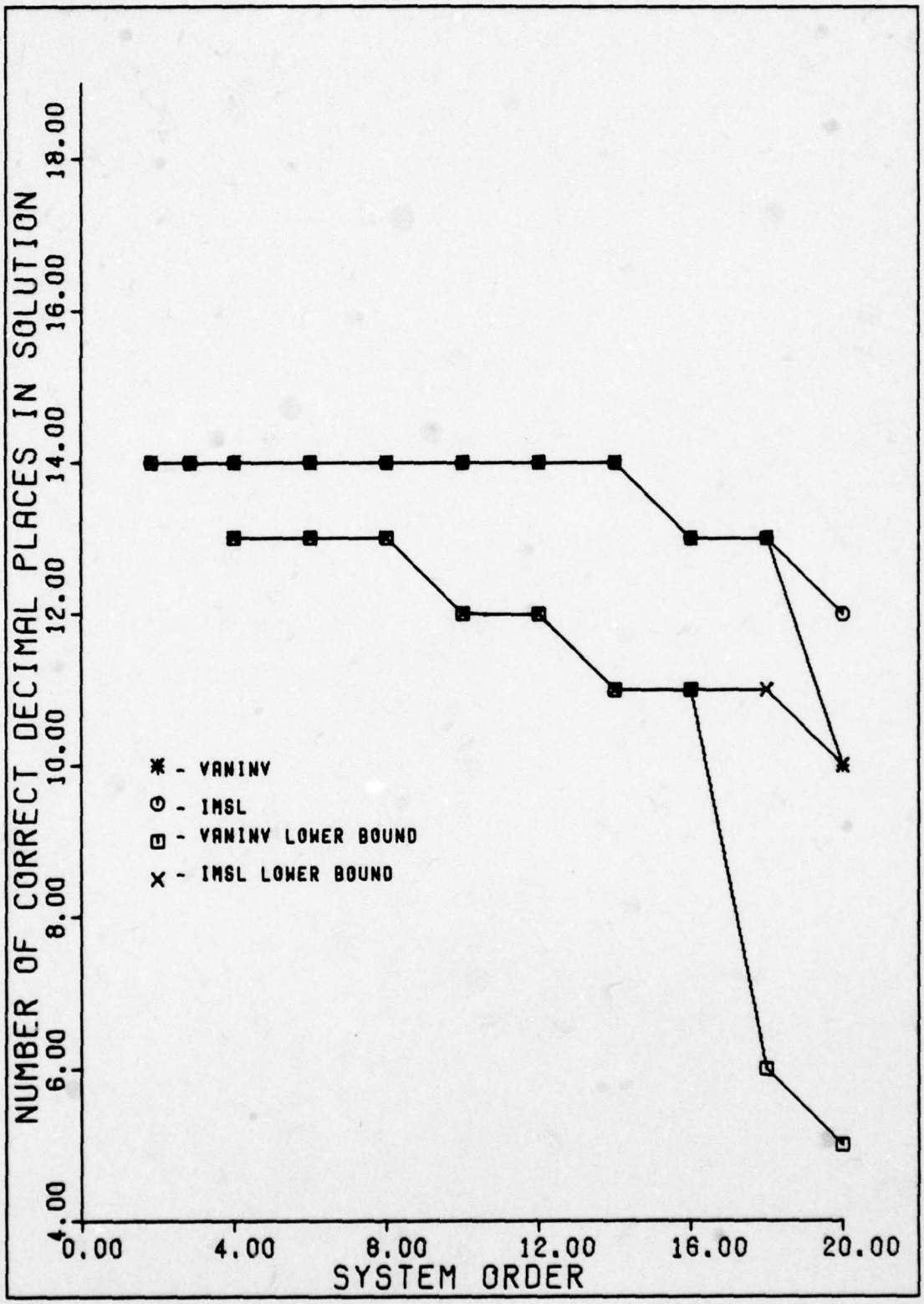

Figure 5. Computation accuracy as a function of system order (real eigenvalues).

at the higher level. The reason for the sudden drop in the minimum bound for VANINV beginning with the 18th order system is not intuitively obvious, but it is interesting to note that it seems to be indicative of a trend as the system order continues to increase.

Accuracy as a Function of $|\lambda_x - \lambda_y|_{min}$

From Figure 6 it can be seen that the accuracy for both VANINV and the IMSL routine drops off as the minimum distance between two eigenvalues goes to zero. Neither routine is significantly better than the other, as the results of Tests 12 through 20 show.

Apparently the second accuracy bound test does not work very well for systems having two eigenvalues close to each other. The closer the eigenvalues become, the less realistic the accuracy bound is.

It is interesting to note that although the accuracy of the IMSL routine decreases as $|\lambda_x - \lambda_y|_{min}$ decreases, the resulting inverse is stable. In each test the inverse and the inverse of the inverse have about the same accuracy, so the solution of the inverse matrix is termed "stable" for this discussion.

Accuracy as a Function of Condition Index

Tests 21 through 27 hold the minimum eigenvalue, $|\lambda_{min}|$, and the closest distance, $|\lambda_x - \lambda_y|_{min}$, between two eigenvalues constant as the condition index, $C = \frac{|\lambda_{max}|}{|\lambda_{min}|}$,

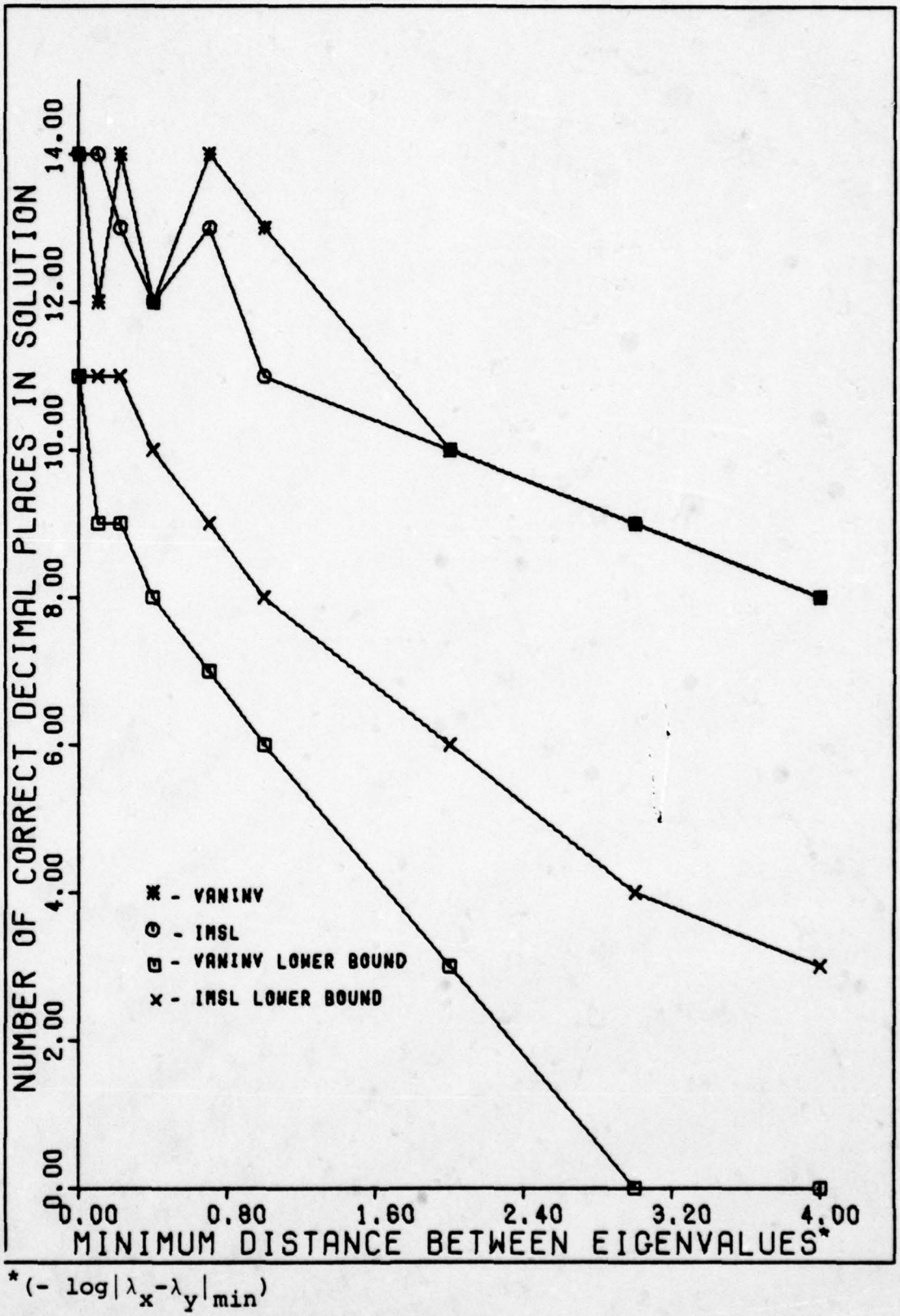

Figure 6. Computation accuracy as a function of the two closest eigenvalues.

increased. The reinversion of the inverse by the IMSL routine indicates that the inversion accuracy decreases as the condition index increases. The actual inversion accuracy could not be determined by inspection of the data, though VANINV and the IMSL routine gave identical results up through a condition index of twenty-five thousand. Beyong that point both inversion routines became unstable, as indicated by the large fluctuations in the elements of the check matrix and the Accuracy Merit Figure. Plots were not made of this test series because of the small amount of plotable data.

Accuracy as a Function of Increasing Condition Index Caused by an Eigenvalue Approaching Zero

These tests were run in two series. The first series, Tests 28 through 37, allowed one eigenvalue to go to zero, thereby causing the eigenvalue condition index, C, to increase. The minimum distance between any two eigenvalues was held constant. As seen in Figure 7, the accuracy of both inversion routines dropped as the condition number increased.

In the second series of tests, numbers 38 through 43, the minimum distance between any two eigenvalues was again held constant, but at a larger value in order to determine if it has any effect on the inversion accuracy. The accuracy could not be determined by inspecting the inverse matrices found. However, based on the accuracy of the reinverted

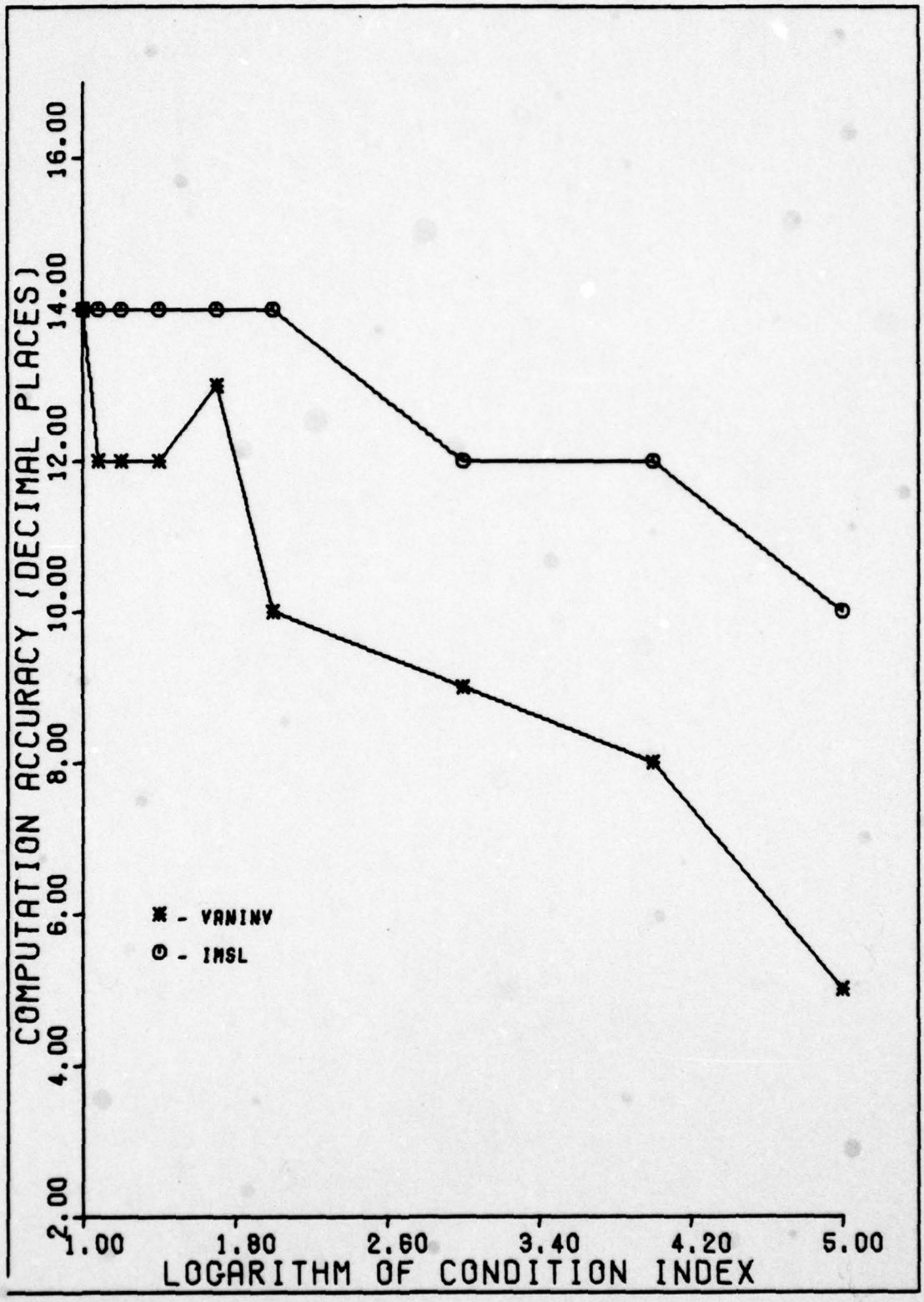

Figure 7. Logarithm of condition index ($\log|\lambda_{max}|/|\lambda_{min}|$).

matrix, the accuracy and stability of the IMSL routines dropped as the condition index grew in each series.

From the results of both series of tests, a high condition index resulting from an eigenvalue near zero is not as detrimental to inversion accuracy or stability as a high condition index caused by a large eigenvalue. In addition, the combination of close eigenvalues and a high condition index has more of a degrading effect on accuracy than either of the two by themselves. It may also be observed that when the eigenvalue actually reaches zero, thereby driving the condition index to infinity, the accuracy rises sharply back to the level realized for small condition indices. This is probably because the computer can represent zero exactly, whereas numbers near zero can only be approximated.

All of the findings of the tests are summarized in the several conclusions of the next section. In addition, recommendations are given for possible improvements to VANINV and for its use in further research regarding system identification processes.

VI Conclusions and Recommendations

Several conclusions were drawn from the results reported in Section V. Based on those conclusions it is recommended that subroutine VANINV, which implements Csáki's generalized algorithm for finding the inverse of a Vandermonde matrix, be refined for computational speed and tested in its intended application, that of finding component matrices for a system identification process. The conclusions and the recommendations are explained further in the remainder of this section. The method of presentation is a listing of the conclusions/recommendations, followed by some comments regarding them.

Conclusions

The following conclusions were made:

1. The IMSL routines are generally slightly faster, and about as accurate or somewhat more accurate than VANINV is in its present form.
2. The accuracy of both routines is reduced by high condition indices, particularly if the high condition indices are caused by large eigenvalues, as opposed to being caused by one eigenvalue near zero.

In regard to computation speed, it should be noted that VANINV is compiled to optimization level two, whereas the IMSL routines are only compiled to optimization level

one. This means that IMSL could possibly run significantly faster than VANINV. However, these tests were restricted to real distinct eigensystems. VANINV has the capability of computing the inverse of Vandermonde matrices with combinations of repeated and/or complex eigenvalues. A fair test of this property would necessitate special conversions to turn the Vandermonde matrix into a purely real matrix that the IMSL routines could handle. This conversion time would then be considered part of the IMSL computation time.

It would be interesting to determine VANINV's accuracy if double precision arithmetic was used. Of course, for any intended application a trade-off analysis should be performed between any increase in accuracy and the cost in terms of additional memory requirements.

Recommendations

It is recommended that VANINV be developed further and implemented in a component matrix approach for system identification. In this way, VANINV's accuracy can be checked in the more meaningful environment of its intended application. An accuracy check is built right into the component matrix approach to system identification: all of the component matrices of the pseudo modes should be null. Also, the computational accuracy should be tested on computers with shorter word length than possessed by the CDC 6600 computer used for these tests. Computers used for on-line control applications probably wouldn't have word lengths with 48 bits of data storage like the CDC 6600 has.

The following procedures are therefore recommended for VANINV itself:

1. Test VANINV with repeated and/or complex eigenvalues to determine their effect on computation time and accuracy.
2. Determine where most of the computation time is used on VANINV and improve the efficiency where practical.
3. Eliminate the calls to subroutines that are made only one or two places in VANINV and replace them with the coded subroutines, since all the subroutines have been debugged.
4. Streamline VANINV by using vector array addressing to reduce array index computation time.
5. Incorporate a row jumping capability into VANINV to eliminate duplicated calculations for repeated and/or complex eigenvalues. This is illustrated in Figure 8, which shows a very common occurrence for the component matrix approach. Every complex conjugate pair of eigenvalues in the original system will be repeated in the $2n \times 2n$ Vandermonde matrix of the augmented system. The resulting calculations that VANINV would perform for these eigenvalues is shown by Figure 8. Note that the lower half of the matrix shown in Figure 8 is simply the complex conjugate of the upper half. Thus considerable computation time could be saved

$$
\begin{bmatrix}
\left.\frac{d}{ds}\frac{N_1(S)}{D_1(S)}\right|_{\lambda_1} & \left.\frac{d}{ds}\frac{N_2(S)}{D_1(S)}\right|_{\lambda_1} & \left.\frac{d}{ds}\frac{N_3(S)}{D_1(S)}\right|_{\lambda_1} & \left.\frac{d}{ds}\frac{N_4(S)}{D_1(S)}\right|_{\lambda_1} \\
\left.\frac{N_1(S)}{D_1(S)}\right|_{\lambda_1} & \left.\frac{N_2(S)}{D_1(S)}\right|_{\lambda_1} & \left.\frac{N_3(S)}{D_1(S)}\right|_{\lambda_1} & \left.\frac{N_4(S)}{D_1(S)}\right|_{\lambda_1} \\
\hline
\left.\frac{d}{ds}\frac{N_1(S)}{D_2(S)}\right|_{\lambda_2} & \left.\frac{d}{ds}\frac{N_2(S)}{D_2(S)}\right|_{\lambda_2} & \left.\frac{d}{ds}\frac{N_3(S)}{D_2(S)}\right|_{\lambda_2} & \left.\frac{d}{ds}\frac{N_4(S)}{D_2(S)}\right|_{\lambda_2} \\
\left.\frac{N_1(S)}{D_2(S)}\right|_{\lambda_2} & \left.\frac{N_2(S)}{D_2(S)}\right|_{\lambda_2} & \left.\frac{N_3(S)}{D_2(S)}\right|_{\lambda_2} & \left.\frac{N_4(S)}{D_2(S)}\right|_{\lambda_2}
\end{bmatrix}
$$

Figure 8. Example of VANINV calculations applied to a fourth order system with a repeated complex conjugate pair of eigenvalues.

by incorporating a row jumper which would simply store the complex conjugate of

$$w_{1\ell}^{(1)} = \frac{d}{ds} \frac{N_\ell(S)}{D_1(S)}\bigg|_{\lambda_1}$$

directly into

$$w_{1\ell}^{(2)} = \frac{d}{ds} \frac{N_\ell(S)}{D_2(S)}\bigg|_{\lambda_2}$$

and thus eliminate half the calculations. Also note that a repeated eigenvalue in the original system would occur four times in the 2n × 2n Vandermonde matrix, as Figure 9 shows. Each column is composed of identical elements except for the number of derivatives that must be taken for evaluation. If a row jumper with some memory were added to the routine, successive derivatives could be taken to evaluate the elements of each column. This would require only 1+1+1 = 3 derivatives for each column shown rather than 3+2+1 = 6 derivatives as VANINV must do presently. (Without the row jumper, VANINV simply evaluates each element as it comes to it, without regard for any calculations done previously.)

In addition to the recommended procedures for VANINV described above, two comments are made regarding the component matrix approach to system identification:

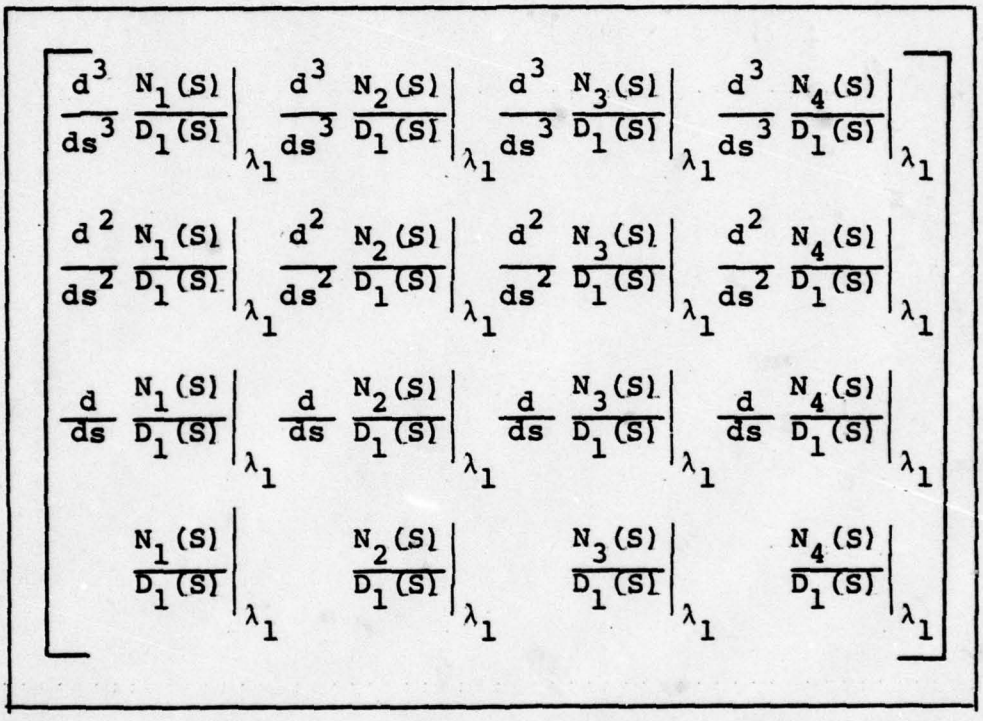

Figure 9. Example of VANINV calculations applied to a fourth order system with like eigenvalues.

1. Many of the matrices resulting from the component matrix approach are very sparse (eg: Eqs (D2-13), (D2-14)), and thus the method lends itself readily to sparse matrix operations such as those available with SOFE (Ref 7).
2. The calculation structure of both the component matrix approach and VANINV lend themselves to parallel calculations. An implementation of these methods on a parallel architecture computer has the potential for being very fast in real time.

Thus it may be seen that a refinement of VANINV may well have a place in future system identification processes.

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VITA

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APPENDIX A

COMPUTER PROGRAMS FOR CSÁKI'S GENERALIZED INVERSE VANDERMONDE ALGORITHM

Preface for Appendix A

This appendix contains the subroutines written to implement the generalized method for finding an inverse Vandermonde matrix that is described in the body of the report. The appendix is formatted to serve as a combination user/programmer guide for further investigation of the proposed alternate method of system identification which is outlined in the main text.

Contents and Subroutine Cross-reference

Subroutine	Subroutines Called																	Description Page	Flowchart Page
	CGAIN	COPYPOL	ESORT	EVAL	EXPANDC	FRACDIF	NFACT	OUT2C	POLYADD	POLYDIF	POLYMC	POLYSUB	RAT	ROOTAY	UNITY	VANINV			
CGAIN																		A-4	A-5
COPYPOL																		A-6	A-7
ESORT †																		A-8 A-9	A-10 A-11
EVAL	X																	A-12	A-13
EXPANDC											X							A-14	A-15
FRACDIF										X	X	X						A-16 A-17	A-18
NFACT ‡																		A-19	A-19
OUT2C †																		A-20	A-21
POLYADD	X														X			A-22	A-23 A-24
POLYDIF	X																	A-25	A-26
POLYMC															X			A-27	A-28
POLYSUB									X									A-29	A-30
RAT																		A-31	A-32
ROOTAY																		A-33	A-34
UNITY																		A-35	A-36
VANINV		X		X	X	X	X						X	X				A-37 thru A-39	A-40 thru A-46

† Not used by any of the subroutines listed, but pertinent to the use of VANINV.

‡ Function subprogram.

Subroutine CGAIN

Subroutine CGAIN multiplies each coefficient of a polynomial by a specified gain. The coefficients and the gain may be complex.

Subroutine statement: SUBROUTINE CGAIN (AR,AI,NSA,AKR,AKI)

Subroutines called: None

Variables:

AR), AI) = Arrays containing the real and imaginary parts of the polynomial coefficients

NSA = Number of storage locations required for the polynomial coefficients

AKR), AKI) = Real and imaginary parts of the polynomial gain

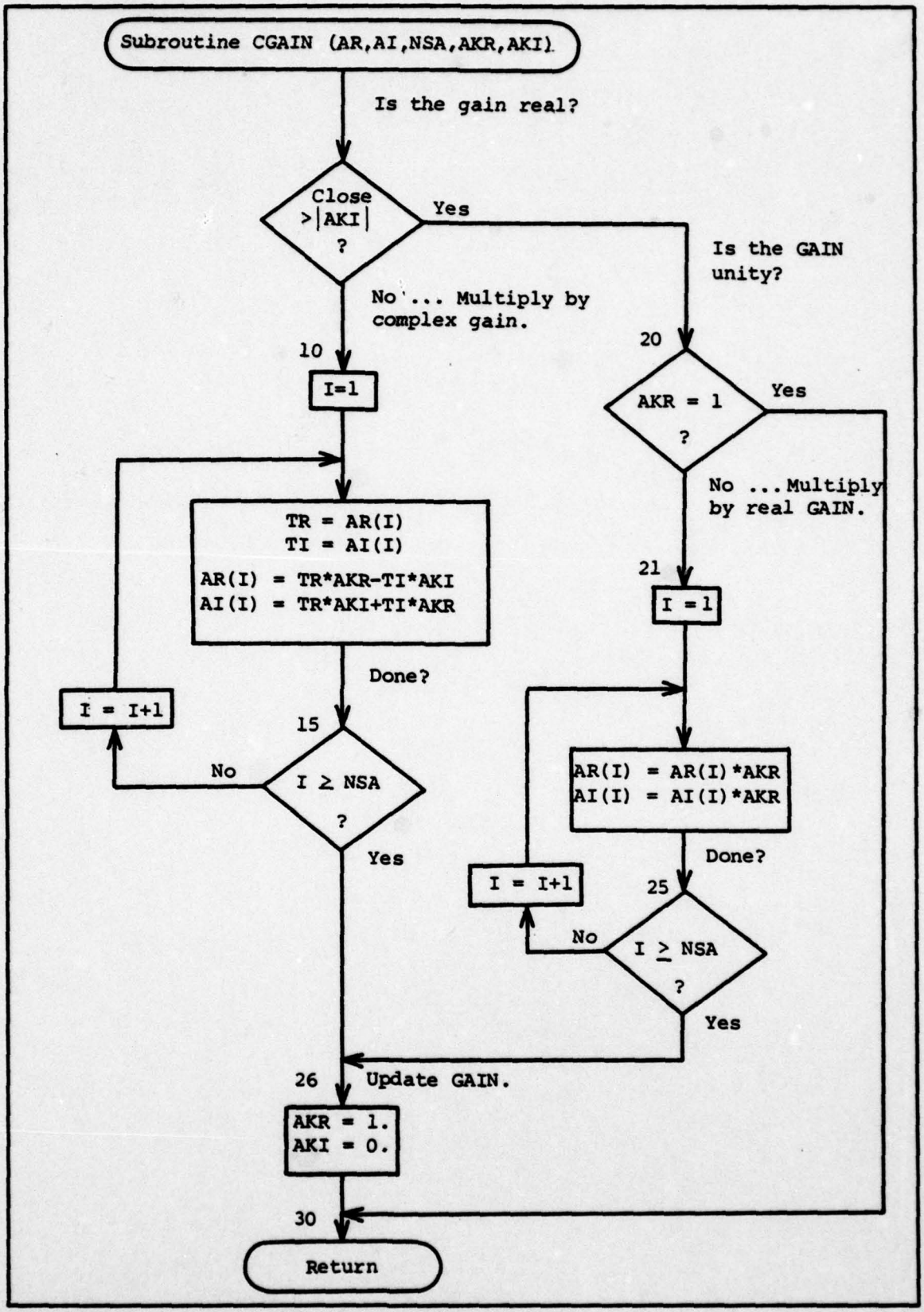

Figure A-1. Flowchart for subroutine CGAIN.

Subroutine COPYPOL

Subroutine COPYPOL copies the coefficients of the input polynomial into the storage locations for the coefficients of the output polynomial. Also, the input gain is copied into the output gain storage. The coefficients and gains may be complex.

Subroutine statement: SUBROUTINE COPYPOL (PIR,PII,GIR,GII,N,POR,POI, GOR,GOI)

Subroutines called: None

Variables:

PIR)
PII) = Arrays containing the real and imaginary parts of the
POR) coefficients of the input (PIx) and output (POx) polynomials
POI)

N = Number of storage locations needed for the polynomial coefficients. (Note that N is one integer value larger than the polynomial order.)

GIR)
GII) = Real and imaginary parts of the gains of the input (GIx)
GOR) and output (GOx) polynomials
GOI)

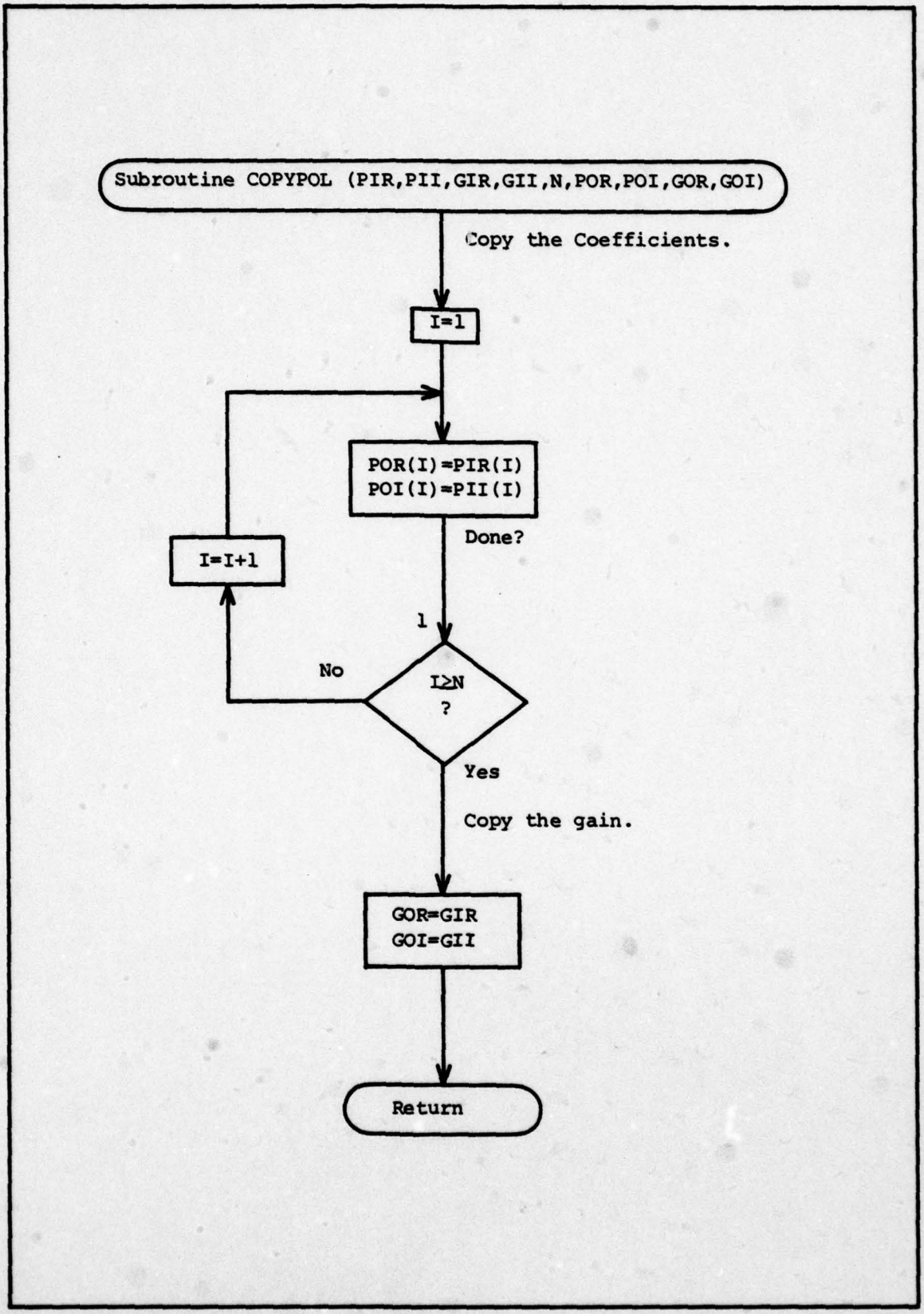

Figure A-2. Flowchart for subroutine COPYPOL.

Subroutine ESORT

Subroutine ESORT sorts a collection of real and complex eigenvalues according to the following specifications:

1. Put in order of decreasing complexness; i.e., complex before real.
2. Order each section in decreasing order of multiplicity.
3. Order complex conjugates with the eigenvalue having the positive imaginary part.
4. For equal multiplicities, order in terms of decreasing magnitude of the real part; or if equal, place eigenvalues with positive real parts before those with negative real parts, and order in terms of the imaginary part.

Subroutine ESORT sweeps through the collection and compares each pair of adjacent eigenvalues to determine whether or not they need to be swapped to get them in the proper relative order. Each time a swap is performed, a "swap counter" is incremented. At the end of each sweep through the eigenvalue array, the swap counter is tested to see if any swaps were made on that sweep. If there were swaps, the counter is reset to zero and another sweep through the array is started. Zero swaps during any sweep through the array indicates that all the eigenvalues are in the proper order, so the subroutine returns the rearranged eigenvalue array.

Subroutine statement: SUBROUTINE ESORT (EIGR,EIGI,KI,M)

Subroutines called: None

Variables:

EIGR), EIGI) = Arrays containing the real and imaginary parts of the collection of eigenvalues

KI = Array containing the corresponding eigenvalue multiplicities

M = Number of different eigenvalues in the array (complex conjugates are considered to be different)

MDIS = "Message DISable" flag, passed in blank common. ESORT normally prints an informative diagnostic if two identical eigenvalues are encountered. If the user doesn't want the message printed, setting MDIS equal to any positive integer will disable the printer. Setting MDIS to zero or a negative integer enables the printer.

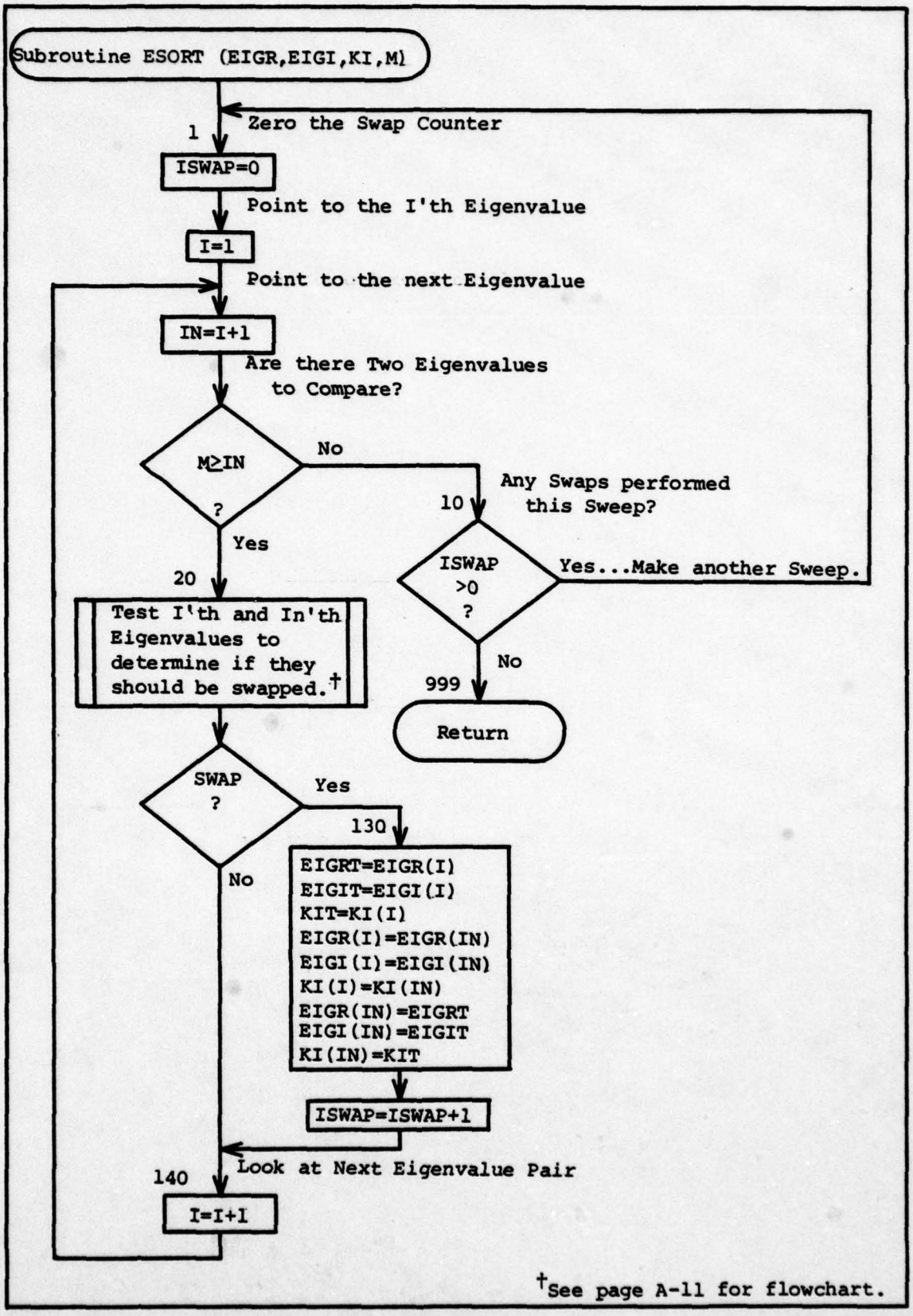

Figure A-3. Flowchart for subroutine ESORT.

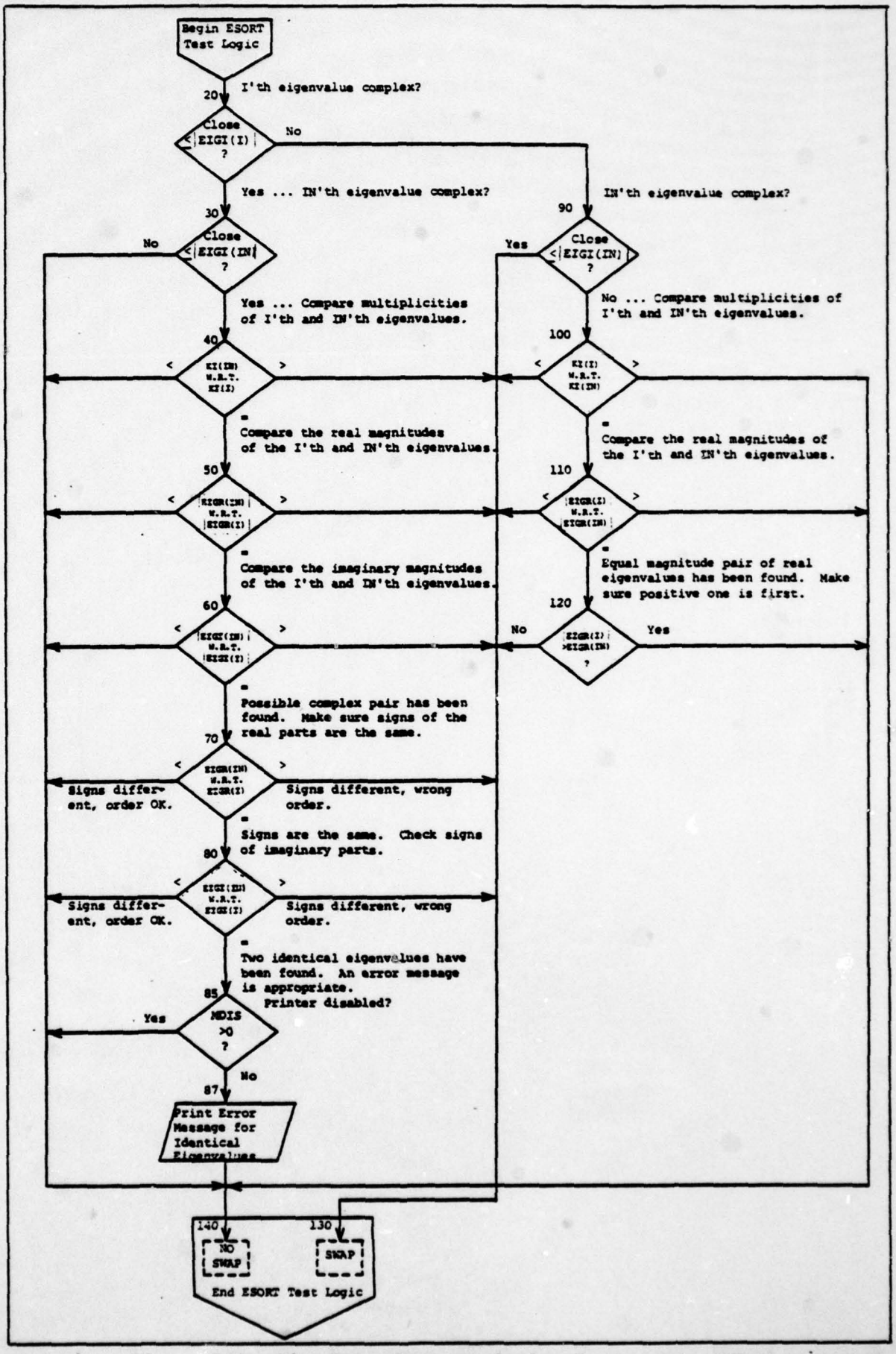

Figure A-4. Flowchart for test logic of subroutine ESORT.

Subroutine EVAL

Subroutine EVAL finds the value of the input polynomial at the specified value of the variable. The input polynomial coefficients and the value of the variable may be complex.

Subroutine statement: SUBROUTINE EVAL (AR,AI,AKR,AKI,EIGR,EIGI, NS,BR,BI)

Subroutines called: CGAIN

Variables:

- AR), AI) = Arrays containing the real and imaginary parts of the coefficients of the input polynomial
- AKR), AKI) = Real and imaginary parts of the gain of the input polynomial
- EIGR), EIGI) = Real and imaginary parts of the value at which the polynomial is to be evaluated (must be chosen by the calling routine)
- NS = Number of storage locations required for the input polynomial; NS=N+1, where N is the order of the polynomial
- BR), BI) = Real and imaginary parts of the returned value

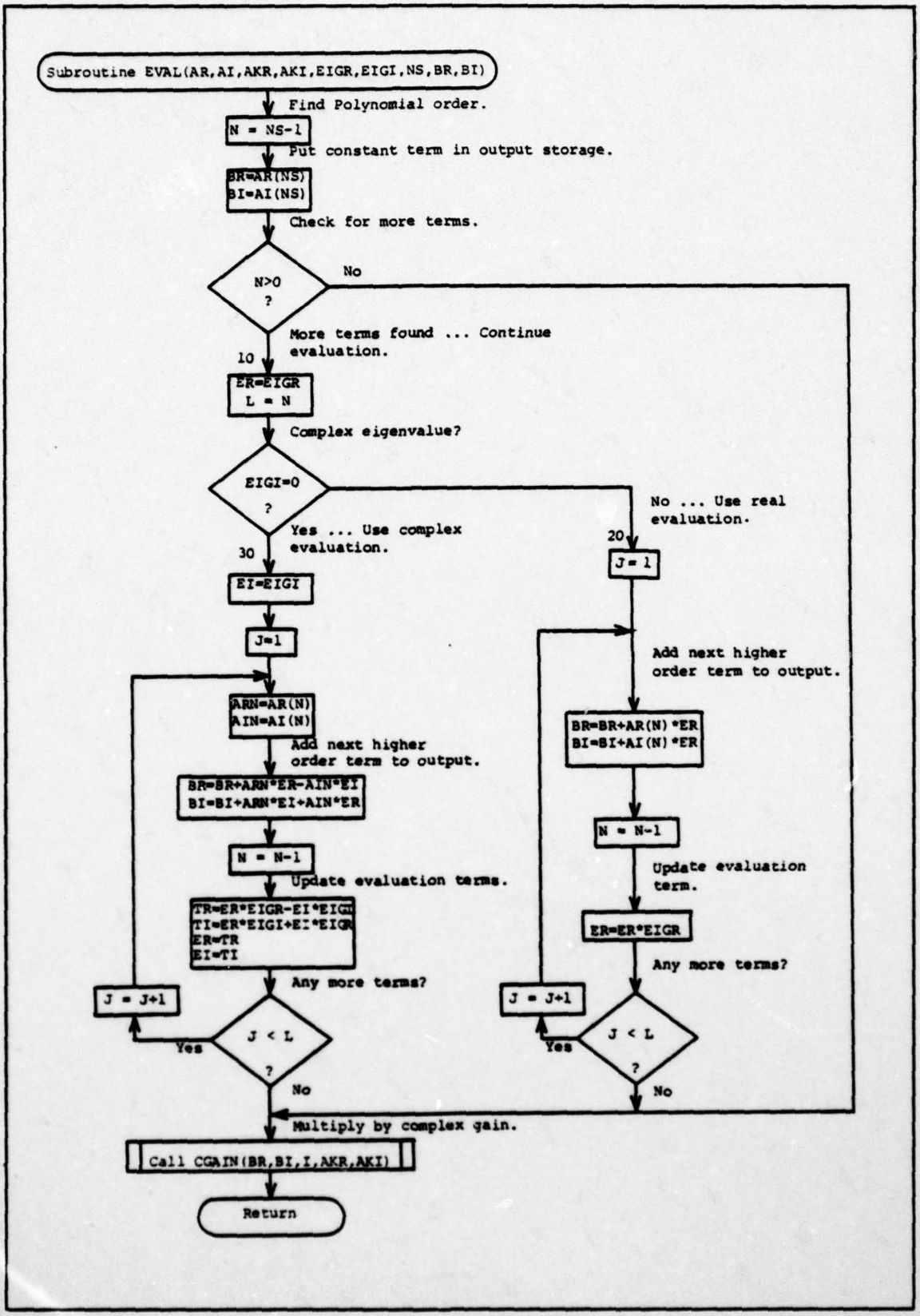

Figure A-5. Flowchart for subroutine EVAL.

Subroutine EXPANDC

Subroutine EXPANDC finds the coefficients for the polynomial expansion of a collection of root factors. EXPANDC reads the roots from a root array and returns the coefficients in a polynomial coefficient array. The coefficient for the highest power of the variable is the first element in the coefficient array. The routine handles both real and complex roots and does not require that complex roots have conjugate pairs. The maximum size of the polynomial found is limited only by the storage allocated for CR and CI in the calling program.

Subroutine statement: SUBROUTINE EXPANDC (ROOTR,ROOTI,NS,BR,BI,CR,CI, CKR,CKI)

Subroutines called: POLYMC

Variables:

ROOTR), ROOTI) = Arrays containing the real and imaginary parts of the roots whose factors are to be expanded into a polynomial

NS = The number of storage locations required for the output polynomial; equal to NF+1 where NF is the number of factors

BR), BT) = NS-dimensioned temporary storage for intermediate results of the polynomial expansion

CR), CI) = NS-dimensioned arrays containing the real and imaginary parts of the coefficients of the output polynomial

CKR), CKI) = Real and imaginary parts of the output polynomial gain multiplier; always equal to 1. and 0., respectively

CLOSE = User specified value used as a null tester. If the magnitude of any value tested is less than CLOSE, that value is set to zero. CLOSE is stored in blank common and must be specified in the main program.

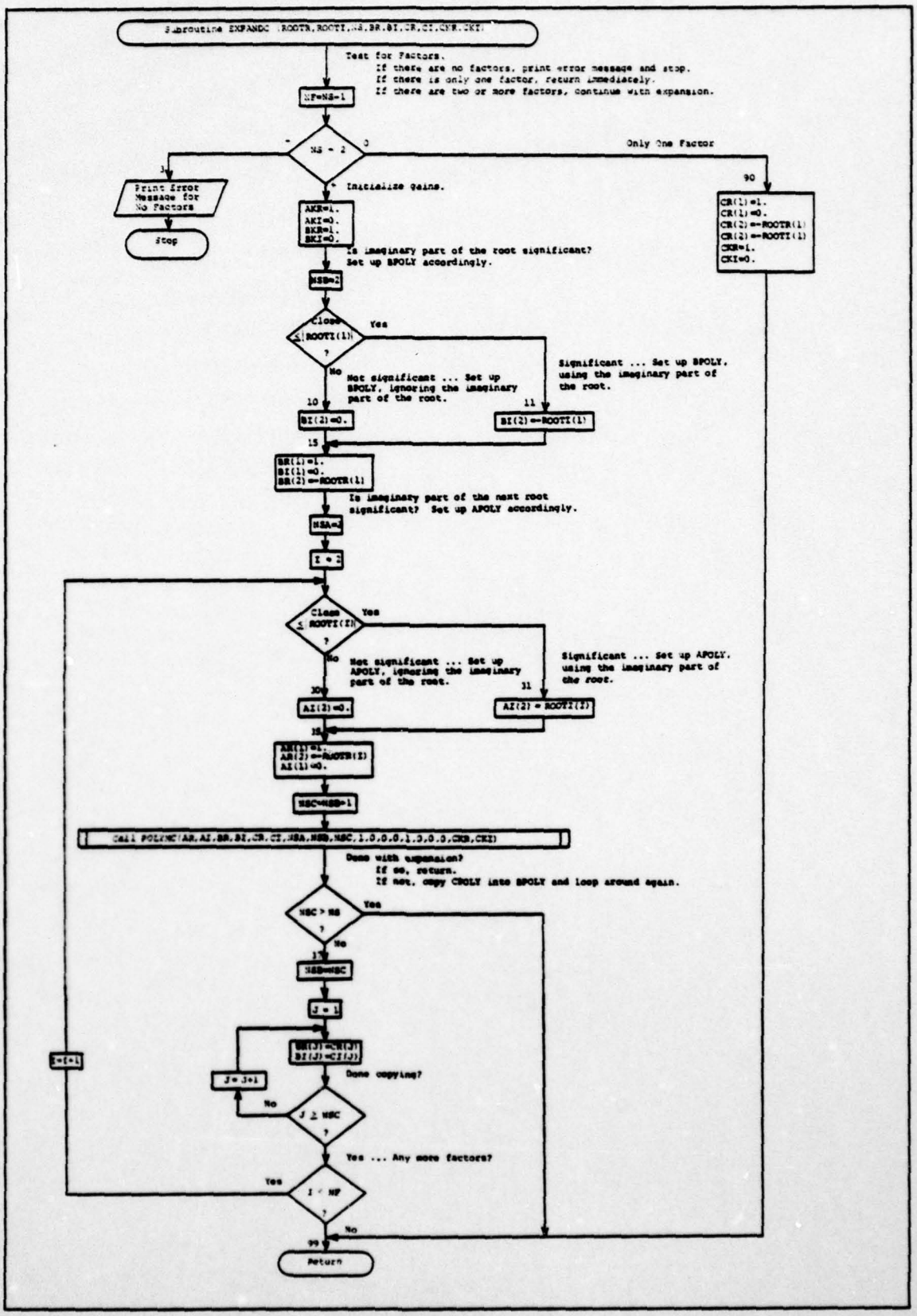

Figure A-6. Flowchart for subroutine EXPANDC.

Subroutine FRACDIF

Subroutine FRACDIF accepts a polynomial fraction and finds its derivative using the quotient rule:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u\frac{dv}{dx} - v\frac{du}{dx}}{v^2}$$

The coefficients of the input numerator and/or denominator polynomial may be complex.

Subroutine statement: FRACDIF (PUR,PUI,PVR,PVI,NUS,NVS,PNR,PNI,PDR, PDI,NNS,NDS,NT,UR,UI,VR,VI,RNR,RNI,DR,DI, TEMP1R,TEMP1I,TEMP2R,TEMP2I,TEMP3R,TEMP3I)

Subroutines called: POLYDIF, POLYMC, POLYSUB

Variables:

PUR,PUI)
PVR,PVI) = Arrays containing the real and imaginary parts of the coefficients of the input polynomials, U and V

NUS)
NVS) = Number of storage locations needed to store the coefficients of the input polynomials

PNR,PNI)
PDR,PDI) = Arrays containing the real and imaginary parts of the coefficients of the output numerator polynomial (PNx) and denominator polynomial (PDx)

NNS)
NDS) = Number of storage locations needed to store the coefficients of the output numerator and denominator polynomials, where:

NNS = NUS + NVS

NDS = 2NVS - 1

NT = Number of temporary storage locations needed for the numerator calculations

NT = NON(L) + NOD(I)

where: NON(L) is storage required for Lth numerator polynomial
NOD(I) is storage required for Ith demoninator polynomial

UR,UI) VR,VI) RNR,RNI) DR,DI)	=	Real and imaginary parts of the gains associated with polynomials U, V, PNx, PDx
TEMP1R,TEMP1I) TEMP2R,TEMP2I) TEMP3R,TEMP3I)	=	Temporary storage locations for numerator calculations

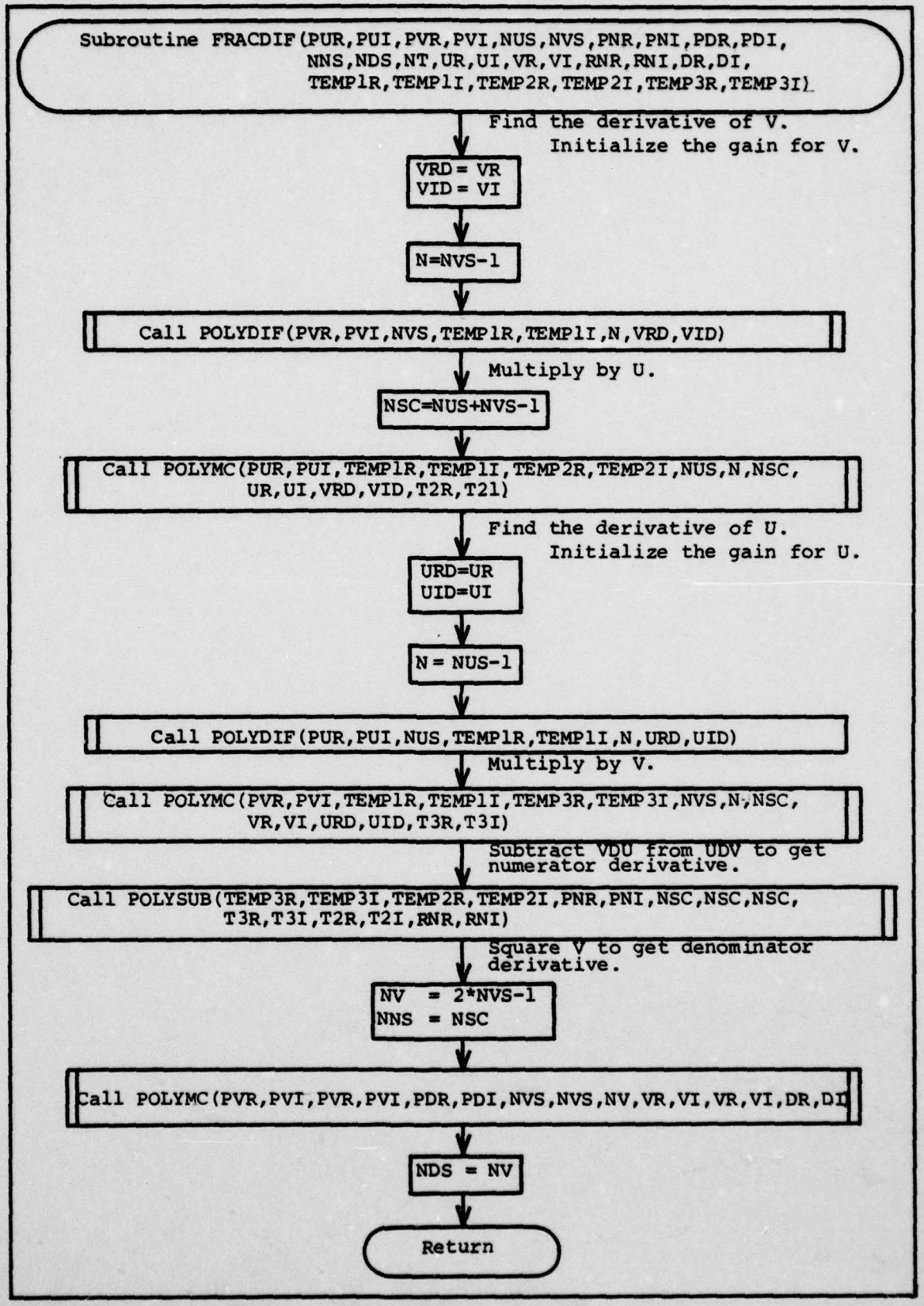

Figure A-7. Flowchart for subroutine FRACDIF.

Function NFACT

Function NFACT finds the factorial value of the argument, (IF), entered. Negative arguments cause an error message to be printed and cause the program to terminate.

Function statement: FUNCTION NFACT(IF)

Subroutine called: None

Variables:

IF = The integer for which the factorial value is desired.

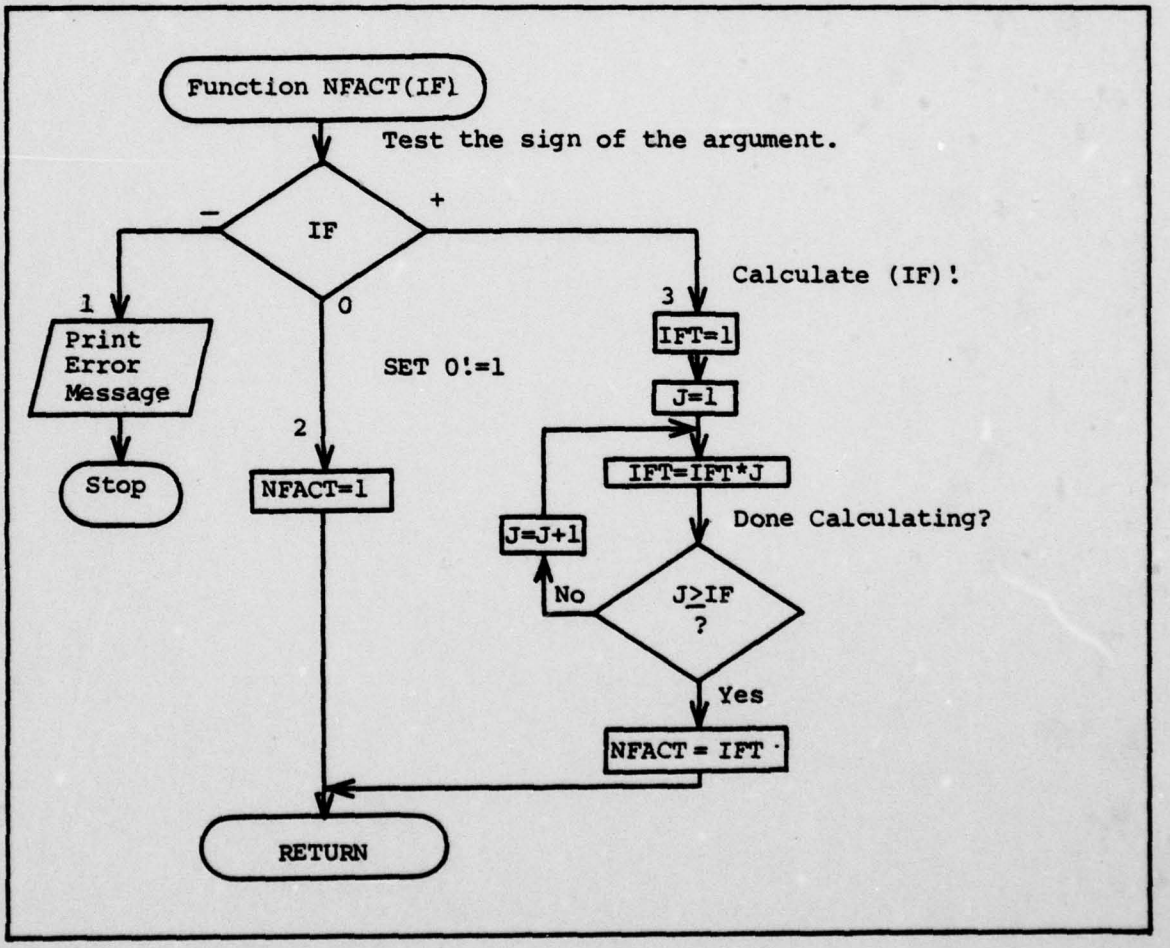

Figure A-8. Flowchart for function subprogram NFACT.

Subroutine OUT2C

Subroutine OUT2C prints out the elements of a two-dimensional array and labels each element with the array name and its position in the array. The real and imaginary parts of the complex array are treated separately.

Subroutine statement: SUBROUTINE OUT2C (NR,NC,AR,AI,IDR,IDI)

Subroutines called: None

Variables:

NR = Number of rows to be printed out

NC = Number of columns to be printed out

AR), AI) = Arrays containing the real and imaginary parts of the array to be printed

IDR), IDI) = Hollerith labels for the real and imaginary parts of each element printed; these must be specified in Hollerith format by the calling program

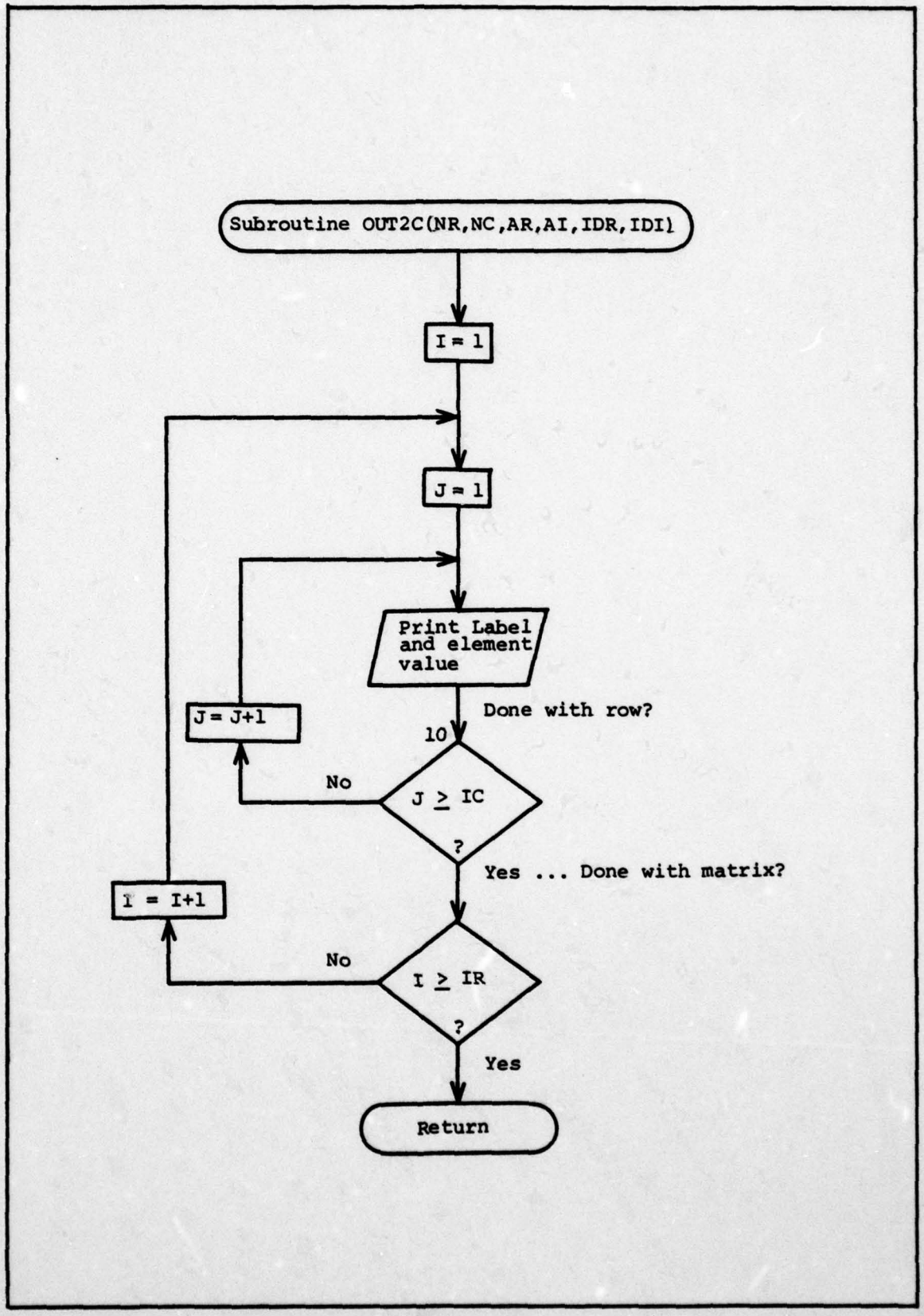

Figure A-9. Flowchart for subroutine OUT2C.

Subroutine POLYADD

Subroutine POLYADD adds two polynomials, POLYA and POLYB, and stores the result in POLYC. The coefficients of the polynomials may be complex. After the addition is complete, the first coefficient of the output is tested to determine if it is unity. If it is not, subroutine UNITY is called to unitize it.

Subroutine statement: SUBROUTINE POLYADD(AR,AI,BR,BI,CR,CI,NSA,NSB,NSC, AKR,AKI,BKR,BKI,CKR,CKI)

Subroutines called: CGAIN, UNITY

Variables:

AR, AI, BR, BI, CR, CI = Arrays containing the real and imaginary parts of the coefficients of POLYA, POLYB, and POLYC

NSA, NSB, NSC = Number of storage locations required for the coefficients of POLYA, POLYB, and POLYC

AKR, AKI, BKR, BKI, CKR, CKI = Real and imaginary parts of the gain terms for POLYA, POLYB, and POLYC

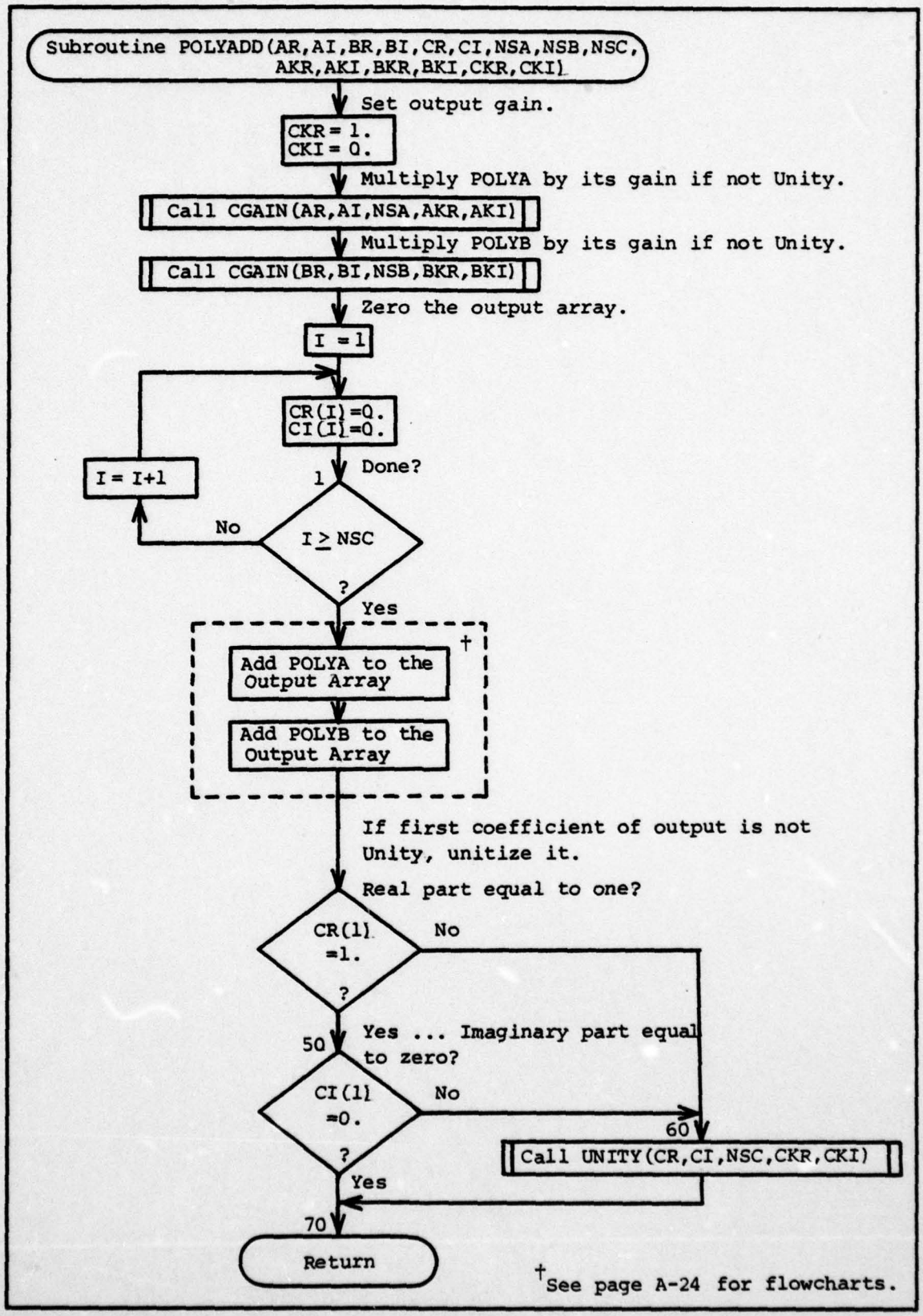

Figure A-10. Flowchart for subroutine POLYADD.

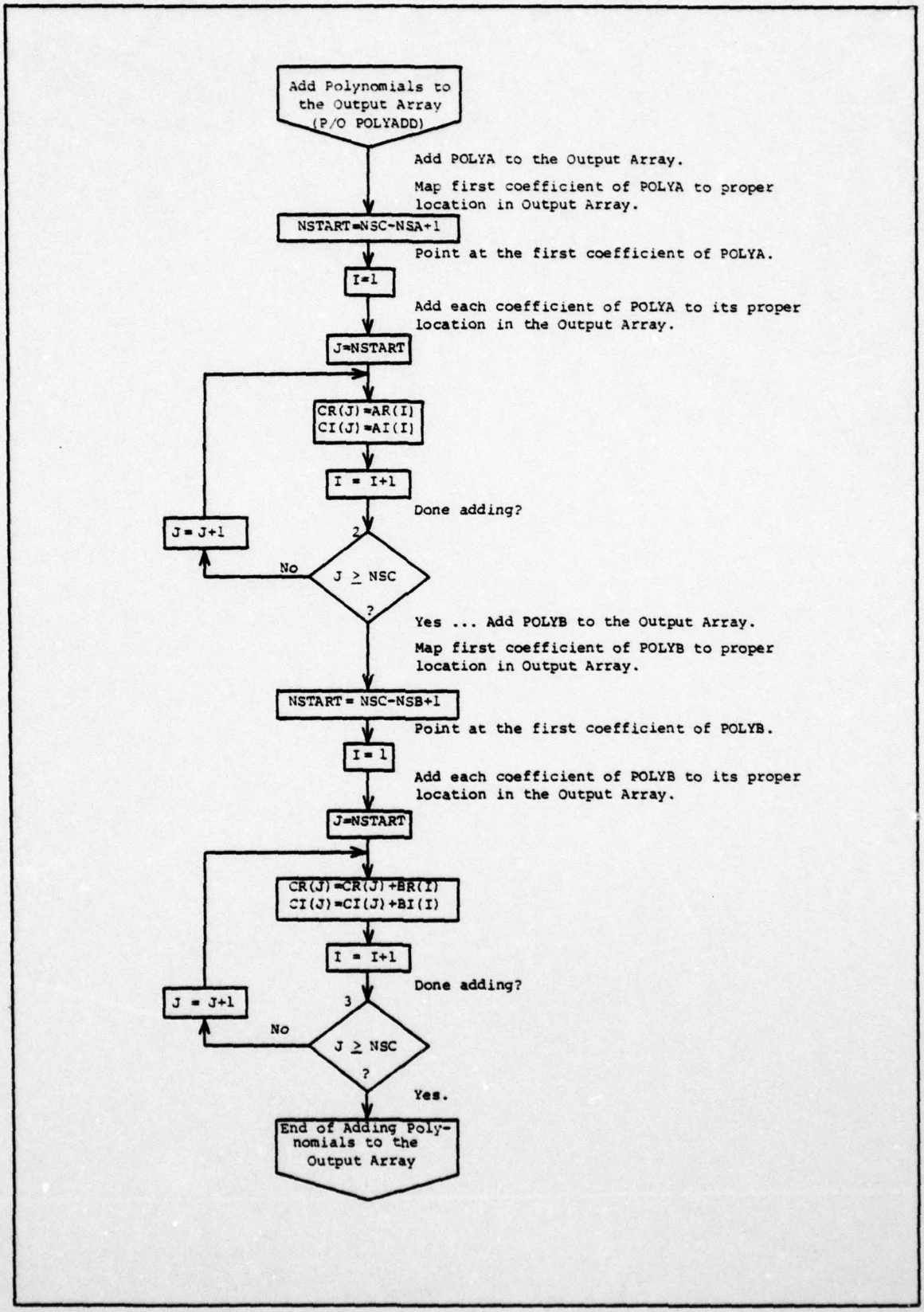

Figure A-11. Flowchart for addition steps of subroutine POLYADD.

Subroutine POLYDIF

Subroutine POLYDIF accepts a polynomial input and returns the derivative. The coefficients of the input polynomial may be complex.

Subroutine statement: SUBROUTINE POLYDIF(POLYR,POLYI,NS,DPOLYR, DPOLYI,N,PKR,PKI)

Subroutines called: CGAIN

Variables:

POLYR), POLYI) = Arrays containing the real and imaginary parts of the coefficients of the input polynomial

DPOLYR), DPOLYI) = Arrays containing the real and imaginary parts of the coefficients of the output (derivative) polynomial

NS), N) = Number of storage locations required for the coefficients of POLY and DPOLY (Note that: NS = NI+1, N = NI; where NI is the order of the input polynomial)

PKR), PKI) = Real and imaginary parts of the polynomial gain

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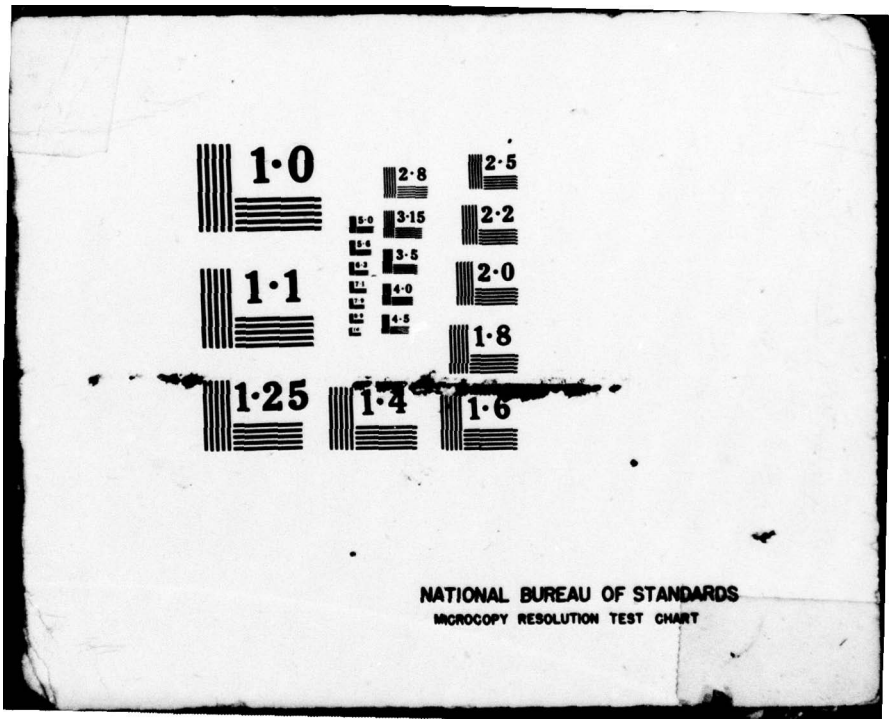
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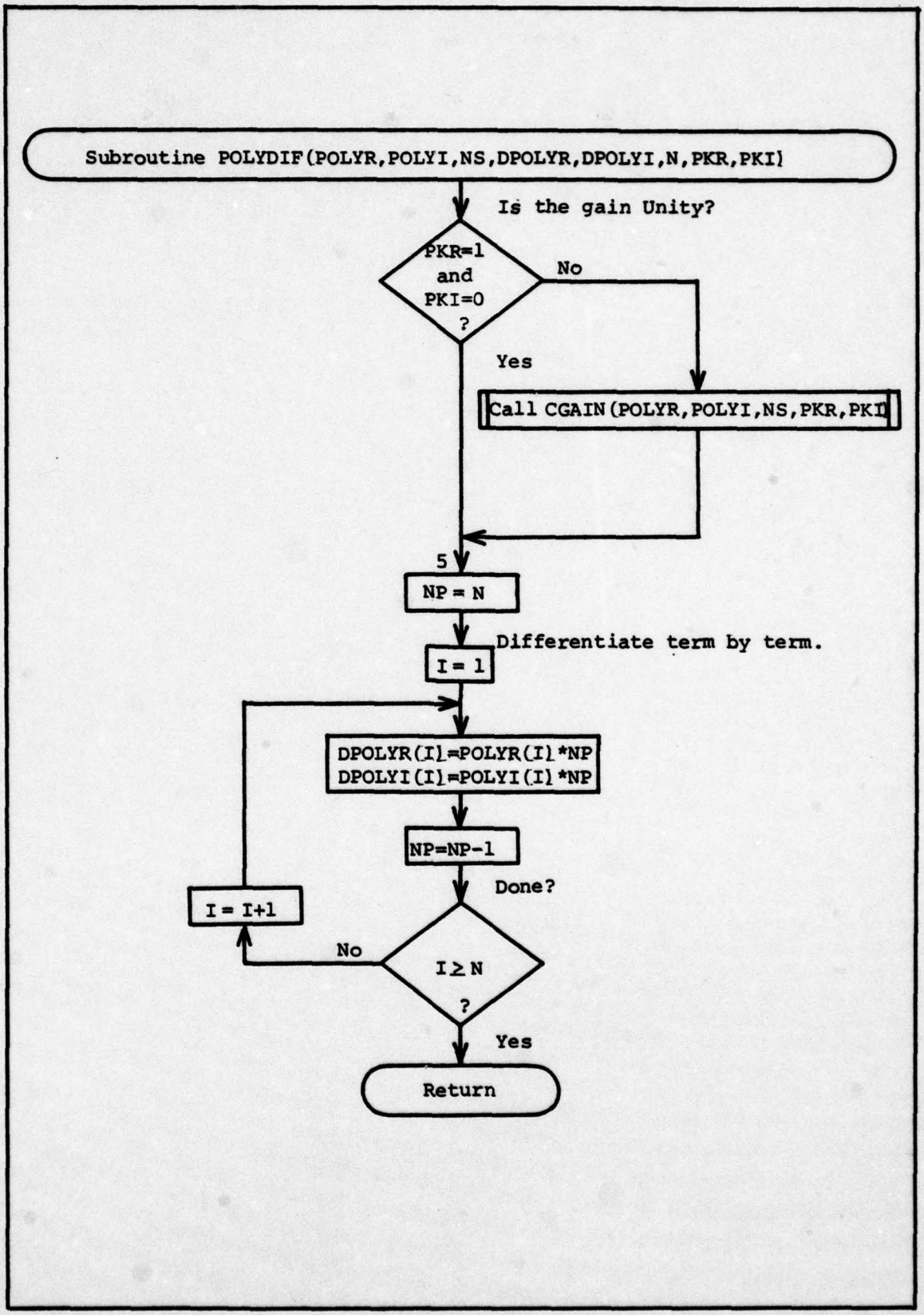

Figure A-12. Flowchart for subroutine POLYDIF.

Subroutine POLYMC

Subroutine POLYMC finds the coefficients of the product of two polynomials. The routine handles all combinations of real and complex input coefficients. The order of the polynomials involved is limited only by the storage allocated for them by the calling program.

Subroutine statement: SUBROUTINE POLYMC(AR,AI,BR,BI,CR,CI,NSA,NSB,NSC, AKR,AKI,BKR,BKI,CKB,CKI)

Subroutines called: UNITY

Variables:

AR, AI = Arrays containing the real and imaginary parts of the coefficients of the first polynomial multiplicand, POLYA

BR, BI = Arrays containing the real and imaginary parts of the coefficients of the second polynomial multiplicand, POLYB

CR, CI = Arrays containing the real and imaginary parts of the coefficients of the polynomial product, POLYC

NSA, NSB, NSC = Number of storage locations required for the coefficients of POLYA, POLYB, POLYC; each is one integer larger than the corresponding polynomial order (Note: NSC = NSA+NSB-1)

AKR, AKI, BKR, BKI, CKR, CKI = Real and imaginary parts of the gains associated with POLYA, POLYB, POLYC

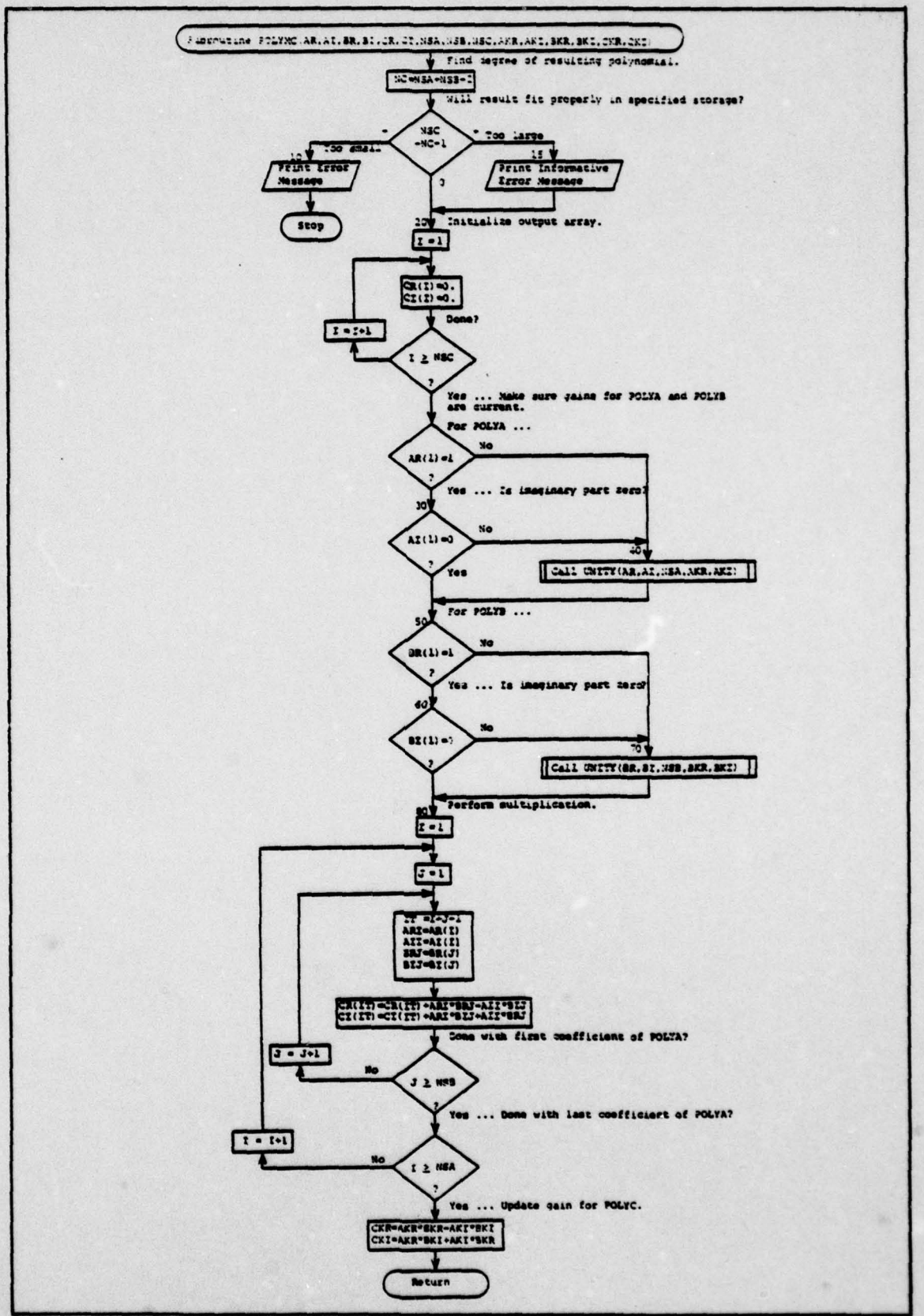

Figure A-13. Flowchart for subroutine POLYMC.

Subroutine POLYSUB

Subroutine POLYSUB subtracts POLYB from POLYA by changing the sign of the gain of POLYB and calling subroutine POLYADD. The coefficients of the polynomials may be complex.

Subroutine statement: SUBROUTINE POLYSUB(AR,AI,BR,BI,CR,CI,NSA,NSB,NSC, AKR,AKI,BKR,BKI,CKR,CKI)

Subroutines called: POLYADD

Variables:

AR, AI, BR, BI, CR, CI = Arrays containing the real and imaginary parts of the coefficients of the polynomials operated upon according to POLYC=POLYA-POLYB

NSA, NSB, NSC = Number of storage locations required for the coefficients of POLYA, POLYB, and POLYC, respectively. Note that NSC equals the larger of the two values NSA and NSB.

AKR, AKI, BKR, BKI, CKR, CKI = Real and imaginary parts of the gain multipliers for POLYA, POLYB, and POLYC, respectively.

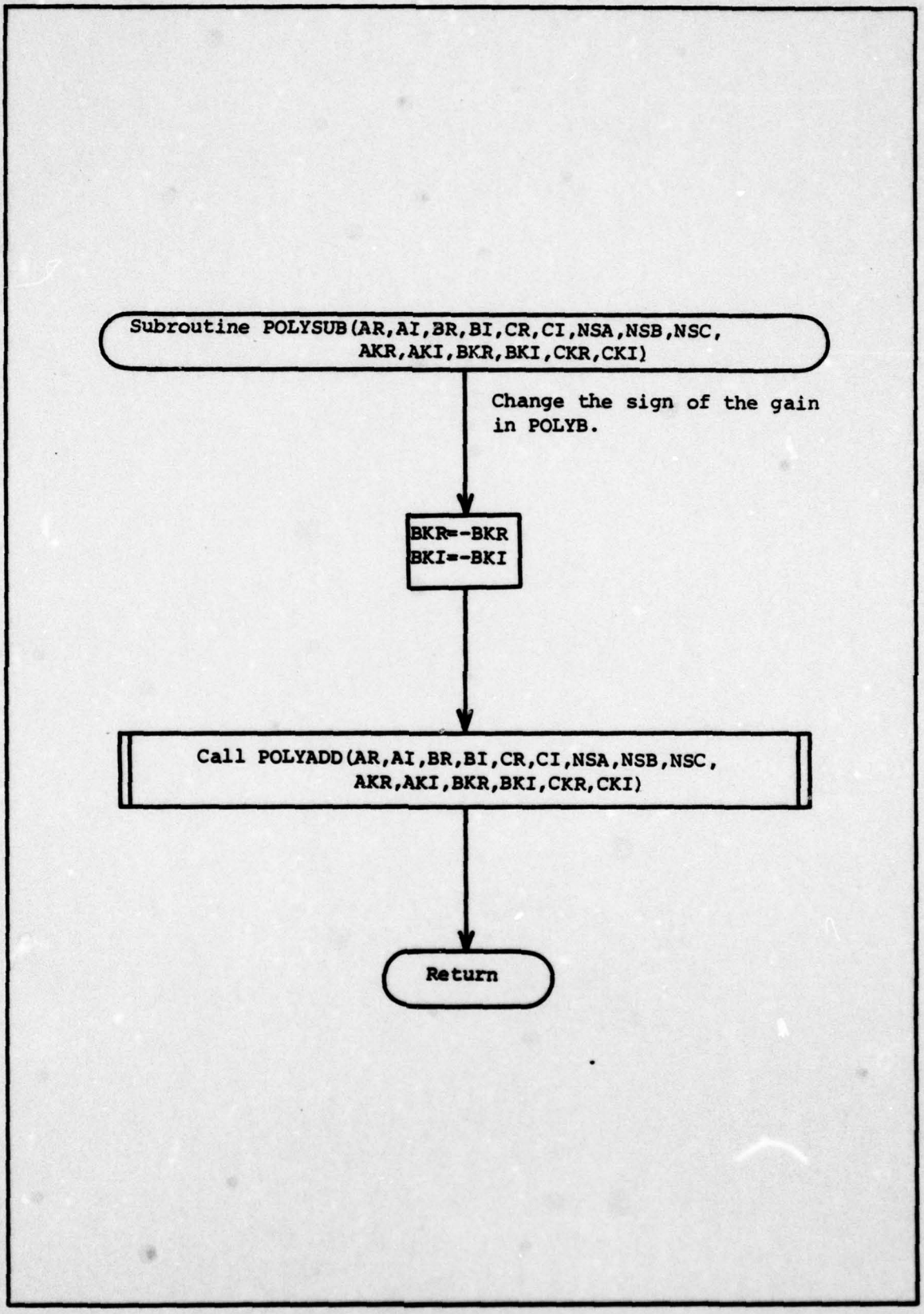

Figure A-14. Flowchart for subroutine POLYSUB.

Subroutine RAT

Subroutine RAT rationalizes a complex fraction by multiplying and dividing the fraction by the complex conjugate of the denominator. The denominator coefficients are tested for nullity prior to the last step of rationalization, and if they are null, an error message is printed and execution is terminated.

Subroutine statement: SUBROUTINE RAT(RNR,RNI,DR,DI,RR,RI)

Subroutines called: None

Variables:

RNR, RNI = Real and imaginary parts of the input numerator

DR, DI = Real and imaginary parts of the input denominator

RR, RI = Real and imaginary parts of the rationalized output

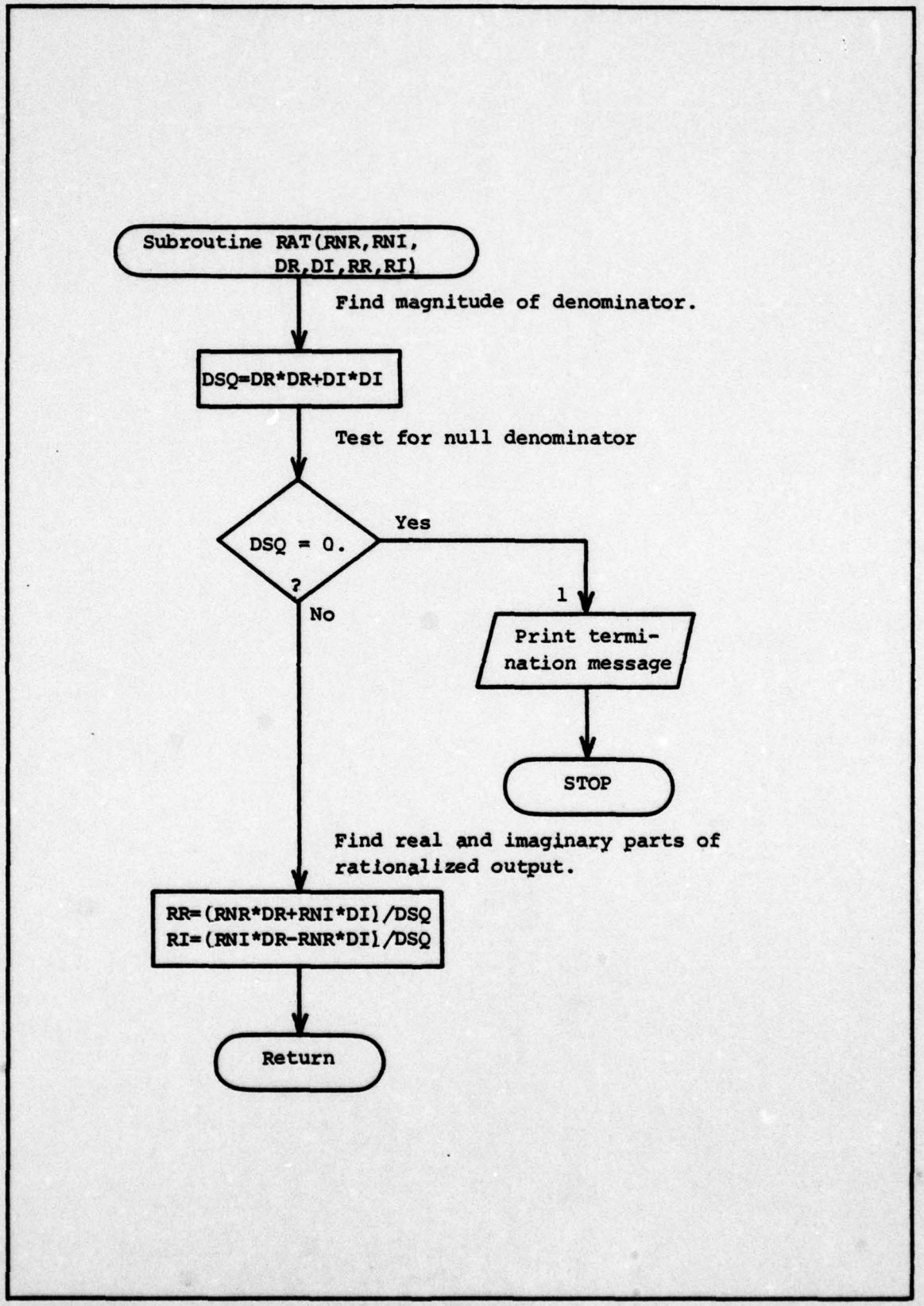

Figure A-15. Flowchart for subroutine RAT.

Subroutine ROOTAY

Subroutine ROOTAY expands a collection of eigenvalues with their associated multiplicities into an array containing one root for each occurrence of each eigenvalue. Both real and complex eigenvalues may be used. The routine forms the root array beginning with the IB'th eigenvalue in the input array and uses all the remaining eigenvalues in the array. Thus the input array must have the eigenvalues arranged so that any eigenvalues to be excluded from the expansion precede the IB'th eigenvalue.

Subroutine statement: SUBROUTINE ROOTAY(EIGR,EIGI,K,M,IS,IB, ROOTR,ROOTI)

Subroutines called: None

Variables:

EIGR) = Arrays containing the real and imaginary parts of the
EIGI) input collection of eigenvalues

K = An array containing the multiplicities of the eigenvalues

M = The number of different eigenvalues

IS = The number of storage locations needed to contain the resulting array of roots;
note that IS = $\sum_{i=1}^{M} K(i)$, the summation of the eigenvalue multiplicities

ROOTR) = Arrays containing the real and imaginary parts of the
ROOTI) roots in the output array

IB = The number of the eigenvalue with which to begin forming the root array.

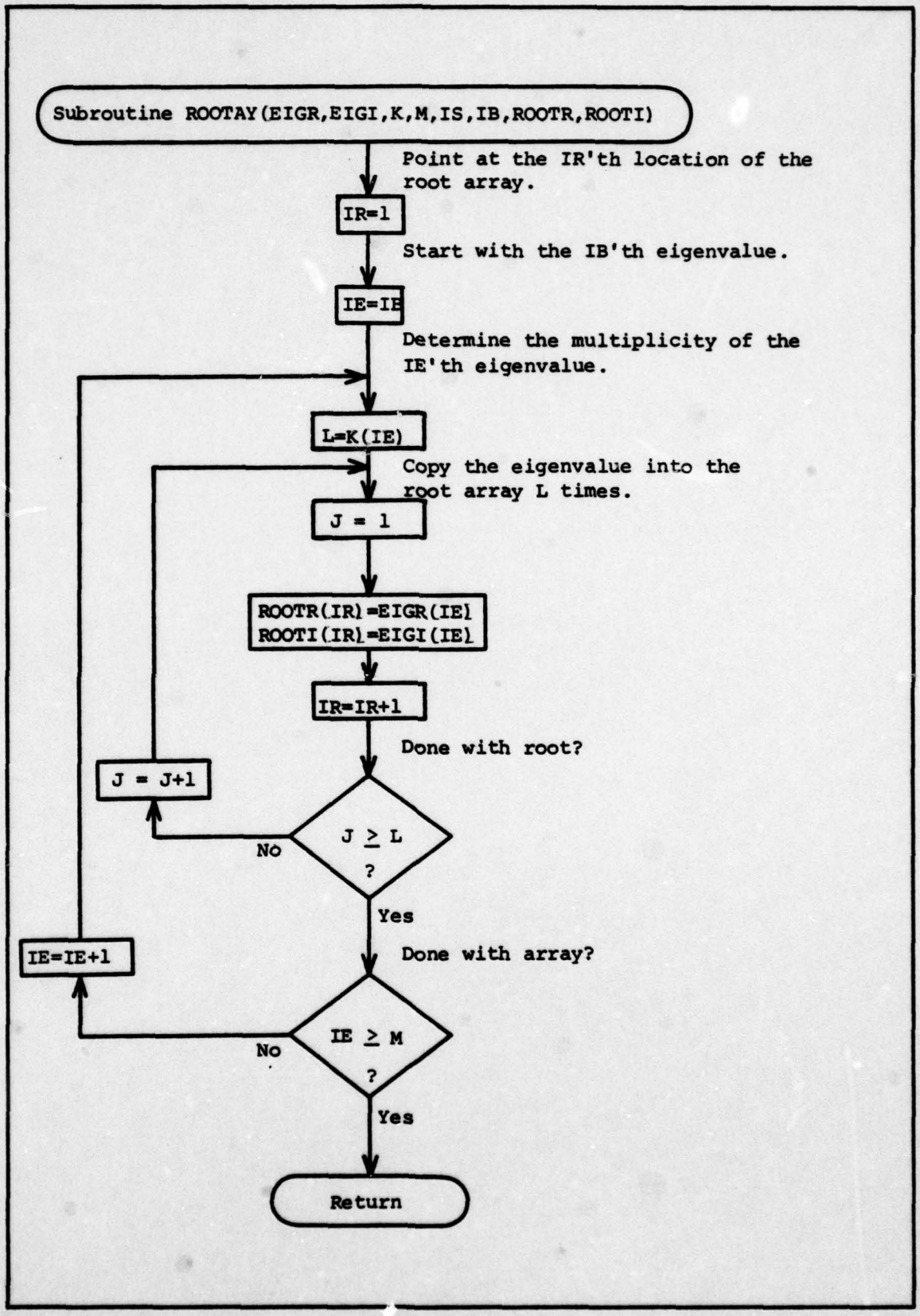

Figure A-16. Flowchart for subroutine ROOTAY.

Subroutine UNITY

Subroutine UNITY operates on the input polynomial to unitize the coefficient of the highest order variable. This is accomplished by extracting the coefficient of the highest order term as a gain multiplier. The polynomial gain is then updated to reflect the extraction. The input coefficients may be real or complex. In the event that the first coefficient of the input polynomial is zero, it is ignored and the next one is tested. If all the coefficients are zero, the gain and constant term are set to zero, and a warning message is printed if the printer has not been disabled.

Subroutine statement: SUBROUTINE UNITY(AR,AI,NS,AKR,AKI)

Subroutines called: None

Variables:

AR), AI) = Arrays containing the real and imaginary parts of the coefficients of the input polynomial

NS = Number of storage locations required for the coefficients of the input polynomial

AKR), AKI) = Real and imaginary parts of the polynomial gain

CLOSE = User specified magnitude used in testing the significance of variables. If the magnitude of the tested variable is less than CLOSE, it is assumed to be zero. CLOSE is stored in blank common.

MDIS = User specified printer disable flag. If MDIS is greater than zero all print steps using the flag are disabled. MDIS is stored in blank common.

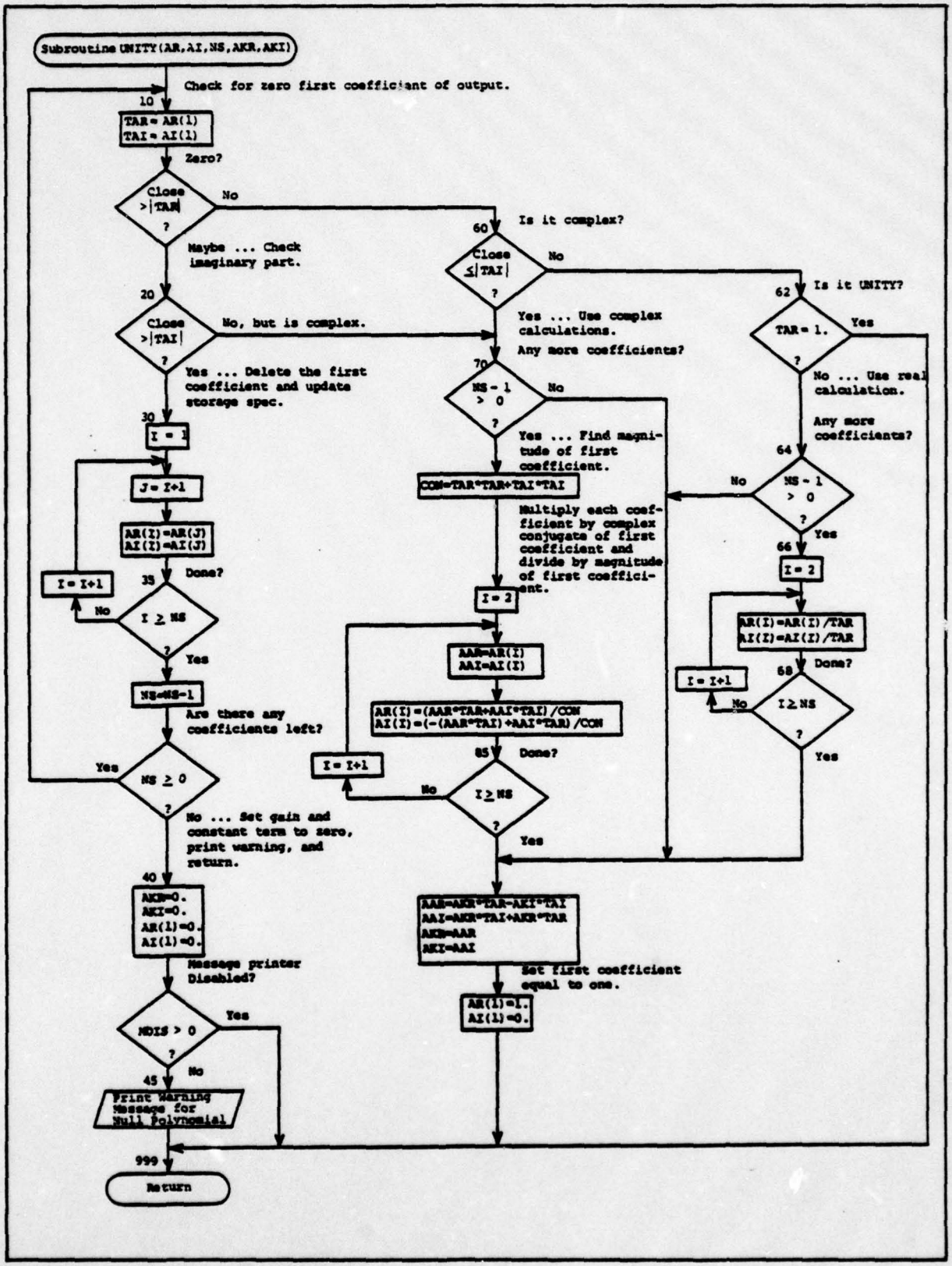

Figure A-17. Flowchart for subroutine UNITY.

Subroutine VANINV

Subroutine VANINV is an implementation of Csáki's generalized algorithm for finding the inverse of the Vandermonde matrix. It accepts an ordered array of real and complex eigenvalues and their multiplicities, and returns the inverse Vandermonde matrix. Prior to calling VANINV, the eigenvalues must be arranged in the order described for subroutine ESORT.

Subroutine statement: SUBROUTINE VANINV(EIGR,EIGI,KI,N,M,NS,MS,WR,WI, NW1,NW2,AR,AI,FR,FI,NR,NI,DR,DI,GR,GI,NOD,NON, PNR,PNI,PDR,PDI,PUR,PUI,PVR,PVI,T1R,T1I,T2R,T2I, T3R,T3I,DKR,DKI)

Subroutines called: COPYPOL,EVAL,EXPANDC,FRACDIF,RAT,ROOTAY

Function subprograms called: NFACT

Variables:

1) Formal parameters

EIGR), EIGI) = Arrays containing the real and imaginary parts of the M different eigenvalues of the N'th order system

KI = Array containing the multiplicities corresponding to the system eigenvalues

N = System order (note that $N = \sum_{i=1}^{M} K(i)$)

M = Number of different eigenvalues (complex conjugates are considered to be different)

NS), MS) = Storage size parameters, equal to N+1 and M+1, respectively

WR), WI) = N × N arrays into which the real and imaginary parts of the inverse Vandermonde matrix are placed by VANINV

NW1), NW2) = Storage size parameters, equal to $(2^{**}(KI_{max}-1))*(N-KI_{max})+1$ and 2N, respectively

AR), AI) = N-dimensioned arrays used for intermediate polynomial calculations (short form of POLYA)

FR)
FI) = NS-dimensioned arrays used for intermediate polynomial calculations (short forms for POLYF and POLYG)
GR)
GI)

NR) = (N × N)-dimensioned arrays used for the real and imaginary parts of the N truncated numerator polynomials
NI)

DR) = (M × MS)-dimensioned arrays used for the real and imaginary parts of the M denominator polynomials
DI)

NOD = M-dimensioned array used to store the polynomial order associated with each of the M denominator polynomials

NON = N-dimensioned array used to store the polynomial order associated with each of the N numerator polynomials

PNR,PNI)
PDR,PDI) = Working storage for polynomials used in differentiating a polynomial fraction by means of the quotient rule:
PUR,PUI)
PVR,PVI)

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u\frac{dv}{dx} - v\left(\frac{du}{dx}\right)}{v^2} = \frac{POLYN}{POLYD}$$

Note that all of these must be dimensioned at least NW1

T1R,T1T)
T2R,T2I) = NW2-dimensioned arrays used for temporary storage
T3R,T3I)

AKR,AKI)
FKR,FKI)
GKR,GKI)
UR,UI) = Real and imaginary parts of the gains associated with POLYA, POLYF, POLYG, POLYU, POLYV, POLYN, POLYD
VR,VI)
RNR,RNI)
RDR,RDI)

GNLR) = Real and imaginary parts of the gains associated with each of the M denominator polynomials. Since in all cases: GNLR=1. and GNLI=0., they are defined with a data statement.
GNLI)

NDIV = Number of derivatives that must be taken of the current inverse Vandermonde element, W(J,L), before evaluation.

NDIVS = Storage for NDIV so initial value can be recalled

RC = Row coefficient for J'th row of I'th block of the inverse Vandermonde matrix

Note that $RC = \frac{1.}{(KI(I) - J)!} \quad \frac{1.}{(NDIVS)!}$

NUS)
NVS)
NNS) = Dummy variables used to pass between subroutines the current size of the storage locations of POLYN and POLYD
NDS)
NT)

RNUMR, RNUMI)
DENR, DENI) = Real and imaginary parts of the polynomial fraction for an element of the inverse Vandermonde prior to rationalization

ZR)
ZI) = Real and imaginary parts of an element of the inverse Vandermonde matrix after rationalization of the fraction but prior to multiplication by the row coefficient

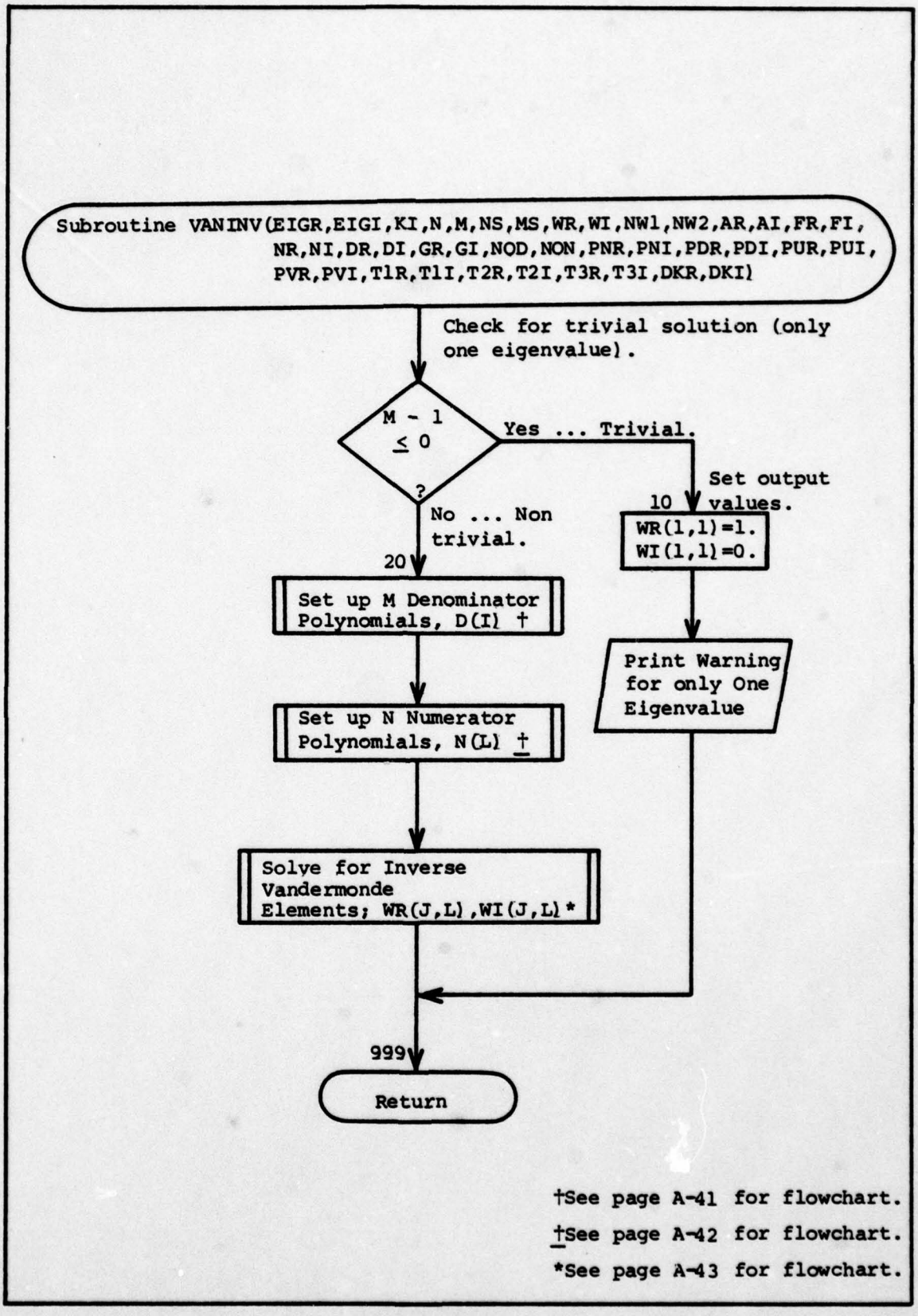

Figure A-18. Flowchart for subroutine VANINV.

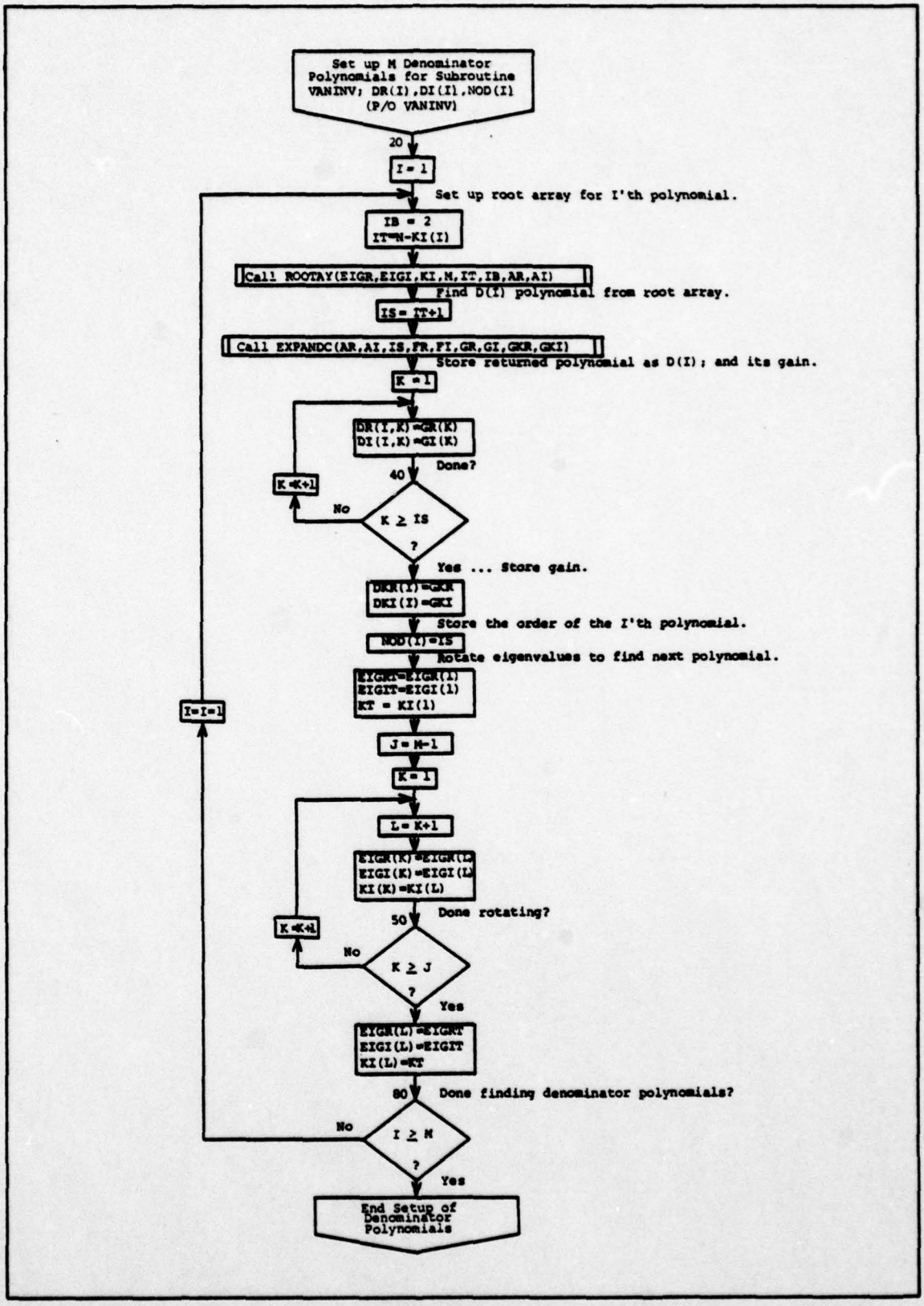

Figure A-19. Flowchart for denominator polynomial setup steps for subroutine VANINV.

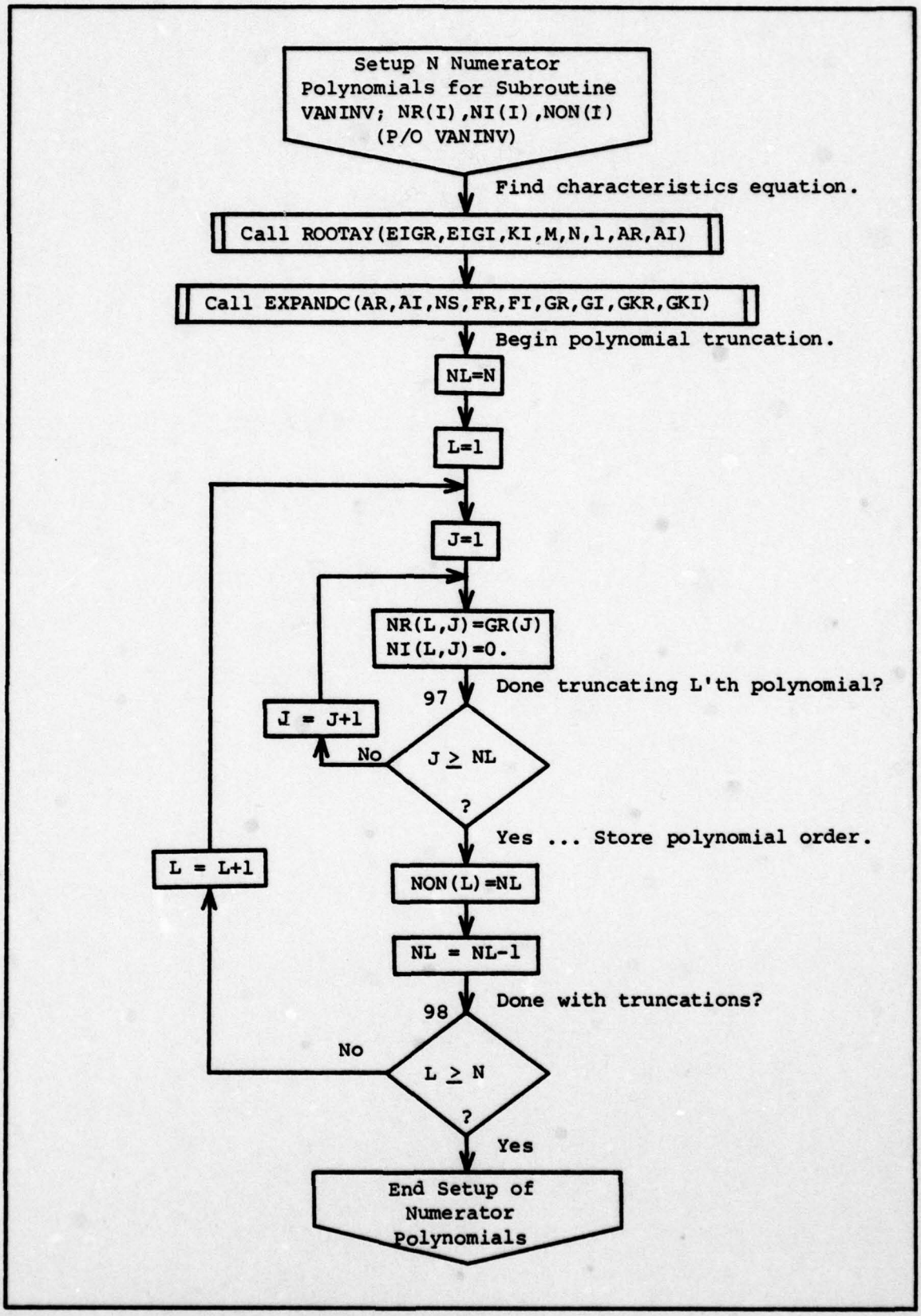

Figure A-20. Flowchart of numerator polynomial setup steps for subroutine VANINV.

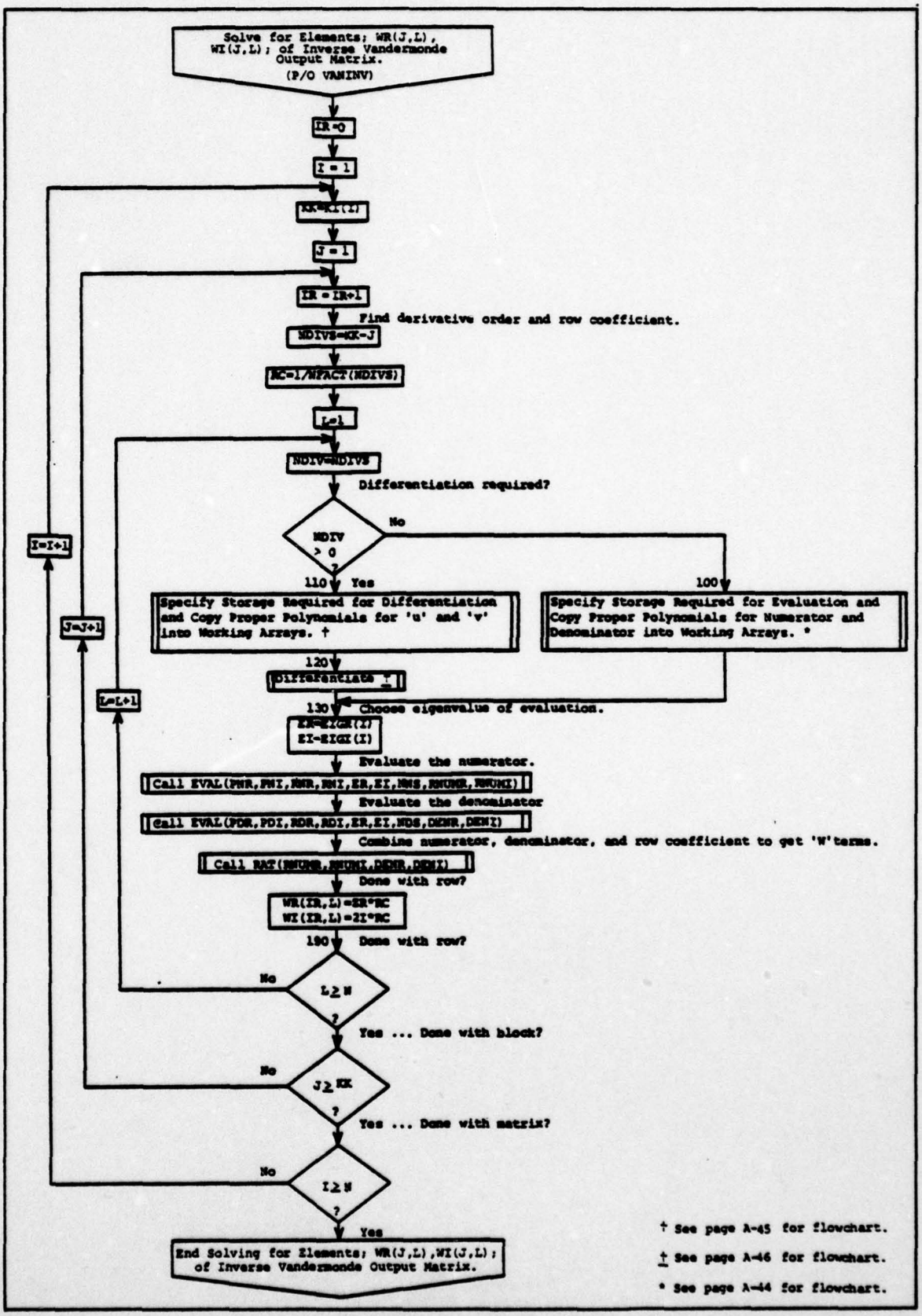

Figure A-21. Flowchart for matrix element computation steps for subroutine VANINV.

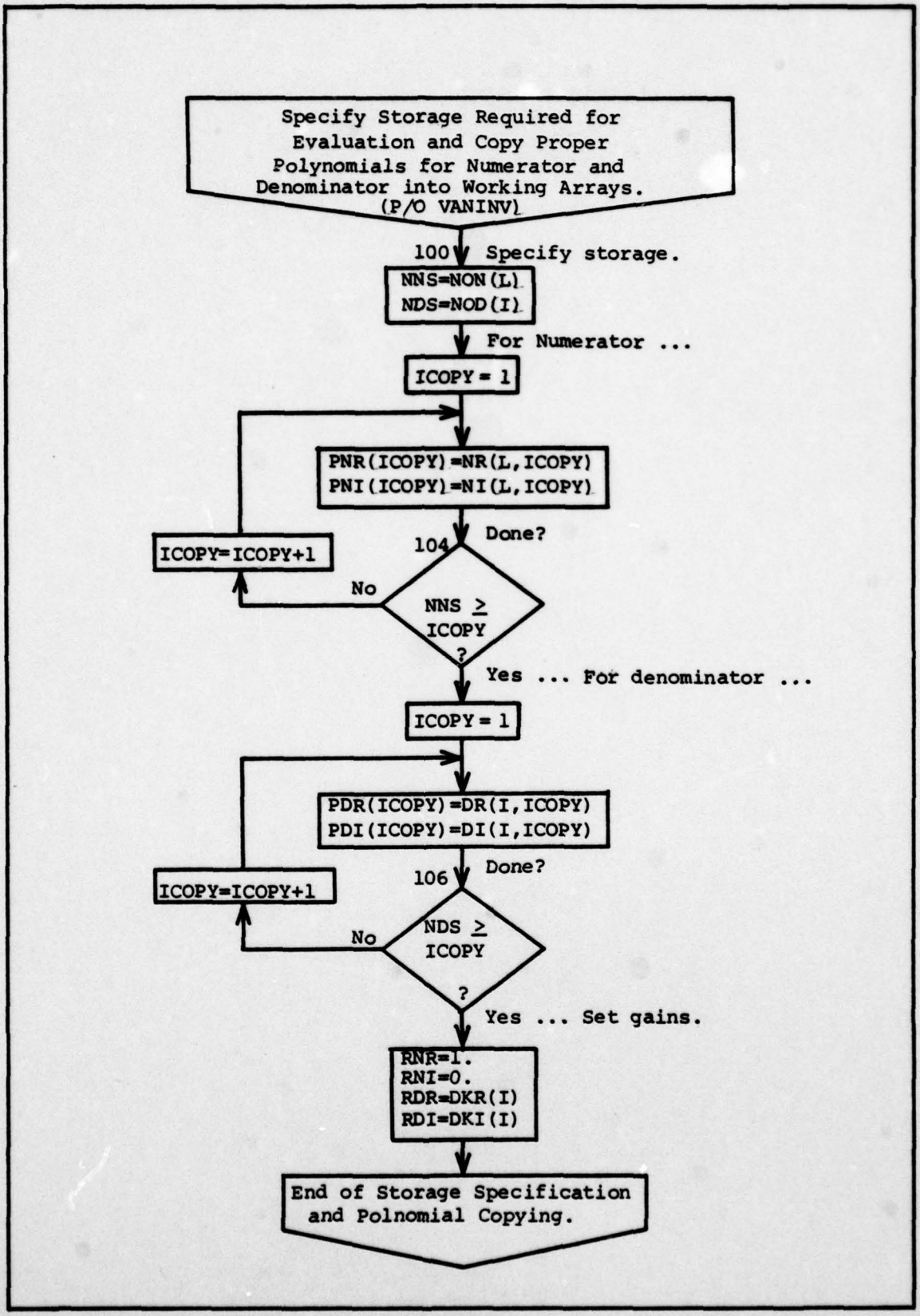

Figure A-22. Flowchart of polynomial transfer steps for subroutine VANINV (if no differentiation required).

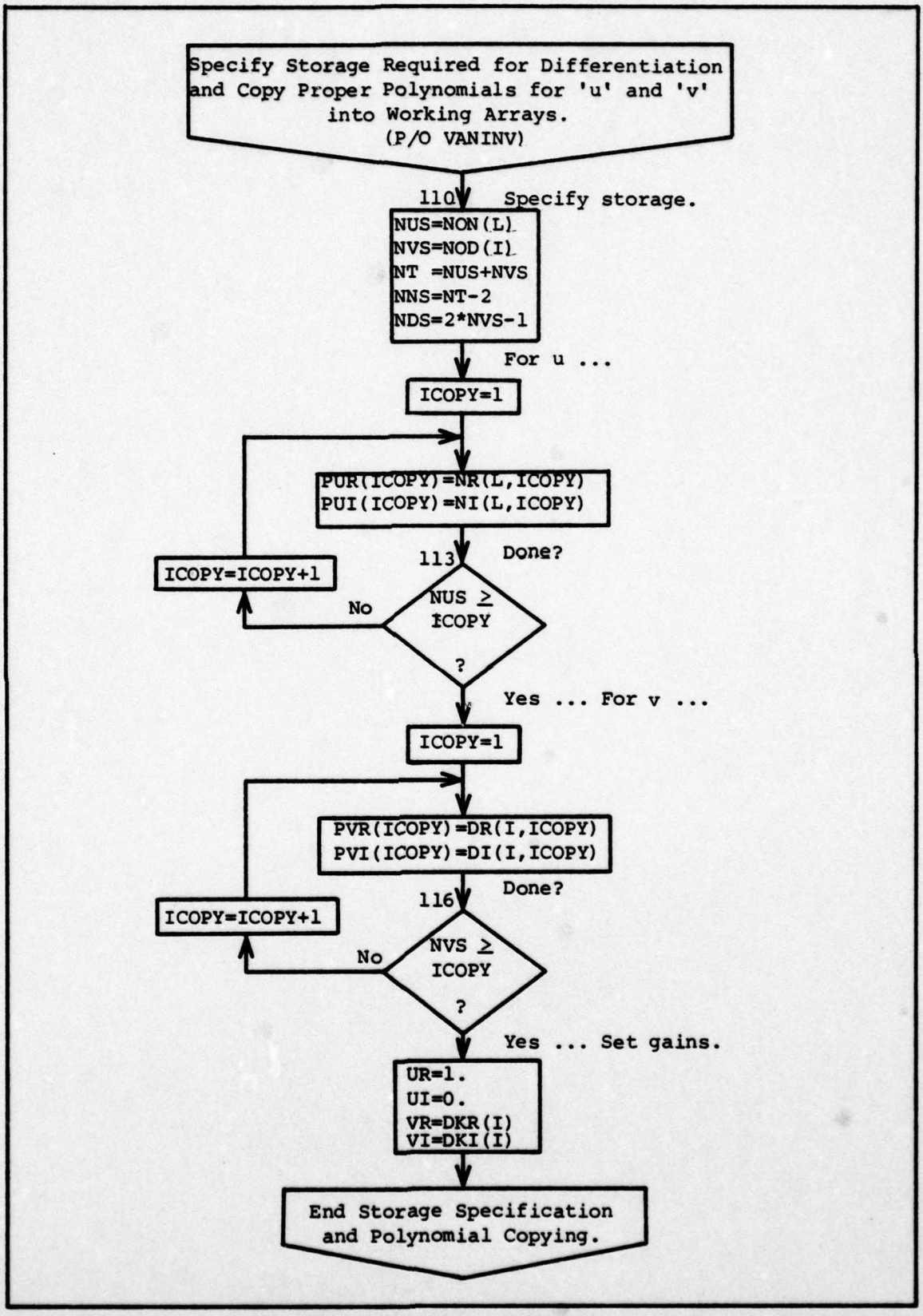

Figure A-23. Flowchart of polynomial transfer steps for subroutine VANINV (if differentiation required).

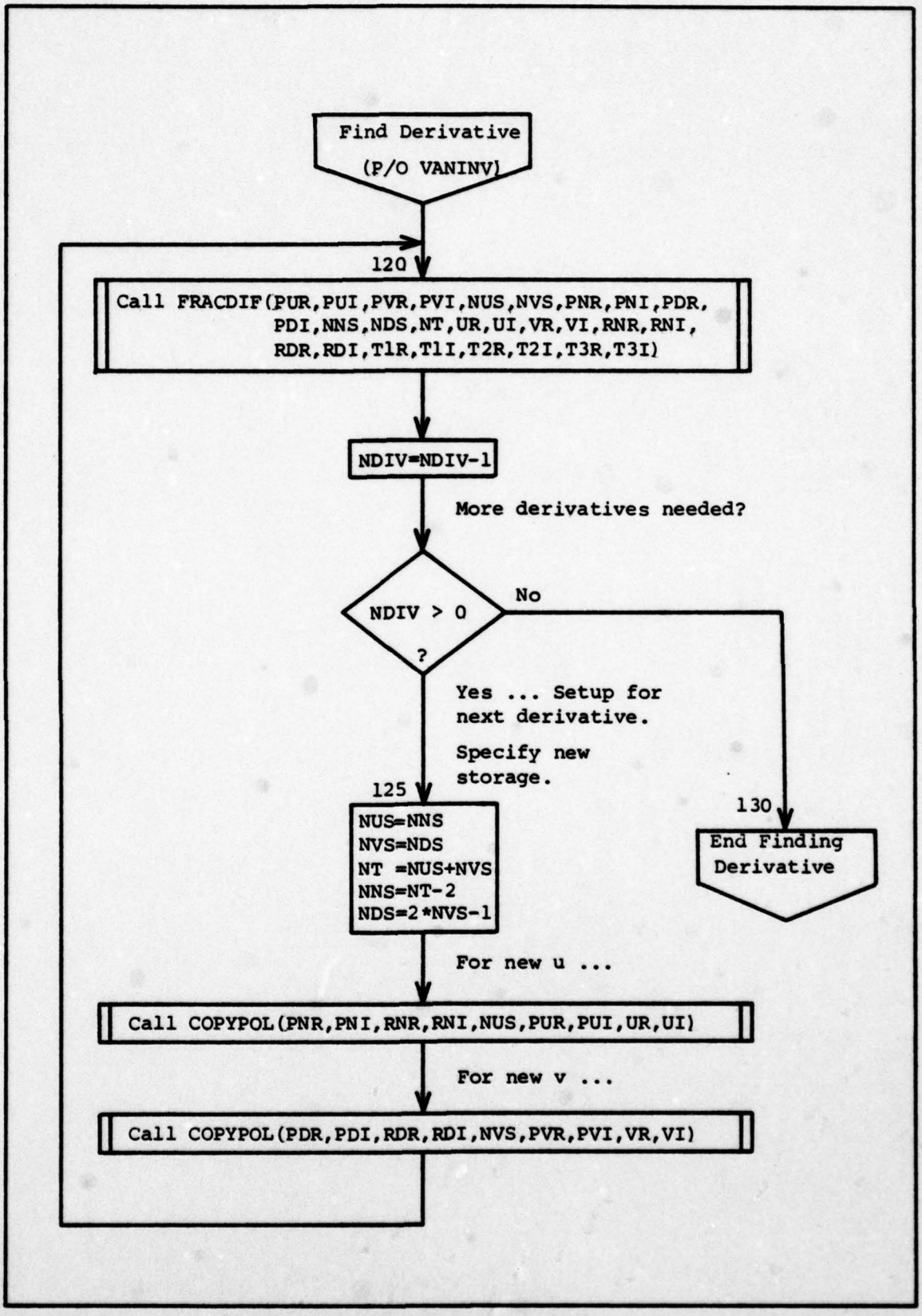

Figure A-24. Flowchart of differentiation steps for subroutine VANINV.

APPENDIX B

CODED PROGRAMS

Preface for Appendix B

Two types of programs are contained in this appendix. Part I contains all the subroutines described in Appendix A. They are listed in alphabetical order and form a complete package of subroutines for implementing Csáki's generalized algorithm as described in Section III of the main text. Part II contains programs and subroutines needed to dimension the arrays for the subroutines of Part I. They also perform input and output operations, and the tests that are described in Section IV of the main text. Since the programs in Part II are not flowcharted, a word description of each is included at the beginning of Part II.

Part I

Contents

```
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ CGAIN
C  SUBROUTINE CGAIN                                                            CGAIN
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ CGAIN
      SUBROUTINE CGAIN(AR,AI,NSA,AKR,AKI)                                      CGAIN
      COMMON CLOSE                                                             CGAIN
      DIMENSION AR(NSA),AI(NSA)                                                CGAIN
C     IS THE GAIN REAL?                                                        CGAIN
      IF(CLOSE-ABS(AKI))10,10,20                                               CGAIN
C     NO...MULTIPLY BY COMPLEX GAIN                                            CGAIN
   10 DO 15 I=1,NSA                                                            CGAIN
      TR=AR(I)                                                                 CGAIN
      TI=AI(I)                                                                 CGAIN
      AR(I)=TR*AKR-TI*AKI                                                      CGAIN
   15 AI(I)=TR*AKI+TI*AKR                                                      CGAIN
      GO TO 26                                                                 CGAIN
C     IS THE GAIN UNITY?                                                       CGAIN
   20 IF(AKR-1)21,30,21                                                        CGAIN
C     NO...MULTIPLY BY REAL GAIN                                               CGAIN
   21 DO 25 I=1,NSA                                                            CGAIN
      AR(I)=AR(I)*AKR                                                          CGAIN
   25 AI(I)=AI(I)*AKR                                                          CGAIN
C     UPDATE GAIN                                                              CGAIN
   26 AKI=0.0                                                                  CGAIN
      AKR=0.0                                                                  CGAIN
   30 RETURN                                                                   CGAIN
      END                                                                      CGAIN
```

```
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ COPYPOL
C  SUBROUTINE COPYPOL                                                          COPYPOL
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ COPYPOL
      SUBROUTINE COPYPOL(PIR,PII,GIR,GII,N,POR,POI,GOR,GOI)                    COPYPOL
      DIMENSION PIR(N),PII(N),POR(N),POI(N)                                    COPYPOL
C     COPY COEFFICIENTS                                                        COPYPOL
      DO 1 I=1,N                                                               COPYPOL
      POR(I)=PIR(I)                                                            COPYPOL
    1 POI(I)=PII(I)                                                            COPYPOL
C     COPY GAIN                                                                COPYPOL
      GOR=GIR                                                                  COPYPOL
      GOI=GII                                                                  COPYPOL
      RETURN                                                                   COPYPOL
      END                                                                      COPYPOL
```

```
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ ESORT
C  SUBROUTINE ESORT                                                            ESORT
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ ESORT
      SUBROUTINE ESORT(EIGR,EIGI,KI,M)                                         ESORT
      COMMON CLOSE,MDIS                                                        ESORT
      DIMENSION EIGR(M),EIGI(M),KI(M)                                          ESORT
C ZERO THE SWAP COUNTER                                                        ESORT
    1 ISWAP=0                                                                  ESORT
      DO 140 I=1,M                                                             ESORT
C POINT AT NEXT EIGENVALUE LOCATION...                                         ESORT
      IN=I+1                                                                   ESORT
C ARE THERE TWO EIGENVALUES TO COMPARE?                                        ESORT
      IF(M-IN)10,20,20                                                         ESORT
C ANY SWAPS PERFORMED THIS PASS?                                               ESORT
   10 IF(ISWAP)11,11,1                                                         ESORT
   11 GO TO 999                                                                ESORT
C                                                                              ESORT
C                    **BEGIN TEST LOGIC**                                      ESORT
C                                                                              ESORT
C  I'TH EIGENVALUE COMPLEX?                                                    ESORT
   20 IF(ABS(EIGI(I))-CLOSE)30,30,30                                           ESORT
C    FOR I'TH EIGENVALUE COMPLEX...                                            ESORT
C      IN'TH EIGENVALUE COMPLEX?                                               ESORT
   30 IF(ABS(EIGI(IN))-CLOSE)140,40,40                                         ESORT
C      COMPARE MULTIPLICITIES OF I'TH & IN'TH EIGENVALUES                      ESORT
   40 IF(KI(IN)-KI(I))140,50,130                                               ESORT
C      COMPARE REAL MAGNITUDES OF I'TH & IN'TH EIGENVALUES                     ESORT
   50 IF(ABS(EIGR(IN))-ABS(EIGR(I)))140,60,130                                 ESORT
C      COMPARE IMAGINARY MAGNITUDES OF I'TH & IN'TH EIGENVALUES                ESORT
   60 IF(ABS(EIGI(IN))-ABS(EIGI(I)))140,70,130                                 ESORT
C      POSSIBLE COMPLEX PAIR HAS BEEN FOUND.  MAKE SURE THE SIGNS OF THE       ESORT
C      REAL PARTS ARE THE SAME.                                                ESORT
   70 IF(EIGR(IN)-EIGR(I))140,80,130                                           ESORT
C      SIGNS OF REAL PARTS ARE THE SAME.  CHECK IMAGINARY PARTS.               ESORT
   80 IF(EIGI(IN)-EIGI(I))140,85,130                                           ESORT
C        PRINT MESSAGE IF MDIS.LE.0                                            ESORT
   85 IF(MDIS)87,87,140                                                        ESORT
```

```
   87 PRINT*,"TWO EIGENVALUES HAVE BEEN FOUND THAT ARE IDENTICAL.  ",     ESORT
     +"CHECK DELETE ROUTINE."                                            ESORT
      GO TO 140                                                          ESORT
C    FOR THE I'TH EIGENVALUE REAL...                                     ESORT
C      IN'TH EIGENVALUE COMPLEX?                                         ESORT
   90 IF(ABS(EIGI(IN))-CLOSE)100,130,130                                 ESORT
C      COMPARE MULTIPLICITIES OF I'TH & IN'TH EIGENVALUES                ESORT
  100 IF(KI(I)-KI(IN))130,110,140                                        ESORT
C      COMPARE REAL MAGNITUDES OF I'TH & IN'TH EIGENVALUES               ESORT
  110 IF(ABS(EIGR(I))-ABS(EIGR(IN)))130,120,140                          ESORT
C      EQUAL MAGNITUDE PAIR OF REAL EIGENVALUES HAS BEEN FOUND.  MAKE    ESORT
G      SURE THE POSITIVE EIGENVALUE IS FIRST.                            ESORT
  120 IF(EIGR(IN)-EIGR(I))140,140,130                                    ESORT
C                                                                        ESORT
C                    **END TEST LOGIC**                                  ESORT
C                                                                        ESORT
C SWAP EIGENVALUES...                                                    ESORT
C                                                                        ESORT
  130 EIGRT=EIGR(I)                                                      ESORT
      EIGIT=EIGI(I)                                                      ESORT
      KIT=KI(I)                                                          ESORT
      EIGR(I)=EIGR(IN)                                                   ESORT
      EIGI(I)=EIGI(IN)                                                   ESORT
      KI(I)=KI(IN)                                                       ESORT
      EIGR(IN)=EIGRT                                                     ESORT
      EIGI(IN)=EIGIT                                                     ESORT
      KI(IN)=KIT                                                         ESOPT
      ISWAP=ISWAP+1                                                      ESORT
C                                                                        ESORT
C MOVE TO NEXT EIGENVALUE PAIR                                           ESORT
C                                                                        ESORT
  140 CONTINUE                                                           ESORT
  999 RETURN                                                             ESORT
      END                                                                ESORT
```

```
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ EVAL
C  SUBROUTINE EVAL                                                              EVAL
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ EVAL
      SUBROUTINE EVAL(AR,AI,AKR,AKI,EIGR,EIGI,NS,BR,BI)                        EVAL
      DIMENSION AR(NS),AI(NS)                                                  EVAL
C     FIND POLYNOMIAL ORDER                                                    EVAL
      N=NS-1                                                                   EVAL
C     PUT CONSTANT TERM IN OUTPUT STORAGE                                      EVAL
      BR=AR(NS)                                                                EVAL
      BI=AI(NS)                                                                EVAL
C     CHECK FOR MORE TERMS                                                     EVAL
      IF(N)50,50,10                                                            EVAL
   10 CONTINUE                                                                 EVAL
C     MORE TERMS FOUND...CONTINUE EVALUATION                                   EVAL
      ER=EIGR                                                                  EVAL
      L=N                                                                      EVAL
C     COMPLEX EIGENVALUE?                                                      EVAL
      IF(EIGI)30,20,30                                                         EVAL
   20 CONTINUE                                                                 EVAL
C     REAL EVALUATION...                                                       EVAL
      DO 25 J=1,L                                                              EVAL
C       ADD NEXT HIGHER ORDER TERMS                                            EVAL
      BR=BR+AR(N)*ER                                                           EVAL
      BI=BI+AI(N)*ER                                                           EVAL
      N=N-1                                                                    EVAL
C       UPDATE EVALUATION TERM                                                 EVAL
   25 ER=ER*EIGR                                                               EVAL
      GO TO 50                                                                 EVAL
   30 CONTINUE                                                                 EVAL
C     COMPLEX EVALUATION...                                                    EVAL
      EI=EIGI                                                                  EVAL
      DO 35 J=1,L                                                              EVAL
      ARN=AR(N)                                                                EVAL
      AIN=AI(N)                                                                EVAL
C       ADD NEXT HIGHER ORDER TERMS                                            EVAL
      BR=BR+ARN*ER-AIN*EI                                                      EVAL
      BI=BI+ARN*EI+AIN*ER                                                      EVAL
```

```
      N=N-1                                                          EVAL
C       UPDATE EVALUATION TERMS                                      EVAL
      TR=ER*EIGR-EI*EIGI                                             EVAL
      TI=ER*EIGI+EI*EIGR                                             EVAL
      ER=TR                                                          EVAL
   35 EI=TI                                                          EVAL
   50 CONTINUE                                                       EVAL
C     MULTIPLY BY COMPLEX GAIN                                       EVAL
      CALL CGAIN(BR,BI,1,AKR,AKI)                                    EVAL
      RETURN                                                         EVAL
      END                                                            EVAL
```

```
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ EXPANDC
C  SUBROUTINE EXPANDC                                                          EXPANDC
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ EXPANDC
      SUBROUTINE EXPANDC(ROOTR,ROOTI,NS,BR,BI,CR,CI,CKR,CKI)                  EXPANDC
      COMMON CLOSE                                                            EXPANDC
      DIMENSION ROOTR(NS),ROOTI(NS),BR(NS),BI(NS),                            EXPANDC
     +CR(NS),CI(NS),AR(2),AI(2)                                               EXPANDC
C                                                                             EXPANDC
C TEST FOR FACTORS.                                                           EXPANDC
C                                                                             EXPANDC
C   IF THERE ARE NO FACTORS, PRINT ERROR MESSAGE AND STOP.                    EXPANDC
C   IF THERE IS ONLY ONE FACTOR, RETURN IMMEDIATELY.                          EXPANDC
C   IF THERE ARE TWO OR MORE FACTORS, CONTINUE WITH EXPANSION.                EXPANDC
C                                                                             EXPANDC
      NF=NS-1                                                                 EXPANDC
      IF(NS-2)3,90,5                                                          EXPANDC
    3 PRINT*,"NO FACTORS FOR EXPANDC ???"                                     EXPANDC
      STOP                                                                    EXPANDC
C                                                                             EXPANDC
C INITIALIZE GAINS                                                            EXPANDC
C                                                                             EXPANDC
    5 AKR=1.                                                                  EXPANDC
      AKI=0.                                                                  EXPANDC
      BKR=1.                                                                  EXPANDC
      BKI=0.                                                                  EXPANDC
C                                                                             EXPANDC
C IS IMAGINARY PART OF THE ROOT SIGNIFICANT?  SET UP BPOLY ACCORDINGLY.       EXPANDC
C                                                                             EXPANDC
      NSB=2                                                                   EXPANDC
      IF(ABS(ROOTI(1))-CLOSE)10,11,11                                         EXPANDC
C     NOT SIGNIFICANT...                                                      EXPANDC
   10 BI(2)=0.                                                                EXPANDC
      GO TO 15                                                                EXPANDC
C     SIGNIFICANT...                                                          EXPANDC
   11 BI(2)=-ROOTI(1)                                                         EXPANDC
   15 BR(1)=1.                                                                EXPANDC
      BI(1)=0.                                                                EXPANDC
```

```
      BR(2)=-ROOTR(1)                                                    EXPANDC
C                                                                        EXPANDC
C IS IMAGINARY PART OF THE NEXT ROOT SIGNIFICANT?  SET UP APOLY          EXPANDC
C ACCORDINGLY.                                                           EXPANDC
C                                                                        EXPANDC
      NSA=2                                                              EXPANDC
      DO 50 I=2,NF                                                       EXPANDC
      IF(ABS(ROOTI(I))-CLOSE)30,31,31                                    EXPANDC
C     NOT SIGNIFICANT...                                                 EXPANDC
   30 AI(2)=0.                                                           EXPANDC
      GO TO 35                                                           EXPANDC
C     SIGNIFICANT...                                                     EXPANDC
   31 AI(2)=-ROOTI(I)                                                    EXPANDC
   35 AR(1)=1.                                                           EXPANDC
      AR(2)=-ROOTR(I)                                                    EXPANDC
      AI(1)=0.                                                           EXPANDC
      NSC=NSB+1                                                          EXPANDC
      CALL POLYMC(AR,AI,BR,BI,CR,CI,NSA,NSB,NSC,1.0,0.0,1.0,0.0,CKR,CKI) EXPANDC
C                                                                        EXPANDC
C DONE WITH EXPANSION?                                                   EXPANDC
C   IF SO, RETURN                                                        EXPANDC
C   IF NOT, COPY CPOLY TO BPOLY AND LOOP AROUND AGAIN.                   EXPANDC
C                                                                        EXPANDC
      IF(NSC-NS)37,37,99                                                 EXPANDC
C     COPY CPOLY TO BPOLY...                                             EXPANDC
   37 NSB=NSC                                                            EXPANDC
      DO 40 J=1,NSC                                                      EXPANDC
      BR(J)=CR(J)                                                        EXPANDC
   40 BI(J)=CI(J)                                                        EXPANDC
   50 CONTINUE                                                           EXPANDC
      GO TO 99                                                           EXPANDC
   99 CR(1)=1.                                                           EXPANDC
      CI(1)=0.                                                           EXPANDC
      CR(2)=-ROOTR(1)                                                    EXPANDC
      CI(2)=-ROOTI(1)                                                    EXPANDC
      CKR=1.                                                             EXPANDC
      CKI=0.                                                             EXPANDC
   99 RETURN                                                             EXPANDC
      END                                                                EXPANDC
```


91

```
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ FRACDIF
C  SUBROUTINE FRACDIF                                                          FRACDIF
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ FRACDIF
      SUBROUTINE FRACDIF(PUR,PUI,PVR,PVI,NUS,NVS,PNR,PNI,PDR,PDI,NNS,          FRACDIF
     +NDS,NT,UR,UI,VR,VI,RNR,RNI,DR,DI,TEMP1R,TEMP1I,TEMP2R,TEMP2I,            FRACDIF
     +TEMP3R,TEMP3I)                                                           FRACDIF
      DIMENSION PUR(NUS),PUI(NUS),PVR(NVS),PVI(NVS),PNR(NNS),PNI(NNS),         FRACDIF
     +PDR(NDS),PDI(NDS),TEMP1R(NT),TEMP1I(NT),TEMP2R(NT),TEMP2I(NT),           FRACDIF
     +TEMP3R(NT),TEMP3I(NT)                                                    FRACDIF
C     FIND THE DERIVITIVE OF V                                                 FRACDIF
C       INITIALIZE GAIN FOR V                                                  FRACDIF
      VRD=VR                                                                   FRACDIF
      VID=VI                                                                   FRACDIF
      N=NVS-1                                                                  FRACDIF
      CALL POLYDIF(PVR,PVI,NVS,TEMP1R,TEMP1I,N,VRD,VID)                        FRACDIF
C     MULTIPLY BY U                                                            FRACDIF
      NSC=NUS+N-1                                                              FRACDIF
      CALL POLYMC(PUR,PUI,TEMP1R,TEMP1I,TEMP2R,TEMP2I,NUS,N,NSC,UR,UI,         FRACDIF
     +VRD,VID,T2R,T2I)                                                         FRACDIF
C     FIND THE DERIVITIVE OF U                                                 FRACDIF
C       INITIALIZE GAIN FOR U                                                  FRACDIF
      URD=UR                                                                   FRACDIF
      UID=UI                                                                   FRACDIF
      N=NUS-1                                                                  FRACDIF
      CALL POLYDIF(PUR,PUI,NUS,TEMP1R,TEMP1I,N,URD,UID)                        FRACDIF
C     MULTIPLY BY V                                                            FRACDIF
      CALL POLYMC(PVR,PVI,TEMP1R,TEMP1I,TEMP3R,TEMP3I,NVS,N,NSC,VR,VI,         FRACDIF
     +URD,UID,T3R,T3I)                                                         FRACDIF
C     SUBTRACT VDU FROM UDV TO GET NUMERATOR DERIVITTIVE                       FRACDIF
      CALL POLYSUB(TEMP3R,TEMP3I,TEMP2R,TEMP2I,PNR,PNI,NSC,NSC,NSC,T3R,        FRACDIF
     +T3I,T2R,T2I,RNR,RNI)                                                     FRACDIF
C     SQUARE V TO GET DENOMINATOR TERM                                         FRACDIF
      NV=2*NVS-1                                                               FRACDIF
      NNS=NSC                                                                  FRACDIF
      CALL POLYMC(PVR,PVI,PVR,PVI,PDR,PDI,NVS,NVS,NV,VR,VI,VR,VI,DR,DI)        FRACDIF
      NDS=NV                                                                   FRACDIF
      RETURN                                                                   FRACDIF
      END                                                                      FRACDIF
```

```
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ NFACT
C  FUNCTION NFACT                                                               NFACT
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ NFACT
      FUNCTION NFACT(IF)                                                        NFACT
      IF(IF)1,2,3                                                               NFACT
    1 PRINT*,"FACTORIAL IS NOT DEFINED FOR NEGATIVE NUMBERS"                    NFACT
      PRINT*,"EXECUTION ABNORMALLY TERMINATED BY FUNCTION NFACT"                NFACT
      STOP                                                                      NFACT
    2 NFACT=1                                                                   NFACT
      RETURN                                                                    NFACT
    3 IFT=1                                                                     NFACT
      DO 4 J=1,IF                                                               NFACT
    4 IFT=IFT*J                                                                 NFACT
      NFACT=IFT                                                                 NFACT
      RETURN                                                                    NFACT
      END                                                                       NFACT
```

```
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ OUT2C
C  SUBROUTINE OUT2C                                                             OUT2C
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ OUT2C
      SUBROUTINE OUT2C(NR,NC,AR,AI,IDR,IDI)                                     OUT2C
      DIMENSION AR(NR,NC),AI(NR,NC)                                             OUT2C
      DO 10 I=1,NR                                                              OUT2C
      DO 10 J=1,NC                                                              OUT2C
      WRITE 5,IDR,I,J,AR(I,J),IDI,I,J,AI(I,J)                                   OUT2C
    5 FORMAT(1X,2(A5,"(",I2,",",I2,")= ",F20.9,5X))                             OUT2C
   10 CONTINUE                                                                  OUT2C
      RETURN                                                                    OUT2C
      END                                                                       OUT2C
```

```
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ POLYADD
C  SUBROUTINE POLYADD                                                          POLYADD
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ POLYADD
      SUBROUTINE POLYADD(AR,AI,BR,BI,CR,CI,NSA,NSB,NSC,AKR,AKI,BKR,BKI,        POLYADD
     +CKR,CKI)                                                                 POLYADD
      DIMENSION AR(NSA),AI(NSA),BR(NSB),BI(NSB),CR(NSC),CI(NSC)                POLYADD
C     SET OUTPUT GAIN                                                          POLYADD
      CKR=1.                                                                   POLYADD
      CKI=0.                                                                   POLYADD
C     MULTIPLY POLYA BY ITS GAIN IF NOT UNITY                                  POLYADD
      CALL CGAIN(AR,AI,NSA,AKR,AKI)                                            POLYADD
C     MULTIPLY POLYB BY ITS GAIN IF NOT UNITY                                  POLYADD
      CALL CGAIN(BR,BI,NSB,BKR,BKI)                                            POLYADD
C     ZERO THE OUTPUT ARRAY                                                    POLYADD
      DO 1 I=1,NSC                                                             POLYADD
      CR(I)=0.0                                                                POLYADD
    1 CI(I)=0.0                                                                POLYADD
C     ADD POLYA TO THE OUTPUT ARRAY                                            POLYADD
      NSTART=NSC-NSA+1                                                         POLYADD
      I=1                                                                      POLYADD
      DO 2 J=NSTART,NSC                                                        POLYADD
      CR(J)=AR(I)                                                              POLYADD
      CI(J)=AI(I)                                                              POLYADD
    2 I=I+1                                                                    POLYADD
C     ADD POLYB TO THE OUTPUT ARRAY                                            POLYADD
      NSTART=NSC-NSB+1                                                         POLYADD
      I=1                                                                      POLYADD
      DO 3 J=NSTART,NSC                                                        POLYADD
      CR(J)=CR(J)+BR(I)                                                        POLYADD
      CI(J)=CI(J)+BI(I)                                                        POLYADD
    3 I=I+1                                                                    POLYADD
C     IF FIRST COEFFICIENT OF OUTPUT IS NOT UNITY,UNITIZE IT                   POLYADD
      IF(CR(1)-1)60,50,60                                                      POLYADD
   50 IF(CI(1))60,70,60                                                        POLYADD
   60 CALL UNITY(CR,CI,NSC,CKR,CKI)                                            POLYADD
   70 RETURN                                                                   POLYADD
      END                                                                      POLYADD
```

```
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ POLYDIF
C  SUBROUTINE POLYDIF                                                          POLYDIF
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ POLYDIF
      SUBROUTINE POLYDIF(POLYR,POLYI,NS,DPOLYR,DPOLYI,N,PKR,PKI)               POLYDIF
      DIMENSION POLYR(NS),POLYI(NS),DPOLYR(N),DPOLYI(N)                        POLYDIF
C     IS THE GAIN UNITY?                                                       POLYDIF
      IF(PKR.EQ.1.AND.PKI.EQ.0) GO TO 5                                        POLYDIF
C       NO...                                                                  POLYDIF
      CALL CGAIN(POLYR,POLYI,NS,PKR,PKI)                                       POLYDIF
C       YES...                                                                 POLYDIF
    5 NP=N                                                                     POLYDIF
      DO 10 I=1,N                                                              POLYDIF
      DPOLYR(I)=POLYR(I)*NP                                                    POLYDIF
      DPOLYI(I)=POLYI(I)*NP                                                    POLYDIF
   10 NP=NP-1                                                                  POLYDIF
      RETURN                                                                   POLYDIF
      END                                                                      POLYDIF
```

```
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ POLYMC
C  SUBROUTINE POLYMC                                                            POLYMC
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ POLYMC
      SUBROUTINE POLYMC(AR,AI,BR,BI,CR,CI,NSA,NSB,NSC,AKR,AKI,BKR,BKI,          POLYMC
     +CKR,CKI)                                                                  POLYMC
      DIMENSION AR(NSA),AI(NSA),BR(NSB),BI(NSB),CR(NSC),CI(NSC)                 POLYMC
C     FIND DEGREE OF RESULTING POLYNOMIAL                                       POLYMC
      NC=NSA+NSB-2                                                              POLYMC
C     WILL RESULT FIT PROPERLY IN SPECIFIED STORAGE?                            POLYMC
      IF(NSC-NC-1)10,20,15                                                      POLYMC
   10 PRINT*,"DEGREE OF RESULT FROM POLYNOMIAL MULTIPLICATION LARGER ",         POLYMC
     +"THAN STORAGE SPECIFIED--EXECUTION HALTED BY SUBROUTINE POLYMC"           POLYMC
      STOP                                                                      POLYMC
   15 PRINT*,"* *"                                                              POLYMC
      PRINT*,"    ***WARNING** DEGREE OF RESULT FROM POLYNOMIAL ",              POLYMC
     +"MULTIPLICATION UNDERFLOWS SPECIFIED STORAGE. CHECK TO VERIFY ",          POLYMC
     +"THAT NSC=NSA+NSB-1 BEFORE SUBROUTINE POLYMC IS CALLED."                  POLYMC
      PRINT*,"* *"                                                              POLYMC
C     INITIALIZE OUTPUT ARRAY                                                   POLYMC
   20 DO 25 I=1,NSC                                                             POLYMC
      CR(I)=0.                                                                  POLYMC
   25 CI(I)=0.                                                                  POLYMC
C     MAKE SURE GAINS FOR POLYA AND POLYB ARE CURRENT                           POLYMC
      IF(AR(1)-1)40,30,40                                                       POLYMC
   30 IF(AI(1))40,50,40                                                         POLYMC
   40 CALL UNITY(AR,AI,NSA,AKR,AKI)                                             POLYMC
   50 IF(BR(1)-1)70,60,70                                                       POLYMC
   60 IF(BI(1))70,80,70                                                         POLYMC
   70 CALL UNITY(BR,BI,NSB,BKR,BKI)                                             POLYMC
C     PERFORM MULTIPLICATION                                                    POLYMC
   80 DO 90 I=1,NSA                                                             POLYMC
      DO 90 J=1,NSB                                                             POLYMC
      IT=I+J-1                                                                  POLYMC
      ARI=AR(I)                                                                 POLYMC
      AII=AI(I)                                                                 POLYMC
      BRJ=BR(J)                                                                 POLYMC
      BIJ=BI(J)                                                                 POLYMC
```

```
      CR(IT)=CR(IT)+ARI*BRJ-AII*BIJ                                    POLYMC
   90 CI(IT)=CI(IT)+ARI*BIJ+AII*BRJ                                    POLYMC
C     UPDATE GAIN FOR POLYC                                            POLYMC
      CKR=AKR*BKR-AKI*BKI                                              POLYMC
      CKI=AKR*BKI+AKI*BKR                                              POLYMC
      RETURN                                                           POLYMC
      END                                                              POLYMC
```

```
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ POLYSUB
C  SUBROUTINE POLYSUB                                                          POLYSUB
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ POLYSUB
      SUBROUTINE POLYSUB(AR,AI,BR,BI,CR,CI,NSA,NSB,NSC,AKR,AKI,BKR,BKI,   POLYSUB
     +CKR,CKI)                                                              POLYSUB
      DIMENSION AR(NSA),AI(NSA),BR(NSB),BI(NSB),CR(NSC),CI(NSC)             POLYSUB
C     CHANGE THE SIGN OF THE GAIN IN POLYB                                   POLYSUB
      BKR=-BKR                                                              POLYSUB
      BKI=-BKI                                                              POLYSUB
      CALL POLYADD(AR,AI,BR,BI,CR,CI,NSA,NSB,NSC,AKR,AKI,BKR,BKI,CKR,     POLYSUB
     +CKI)                                                                  POLYSUB
      RETURN                                                                POLYSUB
      END                                                                   POLYSUB
```

```
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ RAT
C  SUBROUTINE RAT                                                               RAT
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ RAT
      SUBROUTINE RAT(RNR,RNI,DR,DI,RR,RI)                                       RAT
C     FIND MAGNITUDE OF DENOMINATOR                                             RAT
      DSQ=DR*DR+DI*DI                                                           RAT
C     TEST FOR NULL DENOMINATOR                                                 RAT
      IF(DSQ)2,1,2                                                              RAT
    1 PRINT*,"YOU ARE TRYING TO RATIONALIZE A COMPLEX FRACTION WITH A"          RAT
      PRINT*,"NULL DENOMINATOR.  EXECUTION ABNORMALLY TERMINATED BY"            RAT
      PRINT*,"SUBROUTINE RAT"                                                   RAT
      STOP                                                                      RAT
C     FIND REAL AND IMAGINARY PARTS OF RATIONALIZED OUTPUT                      RAT
    2 RR=(RNR*DR+RNI*DI)/DSQ                                                    RAT
      RI=(RNI*DR-RNR*DI)/DSQ                                                    RAT
      RETURN                                                                    RAT
      END                                                                       RAT
```

```
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ ROOTAY
C  SUBROUTINE ROOTAY                                                            ROOTAY
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ ROOTAY
      SUBROUTINE ROOTAY(EIGR,EIGI,K,M,IS,IB,ROOTR,ROOTI)                       ROOTAY
      DIMENSION EIGR(M),EIGI(M),K(M),ROOTR(IS),ROOTI(IS)                       ROOTAY
      IR=1                                                                     ROOTAY
      DO 1 IE=IB,M                                                             ROOTAY
      L=K(IE)                                                                  ROOTAY
      DO 1 J=1,L                                                               ROOTAY
      ROOTR(IR)=EIGR(IE)                                                       ROOTAY
      ROOTI(IR)=EIGI(IE)                                                       ROOTAY
      IR=IR+1                                                                  ROOTAY
    1 CONTINUE                                                                 ROOTAY
      RETURN                                                                   ROOTAY
      END                                                                      ROOTAY
```

```
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ UNITY
C  SUBROUTINE UNITY                                                            UNITY
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ UNITY
      SUBROUTINE UNITY(AR,AI,NS,AKR,AKI)                                       UNITY
      COMMON CLOSE,MDIS                                                        UNITY
      DIMENSION  AR(NS),AI(NS)                                                 UNITY
C CHECK FOR ZERO FIRST COEFFICIENT                                             UNITY
   10 TAR=AR(1)                                                                UNITY
      TAI=AI(1)                                                                UNITY
      IF(CLOSE-ABS(TAR))60,60,20                                               UNITY
C    MIGHT BE ZERO..CHECK IMAGINARY PART                                       UNITY
   20 IF(CLOSE-ABS(TAI))70,70,30                                               UNITY
C    FIRST COEFFICIENT IS ZERO.  DELETE IT AND UPDATE STORAGE SPEC.            UNITY
   30 CONTINUE                                                                 UNITY
      DO 35 I=1,NS                                                             UNITY
      J=I+1                                                                    UNITY
      AR(I)=AR(J)                                                              UNITY
   35 AI(I)=AI(J)                                                              UNITY
      NS=NS-1                                                                  UNITY
C    ARE THERE ANY COEFFICIENTS LEFT?                                          UNITY
      IF(NS)40,40,10                                                           UNITY
C    NO COEFFICIENTS LEFT...SET GAIN AND CONSTANT TERM TO ZERO, PRINT          UNITY
C     WARNING & RETURN                                                        UNITY
   40 AKR=0.                                                                   UNITY
      AKI=0.                                                                   UNITY
      AR(1)=0.                                                                 UNITY
      AI(1)=0.                                                                 UNITY
C     MESSAGE PRINTER DISABLED?                                                UNITY
      IF(MDIS)45,45,999                                                        UNITY
   45 PRINT*,"-*"                                                              UNITY
      PRINT*," ***WARNING*** SUBROUTINE UNITY WAS GIVEN A NULL POLYNOM",       UNITY
     +"IAL UPON WHICH TO OPERATE. THE GAIN AND CONSTANT TERM WERE SET ",       UNITY
     +"TO ZERO AND CONTROL WAS PASSED BACK TO THE CALLING ROUTINE."            UNITY
      PRINT*,"-*"                                                              UNITY
      GO TO 999                                                                UNITY
C    FIRST COEFFICIENT NOT ZERO...IS IT COMPLEX?                               UNITY
   60 IF(CLOSE-ABS(TAI))70,70,62                                               UNITY
C     FIRST COEFFICIENT NOT COMPLEX...IS IT UNITY?                             UNITY
```

```
   62 IF(TAR-1)64,999,64                                                UNITY
C       FIRST COEFFICIENT NOT UNITY...USE REAL CALCULATIONS             UNITY
C         ANY MORE COEFFICIENTS?                                        UNITY
   64 IF(NS-1)90,90,66                                                  UNITY
C           MORE COEFFICIENTS EXIST.  OPERATE ON REMAINING ONES        UNITY
   66 CONTINUE                                                          UNITY
      DO 68 I=2,NS                                                      UNITY
      AR(I)=AR(I)/TAR                                                   UNITY
   68 AI(I)=AI(I)/TAR                                                   UNITY
      GO TO 90                                                          UNITY
C     FIRST COEFFICIENT COMPLEX...USE COMPLEX CALCULATIONS              UNITY
C       ANY MORE COEFFICIENTS?                                          UNITY
   70 IF(NS-1)90,90,80                                                  UNITY
C          MORE COEFFICIENTS EXIST...FIND MAGNITUDE OF THE FIRST ONE    UNITY
   80 CON=TAR*TAR+TAI*TAI                                               UNITY
C          MULTIPLY EACH COEFFICIENT BY THE COMPLEX CONJUGATE OF THE    UNITY
C           FIRST COEFFICIENT & DIVIDE BY THE MAGNITUDE OF THE FIRST    UNITY
C           COEFFICIENT                                                 UNITY
      DO 85 I=2,NS                                                      UNITY
      AAR=AR(I)                                                         UNITY
      AAI=AI(I)                                                         UNITY
      AR(I)=(AAR*TAR+AAI*TAI)/CON                                       UNITY
   85 AI(I)=(-(AAR*TAI)+AAI*TAR)/CON                                    UNITY
C   UPDATE GAIN                                                         UNITY
   90 AAR=AKR*TAR-AKI*TAI                                               UNITY
      AAI=AKR*TAI+AKI*TAR                                               UNITY
      AKR=AAR                                                           UNITY
      AKI=AAI                                                           UNITY
C   SET FIRST COEFFICIENT EQUAL TO ONE                                  UNITY
      AR(1)=1.                                                          UNITY
      AI(1)=0.                                                          UNITY
  999 RETURN                                                            UNITY
      END                                                               UNITY
```

```
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ VANINV
C  SUBROUTINE VANINV                                                            VANINV
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ VANINV
      SUBROUTINE VANINV(EIGR,EIGI,KI,N,M,NS,MS,WR,WI,NW1,NW2,AR,AI,FR,   VANINV
     +FI,NR,NI,DR,DI,GR,GI,NOD,NON,PNR,PNI,POR,PDI,PUR,PUI,PVR,PVI,T1R,  VANINV
     +T1I,T2R,T2I,T3R,T3I,DKR,DKI)                                        VANINV
      REAL NR(N,N),NI(N,N)                                                VANINV
      DIMENSION EIGR(M),EIGI(M),KI(M),AR(N),AI(N),FR(NS),FI(NS),          VANINV
     +DR(M,MS),DI(M,MS),GR(NS),GI(NS),NOD(M),NON(N),PNR(NW1),PNI(NW1),   VANINV
     +POR(NW1),PDI(NW1),PUR(NW1),PUI(NW1),PVR(NW1),PVI(NW1),T1R(NW2),    VANINV
     +T1I(NW2),T2R(NW2),T2I(NW2),T3R(NW2),T3I(NW2),WR(N,N),WI(N,N),       VANINV
     +DKR(N),DKI(N)                                                      VANINV
C                                                                        VANINV
      DATA GNLR,GNLI/1.,0./                                              VANINV
C CHECK FOR TRIVIAL SOLUTION (ONLY ONE EIGENVALUE)                       VANINV
C                                                                        VANINV
      IF(M-1)10,10,20                                                    VANINV
   10 WR(1,1)=1.                                                         VANINV
      WI(1,1)=0.                                                         VANINV
      PRINT*,"  **WARNING**  SUBROUTINE VANINV FOUND A TRIVIAL SYSTEM ", VANINV
     +"WITH ONLY ONE EIGENVALUE:  EIGR= ",EIGR(1)," EIGI= ",EIGI(1),     VANINV
     +"  CHECK SYSTEM VALIDITY AND/OR DIMENSION STATEMENTS IN CALLING ", VANINV
     +"ROUTINE"                                                          VANINV
      GO TO 999                                                          VANINV
C SET UP DENOMINATOR POLYNOMIALS, D(I)                                   VANINV
   20 DO 80 I=1,M                                                        VANINV
C   SET UP ROOT ARRAY                                                    VANINV
      IB=2                                                               VANINV
      IT=N-KI(1)                                                         VANINV
      CALL ROOTAY(EIGR,EIGI,KI,M,IT,IB,AR,AI)                            VANINV
C   FIND D(I) POLYNOMIAL FROM ROOT ARRAY                                 VANINV
      IS=IT+1                                                            VANINV
      CALL EXPANDC(AR,AI,IS,FR,FI,GR,GI,GKR,GKI)                         VANINV
C   STORE RETURNED POLYNOMIAL AS D(I)                                    VANINV
      DO 40 K=1,IS                                                       VANINV
      DR(I,K)=GR(K)                                                      VANINV
   40 DI(I,K)=GI(K)                                                      VANINV
      DKR(I)=GKR                                                         VANINV
```

```
      DKI(I)=GKI                                                        VANINV
      NOD(I)=IS                                                         VANINV
C   ROTATE EIGENVALUES TO FIND NEXT POLYNOMIAL                          VANINV
      EIGRT=EIGR(1)                                                     VANINV
      EIGIT=EIGI(1)                                                     VANINV
      KT=KI(1)                                                          VANINV
      J=M-1                                                             VANINV
      DO 50 K=1,J                                                       VANINV
      L=K+1                                                             VANINV
      EIGR(K)=EIGR(L)                                                   VANINV
      EIGI(K)=EIGI(L)                                                   VANINV
   50 KI(K)=KI(L)                                                       VANINV
      EIGR(L)=EIGRT                                                     VANINV
      EIGI(L)=EIGIT                                                     VANINV
   80 KI(L)=KT                                                          VANINV
C SET UP N'NUMERATOR POLYNOMIALS, N(L)                                  VANINV
C   FIND CHARACTERISTIC EQUATION                                        VANINV
      CALL ROOTAY(EIGR,EIGI,KI,M,N,1,AR,AI)                             VANINV
      CALL EXPANDC(AR,AI,NS,FR,FI,GR,GI,GKR,GKI)                        VANINV
C   BEGIN POLYNOMIAL TRUNCATION                                         VANINV
      NL=N                                                              VANINV
      DO 98 L=1,N                                                       VANINV
      DO 97 J=1,NL                                                      VANINV
      NR(L,J)=GR(J)                                                     VANINV
   97 NI(L,J)=0.                                                        VANINV
C     DONE TRUNCATING L'TH POLYNOMIAL.  STORE POLYNOMIAL ORDER & GO TO  VANINV
C     NEXT POLYNOMIAL                                                   VANINV
      NON(L)=NL                                                         VANINV
   98 NL=NL-1                                                           VANINV
C SOLVE FOR ELEMENTS: WR(J,L), WI(J,L): OF OUTPUT MATRIX                VANINV
C                                                                       VANINV
      IR=0                                                              VANINV
      DO 190 I=1,M                                                      VANINV
      KK=KI(I)                                                          VANINV
      DO 190 J=1,KK                                                     VANINV
      IR=IR+1                                                           VANINV
C   FIND DERIVITIVE ORDER AND ROW COEFFICIENT                           VANINV
                                                                        VANINV
```

```
      NDIVS=KK-J
      RC=1./NFACT(NDIVS)                                                VANINV
      DO 190 L=1,N                                                      VANINV
      NDIV=NDIVS                                                        VANINV
C   DIFFERENTIATION REQUIRED?                                           VANINV
      IF(NDIV)100,100,110                                               VANINV
  100 CONTINUE                                                          VANINV
C     NO...SPECIFY STORAGE REQUIRED FOR EVALUATION AND COPY PROPER      VANINV
C      POLYNOMIALS FOR NUMERATOR & DENOMINATOR INTO WORKING ARRAYS      VANINV
      NNS=NON(L)                                                        VANINV
      NDS=NOD(I)                                                        VANINV
C       FOR NUMERATOR                                                   VANINV
      DO 104 ICOPY=1,NNS                                                VANINV
      PNR(ICOPY)=NR(L,ICOPY)                                            VANINV
  104 PNI(ICOPY)=NI(L,ICOPY)                                            VANINV
C       FOR DENOMINATOR                                                 VANINV
      DO 106 ICOPY=1,NDS                                                VANINV
      PDR(ICOPY)=DR(I,ICOPY)                                            VANINV
  106 PDI(ICOPY)=DI(I,ICOPY)                                            VANINV
C       SET GAINS                                                       VANINV
      RNR=1.                                                            VANINV
      RNI=0.                                                            VANINV
      RDR=DKR(I)                                                        VANINV
      RDI=DKI(I)                                                        VANINV
      GO TO 130                                                         VANINV
  110 CONTINUE                                                          VANINV
C     YES...SPECIFY STORAGE REQUIRED FOR DIFFERENTIATION AND COPY PROPER VANINV
C      POLYNOMIALS FOR 'U' & 'V' INTO WORKING ARRAYS                    VANINV
      NUS=NON(L)                                                        VANINV
      NVS=NOD(I)                                                        VANINV
      NT=NUS+NVS                                                        VANINV
      NNS=NT-2                                                          VANINV
      NDS=2*NVS-1                                                       VANINV
C       FOR U...                                                        VANINV
      DO 113 ICOPY=1,NUS                                                VANINV
      PUR(ICOPY)=NR(L,ICOPY)                                            VANINV
  113 PUI(ICOPY)=NI(L,ICOPY)                                            VANINV
C       FOR V...                                                        VANINV
      DO 116 ICOPY=1,NVS                                                VANINV
```

```
      PVR(ICOPY)=DR(I,ICOPY)                                                VANINV
  116 PVI(ICOPY)=DI(I,ICOPY)                                                VANINV
C       SET GAINS                                                           VANINV
      UR=1.                                                                 VANINV
      UI=0.                                                                 VANINV
      VR=DKR(I)                                                             VANINV
      VI=DKI(I)                                                             VANINV
  120 CONTINUE                                                              VANINV
C     FIND DERIVITIVE                                                       VANINV
      CALL FRACDIF(PUR,PUI,PVR,PVI,NUS,NVS,PNR,PNI,PDR,PDI,NNS,NDS,NT,      VANINV
     +UR,UI,VR,VI,RNR,RNI,RDR,RDI,T1R,T1I,T2R,T2I,T3R,T3I)                  VANINV
      NDIV=NDIV-1                                                           VANINV
C     MORE DERIVITIVES NEEDED?                                              VANINV
      IF(NDIV)130,130,125                                                   VANINV
  125 CONTINUE                                                              VANINV
C       YES...SET UP FOR NEXT DERIVITIVE                                    VANINV
C         SPECIFY NEW STORAGE                                               VANINV
      NUS=NNS                                                               VANINV
      NVS=NDS                                                               VANINV
      NT=NUS+NVS                                                            VANINV
      NNS=NT-2                                                              VANINV
      NDS=2*NVS-1                                                           VANINV
C         FOR NEW U...                                                      VANINV
      CALL COPYPOL(PNR,PNI,RNR,RNI,NUS,PUR,PUI,UR,UI)                       VANINV
C         FOR NEW V...                                                      VANINV
      CALL COPYPOL(PDR,PDI,RDR,RDI,NVS,PVR,PVI,VR,VI)                       VANINV
      GO TO 120                                                             VANINV
  130 CONTINUE                                                              VANINV
C       NO...CHOOSE EIGENVALUE OF EVALUATION                                VANINV
      ER=EIGR(I)                                                            VANINV
      EI=EIGI(I)                                                            VANINV
C   EVALUATE RESULTING NUMERATOR                                            VANINV
      CALL EVAL(PNR,PNI,RNR,RNI,ER,EI,NNS,RNUMR,RNUMI)                      VANINV
C   EVALUATE RESULTING DENOMINATOR                                          VANINV
      CALL EVAL(PDR,PDI,RDR,RDI,ER,EI,NDS,DENR,DENI)                        VANINV
C   COMBINE NUMERATOR, DENOMINATOR, & ROW COEFFICIENT TO GET 'W' TERMS      VANINV
      CALL RAT(RNUMR,RNUMI,DENR,DENI,ZR,ZI)                                 VANINV
```

```
      WR(IR,L)=ZR*RC                                                    VANINV
      WI(IR,L)=ZI*RC                                                    VANINV
  190 CONTINUE                                                          VANINV
  999 RETURN                                                            VANINV
      END                                                               VANINV
```

Part II

Contents

Program READER contains the dimension statements needed to test subroutine VANINV's computational time and accuracy. After reading in the system eigenvalues, READER calls subroutine CATEST to perform the tests. In the form shown, READER works only with real, distinct eigenvalues. Some simple modifications would enable it to accommodate complex and/or repeated eigenvalues.

Subroutine CATEST performs computation accuracy and time tests on subroutine VANINV and the IMSL subroutine LINV2F.

Subroutine OUT2 prints out, in column form, a two dimensional array. It is similar to OUT2C shown in Part I.

Subroutine UERTST disables the warning messages from the IMSL routines. Terminal errors (IER > 128) are not suppressed.

Subroutine VANF forms the Vandermonde matrix from a collection of eigenvalues. In its present form, it cannot handle complex or repeated eigenvalues.

```
      PROGRAM READER(INPUT=/72,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
      COMMON CLOSE,MDIS
      DIMENSION EIGR(20),EIGI(20),KI(20),AR(20),AI(20),FR(21),FI(21),
     +DR(20,21),DI(20,21),GR(21),GI(21),NOD(20),NON(20),PNR(20),PNI(20),
     +PDR(20),PDI(20),PUR(20),PUI(20),PVR(20),PVI(20),T1R(40),T1I(40),
     +T2R(40),T2I(40),T3R(40),T3I(40),VANIR(20,20),VANII(20,20),DKR(20),
     +DKI(20),C(20,20),B(20,20),VANR(20,20),NR(20,20),NI(20,20),
     +WKAREA(500)
      DATA ID,IDI/4HVANR,5HCHECK/
      CLOSE=.0000001
      MDIS=1
      READ*,NO,NUMB,IDGT
      PRINT*,"THERE ARE ",NO," DATA SETS.  EACH WILL BE TESTED FOR ",
     +"COMPUTATION TIME BY PERFORMING ",NUMB," ITERATIONS WITH IDGT= ",
     +IDGT
      PRINT*,"***"
      DO 50 I=1,NO
      READ*,N
      PRINT*,"DATA SET ",I," HAS ",N," EIGENVALUES.  THEY ARE:"
      DO 10 J=1,N
      READ*,EIGR(J)
      EIGI(J)=0.
      KI(J)=1
   10 PRINT*,"EIGR(",J,") = ",EIGR(J)," EIGI(",J,") = ",EIGI(J)," KI(",J,
     +") = ",KI(J)
      PRINT*,"***"
      NS=N+1
      NW2=N+N
      CALL CATEST(EIGR,EIGI,KI,VANIR,VANII,AR,AI,FR,FI,NR,NI,DR,DI,GR,
     +GI,NOD,NON,PNR,PNI,PDR,PDI,PUR,PUI,PVR,PVI,T1R,T1I,T2R,T2I,T3R,
     +T3I,DKR,DKI,NS,N,C,NW2,B,ID,IDI,NUMB,VANR,IDGT,WKAREA)
   50 PRINT*,"***"
      PRINT*,"END"
      END
```

```
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ CATEST
C  SUBROUTINE CATEST                                                           CATEST
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ CATEST
      SUBROUTINE CATEST(EIGR,EIGI,KI,VANIR,VANII,AR,AI,FR,FI,NR,NI,DR,         CATEST
     +DI,GR,GI,NOD,NON,PNR,PNI,PDR,PDI,PUR,PUI,PVR,PVI,T1R,T1I,T2R,T2I,       CATEST
     +T3R,T3I,DKR,DKI,NS,N,C,NW2,B,ID,IDI,NUMB,VANR,IDGT,WKAREA)               CATEST
      DIMENSION EIGR(N),EIGI(N),KI(N),AR(N),AI(N),FR(NS),FI(NS),               CATEST
     +DR(N,NS),DI(N,NS),GR(NS),GI(NS),NOD(N),NON(N),PNR(N),PNI(N),             CATEST
     +PDR(N),PDI(N),PUR(N),PUI(N),PVR(N),PVI(N),T1R(NW2),T1I(NW2),             CATEST
     +T2R(NW2),T2I(NW2),T3R(NW2),T3I(NW2),VANIR(N,N),VANII(N,N),DKR(N),         CATEST
     +DKI(N),C(N,N),B(N,N),VANR(N,N)                                           CATEST
      IDGTS=IDGT                                                               CATEST
      CALL VANF(EIGR,N,VANR)                                                   CATEST
      CALL OUT2(N,N,VANR,ID)                                                   CATEST
      CPSTORE=SECOND(CP)                                                       CATEST
      DO 10 I=1,NUMB                                                           CATEST
   10 CALL VANINV(EIGR,EIGI,KI,N,N,NS,NS,VANIR,VANII,N,NW2,AR,AI,FR,FI,        CATEST
     +NR,NI,DR,DI,GR,GI,NOD,NON,PNR,PNI,PDR,PDI,PUR,PUI,PVR,PVI,T1R,T1I,       CATEST
     +T2R,T2I,T3R,T3I,DKR,DKI)                                                 CATEST
      CPTIME=SECOND(CP)-CPSTORE                                                CATEST
      PRINT*,"*"                                                               CATEST
      PRINT*,"TIME FOR ",NUMB," ITERATIONS USING VANINV IS:  ",CPTIME          CATEST
      PRINT*,"*"                                                               CATEST
      PRINT*,"INVERSE VANDERMONDE FOUND IS:"                                   CATEST
      CALL OUT2(N,N,VANIR,ID)                                                  CATEST
      PRINT*,"*"                                                               CATEST
      CALL VMULFF(VANR,VANIR,N,N,N,N,N,C,N,IER)                                CATEST
      PRINT*,"FOR VMULFF, IER= ",IER                                           CATEST
      DO 20 I=1,N                                                              CATEST
   20 C(I,I)=C(I,I)-1.                                                         CATEST
      DO 25 J=1,N                                                              CATEST
      DO 25 I=1,N                                                              CATEST
      C(I,J)=ABS(C(I,J))                                                       CATEST
   25 VANIR(I,J)=ABS(VANIR(I,J))                                               CATEST
      CALL VMULFF(VANIR,C,N,N,N,N,N,VANII,N,IER)                               CATEST
      PRINT*,"FOR VMULFF(2), IER= ",IER                                        CATEST
      PRINT*,"ACCURACY CHECK MATRIX (CHECK=A**(-1)*E) SHOULD BE NULL."         CATEST
```

```
      CALL OUT2(N,N,VANII,IDI)                                          CATEST
      PRINT*,"**"                                                       CATEST
      SUM=0.                                                            CATEST
      DO 27 J=1,N                                                       CATEST
      DO 27 I=1,N                                                       CATEST
   27 SUM=SUM+VANII(I,J)                                                CATEST
      PRINT*,"ACCURACY MERIT FIGURE FOR VANINV IS:  ",SUM               CATEST
      PRINT*,"**"                                                       CATEST
      CPSTORE=SECOND(CP)                                                CATEST
      DO 40 KOUNT=1,NUMB                                                CATEST
      DO 30 I=1,N                                                       CATEST
      DO 30 J=1,N                                                       CATEST
   30 VANII(I,J)=VANR(I,J)                                              CATEST
      CALL LINV2F(VANII,N,N,B,IDGT,WKAREA,IER)                          CATEST
      IDOUT=IDGT                                                        CATEST
   40 IDGT=IDGTS                                                        CATEST
      CPTIME=SECOND(CP)-CPSTORE                                         CATEST
      PRINT*,"TIME FOR ",NUMB," ITERATIONS USING IMSL IS:  ",CPTIME     CATEST
      PRINT*,"FOR LINV2F, IER= ",IER,"  IDGT WAS RETURNED=  ",IDOUT     CATEST
      PRINT*,"**"                                                       CATEST
      PRINT*,"INVERSE VANDERMONDE MATRIX FOUND IS:"                     CATEST
      CALL OUT2(N,N,B,ID)                                               CATEST
      PRINT*,"**"                                                       CATEST
      IDGT=IDGTS                                                        CATEST
      DO 45 J=1,N                                                       CATEST
      DO 45 I=1,N                                                       CATEST
   45 VANII(I,J)=B(I,J)                                                 CATEST
      CALL LINV2F(VANII,N,N,C,IDGT,WKAREA,IER)                          CATEST
      PRINT*,"FOR LINV2F(2), IER= ",IER,"  IDGT WAS RETURNED= ",IDGT    CATEST
      PRINT*,"REINVERTING THE IMSL INVERSE USING IMSL YIELDS:"          CATEST
      CALL OUT2(N,N,C,ID)                                               CATEST
      PRINT*,"**"                                                       CATEST
      CALL VMULFF(VANR,B,N,N,N,N,N,C,N,IER)                             CATEST
      PRINT*,"FOR VMULFF, IER= ",IER                                    CATEST
      DO 50 I=1,N                                                       CATEST
   50 C(I,I)=C(I,I)-1.                                                  CATEST
      DO 55 J=1,N                                                       CATEST
```

```
      DO 55 I=1,N                                                          CATEST
      C(I,J)=ABS(C(I,J))                                                   CATEST
   55 B(I,J)=ABS(B(I,J))                                                   CATEST
      CALL VMULFF(B,C,N,N,N,N,N,VANII,N,IER)                               CATEST
      PRINT*,"FOR VMULFF(2), IER= ",IER                                    CATEST
      PRINT*,"ACCURACY CHECK MATRIX (CHECK=A**(-1)*E) SHOULD BE NULL."     CATEST
      CALL OUT2(N,N,VANII,IDI)                                             CATEST
      PRINT*,"**"                                                          CATEST
      SUM=0.                                                               CATEST
      DO 60 J=1,N                                                          CATEST
      DO 60 I=1,N                                                          CATEST
   60 SUM=SUM+VANII(I,J)                                                   CATEST
      PRINT*,"ACCURACY MERIT FIGURE FOR IMSL IS:  ",SUM                    CATEST
      PRINT*,"**"                                                          CATEST
      RETURN                                                               CATEST
      END                                                                  CATEST
```

```
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ OUT2
C  SUBROUTINE OUT2                                                              OUT2
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ OUT2
      SUBROUTINE OUT2(NR,NC,A,ID)                                               OUT2
      DIMENSION A(NR,NC)                                                        OUT2
      DO 10 J=1,NC                                                              OUT2
      DO 10 I=1,NR                                                              OUT2
      WRITE 5,ID,I,J,A(I,J)                                                     OUT2
    5 FORMAT(1X,A5,"(",I2,",",I2,")= ",E25.18)                                  OUT2
   10 CONTINUE                                                                  OUT2
      RETURN                                                                    OUT2
      END                                                                       OUT2
```

```
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ UERTST
C  SUBROUTINE UERTST                                                            UERTST
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ UERTST
      SUBROUTINE UERTST(IER,NAME)                                               UERTST
      IF(IER.LT.128) RETURN                                                     UERTST
      PRINT*,"UERTST TERMINAL ERROR FROM--",NAME,"  IER= ",IER                  UERTST
      RETURN                                                                    UERTST
      END                                                                       UERTST
```

```
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ VANF
C  SUBROUTINE VANF                                                              VANF
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ VANF
      SUBROUTINE VANF(EIGR,N,VANR)                                              VANF
      DIMENSION EIGR(N),VANR(N,N)                                               VANF
C     PUT 1'S IN FIRST ROW                                                      VANF
      DO 10 J=1,N                                                               VANF
   10 VANR(1,J)=1.                                                              VANF
C     FILL IN REMAINDER OF EACH COLUMN                                          VANF
      DO 30 J=1,N                                                               VANF
      EI=EIGR(J)                                                                VANF
      TR=1.                                                                     VANF
      DO 30 I=2,N                                                               VANF
      TR=TR*EI                                                                  VANF
   30 VANR(I,J)=TR                                                              VANF
      RETURN                                                                    VANF
      END                                                                       VANF
```

APPENDIX C

SUPPLEMENTARY MATHEMATICAL DEVELOPMENTS

Three mathematical developments are included here to aid in understanding some concepts which were stated in the text. They are placed here in order to avoid long interruptions in the main text. The concepts are:

C1. Sensitivity Derivatives as Part of the Augmented System State Transition Matrix

Using Eq(8), the system plant matrix may be augmented to find the sensitivities of the i'th parameter as

$$\bar{A}_i \equiv \begin{bmatrix} \bar{A} & \vdots & \bar{0} \\ \cdots & \vdots & \cdots \\ A_{(i)} & \vdots & A \end{bmatrix}_{2n \times 2n} \qquad \text{(C1-1)}$$

Notice that the eigenvalues, and hence the system modes, have not changed. The augmented system has n pseudo modes in addition to the original n modes, but they are entirely independent of the original modes in the component matrix approach, and the augmentation therefore does not actually change the system. This desirable feature of the component matrix approach is illustrated in Appendix D, Example 3.

From Eq (C1-1) the state transition matrix for the augmented system can be defined. For time-invariant systems, the exponential can be applied to each section of the partitioned matrix. Thus

$$e^{\begin{bmatrix} A & \vdots & 0 \\ \cdots & \vdots & \cdots \\ A_{(i)} & \vdots & A \end{bmatrix} t} = \begin{bmatrix} e^{At} & \vdots & 0 \\ \cdots & \vdots & \cdots \\ e^{At}_{(i)} & \vdots & e^{At} \end{bmatrix}_{2n \times 2n} \qquad \text{(C1-2)}$$

which shows that the sensitivity derivative of the original state transition matrix is one of the partitioned sections of the state transition matrix for the augmented system.

C2. Determination of State Transition Matrices Using the Cayley-Hamilton Theorem

This section shows how the Cayley-Hamilton theorem can be applied in order to determine a state transition matrix (STM). The use of the Vandermonde matrix and remainder polynomial coefficients is described for this application. Finally, the method is extended to show how it can be used to find the partitioned STM for system sensitivity calculations. Numerical examples of the concepts presented here are included in Appendix D, Examples 1 and 2.

Basic Cayley-Hamilton Theorem Application. The Cayley-Hamilton theorem states that any square matrix satisfies its own characteristic equation. From it, the STM can be shown to be a summation of products of the "remainder polynomial" coefficients, $a_j(t)$, and powers of the system matrix, A, (Ref 9:V-110 through V-115b):

$$e^{At} = \sum_{j=1}^{n} A^{j-1} a_j(t) \qquad \text{(C2-1)}$$

The remainder polynomial coefficients can be uniquely determined from n linear "modal" equations (Ref 9:V-115b):

$$\begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a_n(t) \end{bmatrix} \qquad \text{(C2-2)}$$

where $e^{\lambda_j t}$ are system "modes" and λ_j are system eigenvalues. Eq (C2-2) is equivalent to

$$\underline{e}^{\lambda_j t} = \underline{V}^T \underline{a}_j(t) \qquad \text{(C2-3)}$$

where $\underline{V}$ is the "Vandermonde" matrix. Since $\underline{V}^T$ must be inverted to find the remainder polynomial coefficients, it must not be singular. Therefore, if the system matrix does not have n distinct eigenvalues (ie: if at least one eigenvalue, λ_j, has a multiplicity, m_j, greater than one), V^T is not directly invertable if defined as shown in Eq (C2-2). Thus for systems having repeated eigenvalues, the Vandermonde matrix is defined by taking repeated derivatives, w.r.t. the eigenvalue, of the modal equation for that eigenvalue until there is one independent equation for each occurrence of the eigenvalue (Ref 9:V-116). For example, in a system having an eigenvalue with multiplicity $m_j=3$, the original modal equation must be differentiated twice to get three independent equations:

$$\frac{d}{d\lambda_j} e^{\lambda_j t} = te^{\lambda_j t} = \frac{d}{d\lambda_j} \begin{bmatrix} 1 & \lambda_j & \lambda_j^2 & \cdots & \lambda_j^{n-1} \end{bmatrix} \begin{bmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a_n(t) \end{bmatrix} \qquad \text{(C2-4)}$$

and

$$\frac{d^2}{d\lambda_j^2} e^{\lambda_j t} = t^2 e^{\lambda_j t} = \frac{d^2}{d\lambda_j^2} \begin{bmatrix} 1 & \lambda_j & \lambda_j^2 & \cdots & \lambda_j^{n-1} \end{bmatrix} \begin{bmatrix} a_1(t) \\ a_2(t) \\ \cdot \\ \cdot \\ \cdot \\ a_n(t) \end{bmatrix} \quad \text{(C2-5)}$$

Assuming λ_1 is the repeated eigenvalue in the example, Eq (C2-2) is then

$$\begin{bmatrix} e^{\lambda_1 t} \\ te^{\lambda_1 t} \\ t^2 e^{\lambda_1 t} \\ e^{\lambda_1 t} \\ \cdot \\ e^{\lambda_n t} \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & & \lambda_1^{n-1} \\ 0 & 1 & 2\lambda_1 & \cdots & (n-1)\lambda_1^{n-2} & \\ 0 & 0 & 2 & \cdots (n-2) & (n-1)\lambda_1^{n-3} & \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & & \lambda_2^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ 1 & \lambda_n & \lambda_n^2 & \cdots & & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \\ a_4(t) \\ \cdot \\ a_n(t) \end{bmatrix} \quad \text{(C2-6)}$$

For simplicity, only systems with distinct eigenvalues are used in the remainder of this discussion.

By defining v_{ij} to be the element in the i'th row and j'th column of the inverse of the transposed Vandermonde matrix, the remainder polynomial coefficients may be determined by

$$\begin{bmatrix} a_1(t) \\ a_2(t) \\ . \\ . \\ . \\ a_n(t) \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ . \\ . \\ . \\ e^{\lambda_n t} \end{bmatrix} \qquad \text{(C2-7)}$$

Eq (C2-1) may be expanded to

$$e^{At} = a_1(t)I + a_2(t)A + a_3(t)A^2 + \ldots + a_n(t)A^{n-1} \qquad \text{(C2-8)}$$

Applying the results of Eq (C2-7) to Eq (C2-8) results in

$$\begin{aligned} e^{At} = {} & (v_{11} e^{\lambda_1 t})I + (v_{12} e^{\lambda_2 t})I + \ldots + (v_{1n} e^{\lambda_n t})I \\ & + (v_{21} e^{\lambda_1 t})A + (v_{22} e^{\lambda_2 t})A + \ldots + (v_{2n} e^{\lambda_n t})A \\ & + \quad . \qquad . \qquad . \qquad . \qquad . \\ & + (v_{n1} e^{\lambda_1 t})A^{n-1} + (v_{n2} e^{\lambda_2 t})A^{n-1} + \ldots + (v_{nn} e^{\lambda_n t})A^{n-1} \end{aligned} \qquad \text{(C2-9)}$$

Notice that in Eq (C2-9) the state transition matrix is expressed in terms of products of: (1) elements of the inverse transposed Vandermonde matrix, (2) the system modes, and (3) powers of the original system matrix. The remainder polynomial coefficients, though they were used in the theoretical formulation of the solution, do not even appear in it and thus it is never necessary to actually calculate them. A very significant result of this formulation is that the

original system modes are completely separate and visible, thus physical insight into the system is preserved. This decoupling of modes makes it possible to represent the state transition matrix in terms of the modes, each with a "component matrix" multiplier. An explanation of component matrix formation is given in section C3.

Extended Application of the Cayley-Hamilton Theorem. The procedure outlined in the preceding discussion can be very simply extended to the augmented system state transition matrix, Eq (16), which is used in the sensitivity parameter calculations for the system identification process. Eq (16) is repeated here for the reader's benefit:

$$e^{\begin{bmatrix} A & | & 0 \\ \hline A_{(i)} & | & A \end{bmatrix} t} = \sum_{j=1}^{2n} \begin{bmatrix} A & | & 0 \\ \hline A_{(i)} & | & A \end{bmatrix} a_j(t) \qquad \text{(C2-10)}$$

Note from Eq (C2-10) that the augmented system has the same eigenvalues as the original system since the augmented system matrix is triangular. The only difference is that the eigenvalue multiplicities of the augmented system are double those of the original system. This means that the Vandermonde matrix for the augmented system is $2n \times 2n$, there are $2n$ remainder polynomial coefficients, and there are $2n$ modes. However, half of the modes are simply derivative modes resulting from the doubled eigenvalue multiplicity which came from augmenting the system. These extra modes ultimately have null component matrices (Ref 9:V-122) and may be thought of as placeholders or "pseudo" modes.

Thus it is seen how the Cayley-Hamilton theorem can be applied to determine the state transition matrices of both the original and augmented (sensitivity) systems. The next step is to determine how to

find the "component matrices" which were revealed as part of Cayley-Hamilton theorem application. That determination is the subject of the next section.

C3. Definition and Construction of Component Matrices. Several references have been made to "component matrices" in other sections of this thesis. The purpose of this section is to show what component matrices are and how they are formed.

A component matrix may be defined as a matrix which shows the contribution of a given system mode to the state transition matrix (Ref 15:610). Thus it can be seen from Eq (C2-9) that each column of terms constitutes a component matrix, multiplied by its corresponding mode.

Component matrices are commonly labeled with a double subscript, e.g.: Z_{10}. The first subscript identifies which eigenvalue made the mode with which the component matrix belongs. The second subscript can be nonzero only when the system has repeated eigenvalues. It then identifies the number of derivatives which must be taken of the associated primary mode to obtain the pseudo mode with which the component matrix belongs. (This derivative process is described in Section C2.) Two examples should help clarify this. First consider Z_{10}. This is the component matrix associated with the primary mode of eigenvalue number one: $e^{\lambda_1 t}$. Next consider Z_{82}. This is the component matrix associated with one of the pseudo modes of eigenvalue number eight. Specifically, it is for the pseudo mode found by taking two time derivatives of the primary mode: $\frac{d^2}{dt^2} e^{\lambda_8 t} = t^2 e^{\lambda_8 t}$. Note that this implies eigenvalue eight has a multiplicity of at least three.

Applying the observations made and the subscript conventions just described results in a component matrix formula:

$$Z_{j,k} = \sum_{\ell=1}^{n} v_{\ell,c} A^{\ell-1}; \quad c = \left[\sum_{i=1}^{j} m_i \right] - m_j + k + 1 \qquad \text{(C3-1)}$$

where $0 \leq k \leq m_j - 1$; m_j being the multiplicity of the j'th eigenvalue. The state transition matrix may then be found from

$$e^{At} = \sum_{j=1}^{M} \sum_{k=0}^{m_j-1} (Z_{j,k}) t^k e^{\lambda_j t} \qquad \text{(C3-2)}$$

in which M is the number of different eigenvalues. Some numerical examples of the formation of component matrices are given in Appendix D, Examples 1 and 2.

APPENDIX D

EXAMPLE PROBLEMS

Several numerical examples are provided here to help clarify the application of concepts presented in the text. Examples 1 and 2 work with systems whose eigenvalues were carefully selected for illustrative purposes. Each starts with a set of eigenvalues, $\underline{\lambda}_i$, and a plant matrix, A, which contains a set of system parameters, θ_i. For the system identification process outlined in the text, the quantities sought are the output sensitivities, $\frac{\partial y}{\partial \theta_i}$, as defined by Eq (13). Example 4 illustrates the use of Csáki's generalized algorithm. This is also the process used by subroutine VANINV.

Example 1 - Distinct Eigenvalue System

Suppose n = 3, and

$$\lambda_1 = 4 \quad \theta_1 = -1$$
$$\lambda_2 = -3 \quad \theta_2 = -2 \qquad A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ \theta_1 & \lambda_2 & 0 \\ \theta_2 & 0 & \lambda_3 \end{bmatrix} \qquad \text{(D1-1)}$$
$$\lambda_3 = 1$$

The solution is presented in four steps:

1. Determination of inverse Vandermonde matrix and (n-1) powers of the plant matrix.
2. Calculation of the state transition matrix, e^{At}.
3. Calculation of the parameter sensitivity derivatives, $e^{At}_{(i)}$, of the state transition matrix.
4. Determination of e^{At} and $e^{At}_{(i)}$ by means of component matrices rather than the solution of n linear equations as done for steps 2 and 3.

Inverse Vandermonde Matrix. For distinct eigenvalues, no derivatives need be taken to find the Vandermonde matrix:

$$v^T = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 16 \\ 1 & -3 & 9 \\ 1 & 1 & 1 \end{bmatrix} \qquad \text{(D1-2)}$$

which has an inverse

$$[v^T]^{-1} = \begin{bmatrix} -1/7 & 1/7 & 1 \\ 2/21 & -5/28 & 1/12 \\ 1/21 & 1/28 & -1/12 \end{bmatrix} \qquad \text{(D1-3)}$$

Powers of Plant Matrix. Since n=3, only powers of A through n-1 = 2 need be found. If A^2 is multiplied out, the element in the i'th row and j'th column of the product may be represented by

$$[A^2]_{ij} = \sum_{k=1}^{n=3} \alpha_{ik}\, \alpha_{kj} \qquad \text{(D1-4)}$$

where the α's represent the elements of the original matrix. For this example:

$$A^2 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ \theta_1 & \lambda_2 & 0 \\ \theta_2 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \lambda & 0 & 0 \\ \theta_1 & \lambda_2 & 0 \\ \theta_2 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ \theta_1(\lambda_1+\lambda_2) & \lambda_2^2 & 0 \\ \theta_2(\lambda_1+\lambda_3) & 0 & \lambda_3^2 \end{bmatrix} \qquad \text{(D1-5)}$$

or

$$A^2 = \begin{bmatrix} 4 & 0 & 0 \\ -1 & -3 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ -1 & -3 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 0 & 0 \\ -1 & 9 & 0 \\ -10 & 0 & 1 \end{bmatrix} \qquad \text{(D1-6)}$$

State Transition Matrix. Eq (C2-8) for this example is

$$e^{At} = a_1(t)I + a_2(t)A + a_3(t)A^2 \qquad \text{(D1-7)}$$

The remainder polynomial coefficients, $a_j(t)$, may be found by applying Eq (C2-7):

$$\begin{bmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ e^{\lambda_3 t} \end{bmatrix} \qquad \text{(D1-8)}$$

This results in

$$a_1(t) = -1/7\, e^{4t} + 1/7\, e^{-3t} + e^{t}$$
$$a_2(t) = 2/21\, e^{4t} - 5/28 e^{-3t} + 1/12 e^{t} \qquad \text{(D1-9)}$$
$$a_3(t) = 1/21\, e^{4t} + 1/28 e^{-3t} - 1/12 e^{t}$$

Substituting into Eq (D2-7):

$$e^{At} = (-1/7\, e^{4t} + 1/7\, e^{-3t} + e^{t}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$+ (2/21\, e^{4t} - 5/28\, e^{-3t} + 1/12\, e^{t}) \begin{bmatrix} 4 & 0 & 0 \\ \theta_1 & -3 & 0 \\ \theta_2 & 0 & 1 \end{bmatrix}$$
$$+ (1/21\, e^{4t} + 1/28\, e^{-3t} - 1/12\, e^{t}) \begin{bmatrix} 16 & 0 & 0 \\ \theta_1 & 9 & 0 \\ 5\theta_2 & 0 & 1 \end{bmatrix} \qquad \text{(D1-10)}$$

(The parameter sensitivities, θ_i, are left in for the present so the partials of the state transition matrix w.r.t. the parameter sensitivities can be taken later.)

Adding everything up results in

$$e^{At} = \begin{bmatrix} (e^{4t} & &) & 0 & 0 \\ (\frac{\theta_1}{7}e^{4t} & - \frac{\theta_1}{7}e^{-3t} &) & e^{-3t} & 0 \\ (\frac{\theta_2}{3}e^{4t} & & - \frac{\theta_2}{3}e^{t}) & 0 & e^{t} \end{bmatrix} \qquad \text{(D1-11)}$$

Which when evaluated at the values of θ_i yields

$$e^{At} = \begin{bmatrix} e^{4t} & 0 & 0 \\ (-1/7e^{4t} + 1/7e^{-3t}) & e^{-3t} & 0 \\ (-2/3e^{4t} + 2/3e^{t}) & 0 & e^{t} \end{bmatrix} \quad \text{(D1-12)}$$

Parameter Sensitivity Derivatives. The parameter sensitivity derivatives, $e^{At}_{(i)}$, of the state transition matrix can be found by differentiating Eq (D1-11) w.r.t. the parameters, θ_i. For θ_1:

$$e^{At}_{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ (1/7e^{4t} - 1/7e^{-3t}) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(D1-13)}$$

And for θ_2:

$$e^{At}_{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (1/3e^{4t} - 1/3e^{t}) & 0 & 0 \end{bmatrix} \quad \text{(D1-14)}$$

Use of Component Matrices. There exists one component matrix for each system mode or pseudo mode, therefore it is necessary to find three component matrices for this system. Eq (C3-2), which is repeated here as Eq (D1-15) may be used to find the state transition matrix.

$$e^{At} = \sum_{j=1}^{M} \sum_{k=0}^{m_j-1} (Z_{j,k})\, t^k\, e^{\lambda_j t} \quad \text{(D1-15)}$$

For this example, there are M=3 distinct eigenvalues. There are no pseudo modes, therefore $k = 0$ always. The three component matrices to be found are Z_{10}, Z_{20} and Z_{30}, which are found by using Eq (C3-1), repeated here as (D1-16):

$$Z_{j,k} = \sum_{\ell=1}^{n} v_{\ell,c} A^{\ell-1}; \quad c = \left[\sum_{i=1}^{j} m_i\right] - m_j + k + 1 \qquad \text{(D1-16)}$$

Therefore, substituting from previous calculations;

$$Z_{10} = \sum_{\ell=1}^{n} v_{\ell,1} A^{\ell-1} \qquad \text{(D1-17a)}$$

$$= -\frac{1}{7}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{2}{21}\begin{bmatrix} 4 & 0 & 0 \\ \theta_1 & -3 & 0 \\ \theta_2 & 0 & 1 \end{bmatrix}$$

$$+ \frac{1}{21}\begin{bmatrix} 16 & 0 & 0 \\ \theta_1 & 9 & 0 \\ 5\theta_2 & 0 & 1 \end{bmatrix} \qquad \text{(D1-17b)}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{7}\theta_2 & 0 & 0 \\ \frac{1}{3}\theta_2 & 0 & 0 \end{bmatrix} \qquad \text{(D1-17c)}$$

$$Z_{20} = \sum_{\ell=1}^{n=3} v_{\ell,2} A^{\ell-1} \qquad \text{(D1-18a)}$$

$$= \frac{1}{7}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{5}{28}\begin{bmatrix} 4 & 0 & 0 \\ \theta_1 & -3 & 0 \\ \theta_2 & 0 & 1 \end{bmatrix}$$

$$+ \frac{1}{28}\begin{bmatrix} 16 & 0 & 0 \\ \theta_1 & 9 & 0 \\ 5\theta_2 & \theta & 1 \end{bmatrix} \qquad \text{(D1-18b)}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{7}\theta_1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{(D1-18c)}$$

$$Z_{30} = \sum_{\ell=1}^{n=3} v_{\ell,3} A^{\ell-1} \qquad \text{(D1-19a)}$$

$$= 1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{12}\begin{bmatrix} 4 & 0 & 0 \\ \theta_1 & -3 & 0 \\ \theta_2 & 0 & 1 \end{bmatrix}$$

$$- \frac{1}{12}\begin{bmatrix} 16 & 0 & 0 \\ \theta_1 & 9 & 0 \\ 5\theta_2 & 0 & 1 \end{bmatrix} \qquad \text{(D1-19b)}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{3}\theta_2 & 0 & 1 \end{bmatrix} \qquad \text{(D1-19c)}$$

Substituting the component matrices into Eq (D1-15) results in the state transition matrix:

$$e^{At} = \sum_{j=1}^{M=3} \sum_{k=0}^{0} (Z_{j,0}) t^0 e^{\lambda_j t} \qquad \text{(D1-20a)}$$

$$= z_{10}e^{\lambda_1 t} + z_{20}e^{\lambda_2 t} + z_{30}e^{\lambda_3 t} \qquad \text{(D1-20b)}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{7}\theta_1 & 0 & 0 \\ \frac{1}{3}\theta_2 & 0 & 0 \end{bmatrix} e^{4t} + \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{7}\theta_1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} e^{-3t}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{3}\theta_2 & 0 & 1 \end{bmatrix} e^{t} \qquad \text{(D1-20c)}$$

$$= \begin{bmatrix} (e^{4t} & &) & 0 & 0 \\ (\frac{1}{7}\theta_1 e^{4t} & \frac{1}{7}\theta_1 e^{-3t} &) & e^{-3t} & 0 \\ (\frac{1}{3}\theta_2 e^{4t} & & -\frac{1}{3}\theta_2 e^{t}] & 0 & e^{t} \end{bmatrix} \qquad \text{(D1-20d)}$$

Comparing Eq (D1-20d) with Eq (D1-11) shows that the same answer comes out either way. Note from Eq (D1-20c) how clearly visible the modes are. Their individual influence on the state transition matrix is easily discernable. Also, the influence of the parameters, θ_i, on both the modes and the state transition matrix are easily seen from this formulation.

Note that in actual practice the parameters are numbers rather than variables as shown in this example. They were carried as variables in the example to show how they influence the modes and component matrices. Another important point to note is how direct the component matrix approach turns out to be in the end. Although the development may seem to follow several independent paths, the application is very direct. Once the component matrices are found from the inverse Vandermonde matrix, all the effects of the modes and parameter sensitivities are related to the system by Eq (21).

Example 2 - Augmented System

If n = 2, and

$$\lambda_1 = 2 \qquad \theta_1 = -3 \qquad A = \begin{bmatrix} \lambda_1 & 0 \\ \theta_1^2 & \lambda_2 \end{bmatrix} \qquad \text{(D2-1)}$$

$$\lambda_2 = -1$$

the solution is found as shown below, following the same procedure as Example 1. Before proceeding, however, the augmented system must be formed

$$\bar{A} = \left[\begin{array}{c|c} A & 0 \\ \hline A_{(i)} & A \end{array}\right] = \left[\begin{array}{cc|cc} \lambda_1 & 0 & & \\ \theta_1^2 & \lambda_2 & \multicolumn{2}{c}{0} \\ \hline 0 & 0 & \lambda_1 & 0 \\ 2\theta_1 & 0 & \theta_1^2 & \lambda_2 \end{array}\right] \qquad \text{(D2-2)}$$

Notice that the eigenvalues of the augmented system are the same as those of the original system. The only difference is that the multiplicity of each is doubled.

Inverse Vendermonde Matrix. Since the eigenvalues of the augmented system are not distinct, derivatives must be taken to form the transposed Vandermonde matrix:

$$V^T = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 \\ 0 & 1 & 2\lambda_1 & 3\lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 & \lambda_2^3 \\ 0 & 1 & 2\lambda_2 & 3\lambda_2^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 4 & 12 \\ 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \end{bmatrix} \qquad \text{(D2-3)}$$

which has an inverse

$$[V^T]^{-1} = \begin{bmatrix} 21/81 & -18/81 & 60/81 & 36/81 \\ 36/81 & -27/81 & -36/81 & 0. \\ 9/81 & 0. & -9/81 & -27/81 \\ -6/81 & 9/81 & 6/81 & 9/81 \end{bmatrix} \qquad \text{(D2-4)}$$

Powers of Plant Matrix. For this system with n=4, powers of A through A^3 must be taken:

$$\bar{A}^2 = \left[\begin{array}{cc|cc} \lambda_1 & 0 & 0 & 0 \\ \theta_1^2 & \lambda_2 & 0 & 0 \\ \hline 0 & 0 & \lambda_1 & 0 \\ 2\theta_1 & 0 & \theta_1^2 & \lambda_2 \end{array}\right] \left[\begin{array}{cc|cc} \lambda_1 & 0 & 0 & 0 \\ \theta_1^2 & \lambda_2 & 0 & 0 \\ \hline 0 & 0 & \lambda_1 & 0 \\ 2\theta_1 & 0 & \theta_1^2 & \lambda_2 \end{array}\right] \qquad \text{(D2-5a)}$$

$$= \left[\begin{array}{cc|cc} \lambda_1^2 & 0 & 0 & 0 \\ \theta_1^2(\lambda_1+\lambda_2) & \lambda_2^2 & 0 & 0 \\ \hline 0 & 0 & \lambda_1^2 & 0 \\ 2\theta_1(\lambda_1+\lambda_2) & 0 & \theta_1^2(\lambda_1+\lambda_2) & \lambda_2^2 \end{array}\right] \qquad \text{(D2-5b)}$$

or

$$\bar{A}^2 = \left[\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 9 & -1 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 \\ -6 & 0 & -3 & -1 \end{array}\right] \left[\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 9 & -1 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 \\ -6 & 0 & -3 & -1 \end{array}\right] \qquad \text{(D2-6a)}$$

$$= \left[\begin{array}{cc|cc} 4 & 0 & 0 & 0 \\ 9 & 1 & 0 & 0 \\ \hline 0 & 0 & 4 & 0 \\ -6 & 0 & 9 & 1 \end{array}\right] \qquad \text{(D2-6b)}$$

and

$$\bar{A}^3 = \left[\begin{array}{cc|cc} \lambda_1^2 & 0 & 0 & 0 \\ \theta_1^2(\lambda_1+\lambda_2) & \lambda_2^2 & 0 & 0 \\ \hline 0 & 0 & \lambda_1^2 & 0 \\ 2\theta_1(\lambda_1+\lambda_2) & 0 & \theta_1^2(\lambda_1+\lambda_2) & \lambda_2^2 \end{array}\right] \left[\begin{array}{cc|cc} \lambda_1 & 0 & 0 & 0 \\ \theta_1^2 & \lambda_2 & 0 & 0 \\ \hline 0 & 0 & \lambda & 0 \\ 2\theta_1 & 0 & \theta_1^2 & \lambda_2 \end{array}\right] \tag{D2-7a}$$

$$= \left[\begin{array}{cc|cc} \lambda_1^3 & 0 & 0 & 0 \\ \theta_1^2(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2) & \lambda_2^3 & 0 & 0 \\ \hline 0 & 0 & \lambda_1^2 & 0 \\ 2\theta_1(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2) & 0 & \theta_1^2(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2) & \lambda_2^3 \end{array}\right] \tag{D2-7b}$$

or

$$\bar{A}^3 = \left[\begin{array}{cc|cc} 4 & 0 & 0 & 0 \\ 9 & 1 & 0 & 0 \\ \hline 0 & 0 & 4 & 0 \\ -6 & 0 & 9 & 1 \end{array}\right] \left[\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 9 & -1 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 \\ -6 & 0 & -3 & -1 \end{array}\right] \tag{D2-8a}$$

$$= \left[\begin{array}{cc|cc} 8 & 0 & 0 & 0 \\ 27 & -1 & 0 & 0 \\ \hline 0 & 0 & 8 & 0 \\ -18 & 0 & 27 & -1 \end{array}\right] \tag{D2-8b}$$

Notice from Eqs (D2-2, D2-5, D2-7) that the parameter and its sensitivity remain in their respective positions for all the powers of the augmented matrix. This is an important property because it causes all the modes to remain visible throughout the component matrix process of system identification.

State Transition Matrix. For this example, Eq (C2-8) is

$$e^{\bar{A}t} = a_1(t)I + a_2(t)\bar{A} + a_3(t)\bar{A}^2 + a_4(t)\bar{A}^3 \qquad \text{(D2-9)}$$

Eq (C2-7) results in the following system of linear equations, from which the remainder polynomial coefficients, $a_j(t)$, may be found:

$$\begin{bmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \\ a_4(t) \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} \\ te^{\lambda_1 t} \\ e^{\lambda_1 t} \\ te^{\lambda_1 t} \end{bmatrix} \qquad \text{(D2-10)}$$

By applying Eq (D2-4), the following coefficients are found:

$$\begin{aligned} a_1(t) &= 21/81e^{2t} - 18/81te^{2t} + 60/81e^{-t} + 36/81te^{-t} \\ a_2(t) &= 36/81e^{2t} - 27/81te^{2t} - 36/81e^{-t} \\ a_3(t) &= 9/81e^{2t} - 9/81e^{-t} - 27/81te^{-t} \\ a_4(t) &= -6/81e^{2t} + 9/81te^{2t} + 6/81e^{-t} + 9/81te^{-t} \end{aligned} \qquad \text{(D2-11)}$$

Substituting into Eq (D3-9):

$$e^{\bar{A}t} = (21/81e^{2t} - 18/81te^{2t} + 60/81e^{-t} + 36/81te^{-t})$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$+ (36/81e^{2t} - 27/81te^{2t} - 36/81e^{-t})$$

$$\left[\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ \theta_1^2 & -1 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 \\ 2\theta_1 & 0 & \theta_1^2 & -1 \end{array}\right]$$

$$+ (9/81e^{2t} \qquad -9/81e^{-t} - 27/81te^{-t})$$

$$\left[\begin{array}{cc|cc} 4 & 0 & 0 & 0 \\ \theta_1^2 & 1 & 0 & 0 \\ \hline 0 & 0 & 4 & 0 \\ 2\theta_1 & 0 & \theta_1^2 & 1 \end{array}\right]$$

$$+ (-6/81e^{2t} + 9/81te^{2t} + 6/81e^{-t} + 9/81te^{-t})$$

$$\left[\begin{array}{cc|cc} 8 & 0 & 0 & 0 \\ 3\theta_1^2 & -1 & 0 & 0 \\ \hline 0 & 0 & 8 & 0 \\ 6\theta_1 & 0 & 3\theta_1^2 & -1 \end{array}\right] \qquad \text{(D2-12)}$$

Summing up all the terms yields

$$e^{\bar{A}t} = \left[\begin{array}{cc|cc} e^{2t} & 0 & 0 & 0 \\ \theta_1^2(27/81e^{2t} - 27/81e^{-t}) & e^{-t} & 0 & 0 \\ \hline 0 & 0 & e^{2t} & 0 \\ \theta_1(54/81e^{2t} - 54/81e^{-t}) & 0 & \theta_1^2(27/81e^{2t} - 27/81e^{-t}) & e^{-t} \end{array}\right] \qquad \text{(D2-13)}$$

If Eq (D2-13) is evaluated at $\theta_1 = -3$, the result is

$$e^{\bar{A}t} = \left[\begin{array}{cc|cc} e^{2t} & 0 & 0 & 0 \\ (3e^{2t} - 3e^{-t}) & e^{-t} & 0 & 0 \\ \hline 0 & 0 & e^{2t} & 0 \\ (-2e^{2t} + 2e^{-t}) & 0 & (3e^{2t} - 3e^{-t}) & e^{-t} \end{array}\right] \quad \text{(D2-14)}$$

Parameter Sensitivity Derivatives. This system has only one parameter sensitivity derivative, $e^{At}_{(1)}$, and it can be found by applying Eq (19), which for this example is

$$e^{At}_{(1)} = \sum_{j=1}^{2n=4} A^{j-1}_{(1)} a_j(t) \quad \text{(D2-15)}$$

From Eq (D2-15) it is evident that powers of the original system matrix are needed up through the third power. Therefore

$$A^2 = \begin{bmatrix} \lambda_1 & 0 \\ \theta_1^2 & \lambda_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ \theta & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 \\ \theta_1^2(\lambda_1+\lambda_2) & \lambda_2^2 \end{bmatrix} \quad \text{(D2-16)}$$

or

$$A^2 = \begin{bmatrix} 2 & 0 \\ 9 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 9 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 9 & 1 \end{bmatrix} \quad \text{(D2-17)}$$

and

$$A^3 = \begin{bmatrix} \lambda_1^2 & 0 \\ \theta_1^2(\lambda_1+\lambda_2) & \lambda_2^2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ \theta_1^2 & \lambda_2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1^3 & 0 \\ \theta_1^2(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2) & \lambda_2^3 \end{bmatrix} \quad \text{(D2-18)}$$

or

$$A^3 = \begin{bmatrix} 4 & 0 \\ 9 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 9 & -1 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 27 & -1 \end{bmatrix} \quad \text{(D2-19)}$$

Applying these to Eq (D2-15):

$$e^{At}_{(1)} = \frac{d}{d\theta_1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\left(2/81e^{2t} - 18/81te^{2t} + 60/81e^{-t} + 36/81te^{-t}\right)$$

$$+ \frac{d}{d\theta_1}\begin{bmatrix} 2 & 0 \\ \theta_1^2 & -1 \end{bmatrix}\left(36/81e^{2t} - 27/81te^{2t} - 36/81e^{-t}\right)$$

$$+ \frac{d}{d\theta_1}\begin{bmatrix} 4 & 0 \\ \theta_1^2 & 1 \end{bmatrix}\left(9/81e^{2t} - 9/81e^{-t} - 27/81te^{-t}\right)$$

$$+ \frac{d}{d\theta_1}\begin{bmatrix} 8 & 0 \\ 3\theta_1^2 & -1 \end{bmatrix}\left(-6/81e^{2t} + 9/81te^{2t} + 6/81e^{-t} + 9/81te^{-t}\right)$$

(D2-20a)

$$= \begin{bmatrix} 0 & 0 \\ 2\theta_1 & 0 \end{bmatrix}\left(36/81e^{2t} + 27/81te^{2t} - 36/81e^{-t}\right)$$

$$+ \begin{bmatrix} 0 & 0 \\ 2\theta_1 & 0 \end{bmatrix}\left(9/81e^{2t} - 9/81e^{-t} - 27/81te^{-t}\right)$$

$$+ \begin{bmatrix} 0 & 0 \\ 6\theta_1 & 0 \end{bmatrix}\left(-6/81e^{2t} + 9/81te^{2t} + 6/81e^{-t} + 9/81te^{-t}\right)$$

(D2-20b)

Finally,

$$e^{At}_{(1)} = \begin{bmatrix} 0 & 0 \\ \theta_1(54/81e^{2t} - 54/81e^{-t}) & 0 \end{bmatrix} \qquad \text{(D2-21)}$$

A comparison of Eq (D2-21) and Eq (D2-13) shows that the parameter sensitivity of the original state transition matrix is in fact the lower left block of the partitioned state transition matrix for the augmented (sensitivity) system. Thus the validity of Eq (15) is demonstrated.

Use of Component Matrices. Since the augmented system has two distinct modes and two pseudo modes, a total of four component matrices must be found: Z_{10}, Z_{11}, Z_{20}, Z_{21}. The augmented system state transition matrix and the component matrices may be found using Eqs (C3-2) and (C3-1), respectively. They are repeated here as Eqs (D2-22) and (D2-23).

$$e^{\bar{A}t} = \sum_{j=1}^{M} \sum_{k=0}^{m_j-1} (Z_{j,k}) t^k e^{\lambda_j t} \qquad \text{(D2-22)}$$

$$Z_{j,k} = \sum_{\ell=1}^{n} v_{\ell,c} \bar{A}^{\ell-1}; \quad c = \left[\sum_{i=1}^{j} m_i\right] - m_j + k + 1 \qquad \text{(D2-23)}$$

Applying previous calculations:

$$Z_{10} = \sum_{\ell=1}^{n=4} v_{\ell,1} \bar{A}^{\ell-1} \qquad \text{(D2-25a)}$$

$$= 21/81 \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] + 36/81 \left[\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ \theta_1^2 & -1 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 \\ 2\theta_1 & 0 & \theta_1^2 & -1 \end{array}\right]$$

$$+ 9/81 \left[\begin{array}{cc|cc} 4 & 0 & 0 & 0 \\ \theta_1^2 & 1 & 0 & 0 \\ \hline 0 & 0 & 4 & 0 \\ 2\theta_1 & 0 & \theta_1^2 & 1 \end{array}\right] - 6/81 \left[\begin{array}{cc|cc} 8 & 0 & 0 & 0 \\ 3\theta_1^2 & -1 & 0 & 0 \\ \hline 0 & 0 & 8 & 0 \\ 6\theta_1 & 0 & 3\theta_1^2 & -1 \end{array}\right] \qquad \text{(D2-25b)}$$

$$= \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \frac{27}{81}\theta_1^2 & 0 & 0 \\ \hline 0 & 0 & 1 \\ \hline \frac{54}{81}\theta_1^2 & 0 & \frac{27}{81}\theta_1^2 \end{array}\right] \quad \text{(D2-25c)}$$

$$Z_{11} = \sum_{\ell=1}^{4} v_{\ell,2}\, \bar{A}^{\ell-1} \quad \text{(D2-26a)}$$

$$= -18/81 \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] - 27/81 \left[\begin{array}{c|c|c} 2 & 0 & 0 \\ \theta_1^2 & -1 & 0 \\ \hline 0 & 0 & 2 \\ \hline 2\theta_1 & 0 & \theta_1^2 \end{array}\right]$$

$$+ 9/81 \left[\begin{array}{c|c|c} 8 & 0 & 0 \\ 3\theta_1^2 & -1 & 0 \\ \hline 0 & 0 & 8 \\ \hline 6\theta_1 & 0 & 3\theta_1^2 \end{array}\right] \quad \text{(D2-26b)}$$

$$= \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \quad \text{(D2-26c)}$$

$$Z_{20} = \sum_{\ell=1}^{4} v_{\ell,3}\, \bar{A}^{\ell-1} \quad \text{(D2-27a)}$$

$$= 60/81 \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] - 36/81 \left[\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ \theta_1^2 & -1 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 \\ 2\theta_1 & 0 & \theta_1^2 & -1 \end{array}\right]$$

$$- 9/81 \left[\begin{array}{cc|cc} 4 & 0 & 0 & 0 \\ \theta_1^2 & 1 & 0 & 0 \\ \hline 0 & 0 & 4 & 0 \\ 2\theta_1 & 0 & \theta_1^2 & 1 \end{array}\right] + 6/81 \left[\begin{array}{cc|cc} 8 & 0 & 0 & 0 \\ 3\theta_1^2 & -1 & 0 & 0 \\ \hline 0 & 0 & 8 & 0 \\ 6\theta_1 & 0 & 3\theta_1^2 & -1 \end{array}\right] \qquad \text{(D2-27b)}$$

$$= \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ -\frac{27}{81}\theta_1^2 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ -\frac{54}{81}\theta_1 & 0 & -\frac{27}{81}\theta_1^2 & 1 \end{array}\right] \qquad \text{(D2-27c)}$$

$$Z_{21} = \sum_{\ell=1}^{4} v_{\ell,4} \, \bar{A}^{\ell-1} \qquad \text{(D2-28a)}$$

$$= 36/81 \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] - 27/81 \left[\begin{array}{cc|cc} 4 & 0 & 0 & 0 \\ \theta_1^2 & 1 & 0 & 0 \\ \hline 0 & 0 & 4 & 0 \\ 2\theta_1 & 0 & \theta_1^2 & 1 \end{array}\right]$$

$$+ 9/81 \left[\begin{array}{cc|cc} 8 & 0 & 0 & 0 \\ 3\theta_1^2 & -1 & 0 & 0 \\ \hline 0 & 0 & 8 & 0 \\ 6\theta_1 & 0 & 3\theta_1^2 & -1 \end{array}\right] \qquad \text{(D2-28b)}$$

$$= \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \text{(D2-28c)}$$

It can be seen from the above that if the component matrices are calculated in order of increasing value of the subscript, the calculation of c, the column index for $v_{\ell,c}$, need not be computed. It can be simply initialized to one for the first component matrix, and thereafter incremented by one for each new component matrix calculation.

Substituting the component matrices into Eq (D2-22) yields the state transition matrix for the augmented system:

$$e^{\bar{A}t} = \sum_{j=1}^{M=2} \sum_{k=0}^{m_j-1} (z_{j,k}) t^k e^{\lambda_j t} \qquad \text{(D2-29a)}$$

$$= z_{10} e^{\lambda_1 t} + z_{11} te^{\lambda_1 t} + z_{20} e^{\lambda_2 t} + z_{21} te^{\lambda_2 t} \qquad \text{(D2-29b)}$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ \frac{27}{81}\theta_1^2 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \frac{54}{81}\theta_1 & 0 & \frac{27}{81}\theta_1^2 & 0 \end{array}\right] e^{2t} + [0]te^{2t}$$

$$+ \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ -\frac{27}{81}\theta_1^2 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ -\frac{54}{81}\theta_1 & 0 & -\frac{27}{81}\theta_1^2 & 1 \end{array}\right] e^{-t} + [0]te^{-t} \qquad \text{(D2-29c)}$$

$$= \left[\begin{array}{cc|cc} e^{2t} & 0 & 0 & 0 \\ \theta_1^2(27/8le^{2t} - 27/8le^{-t}) & e^{-t} & 0 & 0 \\ \hline 0 & 0 & e^{2t} & 0 \\ \theta_1(54/8le^{2t} - 54/8le^{-t}) & 0 & \theta_1^2(27/8le^{2t} - 27/8le^{-t}) & e^{-t} \end{array}\right]$$

(D2-29d)

Comparing Eq (D2-29d) with Eq (D2-13) shows that the same solution is found by either method. Again the influence of the system modes and parameters is clearly visible throughout the component matrix method solution process.

Example 3 - Use of Csáki's Generalized Algorithm

An explanation of Csáki's generalized algorithm is found in Section III of the main text. Suppose for this example that a fourth order system has one distinct eigenvalue and one eigenvalue repeated three times: $\lambda_1 = 1$ with multiplicity $k_1 = 3$, and $\lambda_2 = -1$ with multiplicity $k_2 = 1$. The Vandermonde matrix can then be written

$$V = \begin{bmatrix} 1 & 0 & 0 & 1 \\ \lambda_1 & 1 & 0 & \lambda_2 \\ \lambda_1^2 & 2\lambda_1 & 1/2(2) & \lambda_2^2 \\ \lambda_1^3 & 3\lambda_1^2 & 1/2(6\lambda_1) & \lambda_2^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 2 & 1 & 1 \\ 1 & 3 & 3 & -1 \end{bmatrix}$$

(D3-1)

which has an inverse

$$V^{-1} = \begin{bmatrix} 7/8 & 3/8 & -3/8 & 1/8 \\ -6/8 & 2/8 & 6/8 & -2/8 \\ 4/8 & -4/8 & -4/8 & 4/8 \\ 1/8 & -3/8 & 3/8 & -1/8 \end{bmatrix} \qquad \text{(D3-2)}$$

The characteristic polynomial is

$$D(S) = (S-1)^3(S+1) = S^4 - 2S^3 + 0S^2 + 2S - 1 \qquad \text{(D3-3)}$$

There are two denominator polynomials--one for each eigenvalue. They are

$$D_1(S) = \frac{D(S)}{(S-\lambda_1)^{k_1}} = \frac{S^4-2S^3+0S^2+2S-1}{(S-1)^3} = S + 1 \qquad \text{(D3-4)}$$

$$D_2(S) = \frac{D(S)}{(S-\lambda_2)^{k_2}} = \frac{S^4-2S^3+0S^2+2S-1}{(S+1)} = (S-1)^3 = S^3-3S^2+3S-1 \qquad \text{(D3-5)}$$

For every application of Csáki's algorithm there are n truncated polynomials, $N_\ell(S)$. In this example they are:

$$N_1(S) = S^3 - 2S^2 + 0S + 2 \qquad \text{(D3-6)}$$

$$N_2(S) = S^2 - 2S = 0 \qquad \text{(D3-7)}$$

$$N_3(S) = S - 2 \qquad \text{(D3-8)}$$

$$N_4(S) = 1 \qquad \text{(D3-9)}$$

The following structure describes the inverse Vandermonde for this example:

$$V^{-1} = \begin{bmatrix} \\ W_1 \\ \\ \hline \\ W_2 \\ \\ \end{bmatrix} = \left[\begin{array}{cccc} W_{11}^{(1)} & W_{12}^{(1)} & W_{13}^{(1)} & W_{14}^{(1)} \\ W_{21}^{(1)} & W_{22}^{(1)} & W_{23}^{(1)} & W_{24}^{(1)} \\ W_{31}^{(1)} & W_{32}^{(1)} & W_{33}^{(1)} & W_{34}^{(1)} \\ \hline W_{11}^{(2)} & W_{12}^{(2)} & W_{13}^{(2)} & W_{14}^{(2)} \end{array}\right] \qquad \text{(D3-10)}$$

Each element is evaluated as follows, using Eq (25), repeated here as Eq (D3-11):

$$W_{jl}^{(i)} = \frac{1}{(k_i-j)!} \frac{d^{(k_i-j)}}{ds^{(k_i-j)}} \frac{N_l(S)}{D_i(S)} \Bigg|_{S=\lambda_1} \qquad \text{(D3-11)}$$

$$W_{11}^{(1)} = \frac{1}{(3-1)!} \frac{d^2}{ds^2} \frac{N_1(S)}{D_1(S)} \Bigg|_{S=\lambda_1=1} = \frac{1}{2} \frac{d^2}{ds^2} \frac{S^3-2S^2+2}{S+1} \Bigg|_{S=1} = \frac{7}{8} \qquad \text{(D3-12a)}$$

$$W_{12}^{(1)} = \frac{1}{(3-1)!} \frac{d^2}{ds^2} \frac{N_2(S)}{D_1(S)} \Bigg|_{S=\lambda_1=1} = \frac{1}{2} \frac{d^2}{ds^2} \frac{S^2-2S}{S+1} \Bigg|_{S=1} = \frac{3}{8} \qquad \text{(D3-12b)}$$

$$W_{13}^{(1)} = \frac{1}{(3-1)!} \frac{d^2}{ds^2} \frac{N_3(S)}{D_1(S)} \Bigg|_{S=\lambda_1=1} = \frac{1}{2} \frac{d^2}{ds^2} \frac{S-2}{S+1} \Bigg|_{S=1} = \frac{3}{8} \qquad \text{(D3-12c)}$$

$$W_{14}^{(1)} = \frac{1}{(3-1)!} \frac{d^2}{ds^2} \frac{N_4(S)}{D_1(S)} \Bigg|_{S=\lambda_1=1} = \frac{1}{2} \frac{d^2}{ds^2} \frac{1}{S+1} \Bigg|_{S-1} = \frac{1}{8} \qquad \text{(D3-12d)}$$

$$\vdots$$

etc.

After evaluating each element, the matrix formed from Eqs (D3-10) and (D3-11) is the same as the matrix, Eq (D3-2), found by inverting the Vandermonde matrix. Note, however, that Csáki's method does not need to work with the Vandermonde matrix at all. It only needs the system eigenvalues and their multiplicities.

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Efficient inversion of a 2n × 2n Vandermonde matrix is a key requirement of a new algorithm developed ~~by J. Gary Reid (Ref 10,11)~~ for parameter identification in linear time-invariant systems. It has features very desirable for application to large systems with many unknown parameters, such as adaptive flight control systems.

A generalized algorithm for inverting the Vandermonde matrix was proposed ~~by F. C. Ceáki (Ref 1)~~. It was chosen for study because its structure parallels

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that of the first Reid's algorithm. The Csaki's algorithm was coded into a computer subroutine called "VANINV," and 43 eigensystems were used to test its computational accuracy and efficiency. Four potential problem areas tested were: (1) large system orders, (2) eigenvalues near each other, (3) eigenvalues near zero, and (4) eigenvalues with large magnitude differences. For a comparison, the Vandermonde matrix for each system was also inverted using routines from the International Mathematical & Statistical Library (IMSL).

Test results indicated that VANINV is not quite as fast or as accurate as the IMSL routines. However, the tests were limited to systems with real distinct eigenvalues because the IMSL routines cannot handle other types. VANINV in its present form can handle systems having any combination of complex and/or repeated eigenvalues. Therefore, recommendations for further research and several possible means of improving VANINV are outlined.