NUSC Technical Report 5929

NUSC Technical Report 5929

Numerical Solutions of Underwater Acoustic Wave Propagation Problems

Ding Lee
John S. Papadakis
Special Projects Department

25 February 1979

NUSC

Naval Underwater Systems Center
Newport, Rhode Island • New London, Connecticut

PREFACE

This report was prepared under NUSC Project No. A-650-20, "Finite-Difference Solutions to Acoustic Wave Propagation," Principal Investigator, D. Lee, Code 312. The sponsoring activity was the Naval Material Command, Project Manager, T. H. Probus, Code 08T1.

The Technical Reviewer was Dr. H. L. Pond, Code 401.

The authors gratefully acknowledge the many valuable discussions they had with Dr. F. R. DiNapoli, Code 312.

REVIEWED AND APPROVED: 25 February 1979

R. W. Hasse
Head: Special Projects Department

The authors of this report are located at the New London Laboratory, Naval Underwater Systems Center, New London, Connecticut 06320.

(14) NUSC-TR-5929

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: TR 5929
2. GOVT ACCESSION NO.:
3. RECIPIENT'S CATALOG NUMBER:

4. TITLE (and Subtitle): (6) NUMERICAL SOLUTIONS OF UNDERWATER ACOUSTIC WAVE PROPAGATION PROBLEMS.

5. TYPE OF REPORT & PERIOD COVERED: (9) Technical rept.

6. PERFORMING ORG. REPORT NUMBER:

7. AUTHOR(s): (10) Ding/Lee, John S./Papadakis

8. CONTRACT OR GRANT NUMBER(s): (12) 42p.

9. PERFORMING ORGANIZATION NAME AND ADDRESS:
Naval Underwater Systems Center
New London Laboratory
New London, CT

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS:
A-650-20

11. CONTROLLING OFFICE NAME AND ADDRESS:
Naval Material Command
MAT 08T1
Washington, DC

12. REPORT DATE: (11) 25 Feb[illegible]79

13. NUMBER OF PAGES: 37

14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office):

15. SECURITY CLASS. (of this report): UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE:

16. DISTRIBUTION STATEMENT (of this Report):

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report):

18. SUPPLEMENTARY NOTES:

19. KEY WORDS (Continue on reverse side if necessary and identify by block number):

Initial Boundary Value Problems	Explicit and Implicit schemes
Differential Equations	Convergence
Finite-difference	Variable step-size technique
Parabolic Wave Equations	Predictor-and-corrector

20. ABSTRACT (Continue on reverse side if necessary and identify by block number):

Contributions to solving initial boundary value problems for partial differential equations have been made by applying finite-difference methods to solve seismic wave propagation problems. Very little has been done in the area of underwater acoustic wave propagation problems, although a set of properly developed numerical methods could very well solve these problems effectively. These numerical methods can solve not only range-dependent problems but also can handle irregular boundaries with arbitrary boundary conditions. In this report, as a start, two accurate general purpose (over)

DD FORM 1 JAN 73 1473

405 918

GW to next page

20. (Cont'd)

approaches are presented for the solution of variable coefficient parabolic wave equations.

In a finite-difference approach, techniques are derived from both the conventional explicit and implicit schemes, and the associated convergence theory is thoroughly analyzed. The techniques are found to be general purpose and to provide reasonable accuracy.

In an ordinary differential equation approach the parabolic equation is treated as a system of equations in which the second partial derivative with respect to the space variable is discretized by means of a second order central difference (also known as the Method of Lines). Nonlinear multistep (NLMS) and linear multistep (LMS) methods are used as predictor-and-corrector for solving this system. A built-in variable step-size technique gives the desired accuracy. The theory with regard to consistency, stability, and convergence has been very well developed for both the NLMS and LMS methods, thus ensuring the convergence of this procedure.

A practical treatment of irregular boundaries is described with an illustrative example, and a selected set of experimental numerical results are included to demonstrate the validity of these approaches.

TABLE OF CONTENTS

NUMERICAL SOLUTIONS OF UNDERWATER ACOUSTIC WAVE PROPAGATION PROBLEMS

1. INTRODUCTION

Significant contributions to solving seismic wave propagation problems have been made by applying finite difference (FD) methods;[1-3] however, little has been done in applying FD methods to the solution of underwater acoustic wave propagation problems. Some literature[4-6] exists, but further improvement and efficient applications of FD methods to underwater acoustic wave propagation problems are required. The basic problems involved in the application of FD methods — speed, accuracy, and memory capacity -- were never thoroughly analyzed. A study of the efficiency of FD methods in solving general underwater acoustic wave propagation problems was never performed. In addition, the theory involved in the FD methods (which offers important information regarding convergence and error control) was completely neglected.

FD methods are a general purpose scheme and have very few restrictions.[7-9] As a start, we will search for efficient solutions of parabolic wave equations. In this report we will:

1. Formulate both explicit and implicit FD schemes for variable coefficient parabolic equations and examine whether a more efficient FD method can be developed for underwater acoustic wave propagation problems,

2. Discuss the theory of our FD development and estimate the error,

3. Demonstrate the validity of our FD techniques.

In addition to the FD approach, an ordinary differential equation (ODE) approach was taken and was found reasonably effective for handling parabolic equations. Numerical analysts have frequently remarked that FD methods would be enhanced if a variable step length could be adopted to solve partial differential equations. The ODE approach, used in conjunction with the FD approach, certainly offers this advantage. Our method of attack is to discretize the second partial derivative with respect to space variables into FD representations and then to transform the whole parabolic equation into a system of ODEs. The advantage of using this approach is that the theory of numerical solutions of ODEs is very well developed and powerful computer software exists. To gain speed while maintaining accuracy, we incorporate a variable step-size technique with nonlinear multistep (NLMS) and linear multistep (LMS) methods to handle the automatic step-size adjustment. From test results obtained to date, we find that the ODE approach may turn out to be the most desirable.

In addition to the FD material, we will also

4. Discuss the formulation of ODE solutions and state the well developed theory,

5. Describe how to automatically adjust the step size by means of the variable step-size technique.

Prior to the discussion of these numerical approaches, a section addresses some of the theoretical background. After the descriptions of these approaches, the relative merits and disadvantages are discussed. Finally, a set of test results is given to demonstrate the validity of our approaches. The accurate solution of parabolic test examples numerically disclose how well the parabolic equation provides a solution of the sound propagation problem. A special section presents a fresh, general treatment of irregular bottom descriptions with arbitrary boundary conditions. Some conclusions are provided.

The experimental programs are written in ANSI FORTRAN language and have been checked out on the Center's PDP 11/70 computer. Since these programs are not yet finalized, a listing is omitted. A comprehensive document describing the computer model will be provided separately. The theoretical discretization errors are given in this report. Since computational errors are heavily involved, the total error analysis will be presented in the computer model documentation.

2. THEORETICAL BACKGROUND

We begin with a cylindrically symmetric acoustic wave equation in cylindrical coordinates:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial z^2} + k_o^2 n^2(r,z)p = 0 \ , \tag{2-1}$$

where p is the acoustic pressure, k_o is the reference wave number, $n(r,z)$ is the index of refraction, and r is the range variable. Let $p = u(r,z)v(r)$ where $v(r)$ is strongly dependent on r, but $u(r,z)$ is only weakly dependent on r.

By substituting $p = uv$ into equation (2-1), rearranging the terms, and using k_o^2 as a separation constant, we obtain the following reflected field and transmitted field parabolic wave equations:

$$v_{rr} + \frac{1}{r} v_r + k_o^2 v = 0 \tag{2-2}$$

$$u_{rr} + u_{zz} + \left(\frac{1}{r} + \frac{2}{v}\frac{\partial v}{\partial r}\right)u_r + k_o^2 (n^2-1)u = 0. \tag{2-3}$$

After we introduce the far-field approximation ($k_o r >> 1$), the exact solution of equation (2-2) is known:

$$v(r) = H_o^{(1)}(k_o r) \cong \sqrt{\frac{2}{\pi k_o r}}\, e^{i(k_o r - \frac{\pi}{4})} \ . \tag{2-4}$$

Using equation (2-4) to eliminate v from equation (2-3), we obtain

$$u_{rr} + u_{zz} + 2ik_o u_r + k_o^2(n^2-1)u = 0. \tag{2-5}$$

If we assume that inhomogeneities vary slowly with range, the reflected field can be neglected, which leads to the following parabolic wave equation:

$$u_r = \frac{ik_o(n^2-1)}{2} u + \frac{i}{2k_o} u_{zz} \ . \tag{2-6}$$

Performing parabolic approximations to the reduced wave equation will result in various slightly different forms.[2,5,6,10] We choose to deal with equation (2-6) because it is commonly recognized in the underwater acoustics community, and existing PE (parabolic equation) models, such as the split-step[11], are directed toward the solution of equation (2-6).

Note that in the shallow water environment or wherever bottom interaction is important, the exact boundary condition must be satisfied; therefore, more general purpose methods are sought for the solution of equation (2-6). Existing methods using the fast Fourier transformation (FFT) are inappropriate for such problems because these methods reflect the entire medium across the surface; however, below the physical ocean, an absorbing layer is introduced, permitting the infinite transforms to be truncated, so that the FFT becomes applicable. The numerical ODE and the FD approaches are advantageous because they do not introduce an artificial bottom.

3. NUMERICAL SOLUTIONS

3.1 FINITE DIFFERENCE APPROACH

3.1.1 Formulation

We consider

$$u_r = a(k_o,r,z)u + b(k_o,r,z)u_{zz} = Lu. \tag{3-1}$$

Let D be the FD operator, and
δ_z be the central-difference operator in the z-direction.

D and δ_z are related by

$$D = \frac{2}{h} \sin^{-1} \frac{\delta_z}{2} .$$

Where $h = \Delta z$, we use $k = \Delta r$. Therefore,

$$D^2 = \frac{\delta_z^2}{h^2} \left(1 - \frac{1}{12} \delta_z^2 + \frac{1}{90} \delta_z^4 - \ldots \right) . \tag{3-2}$$

We choose to write

$$D^2 = \frac{\delta_z^2}{h^2} \left[1 - \rho(k,h)\right] . \tag{3-3}$$

$\rho(k,h)$ is a parameter to be determined such that the choice of $\rho(k,h)$ will minimize the initial local discretization error for implicit schemes.

Further, we assume that $\rho(k,h) \neq 1$. If $\rho(k,h) = 1$, equation (3-1) reduces to $u_r = au$, which is trivial. In fact, we will demonstrate that

$\rho(k,h) = 0$ gives the Crank-Nicolson formula,
$\rho(k,h) = h^2/6k$ gives the Douglas formula.

We will make an attempt to find a $\rho(k,h)$ such that the order of the error is the smallest, if possible.

3.1.1.1 Explicit Methods. We start with the formulation of explicit methods. Using the Taylor expansion, we can obtain a two-level scheme for equation (3-1):

$$u(r + \Delta r,z) = \left(1 + k\frac{\partial}{\partial r} + \frac{1}{2!} k^2 \frac{\partial^2}{\partial r^2} + \ldots \right) u(r,z) = e^{k \frac{\partial}{\partial r}} u(r,z). \tag{3-4}$$

If we let $z = mh$, $r = nk$, and $u(r,z) = u(nk,mh) = u_m^n$, then equation (3-4) can represent an explicit scheme:

$$u_m^{n+1} = e^{k\frac{\partial}{\partial r}} u_m^n . \tag{3-5}$$

To solve equation (3-1), we could use equation (3-5) and retain only the second order difference; the explicit formula becomes

$$u_m^{n+1} = \left(1 + k\frac{\partial}{\partial r}\right)u_m^n = \left(1 + ak + \frac{b}{h^2}\,k\,\delta_z^2\right)u_m^n . \tag{3-6}$$

In general, we drop the superscripts and subscripts of a and b for economy in writing; however, in the critical places, we will use the indexes to make our formulas clearly readable. Using the second order central difference for δ_z^2, we find that

$$u_m^{n+1} = (1 + ak)u_m^n + \frac{b}{h^2}\,k\left(u_{m+1}^n - 2u_m^n + u_{m-1}^n\right). \tag{3-7}$$

If we write $1 + ak = \alpha_m^n$, $\frac{b}{h^2}\,k = \beta_m^n$, we can express equation (3-7) in a matrix form:

$$\begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ u_m^{n+1} \end{pmatrix} = \begin{pmatrix} \alpha_1-2\beta_1 & \beta_1 & 0 & 0 & \cdots & 0 \\ \beta_2 & \alpha_2-2\beta_2 & \beta_2 & 0 & \cdots & 0 \\ & & \ddots & & & \\ 0 & 0 & \cdots & \beta_m & \alpha_m-2\beta_m & \end{pmatrix} \begin{pmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_m^n \end{pmatrix} + \begin{pmatrix} \beta_1 u_o^n \\ 0 \\ \vdots \\ \beta_m u_{m+1}^n \end{pmatrix} \tag{3-8}$$

3.1.1.2 Implicit Methods. To pursue the formulation of implicit methods, we use

$$e^{-\frac{1}{2}k\frac{\partial}{\partial r}} u_m^{n+1} = e^{\frac{1}{2}k\frac{\partial}{\partial r}} u_m^n . \tag{3-9}$$

To solve equation (3-1), we use

$$\left[1 - \frac{1}{2}k(a + bD^2)\right] u_m^{n+1} = \left[1 + \frac{1}{2}k(a + bD^2)\right] u_m^n. \tag{3-10}$$

Again, a second order central difference is used for D^2 in equation (3-10); we obtain

$$\left\{1 - \frac{1}{2}ka - \frac{1}{2}kb\left[\frac{\delta_z^2}{h^2}(1-\rho(k,h))\right]\right\} u_m^{n+1} = \left\{1 + \frac{1}{2}ka + \frac{1}{2}kb\left[\frac{\delta_z^2}{h^2}(1-\rho(k,h))\right]\right\} u_m^n . \tag{3-11}$$

Simplifying the above, we find that

$$\text{LHS} = \left[1 - \frac{1}{2}ka + bs(1-\rho)\right] u_m^{n+1} - \frac{1}{2}bs(1-\rho)\left(u_{m+1}^{n+1} + u_{m-1}^{n+1}\right) \tag{3-12}$$

$$\text{RHS} = \left[1 + \frac{1}{2}ka - bs(1-\rho)\right] u_m^n + \frac{1}{2}bs(1-\rho)\left(u_{m+1}^n + u_{m-1}^n\right). \tag{3-13}$$

where $s = \frac{k}{h^2}$.

If we write

$$\alpha = 1 - \frac{1}{2}ka, \quad \beta = bs(1-\rho), \quad \rho = \rho(k,h), \quad \gamma = 1 + \frac{1}{2}ka ,$$

$$X_m = X_m^{n+1} = \alpha_m^{n+1} + \beta_m^{n+1} = 1 - \frac{1}{2}ka_m^{n+1} + b_m^{n+1}\, s(1-\rho),$$

$$Y_m = Y_m^n = \gamma_m^n - \beta_m^n = 1 + \frac{1}{2}ka_m^n - b_m^n\, s(1-\rho),$$

and equate both sides (LHS = RHS), we find that

$$-\frac{1}{2}\beta_m^{n+1} u_{m+1}^{n+1} + \left(\alpha_m^{n+1} + \beta_m^{n+1}\right) u_m^{n+1} - \frac{1}{2}\beta_m^{n+1} u_{m-1}^{n+1}$$

$$= \frac{1}{2}\beta_m^n u_{m+1}^n + \left(\gamma_m^n - \beta_m^n\right) u_m^n + \frac{1}{2}\beta_m^n u_{m-1}^n . \tag{3-14}$$

Equation (3-14) can be written in matrix form as:

$$\begin{pmatrix} X_1 & -\frac{1}{2}\beta_1^{n+1} & 0 & 0 & \cdots & 0 \\ -\frac{1}{2}\beta_2^{n+1} & X_2 & -\frac{1}{2}\beta_2^{n+1} & 0 & \cdots & 0 \\ & & \cdot & & & \\ & & \cdot & & & \\ & & \cdot & & & \\ 0 & 0 & 0 & 0 & -\frac{1}{2}\beta_m^{n+1} & X_m \end{pmatrix} \begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \cdot \\ \cdot \\ \cdot \\ u_m^{n+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\beta_1^{n+1}\, u_o^{n+1} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \frac{1}{2}\beta_m^{n+1}\, u_{m+1}^{n+1} \end{pmatrix}$$

$$+ \begin{pmatrix} Y_1 & \frac{1}{2}\beta_1^{n} & 0 & 0 & \cdots & 0 \\ \frac{1}{2}\beta_2^{n} & Y_2 & \frac{1}{2}\beta_2^{n} & 0 & \cdots & 0 \\ & & \cdot & & & \\ & & \cdot & & & \\ & & \cdot & & & \\ 0 & 0 & 0 & \cdots & \frac{1}{2}\beta_m^{n} & Y_m \end{pmatrix} \begin{pmatrix} u_1^{n} \\ u_2^{n} \\ \cdot \\ \cdot \\ \cdot \\ u_m^{n} \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\beta_1^{n}\, u_o^{n} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \frac{1}{2}\beta_m^{n}\, u_{m+1}^{n} \end{pmatrix} \quad (3\text{-}15)$$

The two components of the first column vector on the RHS are two boundary points, and the two components of the last column vector on the RHS are two boundary points on the initial line. (Note: If we select $\rho = 0$, equation (3-15) reduces to the Crank-Nicolson scheme; if we select $\rho = h^2/6k$, equation (3-15) reduces to the Douglas scheme.) At this point, we cannot yet make a choice of ρ to arrive at a new method. We defer this until after we have developed the consistency of the method.

3.1.2 Consistency. The conventional definition of consistency states that an FD approximation to a parabolic equation is consistent if

$$\frac{\text{Truncation error}}{k} \to 0 \text{ as } h,\ k \to 0.$$

However, we adopt the definition of consistency in the sense of Keller[12] as follows:

Let

$\mathcal{L}^o[u;h,k]$ be an FD approximation to equation (3-1),

$\mathcal{L}[u]$ be the true operator.

Define

$$\tau[u;h,k] = \mathcal{L}[u] - \mathcal{L}^o[u;h,k].$$

If

$$\lim_{h,k \to 0} \tau[u;h,k] \to 0,$$

we say that the method is consistent, meaning that the FD operator is consistent with the true operator.

Now we proceed to develop the concept of consistency and to obtain the "initial local discretization error" (usually called the "truncation error").

Expanding u_m^{n+1}, u_{m+1}^n, u_{m-1}^n upon u_m^n, using the Taylor expansion, and substituting the results into equation (3-7); we obtain

$$u_m^{n+1} - (1+ak)u_m^n - \frac{b}{h^2}k\left(u_{m+1}^n - 2u_m^n + u_{m-1}^n\right)$$

$$= \left\{ k\left(\frac{\partial u}{\partial t}\right)_m^n - kau_m^n - kb\left(\frac{\partial^2 u}{\partial z^2}\right)_m^n \right\}$$

$$+ \left[\frac{k^2}{2}\left(\frac{\partial^2 u}{\partial r^2}\right)_m^n - \frac{1}{12}\,bkh^2\left(\frac{\partial^4 u}{\partial z^4}\right)_m^n \right] + \frac{k^3}{3!}\left(\frac{\partial^3 u}{\partial r^3}\right)_m^n + \cdots . \qquad (3\text{-}16)$$

The terms inside the $\{\ \}$ of equation (3-16) = 0 because they satisfy equation (3-1). Let E[e] indicate the principal part of the initial local discretization error. Then, we have

$$E[e] = O(k^2 + kh^2).$$

As $h,k \to 0$, $\lim \tau[u;h,k] \to 0$; therefore, method (3-7) is consistent.

Similarly, expanding u_{m+1}^{n+1}, u_{m-1}^{n+1}, u_{m+1}^{n}, u_{m-1}^{n} upon u_m^n, using the Taylor expansion, substituting the results into equation (3-14), and simplifying, we obtain

$$\left\{-\frac{1}{2}\beta_m^{n+1} u_{m+1}^{n+1} + \left(\alpha_m^{n+1} + \beta_m^{n+1}\right) u_m^{n+1} - \frac{1}{2}\beta_m^{n+1} u_{m-1}^{n+1}\right\}$$

$$-\left\{\frac{1}{2}\beta_m^{n} u_{m+1}^{n} + \left(\gamma_m^{n} - \beta_m^{n}\right) u_m^{n} + \frac{1}{2}\beta_m^{n} u_{m-1}^{n}\right\}$$

$$= \left\{-kau_m^n + k\left(\frac{\partial u}{\partial r}\right)_m^n - bsh^2\left(\frac{\partial^2 u}{\partial z^2}\right)_m^n\right\}$$

$$+ \left[-\frac{1}{2}kak\left(\frac{\partial u}{\partial r}\right)_m^n + \frac{k^2}{2!}\left(\frac{\partial^2 u}{\partial r^2}\right)_m^n - bs\frac{h^2}{2!}k\left(\frac{\partial^3 u}{\partial z^2 \partial r}\right)_m^n\right]$$

$$+ \left(-\frac{1}{2}ka\frac{k^2}{2!}\left(\frac{\partial^2 u}{\partial r^2}\right)_m^n + \frac{k^3}{2!}\left(\frac{\partial^3 u}{\partial r^3}\right)_m^n - bs\frac{h^2}{2!}\frac{k^2}{2!}\left(\frac{\partial^4 u}{\partial z^2 \partial r^2}\right)_m^n\right) - \frac{k^3}{12}\left(\frac{\partial^3 u}{\partial r^3}\right)_m^n$$

$$+ \rho bsh^2\left(\frac{\partial^2 u}{\partial z^2}\right)_m^n + \rho bs\frac{h^2}{2!}k\left(\frac{\partial^3 u}{\partial z^2 \partial r}\right)_m^n + \rho bs\frac{h^2}{2!}\frac{k^2}{2!}\left(\frac{\partial^4 u}{\partial z^2 \partial r^2}\right)_m^n$$

$$- bs(1-\rho)\frac{h^4}{12}\left(\frac{\partial^4 u}{\partial z^4}\right)_m^n + \ldots . \qquad (3\text{-}17)$$

The terms inside the $\{\ \}$ of the RHS of equation (3-17) = 0 because they satisfy equation (3-1). If a and b are range independent, we see that the terms inside the $[\ ]$ and the $(\ )$ of the RHS of equation (3-17) all = 0 because the terms inside the $[\ ]$ satisfy $u_{rr} - au_r - bu_{zzr} = 0$ and the terms inside the $(\ )$ satisfy $u_{rrr} - au_{rr} - bu_{zzrr} = 0$. Let E[I] indicate the principal part of the initial local discretization error of the implicit scheme of equation (3-14).

For $\rho = 0$, $\quad E[I] = O(k^3 + kh^2)$, a Crank-Nicolson error;

$\rho = O(kh^2)$, $E[I] = O(k^3 + kh^4)$, a Douglas error.

It does not seem likely that a ρ can be chosen such that E[I] is smaller than the Douglas error. However, we see that $\lim_{h,k \to 0} \tau\,[u;h,k] \to 0$; therefore, the implicit methods of formula (3-15) are consistent for range-independent coefficients a and b. For the range-dependent case, the terms inside the $[\ ]$

and the ( ) of the RHS of equation (3-17) do not vanish, so that we will obtain a larger E[I]. For all ρ's, we can see that

$$E[I] = O(k^2 + kh^2) .$$

However, $\lim_{h,k\to 0} \tau[u;h,k] \to 0$; therefore, the implicit methods of formula (3-15) are consistent also for range-dependent coefficients.

3.1.3 Stability. The general concept of stability states that the difference between the theoretical and numerical solutions remains bounded as the range step n increases, provided the range increment k remains fixed for all space steps m. To find out whether a method is stable or not, we examine the satisfaction of the stability condition; this condition can be derived by means of familiar methods such as Von Neumann's, the matrix, or the Fourier series.

We shall first derive the stability condition for the explicit methods, equation (3-7). We apply Von Neumann's criterion of stability to equation (3-7) by seeking a solution in the form $e^{\alpha r}\, e^{i\omega z}$. Substituting the solution into equation (3-7), we get

$$\xi^{n+1}\, e^{i\omega mh} = \xi^n(1 + a_m^n k - 2b_m^n s)e^{i\omega h} + \xi^n\, b_m^n\, s \left[e^{i\omega(m+1)h} + e^{i\omega(m-1)h}\right] , \qquad (3\text{-}18)$$

where $\xi = e^{\alpha k}$.

Simplification of equation (3-18) gives

$$\xi = 1 + a_m^n k - 4s\, b_m^n \sin^2\left(\frac{\omega h}{2}\right) .$$

$|\xi| \leq 1$ is required to give the stability condition; i.e.,

$$\left| 1 + a_m^n k - 4s\, b_m^n \sin^2\left(\frac{\omega h}{2}\right)\right| \leq 1. \qquad (3\text{-}19)$$

When we use formula (3-7) to solve the example equation, $u_r = u_{zz}$, we arrive at Mitchell's[9] results:

$$-1 \leq 1 - 4\frac{k}{h^2} \sin^2\left(\frac{\omega h}{2}\right) \leq 1 ,$$

which implies $s = \frac{k}{h^2} \leq \frac{1}{2}$ for stability.

In general, $a(k_o,r,z)$ and $b(k_o,r,z)$ are functions of r and z; this requires a thorough examination of condition (3-19). In the case $a(k_o,r,z) = 0$, we need

$$-1 \leq 1 - 4sb_m^n \sin^2\left(\frac{\omega h}{2}\right) \leq +1 . \tag{3-20}$$

As long as $s, b_m^n > 0$, the RHS inequality of equation (3-20) is trivially satisfied. The LHS gives

$$sb_m^n \leq \frac{1}{2 \sin^2\left(\frac{\omega h}{2}\right)} . \tag{3-21}$$

In the case where b_m^n is purely imaginary, we need

$$\left|1 - 4sb_m^n \sin^2\left(\frac{\omega h}{2}\right)\right| \leq 1. \tag{3-22}$$

Condition (3-22) does not hold for $s > 0$, b_m^n purely imaginary; therefore, formula (3-7) is not stable for problems with zero coefficient $a(k_o,r,z)$ and a purely imaginary $b(k_o,r,z)$.

When $a(k_o,r,z) \neq 0$, we need inequality (3-19) to hold. In our applications, $a(k_o,r,z)$ and $b(k_o,r,z)$ are both purely imaginary. Let $a = ia_R$, $b = ib_R$; we then have

$$\left|1 + i\left[a_R k - 4sb_R \sin^2\left(\frac{\omega h}{2}\right)\right]\right| \leq 1. \tag{3-23}$$

Obviously, inequality (3-23) does not hold. To hold the inequality, we must have

$$a_R k - 4sb_R \sin^2\left(\frac{\omega h}{2}\right) = 0,$$

which implies that the condition of stability is

$$h^2 = 4 \frac{b_R}{a_R} \sin^2\left(\frac{\omega h}{2}\right) . \tag{3-24}$$

Equation (3-24) holds for $h = 0$, which implies the instability of scheme (3-7). To overcome this difficulty, a parameter λ can be introduced to obtain an equivalent equation of scheme (3-7); namely,

$$v_r = \left[a(k_o,r,z) - \lambda\right]v + b(k_o,r,z)v_{zz}, \tag{3-7}'$$

where $v = e^{-\lambda r} u$. Then, a similar inequality of equation (3-19) is obtained; namely,

$$\left| (1+\lambda) + i\left[a_R k - 4sb_R \sin^2\left(\frac{\omega h}{2}\right)\right] \right| \leq 1. \qquad (3\text{-}19)'$$

The values k, h, and λ can be chosen such that equation (3-19)' is satisfied; therefore, scheme (3-7)' is conditionally stable.

Now let us turn to the stability of implicit methods, equation (3-14). When $a(k_o,r,z) = 0$, $b(k_o,r,z) = 1$, and $\rho = 0$; equation (3-14) reduces to the example equation $u_r = u_{zz}$ and our formula reduces to the Crank-Nicolson formula. Using Von Neumann's method again, we find that the stability condition is

$$-1 \leq \frac{1 - s\left[1 - \cos(\omega h)\right]}{1 + s\left[1 - \cos(\omega h)\right]} \leq 1. \qquad (3\text{-}25)$$

We see that condition (3-25) is satisfied for $s > 0$. In fact, $s = k/h^2$ is always > 0; therefore, we see that the Crank-Nicolson formula is unconditionally stable for solving $u_r = u_{zz}$.

However, for the equation with complex coefficients, we cannot take the unconditional stability for granted when we are using the generalized Crank-Nicolson formula; the stability of formula (3-14) needs a thorough investigation. We shall apply Von Neumann's method to formula (3-14) to derive a condition and examine the condition in detail.

Define $\xi = e^{\alpha k} e^{i\omega h}$. Substituting ξ into formula (3-14), we get

$$-\frac{b}{2} s e^{\alpha(n+1)k} e^{i\omega(m+1)h} + X e^{\alpha(n+1)k} e^{i\omega mh} - \frac{b}{2} s e^{\alpha(n+1)k} e^{i\omega(m-1)h}$$

$$= \frac{b}{2} s e^{\alpha nk} e^{i\omega(m+1)h} + Y e^{\alpha nk} e^{i\omega mh} + \frac{b}{2} s e^{\alpha nk} e^{i\omega(m-1)h}.$$

Simplifying, we find that

$$\xi = \frac{Y + \frac{b}{2} s \left[2\cos(\omega h)\right]}{X - \frac{b}{2} s \left[2\cos(\omega h)\right]}, \qquad (3\text{-}26)$$

which gives the condition

$$\left| \frac{1 - bs\left[1 - \cos(\omega h)\right] - \frac{1}{2} ak}{1 + bs\left[1 - \cos(\omega h)\right] + \frac{1}{2} ak} \right| \leq 1. \qquad (3\text{-}27)$$

Where $a(k_o,r,z)$, $b(k_o r,z) > 0$, inequality (3-27) is satisfied for all $s > 0$; therefore, the unconditional stability obtains. In our applications, $a(k_o,r,z)$ and $b(k_o,r,z)$ are both purely imaginary. Specifically, we have

$$\left| \frac{1 - \frac{i}{2k_o} s \left[1 - \cos(\omega h)\right] - \frac{k}{2} \left\{ \frac{ik_o}{2} \left[n^2(r,z) - 1\right] \right\}}{1 + \frac{i}{2k_o} s \left[1 - \cos(\omega h)\right] + \frac{k}{2} \left\{ \frac{ik_o}{2} \left[n^2(r,z) - 1\right] \right\}} \right| \leq 1 . \qquad (3\text{-}28)$$

Equation (3-28) can be expressed in the short form

$$\left| \frac{1 - i\chi}{1 + i\chi} \right| \leq 1, \qquad (3\text{-}29)$$

where

$$\chi = \frac{s}{2k_o} \left[1 - \cos(\omega h)\right] + \frac{kk_o}{4} \left[n^2(r,z) - 1\right] .$$

From equation (3-29) we see that

$$\left| \frac{1 - i\chi}{1 + i\chi} \right| \quad \text{always} = 1 \text{ for all real } \chi .$$

Therefore, the FD scheme, equation (3-14), is stable when it is applied to solve our parabolic wave equation.

3.1.4 Convergence. We have developed the consistency and the stability of both explicit and implicit FD schemes, formulas (3-7) and (3-14). Now we want to show that our FD schemes are convergent. Let

"t.s." stand for the theoretical solution of equation (3-1),
"n.s." stand for the numerical solution of equation (3-1),
"f.s." stand for the finite-difference solution of equation (3-1).

The norm inequality shows that

$$||\text{t.s.} - \text{n.s.}|| \leq ||\text{t.s.} - \text{f.s.}|| + ||\text{f.s.} - \text{n.s.}|| .$$

Note that by applying the consistency to the first norm of the RHS and by applying the stability with the error control to the second norm of the RHS, we have established the convergence.

3.1.5 Discretization Errors. As a result of their consistency, the local initial discretization errors of explicit and implicit FD methods are given by formulas (3-16) and (3-17), whose principal parts possess the following expressions:

$$E[e] = \frac{k^2}{2} \left(\frac{\partial^2 u}{\partial r^2} \right)_m^n - \frac{1}{12} bkh^2 \left(\frac{\partial^4 u}{\partial z^4} \right)_m^n = O(k^2 + kh^2). \qquad (3\text{-}30)$$

In the range-independent case,

$$E[I] = -\frac{k^3}{12}\left(\frac{\partial^3 u}{\partial r^3}\right)_m^n - \frac{1}{12}\,bkh^2\,(1-\rho)\left(\frac{\partial^4 u}{\partial z^4}\right)_m^n = 0\left(k^3 + (1-\rho)kh^2\right). \tag{3-31}$$

For the range-dependent case,

$$E[I] = -\frac{k^2}{2}\,a_m^n\left(\frac{\partial u}{\partial r}\right)_m^n - b\frac{kh^2}{12}\,(1-\rho)\left(\frac{\partial^4 u}{\partial z^4}\right)_m^n = 0\left(k^2 + (1-\rho)kh^2\right). \tag{3-32}$$

3.2 ORDINARY DIFFERENTIAL EQUATIONS APPROACH

3.2.1 Formulation

Consider

$$u_r = a(k_o,r,z)u + b(k_o,r,z)u_{zz}\,. \tag{3-33}$$

Discretize the u_{zz} portion by a second order central difference:

$$u_r = a(k_o,r,z)u + \frac{b}{h^2}\,(u_{m+1} - 2u_m + u_{m-1}). \tag{3-34}$$

$$\frac{du_m}{dr} = a_m u_m + \frac{b_m}{h^2}\,(u_{m+1} - 2u_m + u_{m-1})\,. \tag{3-35}$$

$$\begin{pmatrix} \frac{du_1}{dr} \\ \frac{du_2}{dr} \\ \vdots \\ \frac{du_m}{dr} \end{pmatrix} = \begin{pmatrix} a_1 - 2\frac{b_1}{h^2} & \frac{b_1}{h^2} & 0 & \cdots & 0 \\ \frac{b_2}{h^2} & a_2 - 2\frac{b_2}{h^2} & \frac{b_2}{h^2} & \cdots & 0 \\ \vdots & & \vdots & & \\ 0 & \cdots & \frac{b_m}{h^2} & & a_m - 2\frac{b_m}{h^2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} + \begin{pmatrix} \frac{b_1}{h^2}u_o \\ 0 \\ \vdots \\ \frac{b_m}{h^2}u_{m+1} \end{pmatrix} \tag{3-36}$$

u_o and u_{m+1} are surface and bottom boundary points.

Decompose the above matrix form into

$$\begin{pmatrix} \frac{du_1}{dr} \\ \frac{du_2}{dr} \\ \cdot \\ \cdot \\ \cdot \\ \frac{du_m}{dr} \end{pmatrix} = \begin{pmatrix} a_1\frac{2b_1}{h^2} & 0 & \cdots & 0 \\ 0 & a_2\frac{2b_2}{h^2} & \cdots & 0 \\ & & \cdot & \\ & & \cdot & \\ & & \cdot & \\ 0 & & \cdots & a_m\frac{2bm}{h^2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_m \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & \frac{b_1}{h^2} & 0 & \cdots & 0 & 0 \\ \frac{b_2}{h^2} & 0 & \frac{b_2}{h^2} & \cdots & 0 & 0 \\ & & \cdot & & & \\ & & \cdot & & & \frac{b_{m-1}}{h^2} \\ & & \cdot & & & \\ 0 & \cdot & \cdot & \cdot & \frac{b_m}{h^2} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_m \end{pmatrix} + \begin{pmatrix} \frac{b_1}{h^2}u_o \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \frac{b_m}{h^2}u_{m+1} \end{pmatrix} \tag{3-37}$$

which is in the form

$$u' = Au + g(r,z,u) \tag{3-38}$$

3.2.2 Methods Under Consideration (NLMS and LMS)

The above form (equation (3-38)) is the equation to which the NLMS methods are applicable. To simplify the application, we apply an NLM-1-step predictor and a corrector with a built-in variable step size.

As a predictor, the solution is

$$u^{n+1} = e^{Ah}u^n + h\left[-(Ah)^{-1}(I-e^{Ah})\right] g^n(r_n,z,u^n). \tag{3-39}$$

As a corrector, the solution is

$$u^{n+1} = e^{Ah}u^n + h\left[-(Ah)^{-2}\right]\left\{\left[-I + (I-Ah)e^{Ah}\right] g^n + (I + Ah - e^{Ah})\, g^{n+1}(r_{n+1},z,u^{n+1})\right\}. \tag{3-40}$$

A PC^3 procedure is built-in; i.e., the procedure predicts and corrects at most three times.

NLMS methods are designed to be effective in solving equations whose g is either slowly varying in r or is a low order polynomial in r and whose eigenvalues of A have negative real parts and differ greatly in magnitude. The selection of step-size h can be made approximately by

$$\left|\left|\frac{\partial g}{\partial u}\right|\right| \cdot \left|\left|\phi_{kk}(Ah)\right|\right| \cdot h < 1,$$

where ϕ_{kk} has been defined in formula (3-40).

When g is a constant, any h can satisfy the above inequality; therefore, NLMS methods allow the use of a large step size. In this case, an accurate computation of e^{Ah} will give accurate results with fast speed. Therefore, NLMS methods have an excellent application to underwater acoustic range-independent problems with plane parallel boundary conditions where the field vanishes on both boundaries.

3.2.3 Stability, Consistency, and Convergence

The theory with respect to consistency, stability, and convergence has been well developed.[13] We summarize the theory below.

NLMS methods take the expression

$$\sum_{i=0}^{k} \alpha_i\, e^{Ah(k-i)} u_{n+i} = h \sum_{i=0}^{k} \phi_{ki}(Ah) g_{n+i}. \tag{3-41}$$

The characteristic polynomial of NLMS methods is

$$\rho(\lambda,\zeta) = e^{\lambda Ah} \sum_{i=0}^{k} \alpha_i \zeta^i. \tag{3-42}$$

Since we select k=1, α_k=1, and $\alpha_{k-1} = -1$, the root of $\rho(\lambda,\zeta)$ is 1 and simple; therefore, the NLMS methods are stable.

The consistency condition is self-contained because NLMS methods are formulated to yield consistency. Then, by the convergence theorem, stability + consistency → convergence.

We also tried LMS methods, which can be simply obtained from NLMS methods by letting $||A|| \to 0$ so that the theory is automatically applicable.

The predictor we used is

$$u^{n+1} = u^n + hf^n(r,z,u^n), \tag{3-43}$$

which is known as the Adams-Bashforth method and also carries the name Euler method.

The corrector we used is

$$u^{n+1} = u^n + \frac{h}{2}\left[f^n + f^{n+1}(r,z,u^{n+1})\right], \tag{3-44}$$

which is known as the Adams-Moulton method or the trapezoidal rule. f here is the RHS of equation (3-36).

3.2.4 Discretization Errors

The initial local discretization errors of LMS and NLMS methods have been worked out in detail[13,14]. The error terms are listed below in relation to the methods we have applied.

E[AB] = Adams-Bashforth error

$$= h^{p+2}\, u^{(p+2)}(\xi)\left[(-1)^{p+1}\int_0^1 \binom{-s}{p+1} ds\right] = O(h^{p+2}) \tag{3-45}$$

E[AM] = Adams-Moulton error

$$= h^{p+2}\, u^{(p+2)}(\xi)\left[(-1)^{p+1}\int_{-1}^0 \binom{-s}{p+1} ds\right] = O(h^{p+2}) \tag{3-46}$$

E[NLMS] = Nonlinear multistep error

$$= (\text{constant})||g^{(p+1)}(r,u)||h^{p+2} = O(h^{p+2}), \tag{3-47}$$

where p = order of the method,

$$s = \frac{t - t_n}{h}.$$

3.2.5 Variable Step-Size Technique and Error Controls

Definitions: PC^m stands for "predict-and-correct-m-times,"

ε is the user required tolerance.

A variable step-size employs the PC^m procedure to satisfy the user's required tolerance.

Applying a predictor (explicit method), one can use

$$u_{n+k} = \frac{1}{\alpha_k} \left[- \sum_{i=0}^{k-1} \alpha_i e^{Ah(k-i)} u_{n+i} + h \sum_{i=0}^{k-1} \phi_{ki}(Ah) g_{n+i} \right] , \qquad (3\text{-}48)$$

where u_{n+k} is expressed in terms of previous u_{n+i} values.

If a corrector (implicit method) is used, one must have

$$u_{n+k} = \frac{1}{\alpha_k} \left[- \sum_{i=0}^{k-1} \alpha_i e^{Ah(k-i)} u_{n+i} + h \sum_{i=0}^{k} \phi_{ki}(Ah) g_{n+i} \right] , \qquad (3\text{-}49)$$

which is of the form

$$u = G(u), \qquad (3\text{-}50)$$

where

$$u = u_{n+k}.$$

The successive iterative form gives

$$u^{(\nu+1)} = G(u^{(\nu)}) \qquad (3\text{-}51)$$

for any initial vector $u^{(0)}$.

Let $G(u)$ be defined for $||u|| < \infty$, and let there exist a constant K such that $0 \leq K < 1$. Then $G(u)$ satisfies the condition

$$||G(u^*) - G(u)|| \leq K ||u^* - u|| . \qquad (3\text{-}52)$$

Using the definition of $G(u)$ — formula (3-49) — and the fact that $g(r,z,u)$ satisfies the Lipschitz condition with a Lipschitz constant L, one can see that equation (3-52) is satisfied by

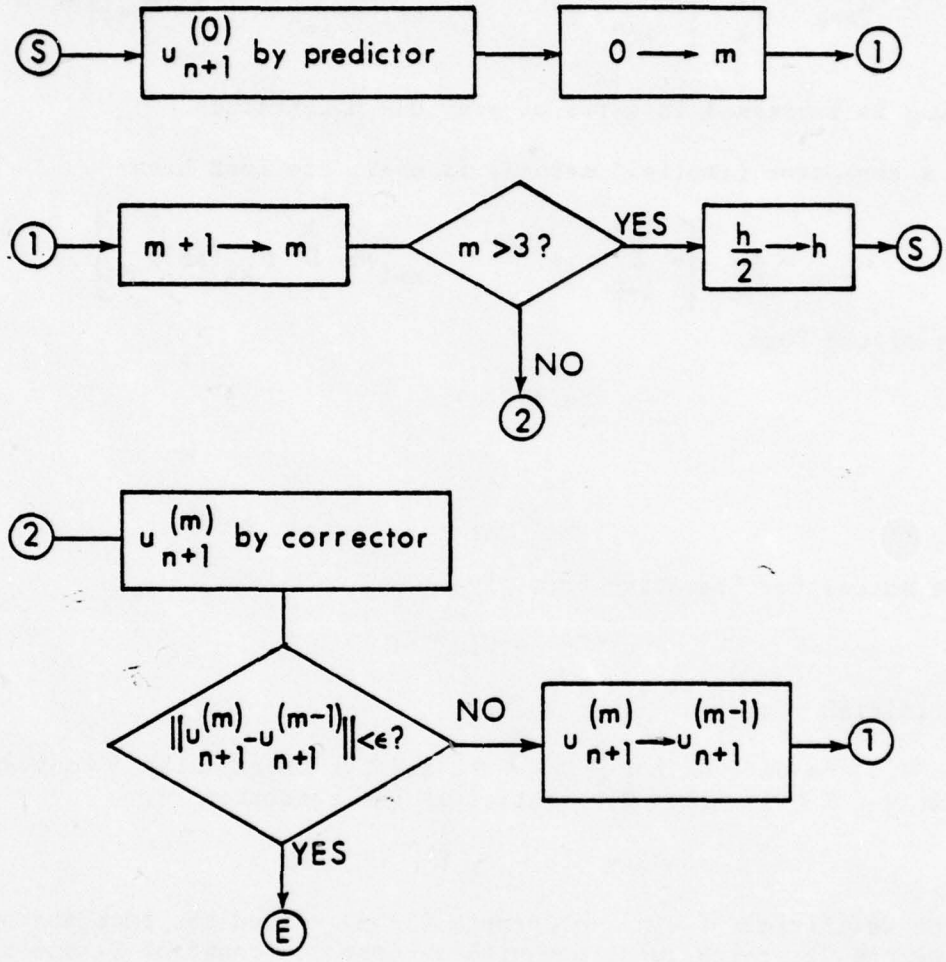
$$K = \frac{h||\phi_{kk}(Ah)||}{\alpha_k} L . \qquad (3\text{-}53)$$

For the iterative procedure, equation (3-51), to converge for arbitrary $u^{(0)}$, K is required to be less than 1:

$$K < 1 \rightarrow \frac{h||\phi_{kk}(Ah)||}{\alpha_k} L < 1 . \qquad (3\text{-}54)$$

This is the condition of the corrector's convergence. Conventionally, we select $\alpha_k = 1$ for computational convenience.

Now we describe how we develop the variable step-size technique. If a predictor is accurate, only one correction is needed. In our implementation we limit $m \leq 3$; thus, we have a PC^3 procedure, which can be described by the following diagram.



Ⓔ, the latest $u_{n+1}^{(m)}$, meets the corrector's convergence; therefore, $u_{n+1}^{(m)}$ is the solution. When the corrector's convergence is met, we double the step size and go to Ⓢ to continue the solution. When the third try fails to meet the convergence, we halve the step size and go to Ⓢ to restart the solution. Since there exists an h that leads to the corrector's convergence, this procedure is a convergent procedure; however, when h is extremely small such that t+h = h in the machine, the program will provide a message and halt the computation.

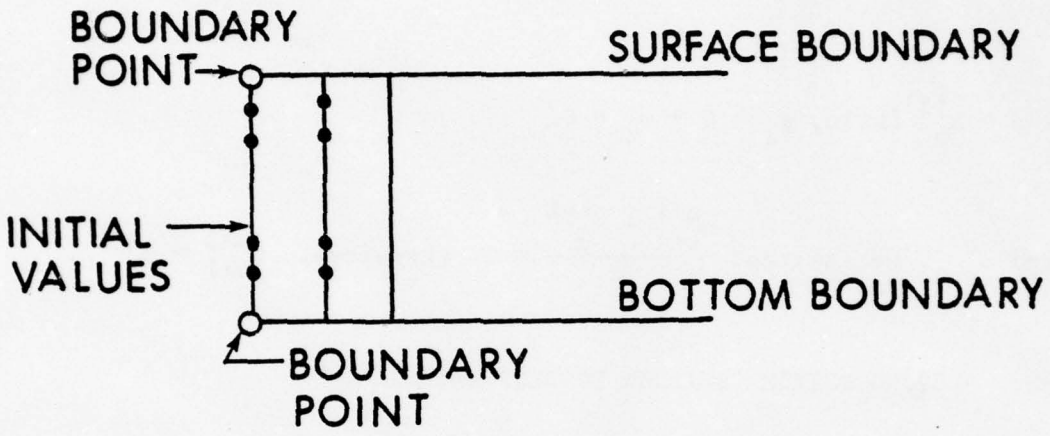
When it is less than the user required tolerance, the accuracy is controlled by examining the norm between two consecutive computations.

4. A NUMERICAL TREATMENT OF BOUNDARIES

Consider the solution of the parabolic wave equation of the form

$$u_r = a(k_o,r,z)u + b(k_o,r,z)u_{zz} \quad . \tag{4-1}$$

Assume that the numerical methods (FD and ODE) are to be used to solve the problem in the rectangular region:

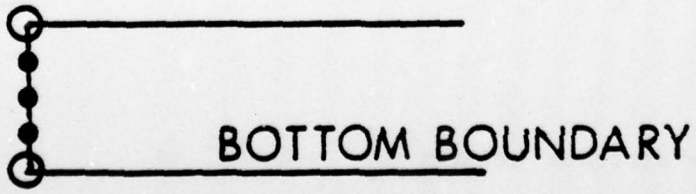


Our methods require the initial conditions and two boundary points before we can proceed to find the solution at the next range line. With this input information, we can classify the boundary descriptions for three different cases, and discuss the treatment of each case separately.

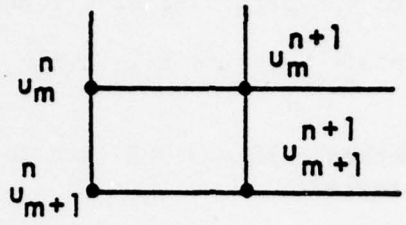
We further assume that the boundary condition is a Neumann condition such that $p_N = 0$, where p is the solution of $\nabla^2 p + k^2 p = 0$.

CASE 1: FLAT BOTTOM BOUNDARY

In this case,



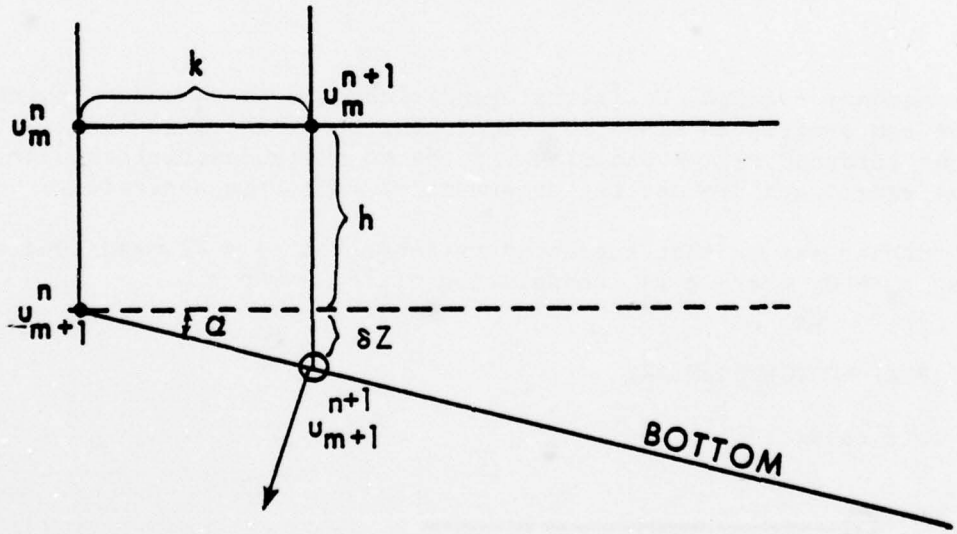
$$p_N = 0 \rightarrow \frac{\partial p}{\partial z}\cos\alpha + \frac{\partial p}{\partial r}\sin\alpha = 0 \text{ and } \alpha = 0 \qquad (4\text{-}2)$$



Since $p = H_o^{(1)}(kr)u$, $p_z = 0 \rightarrow u_z = 0$.

To find u_{m+1}^{n+1}, we see that $\frac{u_{m+1}^{n+1} - u_m^{n+1}}{\Delta z} = 0$; therefore, $u_{m+1}^{n+1} = u_m^{n+1}$.

CASE 2: SLOPING BOTTOM (SHALLOW TO DEEP WATER)



By trigonometry, the distance between u_m^{n+1} and u_{m+1}^{n+1} is

$$h + \delta z = h + k \cdot \mathrm{tg}\alpha \, . \qquad (4\text{-}3)$$

We find u_{m+1}^{n+1} as follows:

$$\frac{\partial p}{\partial z}\cos\alpha - \frac{\partial p}{\partial r}\sin\alpha = 0 \quad \text{at } u = u_{m+1}^{n+1} . \tag{4-4}$$

Also at $u = u_{m+1}^{n+1}$, u must satisfy parabolic equation (4-1); therefore, equation (4-4) becomes

$$H_o^{(1)}(kr)u_z\cos\alpha - \left[(H_o^{(1)}(kr))_r u + H_o^{(1)}(kr)u_r\right]\sin\alpha = 0 .$$

And therefore,

$$H_o^{(1)}(kr)\cos\alpha u_z - \left[H_o^{(1)}(kr)\right]_r \sin\alpha u - H_o^{(1)}(kr)\sin\alpha(au+bu_{zz}) = 0.$$

Simplifying, we obtain

$$u_{zz} - \frac{\cot\alpha}{b}u_z + \left[\frac{(H_o^{(1)}(kr))_r}{H_o^{(1)}(kr)} + a\right]\frac{1}{b}u = 0, \tag{4-5}$$

which is a second-order scalar ODE.

Knowing δz, u_m^{n+1}, we can find $\left(\frac{\partial u}{\partial z}\right)_m^{n+1}$ from

$$\frac{u_m^{n+1} - u_{m-1}^{n+1}}{\delta z} = \left(\frac{\partial u}{\partial z}\right)_m^{n+1} . \tag{4-6}$$

Then we can use numerical methods to solve equation (4-5). We make the following transformation, using the ODE package already incorporated in the system. Let

$$p_1 = -\frac{\cot\alpha}{b} ,$$

$$p_2 = \left[\frac{(H_o^{(1)}(kr))_r}{H_o^{(1)}(kr)} + a\right] \Big/ b .$$

We have

$$\frac{d^2u}{dz^2} + p_1(z)\frac{du}{dz} + p_2(z)u = 0$$

$$\frac{du}{dz} = v$$

$$\frac{dv}{dz} = \frac{d^2u}{dz^2} = - p_1(z)v - p_2(z)u;$$

therefore,

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -p_2(z) & -p_1(z) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} .$$

If $p_1(z)$ and $p_2(z)$ are constants in z, we can solve equation (4-5) exactly; i.e.,

$$s^2 + p_1 s + p_2 = 0 \rightarrow s = \frac{-p_1 \pm \sqrt{p_1^2 - 4p_2}}{2} .$$

The solution is

$$u = c_1 e^{\left(\frac{-p_1}{2} + \frac{1}{2}\sqrt{p_1^2 - 4p_2}\right)z} + c_2 e^{\left(\frac{-p_1}{2} - \frac{1}{2}\sqrt{p_1^2 - 4p_2}\right)z}. \qquad (4\text{-}7)$$

If we use $u(z_o) = u_o$, $u'(z_o) = u'_o$, we can determine c_1, c_2 exactly. Then the solution u can be calculated exactly, and u_{m+1}^{n+1} can be solved.

Once u_{m+1}^{n+1} is found, we can proceed if we use proper care. Note that if the distance between u_m^{n+1} and u_{m+1}^{n+1} is not = $2(\delta z)$, we cannot proceed with the present method because the depth partition at this time level is not uniform. A special treatment is needed at some specific advanced time levels. To handle this, we label u_{m+1}^{n+1} as u_{m+2}^{n+1} and define u_{m+1}^{n+1} as the mid-point between u_m^{n+1} and u_{m+2}^{n+1}. We determine u_{m+1}^{n+1} in such a way that the

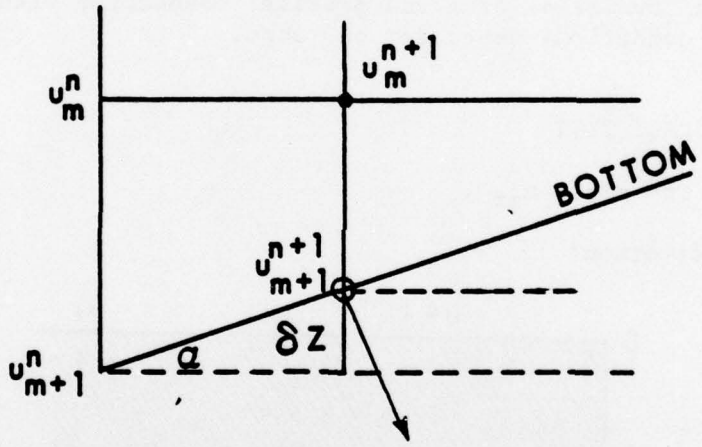
distance between u_m^{n+1} and u_{m+2}^{n+1} is $2(\delta z)$. This treatment ensures a uniform partition in the depth direction. The u_{m+1}^{n+1} can be found the same way by solving equation (4-5).

A question arises: How far can we proceed in determining u_{m+1}^{n+1}? To ease the programming requirements and to maintain accuracy, we determine Δr by

$$\frac{\delta z}{\Delta r} = tg\alpha \rightarrow \Delta r = \delta z \cdot cot\alpha.$$

When we have proceeded Δr distance in range, it is time to determine u_{m+1}^{n+1}.

CASE 3: SLOPING BOTTOM (DEEP TO SHALLOW WATER)



The treatment of case 2 can be applied to case 3. The only differences are α and δz.

5. EXPERIMENTAL NUMERICAL RESULTS

To implement these numerical methods, experimental computer programs have been developed in ANSI FORTRAN language. The computations were carried out on the Acoustic and Environmental Research Division's (Code 312) PDP 11/70 computer. Since these programs are not yet finalized, we will not describe them in this report. A separate document will be issued to describe the computer model in detail.

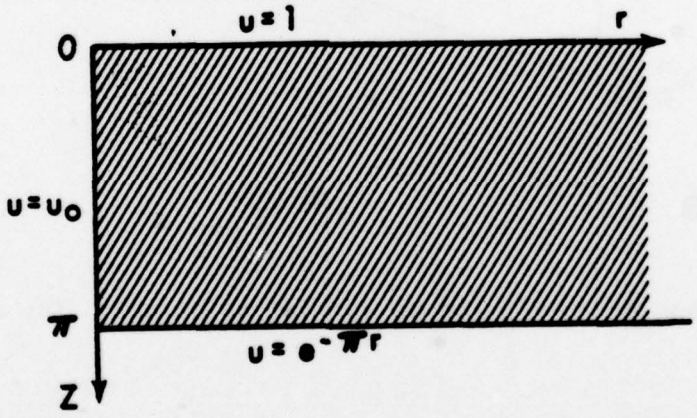
A number of parabolic problems whose solutions are known were used to test our numerical methods; the numerical results show a reasonable accuracy. Three of the test problems are included in this report to demonstrate the validity of the methods. Note that problem 1 (used to test accuracy) does not have any physical significance.

We generally classify underwater acoustic wave propagation problems in two categories: range-independent problems and range-dependent problems. Range-independent problems have coefficients and plane parallel boundary conditions independent of range. Range-dependent problems include either irregular bottom boundaries or plane parallel boundaries with coefficients and/or boundary conditions dependent on range.

PROBLEM 1: ACCURACY TEST

Equation: $u_r = (r^2 - z)u + u_{zz}$.

Region of consideration:



$0 \le r \le 1$

$0 \le z \le \pi$

Conditions: $u_o = u(o,z) = 1$

$u(r,o) = 1$

$u(r,\pi) = e^{-\pi r}$

Exact solution: $u = e^{-rz}$.

Numerical Solutions:

Results are tabulated at a range of 1 mile.

Depth (mile)	ODE	Implicit FD	Exact Solution
0.2	0.81884402	0.81880862	0.81873077
0.5	0.60672331	0.60663551	0.60653067
0.8	0.44953302	0.44939151	0.44932896
time	1^m22^s	1^m17^s	

A variable step-size technique is used for the ODE method; the user required tolerance is of the order h^{-4}. The FD method does not have the variable step-size capability; it uses the optimal step-size (h=0.001) determined by the ODE program.

PROBLEM 2: A SHALLOW WATER WAVE PROPAGATION IN A RECTANGULAR REGION WITH A RIGID BOTTOM[15]

Equation:

$$u_r = \frac{ik_o}{2}(n^2-1)u + \frac{i}{2k_o}u_{zz}; \quad k_o = \frac{2\pi f}{c_o}.$$

Region of consideration:

Input parameters and initial boundary conditions:

Initial field values: supplied by shallow water model[15]

Surface condition: $u(r,0) = 0$

Bottom boundary condition: $u_N = 0$

Source depth: 32 ft

Initial sound speed = c_o = 4950 ft/sec

Bottom depth = 64 ft

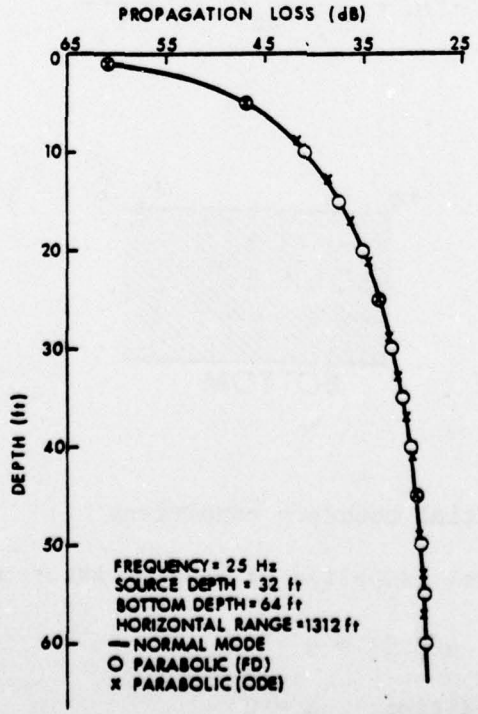
Frequency = 25 Hz

Sound speed profile:

z (ft)	SSP (ft/sec)
0	4950
64	5000
>64	90000

Numerical solutions: Problem started at range = 32 ft and terminated at range = 1313 ft. Three solutions were obtained; a normal mode[15], a variable step-size ODE, and an explicit FD. The FD program used a step-size 10 times smaller than that used for ODE. The FD solution was obtained using double-precision arithmetic.

Graphical outputs: Three numerical solutions are plotted on the graph below: depth (ft) versus propagation loss (dB).



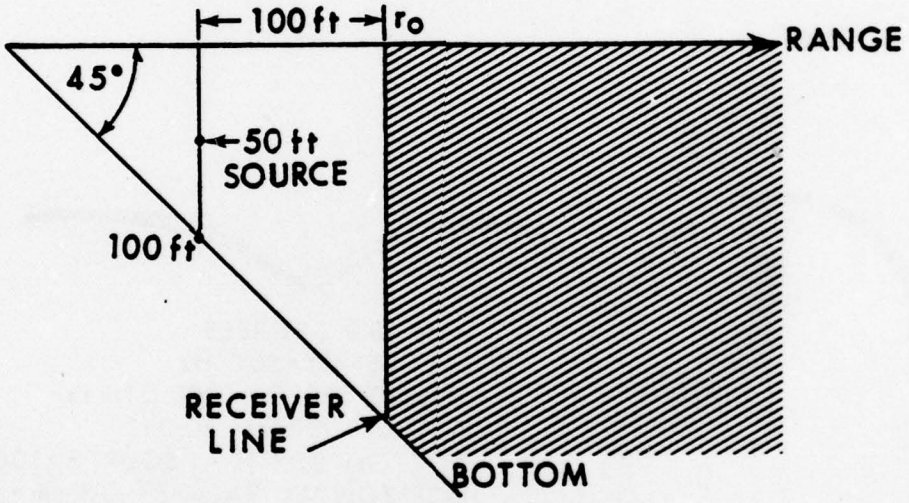
PROBLEM 3: SLOPING BOTTOM (SHALLOW TO DEEP WATER ENVIRONMENT)[16]

Problem Background: The solution for the acoustic field in a homogeneous medium bounded by a wedge can be obtained by the method of images. The solution in mathematical form and a computer program (WEDGE) to evaluate the exact solution are described in reference 16. A homogeneous medium, characterized by a sound speed c_o, bounded by a horizontal surface and sloping bottom that makes an angle of 45 degrees with the horizontal surface is described below under Region of consideration.

Equation: The parabolic equation represents the wave propagation in the range direction after the parabolic decomposition is found to be

$$u_r = \frac{i}{2k_o} u_{zz} .$$

Region of consideration:



A point source is placed 50 ft below the surface, and the receiver is located 100 ft from the source. The depth of the wedge at the source is 100 ft, the frequency equals 80 Hz, and c_o equals 5000 ft/sec. The exact solution above indicates the solution of the acoustic wave equation. We attempt to find the parabolic equation solution u in the shaded region, and then we attempt to compare $H_o^{(1)}(kr)u$ against the acoustic wave equation solution computed by the WEDGE program.

Conditions: Initial $u(r_o,z)$ values are supplied by the WEDGE program.

$u(r,o) = 0$

$u(r,\text{bottom}) = u_N = 0.$

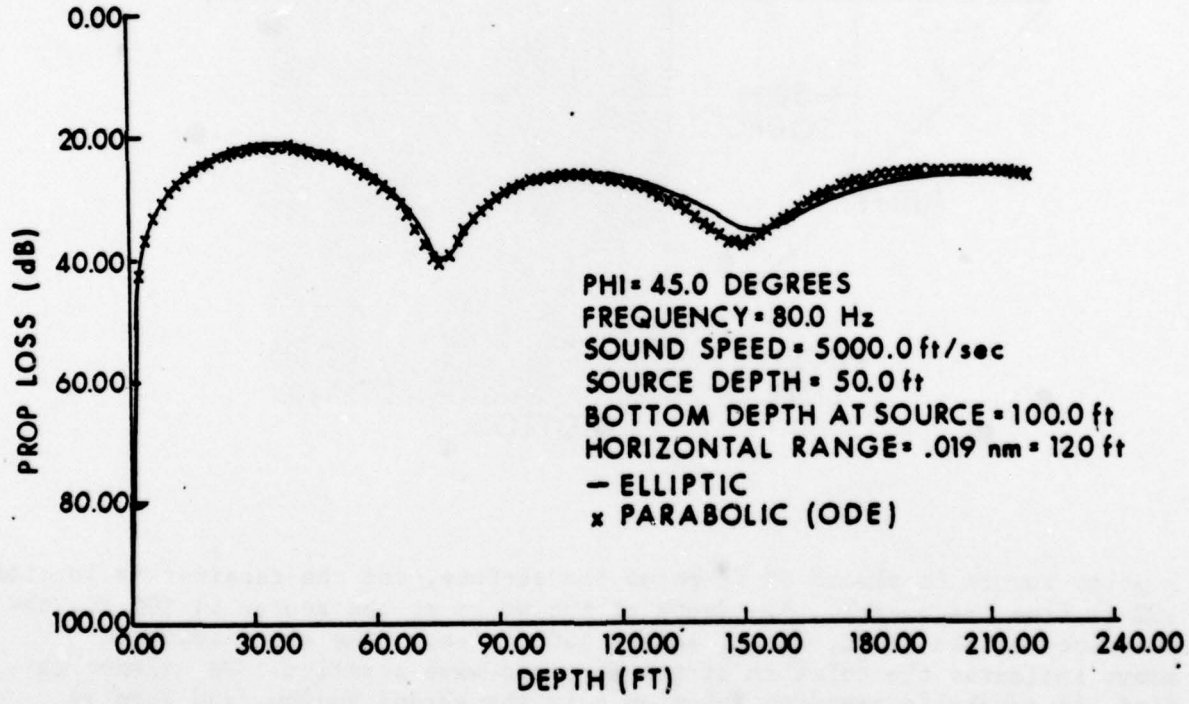
Exact solutions:

The exact solution of the acoustic wave equation is obtained from the WEDGE program.

The exact solution of the parabolic equation is not known.

Numerical solutions:

Numerical solutions are produced by the variable step-size ODE method. The Neumann bottom condition was treated by formula (4-5). Solutions of the parabolic equation are multiplied by $H_o^{(1)}(kr)$ to give an approximate solution of the acoustic wave equation. The graph below is presented in dB-scale, plotting PL(dB) versus DEPTH (ft).



Note that we do not expect the solution of the parabolic equation, multiplied by an appropriate $H_o^{(1)}(kr)$, to agree closely with the solution of the acoustic wave equation since the parabolic solution is just an approximation of the convolution. The introduction of an additional boundary point and the approximation of a boundary condition by a combination of normal derivative and the parabolic equation, which results in a second order ODE of the initial value problem, may produce less accurate results.

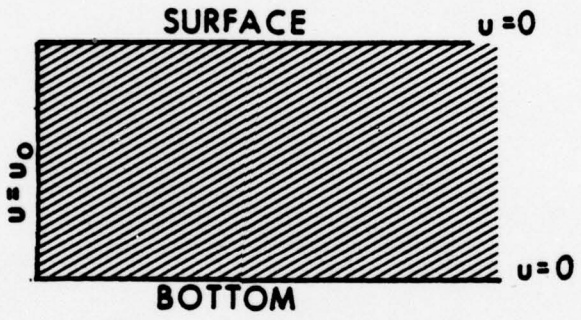
6. CONCLUSIONS

The above ODE and FD methods were developed in order to solve the parabolic wave equation with arbitrary bottom and arbitrary boundary conditions, which the existing split-step algorithm cannot handle. These methods have been shown to be general purpose and to provide the desired accuracy.

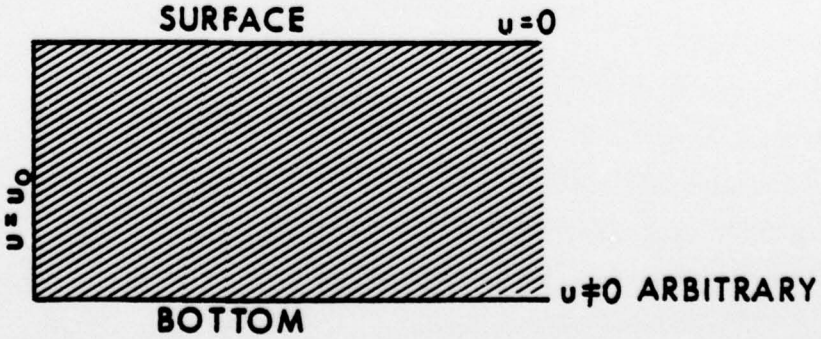
At this stage of their development, both ODE and explicit FD methods require small range step-size for accuracy. Implicit FD methods have favorable stability, but the explicit FD methods do not have this property. The implicit FD methods are faster than the variable step-size ODE methods and are equally accurate.

A categorization of the different environments and various boundary conditions yields the following four cases:

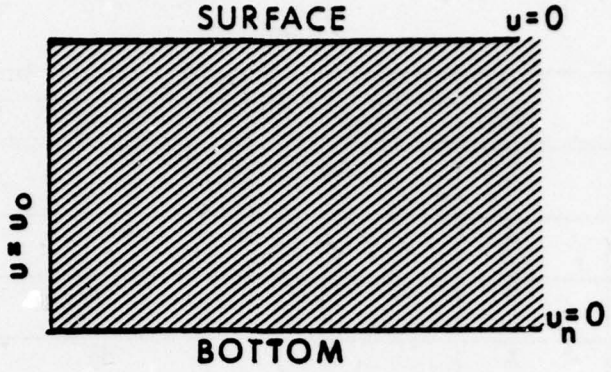
CASE 1: PLANE PARALLEL CONDITIONS WITH u = 0 at the BOTTOM



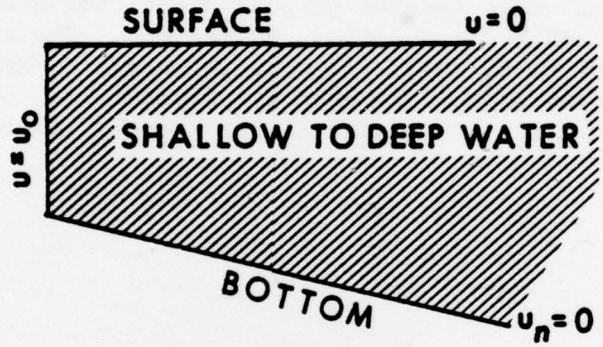
CASE 2: PLANE PARALLEL CONDITIONS WITH u ≠ 0, ARBITRARY AT THE BOTTOM

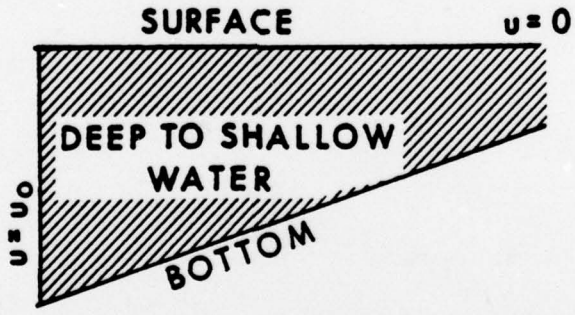


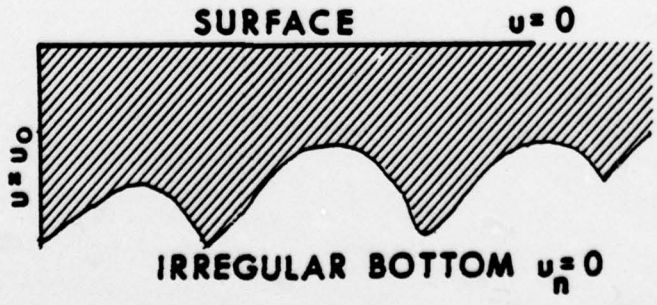
CASE 3: PLANE PARALLEL CONDITIONS WITH $u_N = 0$ AT THE BOTTOM



CASE 4: IRREGULAR BOTTOM WITH NEUMANN BOUNDARY CONDITION







The applicability of these methods is summarized in the following table, where NA stands for "Not Applicable"

METHOD / CASE	ODE	FD		SPLIT-STEP (FFT)
		EXPLICIT	IMPLICIT	
1	X	X	X	X
2	X	X	X	NA
3	X	X	NA	NA
4	X	X	NA	NA

It is evident that most general purpose algorithms used to solve parabolic wave equations are ODE or explicit FD methods. The explicit FD methods are restricted to the use of small step-size in order to achieve reasonable accuracy; they are inferior to ODE methods. In addition, the built-in corrector required by explicit FD methods is very costly in speed and memory capacity.

7. REFERENCES

1. K. R. Kelly et al., "Synthetic Seismograms: A Finite-Difference Approach," Journal of Geophysics, vol. 41, no. 1, 1976, pp. 2-27.

2. J. F. Claerbout, "Coarse Grid Calculations of Waves in Inhomogeneous Media with Application to Delineation of Complicated Seismic Structure," Journal of Geophysics, vol. 35, 1970, pp. 407-418.

3. D. M. Boore, "Finite-difference Solutions to the Equations of Elastic Wave Propagation, with Applications to Love Waves over Dipping Interfaces," Ph.D. thesis, Massachusetts Institute of Technology, 1970.

4. M. C. Smith, "Underwater Acoustic Propagation Prediction by the Alternating-Direction Implicit-Explicit Computational Methods," Journal of the Acoustical Society of America, vol. 46, no. 1 (part 2), 1969, pp. 233-237.

5. S. T. McDaniel, "Propagation of Normal Mode in Parabolic Approximation," Journal of the Acoustical Society of America, vol. 57, no. 2, 1975, pp. 307-311.

6. S. T. McDaniel, "Parabolic Approximations for Underwater Sound Propagation," Journal of the Acoustical Society of America, vol. 58, no. 6, 1975, pp. 1178-1185.

7. W. F. Ames, Numerical Methods for Partial Differential Equations, Academic Press, New York, 1977.

8. G. E. Forsythe and W. R. Wasow, Finite-Difference Methods For Partial Differential Equations, John Wiley & Sons, Inc., New York, 1960.

9. A. R. Mitchell, Computational Methods in Partial Differential Equations, John Wiley & Sons, Inc., New York, 1976.

10. J. Corones, "Bremmer Series That Correct Parabolic Approximations," Journal of Mathematical Analysis and Applications (to be published).

11. R. H. Hardin and F. D. Tappert, "Applications of the Split-Step Fourier Method to the Numerical Solution of Nonlinear and Variable Coefficient Wave Equations," SIAM Rev. 15, 1973, p. 423.

12. H. B. Keller, Numerical Methods for Two-Point Boundary-Value Problems, Blaisdell Publishing Co., Waltham, MA, 1968.

13. D. Lee, Nonlinear Multistep Methods for Solving Initial Value Problems in Ordinary Differential Equations, Ph.D. thesis, NUSC Technical Report 4775, Naval Underwater Systems Center, New London, CT, 1974.

REFERENCE (Cont'd)

14. P. Henrici, Discrete Variable Methods in Ordinary Differential Equations, John Wiley & Sons, Inc., New York, 1962.

15. W. G. Kanabis, "A Shallow Water Acoustic Model for an Ocean Stratified in Range and Depth," vol. I, NUSC Technical Report No. 4887-I, Naval Underwater Systems Center, New London, CT, 1975.

16. R. B. Lauer, R. L. Deavenport, and D. M. Potter, "The Acoustic Field in a Homogeneous Wedge as Given by the Method of Images," NUSC Technical Memorandum No. TA11-88-75, Naval Underwater Systems Center, New London, CT, 1975.

INITIAL DISTRIBUTION LIST

Addressee	No. of Copies
NRL, Code 8104 (S. Hanish)	1
NORDA, Code 200 (K. V. Mackenzie)	1
NAVELECSYSCOM, ELEX 320 (J. Cybulsk)	1
NAVSURFWPNCEN (G. C. Gaunaurd)	1
NAVOCEANSYSCEN, Code 3311B (Dr. Homer Bucker)	1
NAVPGSCOL, Code 58	1
ARL/PENN STATE, STATE COLLEGE (Dr. S. McDaniel)	1
ARL, UNIV of Texas (S. Rutherford, R. Gonzalez)	2
DDC, ALEXANDRIA	12
Dr. N. C. Dutta, Shell Oil Co., P.O. Box 481, Houston, Texas 77001	1
A. J. Haug, Planning System Inc., Suite 600, 7900 Westpark Dr., McLean, VA 22101	1
E. M. Kerwin, BBN Inc., 50 Moulton St., Cambridge, MA 02138	1
Gene E. Shaedel, KAY Elemetrics Corp., 12 Maple Ave., Pine Brook, NJ 07058	1
Prof. Jacob Yaniv, School of Engineering, Tel-Aviv University, Ramat-Aviv, Israel	1
Chuck Spofford, S.A.I., McLean, VA	1
Prof. L. E. Estes, Electrical Engineering Dept., S.M.U., Dartmouth, MA 02747	1
Prof. H. Uberall, Physics Department, Catholic University of America, Washington, D.C. 20064	1
L. W. Brooks, P.O. Box 1141, Salida, CO 81201	1
R. G. Pridham, Box 146, Raytheon Company, Portsmouth, RI	1
Dave Thomson, Defense Research Est. Pacific Forces, Mail Office, Victoria, B.C., Canada VOS-IBO	1
Masatada Fukushima, Electronic Development Dept., Totsuka Works, Hitachi, Ltd., 216 Totsuka-Machi, Totsuka-ku, Yokohama, 244, Japan	1
Engineering Societies Library, United Engineering Center, 345 East 47 Street, NYC 10017	1