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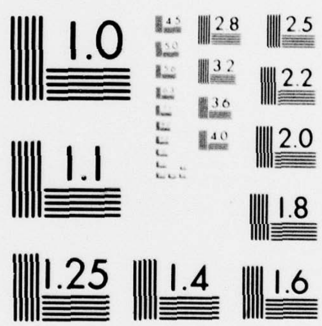

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ON THE DURATION OF THE PROBLEM OF THE POINTS

by
SHELDON M. ROSS, MEHRDAD SHAHSHAHANI and GIDEON WEISS

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ON THE DURATION OF THE PROBLEM OF THE POINTS

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ABSTRACT

We consider an r-player version of the famous problem of the points which was the stimulus for the correspondence between Pascal and Fermat in the seventeenth century. At each play of a game, exactly one of the players wins a point - player i winning with probability p_i. sub i
The game ends the first time a player has accumulated his required number of points - this requirement being n_i for player i. Our main result is to show that N, the total number of plays, is an increasing failure rate random variable. In addition, we prove some Schur convexity results regarding $P(N \leq k)$ as a function of $\underline{p}$ (for $n_i \equiv n$) and as a function of $\underline{n}$ (for $p_i \equiv 1/r$).

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ON THE DURATION OF THE PROBLEM OF THE POINTS

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0. INTRODUCTION AND SUMMARY

We are given an r-sided coin and numbers $n_1, n_2, \ldots, n_r$ along with instructions to continue to flip the coin until side i has appeared n_i times for at least one i. Each flip of the coin is assumed, independently of other flips, to land on side i with probability p_i. Let the random variable N denote the number of flips that are performed. We are interested in studying the properties of N.

In Section 1, we derive expressions for the mean and variance of N and in Section 2 we show that N has the increasing failure rate property--namely that $P\{N = k + 1 \mid N > k\}$ *is monotone nondecreasing* in k, $k = 0,1,2, \ldots$. In Section 3, we show that $P\{N \leq k\}$ is a Schur convex function of $(n_1, \ldots, n_r)$ when $p_i \equiv 1/r$ and is a Schur concave function of $(p_1, \ldots, p_r)$ when $n_i \equiv n$.

1. MEAN AND VARIANCE OF N

Assume that the flips are not performed at fixed times but rather at times chosen in accordance with a Poisson process with rate $\lambda = 1$. In addition, let us imagine that this process of coin-flipping continues indefinitely (even after some side has appeared the required number of times). Letting T_i denote the time of the n_ith appearance of side i, then T, the length of time of the experiment, can be expressed as

$$T = \min_{i=1,\dots,r} T_i . \tag{1}$$

Now it follows from well-known facts about the Poisson process that the T_i are independent gamma random variables with respective parameters (n_i, p_i), $i = 1, \dots, r$. Hence,

$$\begin{aligned} E[T] &= \int_0^\infty P\{T > t\}dt \\ &= \int_0^\infty \prod_{i=1}^r \int_t^\infty p_i e^{-p_i s} \frac{\left(p_i s\right)^{n_i - 1}}{(n_i - 1)!} ds . \end{aligned} \tag{2}$$

Now the relationship between T and N, the number of flips required, is that

$$T = \sum_{i=1}^N X_i \tag{3}$$

where the X_i are independent exponential random variables having rate 1 which are also independent of N. Thus, from (3), we have

$$E[T] = E[N]$$

which in conjunction with (2) yields an expression for $E[N]$.

The variance of N can also be obtained in a similar fashion. Namely from (3), upon conditioning, we obtain

$$\mathrm{Var}\,[T] = E[N]\,\mathrm{Var}\,[X] + E^2[X]\,\mathrm{Var}\,[N]$$

implying that

$$\mathrm{Var}\,[N] = \mathrm{Var}\,[T] - E[N]$$

and $\mathrm{Var}\,[T]$ can be obtained from the representation (2).

2. N IS AN INCREASING FAILURE RATE (IFR) RANDOM VARIABLE

Theorem:

N is an increasing failure rate random variable in the sense that

$P\{N = k + 1 \mid N > k\}$ is nondecreasing in k, $k = 0,1, \ldots$.

Proof:

The proof is by induction on r. For $r = 1$, $N = n_1$ is constant and hence is IFR. We now assume IFR for $r - 1$; in particular, we assume that N', the number of required flips for an experiment with a coin having sides $2, \ldots, r$, integers $n_2, \ldots, n_r$, and probabilities $\frac{p_2}{1 - p_1}, \ldots, \frac{p_r}{1 - p_1}$, is IFR.

Letting D_k denote the side obtained on the kth flip and $X_i(k)$ the *number of occurrences* of side i in the first k flips, we note that

$$P(N = k + 1 \mid N > k) = \sum_{i=1}^{r} P(D_{k+1} = i \, , X_i(k) = n_i - 1 \mid N > k)$$

$$= \sum_{i=1}^{r} p_i P(X_i(k) = n_i - 1 \mid N > k)$$

since D_{k+1} is independent of $X_i(k)$ and of the event $N > k$.

Thus, it is enough to show that for all i, $P(X_i(k) = n_i - 1 \mid N > k)$ is nondecreasing in k. Obviously, it is enough to consider $i = 1$, and as $P(X_1(k) = n_1 - 1 \mid N > k) = 0$ for $k = 0,1, \ldots, n_1-2$, we need only consider $k \geq n_1 - 1$.

We use the definition of N' to write

$$P(X_1(k) = n_1 - 1 \mid N > k) = \frac{P(X_1(k) = n_1 - 1 \,,\, N > k)}{\sum_{j=0}^{n_1-1} P(X_1(k) = j \,,\, N > k)}$$

$$= \frac{P(X_1(k) = n_1 - 1)P(N > k \mid X_1(k) = n_1 - 1)}{\sum_{j=0}^{n_1-1} P(X_1(k) = j)P(N > k \mid X_1(k) = j)}$$

$$= \frac{P(X_1(k) = n_1 - 1)P(N' > k - n_1 + 1)}{\sum_{j=0}^{n_1-1} P(X_1(k) = j)P(N' > k - j)} .$$

To show that this expression is nondecreasing in k, it is enough to show that for $0 \le j \le n_1 - 1$,

$$\frac{P(X_1(k) = j)P(N' > k - j)}{P(X_1(k) = n_1 - 1)P(N' > k - n_1 + 1)}$$

is nonincreasing in k (where $k \ge n_1 - 1$).

The assumption that N' is IFR implies $P(N' > k - j)/P(N' > k - n_1 + 1)$ is nonincreasing in k. Finally, for $0 \le j \le n_1 - 1 \le k$,

$$\frac{P(X_1(k) = j)}{P(X_1(k) = n_1 - 1)} = \frac{\binom{k}{j} p_1^j (1 - p_1)^{k-j}}{\binom{k}{n_1 - 1} p_1^{n_1-1} (1 - p_1)^{k-n_1+1}}$$

$$= \left(\frac{1 - p_1}{p_1}\right)^{n_1-1-j} \frac{(n_1 - 1)!(k - n_1 + 1)!}{j!(k - j)!}$$

is immediately seen to be nonincreasing in k. ||

3. SCHUR CONVEXITY OF $P\{N \le k\}$

For an r vector $\underline{x} = (x_1, \ldots, x_r)$, we denote by $x_{(i)}$ the i^{th} largest component of $\underline{x}$. We say that the permutation invariant function f is a Schur convex function if $f(\underline{x}) \ge f(\underline{y})$ whenever $\underline{x}$ majorizes $\underline{y}$ $\left(\text{written } \underline{x} \underset{m}{\ge} \underline{y}\right)$ where $\underline{x} \underset{m}{\ge} y$ if $\sum_{i=1}^{j} x_{(i)} \ge \sum_{i=1}^{j} y_{(i)}$, $j = 1, \ldots, r-1$ and $\sum_{1}^{r} x_{(i)} = \sum_{1}^{r} y_{(i)}$. If the inequality between $f(\underline{x})$ and $f(\underline{y})$ is reversed, we say that f is Schur concave.

Proposition 1:

If $n_i \equiv n$, $i = 1, \ldots, r$, then $P\{N \le k\}$ is a Schur convex function of $\underline{p} = (p_1, \ldots, p_r)$ for each k.

Proof:

Consider first the case when r equals 2. Then

$$P\{N \le k\} = \begin{cases} 0 \,, & 0 \le k < n \\ \sum_{j=n}^{k} \binom{k}{j}[p^j(1-p)^{k-j} + (1-p)^j p^{k-j}] \,, & n \le k < 2n-1 \\ 1 \,, & 2n-1 \le k \,. \end{cases}$$

Differentiating with respect to p, when $n \le k < 2n-1$, we obtain

$$\frac{d}{dp} P\{N \le k\} = k\binom{k-1}{n-1}[P(1-p)]^{k-n}[P^{2n-1-k} - (1-p)^{2n-1-k}]$$

which is positive for $p > 1/2$ thus implying that $P\{N \le k\}$ is Schur convex when $r = 2$.

For the case of general r, it is well known (see [1], p. 47) that it suffices to show that

$$P\{N(p_1, \ldots, p_r) \leq k\} \geq P\{N(q_1, q_2, p_3, \ldots, p_r) \leq k\}$$

when $p_1 \geq p_2$, $q_1 \geq q_2$, $p_1 > q_1$, $p_1 + p_2 = q_1 + q_2$. Now, let us suppose that the two experiments (the first in which the outcomes occur with probabilities $(p_1, p_2, p_3, \ldots, p_r)$ and the other in which they occur with probabilities $(q_1, q_2, p_3, \ldots, p_r)$) are performed by first flipping a coin having $r - 1$ possible outcomes with probabilities $(p_1 + p_2, p_3, \ldots, p_r)$. If the outcome having probability p_i, $i = 3, \ldots, r$ occurs, then we say that outcome i was the result for both experiments. If the outcome having probability $p_1 + p_2$ occurs, then for experiment 1 we determine its outcome (either 1 or 2) by flipping a coin having respective probabilities $\dfrac{p_1}{p_1 + p_2}$ and $\dfrac{p_2}{p_1 + p_2}$; whereas in experiment 2 we flip a coin whose probabilities are $\dfrac{q_1}{q_1 + q_2}$ and $\dfrac{q_2}{q_1 + q_2}$. Now, by conditioning on the number of the first k flips that the coin (with $r - 1$ possible outcomes) results in outcome 1, we reduce the problem to the case $r = 2$, and so the proof is complete.||

Let us assume that the p_i are constant, then there is also a Schur result when the n_i are allowed to be distinct.

<u>Proposition 2</u>:

If $p_i = 1/r$, $i = 1, \ldots, r$, then $P\{N \leq k\}$ is a Schur convex function of $\underline{n} = (n_1, \ldots, n_r)$ for each k.

Proof:

Again consider first the case when $r = 2$. As

$$P\{N \leq k\} = \left(\frac{1}{2}\right)^k \left[\sum_{j=n_1}^{k} \binom{k}{j} + \sum_{j=n_2}^{k} \binom{k}{j}\right], \quad k \leq n_1 + n_2 - 1$$

we must show that

$$\sum_{j=n_1}^{k} \binom{k}{j} + \sum_{j=n_2}^{k} \binom{k}{j} \geq \sum_{j=n_1-1}^{k} \binom{k}{j} + \sum_{j=n_2+1}^{k} \binom{k}{j}$$

where $k \leq n_1 + n_2 - 1$, $n_1 > n_2$. The above inequality reduces to

$$\binom{k}{n_2} \geq \binom{k}{n_1 - 1}$$

which is easily verified to hold under the above conditions. The general case follows exactly as in Proposition 1.||

4. FINAL COMMENTS

The model considered has applications in reliability theory. Namely, consider an r component system in which each component is subject to shocks. Every shock affects exactly one of the components--it affects component i with probability p_i. Component i can absorb at most $n_i - 1$ shocks before failing (one possibility being that with probability p_i a shock knocks out the component in position i for which there are a total of $n_i - 1$ spares). Assuming that the system structure is a series structure which means that the system is failed when at least one component is failed, it follows that N represents the number of shocks required to cause system failure.

Of course, the model is an r-player version of the famous problem of the points which was the stimulus for the interchange of letters between Pascal and Fermat in the seventeenth century. They were mainly concerned with the probability of each player winning when $n = 2$ (winning means that n_1 type 1 events occur before n_2 type 2 events). In the r-player version, the probability of player j winning can be expressed as

$$P\{j \text{ wins}\} = P\{T_j = \min(T_1, \ldots, T_r)\}$$

$$= \prod_{i=1}^{r} \frac{p_i^{n_i}}{(n_i - 1)!} \int_0^\infty \left[\prod_{i \neq j} \int_t^\infty e^{-p_i s} s^{n_i - 1} ds \right] e^{-p_j t} t^{n_j - 1} dt .$$

REFERENCE

[1] Hardy, G. H., J. E. Littlewood and G. Polya, INEQUALITIES, Cambridge University Press, Cambridge, MA, 1952.