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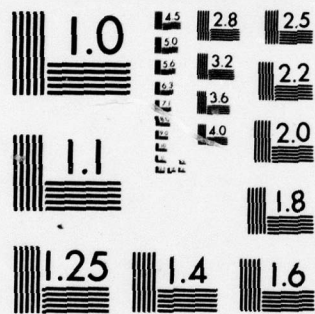
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## ESTIMATING THE TAILS OF A DISTRIBUTION OF KNOWN FORM

by

Ingram Olkin

TECHNICAL REPORT NO. 191

December 1978

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Gerald J. Lieberman, Project Director

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# ESTIMATING THE TAILS OF A DISTRIBUTION OF KNOWN FORM

Ingram Olkin

## 1. Introduction

A problem that has received considerable attention is that of estimating the tail probability of a distribution belonging to a specified family. This area received its impetus in a paper by Lieberman and Resnikoff (1955), who provided an application to acceptance sampling, and by Vajda (1955), who was concerned with an insurance model. Actually, there was an earlier paper by Kolmogorov (1950) in Russian; this paper was relatively unknown until later when a translation appeared in 1962. The history of the subject is confounded by the fact that several different points of origin and paths have led to the same goal. The present paper provides a survey of some of the results in this field and shows how the various developments relate to one another. In the process some new results are obtained.

To fix notation write  $p(x, \theta)$  to denote a parametric family. Here  $x$  and/or  $\theta$  may be a scalar or a vector. Then the tail probability may be written as an expectation:

$$R(\theta) = \int_{-\infty}^{\infty} A(t) p(t, \theta) dt .$$

The choice

$$A(t) = \begin{cases} 1, & \text{if } L \leq t \leq U, \\ 0, & \text{otherwise,} \end{cases}$$

then yields a tail probability if  $L = -\infty$  or  $U = \infty$ .

Given a sample of size  $n$  from  $p(x, \theta)$ , we wish to estimate  $R(\theta)$ . Three classes of estimators of  $R(\theta)$  have generally been considered: maximum likelihood estimators (MLE), minimum variance unbiased estimators (MVUE), and Bayesian estimators (BE).

The determination of the MLE is usually straightforward in that if  $\hat{\theta}$  is the MLE of  $\theta$ , then for each fixed  $t$ ,  $p(t, \hat{\theta})$  is the MLE of  $p(t, \theta)$ , and  $R(\hat{\theta})$  is the MLE of  $R(\theta)$ .

This simplicity is lost when dealing with unbiased estimators. If  $\tilde{\theta}$  is an unbiased estimator of  $\theta$  then  $p(t, \tilde{\theta})$  need not be an unbiased estimator of  $p(t, \theta)$ , and  $R(\tilde{\theta})$  need not be an unbiased estimator of  $R(\theta)$ . However, if  $h(t, u(X_1, \dots, X_n))$  is an unbiased estimator of  $p(t, \theta)$  for each fixed  $t$ , then

$$\int_{-\infty}^{\infty} A(t) h(t, u(X_1, \dots, X_n)) dt$$

is an unbiased estimator of  $R(\theta)$ . Consequently,  $R(\theta)$  may be estimated unbiasedly by estimating  $p(t, \theta)$  unbiasedly, or it may be estimated unbiasedly directly.

There is indeed yet another route. In the case of densities,  $p(t, \theta) = dF(t, \theta)/dt$ , where  $F(t, \theta)$  is the cumulative distribution function. Consequently, by estimating  $F(t, \theta)$  unbiasedly,  $p(t, \theta)$  can be estimated unbiasedly since (under certain regularity conditions) the derivative of the unbiased estimator is an unbiased estimator of the density. Estimating the c.d.f. rather than the density has the advantage that an unbiased estimator of  $F(t, \theta)$  may exist, whereas an

unbiased estimator of  $p(t, \theta)$  may not exist.

In the discussion we have assumed that the data available is a sample from  $p(x, \theta)$  and  $p(t, \theta)$  is to be estimated for each fixed  $t$ . However, in some instances the sample may be from a related distribution  $p^*(x, \theta)$  rather than from  $p(x, \theta)$ . For example, the underlying distribution may be a  $\mathcal{N}(\mu, \sigma^2)$  distribution but the sample is from a  $\mathcal{N}(\mu, g(\sigma^2))$  distribution. Clearly the  $\mathcal{N}(\mu, \sigma^2)$  density will not be estimable for all functions  $g$ . The function  $g(\sigma^2) = k\sigma^2$  arises naturally in some sampling procedures. In this case Ghurye and Olkin (1969) obtain results for a variety of families of distributions.

Notation. We adopt the notation

$$(1.1) \quad \psi(a) = \begin{cases} a, & \text{if } a \geq 0, \\ 0, & \text{if } a < 0. \end{cases}$$

When  $A$  is a symmetric matrix, we write

$$(1.2) \quad \psi(A) = \begin{cases} \det A, & \text{if } A \text{ is positive semi-definite,} \\ 0, & \text{otherwise.} \end{cases}$$

## 2. Applications

In this section a review is given of several applications for which tail probabilities are of interest.

Insurance Claims. An application to insurance claims was formulated by Vajda (1951). Consider an insurance company that handles casualty



insurance. The total amount of claims per year varies from year to year. Assume that the claims  $X_1, \dots, X_k$  for each of  $k$  years are independent, identically distributed from a distribution  $p(x, \theta)$ . (This means that the value of money remains constant and the volume of business remains constant. These assumptions can be avoided by inflating or deflating the claims appropriately.)

Because the insurance claims can vary considerably, the company could be in financial straits if any claim exceeds a critical value  $c$ . Thus, the company may reinsure itself against the contingency that a claim is greater than  $c$ . This is called "stop-loss reinsurance". For this model the net reinsurance premium is

$$R(\theta) = \int_c^{\infty} (x-c) p(x, \theta) dx.$$

Unbiased estimation of  $R(\theta)$  is developed by Vajda (1955); the bias in estimating  $R(\theta)$  via the MLE is studied by Conolly (1955).

Reliability Context. Barlow and Proschan (1965) provide an extensive study of the mathematical theory of reliability. They quote a definition taken from the radio-electronics-television manufacturers association, namely, that "Reliability is the probability of a device performing its purpose adequately for the period of time intended under the operating conditions encountered." Let  $X(u) = 1$  if the device is performing adequately at time  $u$ , and  $X(u) = 0$  otherwise. Then  $P\{X(T) = 1\}$  is

the probability that the device performs adequately over the intended period  $[0, T]$  and represents the reliability.

Sampling Inspection. Lieberman and Resnikoff (1955) study sampling plans for inspection by variables. In lot-by-lot acceptance sampling, a random sample is drawn from a lot and each item in the lot is classified as defective or non-defective. The entire lot is then accepted or rejected depending on the number of defectives. In inspection procedures by variables, the item is measured as to a variable quality characteristic, and the lot is accepted or rejected depending on these measurements.

Upper and/or lower specification limits  $U$  and  $L$  are provided and the item is considered defective if it falls outside of the interval  $[L, U]$ , and is considered acceptable if it falls within the interval. The form of the underlying distribution is known to be in some parametric family.

Pollution Data. Suppose that we have  $p$  chemicals or pollutants and obtain measurements  $X_1, X_2, \dots, X_p$  of some characteristic of the pollutants. For the  $j$ th pollutant there is a regulated threshold  $c_j$  such that if  $X_j \geq c_j$ , then there is a violation of the regulations. Then we are concerned with the events  $\bigcup_{j=1}^p \{X_j \geq c_j\}$  or  $\bigcap_{j=1}^p \{X_j \geq c_j\}$ .

Transformation of Variables. A variety of transformations are used in statistical analyses, e.g., (i)  $z = (x-a)^{1/2}$ , (ii)  $z = \arcsin x^{1/2}$ ,

(iii)  $z = \log x$ , etc. Neyman and Scott (1960) are concerned with correcting for the bias in estimating  $EX$  based on observations on  $z$ . In the above examples, (i)  $x = z^2 + a$ , (ii)  $x = \sin^2 z$ , (iii)  $x = \exp z$ . In general we have  $z = f(x)$  and  $x = f^{-1}(z)$ . For the normal family, conditions on  $f$  for the existence of unbiased estimators are given by Neyman and Scott (1960) and by Schmetterer (1960). Hoyle (1968) considers a closely related problem.

### 3. Estimation Procedures

Several key methods have been used to obtain unbiased estimators. We give a brief review of the essential ideas in each method.

(i) Conditional Expectation: The Rao-Blackwell Procedure. This method is central to almost all techniques. If  $U(X)$  is an unbiased estimator of  $\theta$  and if  $\{p(x, \theta), \theta \in \Theta\}$  admits a complete sufficient statistic  $T(X)$ , then  $E(U(X)|T(X))$  is the UMVUE of  $\theta$ .

Since most of the standard families considered admit a complete sufficient statistic, this method has had widespread applicability. Other methods have been proposed for special problems mainly because conditional expectations are not always easily computed. However, it should be noted that almost all results obtained could have been derived using this procedure.

This procedure is particularly suited to estimating the tail probability since the indicator function serves as a natural choice for the candidate unbiased estimator.



In the case of estimating densities, if  $\{p(x, \theta), \theta \in \Theta\}$  has an unbiased estimator, then the conditional density (if it exists) of  $X_1 | T(X_1, \dots, X_n)$  evaluated at  $x$  is the UMVUE of  $\{p(x, \theta), \theta \in \Theta\}$ . This conditional density can be determined directly from the joint density of the sample, or (under suitable conditions) it can be obtained from the conditional c.d.f. by differentiation, namely,  $\partial/\partial x P_\theta\{X_1 \leq x | T(X)\}$ . Consequently, estimating a c.d.f. (or reliability) unbiasedly will also yield an unbiased estimator of the density, if it exists.

(ii) Transform Theory. For many problems involving the exponential family, the determination of an unbiased estimator of a function  $g(\theta)$  requires the solution of an integral equation

$$(3.1) \quad \int h(x) p(x, \theta) dx = g(\theta) .$$

If the left-hand side can be rewritten as a transform (e.g., Laplace, Mellin, etc.) then the solution can be obtained from transform theory. Washio, Morimoto, and Ikeda (1956) and Tate (1959) make extensive use of transform theory.

Mehran (1973) notes a relation to the theory of convolutions. Suppose  $p(x; \theta) = f(x - \theta)$ , then (3.1) becomes

$$(3.2) \quad \int h(x) f(x - \theta) dx = g(\theta) .$$

But (3.2) is in the form of a convolution so that the inversion



formula yields

$$h(x) = g(x) + \sum_{m=1}^{\infty} a_m g^{(m)}(x) ,$$

where the  $a_j$  are terms in the Taylor expansion at  $E(\exp -Xt)$ .

(iii) Ancillary Statistics. The idea of using an ancillary statistic is contained in papers by Sathe and Varde (1969) and Eaton and Morris (1970). Suppose  $T(X)$  is a complete sufficient statistic,  $A(X)$  is an ancillary statistic, and  $h(X) = W(A(X), T(X))$ . Then

$$E_A W(A, T) \equiv h^*(T),$$

where the expectation is with respect to the marginal distribution of  $A$ , is the MVUE of

$$\varphi(\theta) = E_{\theta} h(x) .$$

The result above is given by Eaton and Morris (1970).

(iv) The Jackknife. Since the jackknife is a technique to reduce bias, it can be used in conjunction with conditional expectation to yield MVUE. The essential idea is contained in Gray, Watkins and Schucany (1973). Suppose that  $\hat{\theta}$  is an estimator of  $\theta$ , then we estimate  $f(\theta)$  by  $f(\hat{\theta})$ . Further, suppose that  $f(\hat{\theta})$  has a bias expansion of  $k$  terms, then we can apply a  $k^{\text{th}}$  order generalized

jackknife to eliminate the bias and then use the Rao-Blackwell theorem.

More specifically, suppose  $X_1, \dots, X_n$  is a random sample of size  $n$  and  $\hat{\theta}_1, \dots, \hat{\theta}_{k+1}$  are estimators of  $\theta$  such that

$$E\hat{\theta}_j = \theta + \sum_{i=1}^k a_{ij} b_i(\theta), \quad j = 1, \dots, k+1.$$

If  $\hat{\theta} = (\hat{\theta}_1, \dots, \hat{\theta}_{k+1})$ ,  $A = (a_{ij})$ ,  $e = (1, \dots, 1)$ , and

$$G(\hat{\theta}_1, \dots, \hat{\theta}_{k+1}) = \det \begin{pmatrix} \hat{\theta} \\ A \end{pmatrix} / \det \begin{pmatrix} e \\ A \end{pmatrix}$$

is defined, then

$$EG(\hat{\theta}_1, \dots, \hat{\theta}_{k+1}) = \theta.$$

The usual jackknife is obtained by special choices of the  $\hat{\theta}_j$ .

If the MLE is used as the initial estimators  $\hat{\theta}$  in the above procedure, then under certain circumstances, the UMVUE is the MLE plus a correction for bias. When the result is in the form of an infinite series, then approximations are obtained by truncating the series.

(v) Location - Scale Parameter Families. When concerned with location and scale parameter families, some simplification can be obtained by splitting the problem into two subproblems, where the first problem is to carry out the estimation assuming that the scale parameter is fixed, and the second is to estimate an appropriate function with

the scale-parameter unknown. Ghurye and Olkin (1969) use this principle based on the following lemma.

Lemma: Let  $S$  and  $T$  be independent statistics with a joint c.d.f.  $F(t; \sigma, \tau) G(s; \sigma)$  depending on the parameters  $(\sigma, \tau) \in \Theta_1 \times \Theta_2$ . If for each  $\sigma \in \Theta_1$ ,  $E_b(\sigma, T) = h(\sigma, \tau)$ ,  $\tau \in \Theta_2$ , and if for each  $\tau \in \Theta_2$ ,  $E_a(s, \tau) = b(\sigma, \tau)$ ,  $\sigma \in \Theta_1$ , then  $a(S, T)$  is an unbiased estimator of  $\{h(\sigma, \tau), (\sigma, \tau) \in \Theta_1 \times \Theta_2\}$ .

#### 4. Estimation of Density Functions, Cumulative Distribution Functions, and Tail Probabilities.

The number of densities studied is large, and a variety of procedures have been developed for special distributions or for special classes of distributions. We here describe some of these results.

##### 4.1. The Normal Distribution: Univariate and Multivariate Case.

In the case of the normal distribution with unknown mean and known variance and with unknown mean and variance, results have been obtained by Kolmogorov (1950), Lieberman and Resnikoff (1955), Vajda (1955) and Healy (1956). The procedure in all cases is an application of the Rao-Blackwell theorem. Subsequent papers by Barton (1961) and Basu (1964) also yield these results.

The UMVUE of the  $\mathcal{N}(t; \mu, \sigma^2)$  density function is

$$(4.1) \quad \frac{\sqrt{n}}{\sqrt{n-1}} \frac{1}{\sqrt{2\pi}\sigma} \exp - \frac{1}{2\sigma^2} \frac{n(t-\bar{x})^2}{n-1}$$



when  $\sigma$  is known, and is

$$(4.2) \quad \frac{\sqrt{n}}{\sqrt{n-1}} \frac{v^{-1}}{B(\frac{n-2}{2}, \frac{1}{2})} \psi \left( 1 - \frac{(t-\bar{x})^2}{v^2} \frac{n}{n-1} \right)^{\frac{n-4}{2}}$$

when  $\sigma$  is unknown. Here  $\bar{x} = \sum_{i=1}^n x_i/n$ ,  $v^2 = \sum_{i=1}^n (x_i - \bar{x})^2$ , and  $\psi$  is defined in (1.1).

This result has been generalized to the multivariate case by Ghurye and Olkin (1969) and by Lumel'skiĭ and Sapazhnikov (1969): the UMVUE of the  $N(t; \mu, \Sigma)$  density function is

$$(4.3) \quad \frac{|\Sigma|^{-1/2}}{[2\pi(n-1)/n]^{p/2}} \exp \left[ -\frac{n}{2(n-1)} (t-\bar{x})' \Sigma^{-1} (t-\bar{x}) \right]$$

when  $\Sigma$  is known; and is

$$(4.4) \quad \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-p-1}{2}) \pi^{\frac{p}{2}} (n-1)^{\frac{p}{2}}} \frac{1}{|S|^{1/2}} \left[ \psi \left( 1 - \frac{(t-\bar{x})' S^{-1} (t-\bar{x})}{n-1} \right) \right]^{(n-p-3)/2},$$

when  $\Sigma$  is unknown. Here  $t = (t_1, \dots, t_p)$ ,  $\bar{x} = (\bar{x}_1, \dots, \bar{x}_p)$ ,  $S = (s_{ij})$ ,  $s_{ij} = \sum_{\alpha=1}^n (x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j)/n$ , and  $\psi(a)$  is defined by (1.1).

Ghurye and Olkin also consider the case when  $\Sigma = \sigma^2 \Sigma_0$ , where  $\sigma^2$  is unknown and  $\Sigma_0$  is known, and the case  $\Sigma = \sigma^2 [(1-\rho)I + \rho e'e]$ , where  $\sigma^2$  and  $\rho$  are unknown. This latter format is the intraclass correlation model. The method of Lumel'skiĭ and Sapazhnikov is a variant of the Rao-Blackwell theorem, whereas the method of Ghurye and Olkin is based on a reduction of each problem into two subproblems (see

Lemma in Section 3(v)). This involves obtaining an estimate with the scale parameter fixed, and then estimating a derived function with the scale-parameter unknown.

#### 4.2. The Normal Distribution: Multivariate Case with a Linearly Restricted Mean.

Suppose we wish to estimate the normal density

$$(4.1) \quad |\Sigma|^{-m/2} \exp \left[ -\frac{1}{2} \sum_{j=1}^r (c_j - \theta B_j)' \Sigma^{-1} (c_j - \theta B_j) \right],$$

where  $\theta$  is an unknown  $\ell$ -dimensional vector,  $B_1, \dots, B_r$  are given  $\ell \times k$  matrices and the vectors  $c_1, \dots, c_r$  are  $k$ -dimensional. The covariance matrix  $\Sigma$  may be known, may be known up to a scalar multiple, may have the intraclass correlation format, or may be unknown.

There is, however, an additional point, namely, that the  $k$ -dimensional observations  $X_1, \dots, X_n$  may come from a normal population with means, namely,  $\mathcal{L}(X_i) = \mathcal{N}(\theta A_i, \Sigma)$ , and  $A_i$  are known  $\ell \times k$  matrices,  $\text{rank}(A_1, \dots, A_n) = 1$ . Results for these four cases are obtained by Ghurye and Olkin (1969):

(i)  $\Sigma = \Sigma_0$  Known. The UMVUE of (4.1) is

$$\frac{\exp \left[ -\frac{1}{2} (c - zQ^{-1}B)' (I - B'Q^{-1}B)^{-1} (c - zQ^{-1}B) \right]}{|\Sigma_0|^{m/2} |I - B'Q^{-1}B|^{1/2}},$$

where  $z = \sum_{i=1}^n x_i \Sigma_0^{-1} A_i$ ,  $Q = A_1 \Sigma_0^{-1} A_1' + \dots + A_n \Sigma_0^{-1} A_n'$ ,

$$B = (B_1 \Sigma_0^{-1/2}, \dots, B_r \Sigma_0^{-1/2}) .$$

(ii)  $\Sigma = \sigma^2 \Sigma_0$ ,  $\Sigma_0$  Known. The UMVUE of (4.1) is

$$\frac{2^{mk/2} \Gamma(v/2) [\psi(s - (c - zQ^{-1}B)(I - B'Q^{-1}B)^{-1}(c - zQ^{-1}B)')] (v - mk - 2)/2}{\Gamma(\frac{v-mk}{2}) |\Sigma_0|^{m/2} |I - B'Q^{-1}B|^{1/2} s^{(v-2)/2}},$$

where  $s = \sum_{i=1}^n x_i \Sigma_0^{-1} x_i' - zQ^{-1}z' \sim \sigma^2 \chi_v^2$ .

Remark: Normally when  $z$  and  $s$  arise from a sample of size  $n$ , the parameter  $v = nk - l$ .

(iii)  $\Sigma = \sigma^2 [(1-\rho)I + \rho e'e]$ . In (4.1) let  $B_j = b_j'e$ , where the  $b_j$  are  $l$ -dimensional vectors. The observations  $X_1, \dots, X_n$  are distributed as  $\mathcal{N}(\theta a_i'e, \Sigma)$ ,  $i = 1, \dots, n$ , where the  $a_i$  are  $l$ -dimensional vectors. The UMVUE of (4.1) is

$$\frac{\tilde{y}_1 \tilde{y}_2}{|I - B' A_0^{-1} B|^{1/2}},$$

where

$$B = (b_1'e, \dots, b_r'e), \quad A_0 = k \Sigma a_i'a_i,$$



$$\tilde{\gamma}_1 = \frac{2^{m/2} \Gamma\left(\frac{n\ell}{2}\right)}{\Gamma\left(\frac{n-m-\ell}{2}\right)} \frac{[\psi(u_1 - (\bar{c} - zA_0^{-1/2} B)(I - B'A_0 B)^{-1}(\bar{c} - zA_0^{-1/2} B)')]^{(n-m-\ell-2)/2}}{u_1^{(n-\ell-2)/2}},$$

$$u_1 = \frac{\Sigma(x_i e')^2}{k} - zz', \quad z = x(e'a_1, \dots, e'a_n) A_0^{-1/2}, \quad \bar{c} = (c_1 e'e, \dots, c_r e'e)/k,$$

$$\tilde{\gamma}_2 = \frac{2^{m(k-1)/2} \Gamma\left(\frac{n(k-1)}{2}\right)}{\Gamma\left(\frac{(n-m)(k-1)}{2}\right)} \frac{[\psi(x(I - \frac{e'e}{k})x' - c(I - \frac{e'e}{k})c')]^{((n-m)(k-1)-2)/2}}{[x(I - \frac{e'e}{k})x']^{(n(k-1)-2)/2}}.$$

(iv)  $\Sigma$  Unknown: The observations consist of independent statistics  $z$  and  $S$ , where  $z : 1 \times k$ ,  $S : k \times k$  with  $\mathcal{L}(z) = \mathcal{N}(\xi, \lambda \Sigma)$ ,  $\mathcal{L}(S) = \mathcal{W}(k, \nu, \Sigma)$ . Then

$$2^{km/2} \frac{k \Gamma\left(\frac{\nu-i+1}{2}\right)}{1 \Gamma\left(\frac{\nu-m-i+1}{2}\right)} (1-\lambda)^{k/2} [\psi(S - \frac{(c-z)'(c-z)}{1-\lambda})]^{(\nu-k-m-1)/2}$$

is the UMVUE of

$$|\Sigma|^{-m/2} \exp \left[ -\frac{1}{2} (c-\theta)' \Sigma^{-1} (c-\theta) \right].$$

This result is contained in Ghurye and Olkin (1969) and Lumel'skiĭ and Sapozhnikov (1969).



#### 4.3. Reliability Function for Normal Models.

In stress versus strength reliability analyses we have a measurement  $X$  of strength and a measurement  $Y$  of stress. A failure occurs if  $X \leq Y$  and the device survives if  $X > Y$ . The distribution is generally assumed to be normal with an unknown mean and variance. The distribution of  $Y$  may be normal with known or unknown parameters, or may be a known truncated normal distribution.

Regardless of the underlying assumptions, the reliability is given by

$$R = P\{X > Y\} .$$

Lipow and Eidemiller (1964) present two examples involving proof-pressure testing of an empty solid propellant rocket motor case. In this application the testing occurs to a given level of pressure  $y_0$ . Consequently, they assume that  $X \sim \mathcal{N}(\mu_1, \sigma_1^2)$  and  $Y$  is a  $\text{TN}(\mu_2, \sigma_2^2)$  truncated at  $y_0$ . They then show that  $R$  may be computed from tables of the bivariate normal distribution. When no truncation occurs the reliability is given by

$$R(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2) = \Phi\left(\frac{\mu_1 - \mu_2}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right).$$

When  $Y$  has a known distribution, say  $\mathcal{N}(0,1)$ , in which case the reliability is

$$R(\mu, \sigma^2) = \Phi\left(\frac{\mu}{\sqrt{1 + \sigma^2}}\right).$$

Church and Harris (1970) study the MLE of  $R(\mu, \sigma^2)$ , and Downton (1973), using the Rao-Blackwell theorem, obtains the UMVUE of  $R(\mu, \sigma^2)$  in the case  $\sigma$  is unknown to be

$$(4.5) \quad \int_{-1}^1 \phi(\bar{x} + tv) \frac{(1-t^2)^{\frac{n-2}{2}-1}}{B\left(\frac{n-2}{2}, \frac{1}{2}\right)} dt,$$

where  $\bar{x} = \Sigma x_i/n$ ,  $v^2 = \Sigma (x_i - \bar{x})^2/n$ .

When the distributions of  $X$  and  $Y$  are both unknown and samples  $x_1, \dots, x_n$  and  $y_1, \dots, y_m$  from  $X$  and  $Y$  are available, the UMVUE was obtained by Downton (1973). Define

$$\bar{x} = \Sigma x_i/n, \bar{y} = \Sigma y_i/m, v_1 = \Sigma (x_i - \bar{x})^2/n, v_2 = \Sigma (y_i - \bar{y})^2/m$$

$$A = (\bar{x} - \bar{y})/v_2, B = v_1/v_2.$$

The UMVUE of  $R(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2)$  is 0 if  $(\bar{x} - \bar{y})/(v_1 + v_2) \geq 1$ , it is 1 if  $(\bar{x} - \bar{y})/(v_1 + v_2) \leq -1$ , and it is

$$(4.6) \quad \int_{-1}^1 \int_{-1}^{A+Bw} \frac{(1-u^2)^{\frac{n-2}{2}-1} (1-w^2)^{\frac{m-2}{2}-1}}{B\left(\frac{m-2}{2}, \frac{1}{2}\right) B\left(\frac{n-2}{2}, \frac{1}{2}\right)} du dw$$

if  $|(\bar{x} - \bar{y})/(v_1 + v_2)| \leq 1$ .

Perhaps a more realistic version of the first model above is when the strength of a material is determined from  $p$  different but correlated tests, rather than from a single test. That is, we now

have a vector  $(X_1, \dots, X_p)$  of measurements of strength, the distribution of which is  $\mathcal{N}(\mu, \Sigma)$ . Analogously, there is a vector  $(Y_1, \dots, Y_p)$  of stresses with known distribution  $\mathcal{N}(0, I)$ . A failure occurs if any  $X_i \leq Y_i$  and the device functions if  $X_i > Y_i$  for all  $i$ . Then

$$(4.7) \quad R(\mu, \Sigma) = P\{X_1 > Y_1, \dots, X_p > Y_p\} \\ = \int_0^\infty \dots \int_0^\infty \frac{\exp[-\frac{1}{2}(t-\mu)(I+\Sigma)^{-1}(t-\mu)']}{(2\pi)^{p/2} |I+\Sigma|^{1/2}} dt.$$

When the distribution of  $Y$  is unknown, samples  $X \sim \mathcal{N}(\mu, \Sigma)$  and  $Y \sim \mathcal{N}(\nu, \psi)$  are available. Then

$$(4.8) \quad R(\mu, \nu, \Sigma, \psi) = P\{X_1 > Y_1, \dots, X_p > Y_p\} \\ = \int_0^\infty \dots \int_0^\infty \frac{\exp[-\frac{1}{2}(t-\mu+\nu)(\Sigma+\psi)^{-1}(t-\mu+\nu)']}{(2\pi)^{p/2} |\Sigma+\psi|^{1/2}} dt.$$

We now obtain UMVUE of  $R(\mu, \Sigma)$  and  $R(\mu, \nu, \Sigma, \psi)$  in a slightly more general context.

Case: Known Covariance Matrix. From an observation  $z \sim \mathcal{N}(\mu, A)$ , where  $A$  is known, what is the UMVUE of

$$(4.9) \quad g(t, \mu) = |B|^{-1/2} \exp[-\frac{1}{2}(t-\mu) B^{-1}(t-\mu)'] ,$$



where  $A < B$ , i.e.,  $B - A$  is positive definite, and  $B$  is known. This is equivalent to solving the integral equation

$$(4.10) \quad \int h(z;t) \frac{\exp[-\frac{1}{2}(z-\mu)' A^{-1}(z-\mu)']}{(2\pi)^{p/2}} dz = g(t,\mu) .$$

Letting  $v = (t-\mu)' B^{-1/2}$ ,  $w = z-t$ , and  $s = wA^{-1} B^{1/2}$  in (4.6) yields the integral equation

$$(4.11) \quad \int k(s) \exp[-vs' - \frac{1}{2} vCv'] ds = 1 ,$$

where  $C = B^{1/2} A^{-1} B^{1/2} - I$ , and

$$k(s) = |A|^{1/2} h(s\Lambda^{-1/2} A+t) \exp[-\frac{1}{2} sB^{-1/2} AB^{-1/2} s'] .$$

The solution of (4.7) is

$$k(s) = |C|^{-1/2} \exp[-\frac{1}{2} sC^{-1} s'] ,$$

so that

$$h(z;t) = \frac{|A|^{1/2} |B|^{1/2}}{|B-A|^{1/2}} \exp[-\frac{1}{2} (z-t)' (B-A)^{-1} (z-t)']$$

is the UMVUE of  $g(t,\mu)$  in (4.9).

Note that  $C > 0$  is required. But  $C > 0$  if and only if  $B^{1/2} A^{-1} B^{1/2} > I$ , which is equivalent to  $B > A$ .

Special Case. When  $z$  is the sample mean, then  $A = \Sigma/n$  and from (4.7)

$B = I + \Sigma$ , so that the requirement becomes  $I + \Sigma > \Sigma/n$ , which always holds.

Remark: An alternative approach is to first estimate the conditional probability  $P\{X > Y | Y = y\}$  and then integrate over  $y$ . Since this procedure makes use of some previously known results, it yields a direct answer.

Case: Unknown Covariance Matrix. If  $\mathcal{L}(z) = \mathcal{N}(\mu, \Sigma/N)$ ,  $\mathcal{L}(S) = W(\Sigma, p, n)$ ,

$S = (s_{ij})$ ,  $s_{ij} = \Sigma(x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j)/n$ , then from (4.4)

$$h(t; z, S) = c(p, n) |S|^{-1/2} \left[ \psi \left( 1 - \frac{(t-z) S^{-1} (t-z)'}{n-1} \right) \right]^{(n-p-3)/2},$$

where  $c(p, n) = \Gamma(\frac{n-1}{2}) / [\Gamma(\frac{n-p-1}{2}) \pi^{p/2} (n-1)^{p/2}]$ , is the UMVUE at the normal density function  $\mathcal{N}(\mu, \Sigma)$  at a point  $t$ .

Then

$$\int_{y_1}^{\infty} \cdots \int_{y_p}^{\infty} h(t; z, S) dt$$

is the UMVUE of the conditional probability  $P\{X_1 > Y_1, \dots, X_p > Y_p | Y_1 = y_1, \dots, Y_p = y_p\}$ . Consequently, the UMVUE of the unconditional probability  $P\{X_1 > Y_1, \dots, X_p > Y_p\}$  is

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\exp[-\frac{1}{2} yy']}{(2\pi)^{p/2}} dy \int_{y_1}^{\infty} \cdots \int_{y_p}^{\infty} c(p, n) |S|^{-1/2} \left[ \psi \left( 1 - \frac{(t-z) S^{-1} (t-z)'}{n-1} \right) \right]^{(n-p-3)/2} dt.$$

Making a change of variables  $t - y = w$  and  $(y-z+w) S^{-1/2} / \sqrt{n-1} = u$  and changing the order of integration, the UMVUE reduces to

$$(4.12) \quad \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-p-1}{2})\pi^{p/2}} \int_0^\infty \cdots \int_0^\infty \phi_p \left( z + uS^{1/2} \sqrt{n-1} \right) [\psi(1-uu')]^{(n-p-3)/2} du ,$$

where  $\phi_p(a_1, \dots, a_p) = \prod_1^p \phi(a_i)$  and  $\phi(a)$  is the standard normal c.d.f. When  $p = 1$ , (4.12) reduces to (4.5), which is the result of Downton (1973).

In the case that the distribution of  $Y$  is unknown, we require a slightly more complicated result. Suppose  $h(t; z_1, S_1)$  is a UMVUE of the  $\mathcal{N}(\mu, \Sigma)$  density at a point  $t$ , and  $g(y; z_2, S_2)$  is a UMVUE of the  $\mathcal{N}(\nu, \psi)$  density at a point  $y$ . Then

$$(4.13) \quad \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty g(y; z_2, S_2) dy \int_{y_1}^\infty \cdots \int_{y_p}^\infty h(t; z_1, S_1) dt$$

is the UMVUE of  $P\{X_1 > Y_1, \dots, X_p > Y_p\}$ .

Substitution of (4.4) for  $g$  and  $h$  in (4.13) yields

$$(4.14) \quad \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \frac{c(m, p)}{|S_2|^{1/2}} \left[ \psi \left( 1 - \frac{(y-z_2)S_2^{-1}(y-z_2)'}{m-1} \right) \right]^{(m-p-3)/2} dy$$

$$\int_{y_1}^\infty \cdots \int_{y_p}^\infty \frac{c(n, p)}{|S_1|^{1/2}} \left[ \psi \left( 1 - \frac{(t-z_1)S_1^{-1}(t-z_1)'}{n-1} \right) \right]^{(n-p-3)/2} dt .$$

Letting  $t-y = t^*$  and interchanging the order of integration leads to the expression

$$\frac{A(m, p)A(n, p)}{|S_1|^{1/2}|S_2|^{1/2}} \int_{1 > qq'} \int_{\Omega} [\psi(1-qq')]^{(m-p-3)/2} \psi(1-ww')^{(n-p-3)/2} dq dw$$



where  $\Omega = \{w : wV_1^{1/2} - qV_2^{1/2} + z_1 - z_2 < 0\}$ . The univariate case is obtained from (4.12) with  $p = 1$  and is the same as (4.3) obtained by Downton (1973).

In the context of stress versus strength reliability analyses,  $X$  and  $Y$  are independent. However, in other contexts  $X$  and  $Y$  may be dependent; in which case

$$R = P\{X_1 > Y_1, \dots, X_p > Y_p\}$$

$$= \int_0^\infty \dots \int_0^\infty \frac{\exp[-\frac{1}{2}(t-\mu+v)(\Sigma_{11}+\Sigma_{22}-\Sigma_{12}-\Sigma_{21})^{-1}(t-\mu+v)']}{(2\pi)^{p/2} |\Sigma_{11}+\Sigma_{22}-\Sigma_{12}-\Sigma_{21}|^{1/2}} dt.$$

Now if  $(\bar{x}, \bar{y}) \sim \mathcal{N}(\mu, v), \Sigma/n), S \sim W(\Sigma, 2p, n)$ , then

$$z = \bar{x} - \bar{y} \sim \mathcal{N}(\mu-v, (\Sigma_{11}+\Sigma_{12}+\Sigma_{21}+\Sigma_{22})/n),$$

$$V = S_{11}+S_{22}+S_{12}+S_{21} \sim W(\Sigma_{11}+\Sigma_{12}+\Sigma_{21}+\Sigma_{22}, p, n),$$

so that we may employ our previous results, namely,

$$(4.17) \quad \int_0^\infty \dots \int_0^\infty \frac{c(p, n)}{|v|^{1/2}} \left[ \psi \left( 1 - \frac{(t-z)V^{-1}(t-z)'}{n-1} \right) \right]^{(n-p-3)/2} dt$$

is the MVUE of  $R$ . Unfortunately, the evaluation of this multiple integral is numerically complicated.



#### 4.4. The Exponential, Gamma and Weibull Families of Distribution.

The exponential, gamma, and Weibull distributions have been used extensively in reliability theory. In part this is due to the fact that they have nondecreasing failure rates. In addition to the three named distributions, censored versions have also been considered.

The UMVUE of the gamma density

$$\frac{u^{p-1} e^{-u/\theta}}{\Gamma(p) \theta^p}$$

at a point  $u$ , based on a sample  $x_1, \dots, x_n$  from a gamma distribution with parameters  $(q, \theta)$ ,  $nq > p$ , is

$$\frac{1}{B(p, nq-p)} \frac{u^{p-1}}{(\bar{nx})^p} \psi \left[ \left( 1 - \frac{u}{\bar{nx}} \right) \right]^{nq-p-1}.$$

Therefore the UMVUE of  $R(\theta) = P\{X > t\}$  is given by the incomplete beta function  $I_{t/(\bar{nx})}(p, (n-1)p)$  if  $t/(\bar{nx}) \leq 1$ , and is unity if  $t/(\bar{nx}) > 1$ .

This result was obtained (in a different form) by Tate (1959) and by Basu (1964). The case  $p = 1$  reduces to the exponential distribution in which case the UMVUE is

$$\left( 1 - \frac{t}{\bar{nx}} \right)^{n-1},$$

for  $\bar{nx} > t$  and is 0 for  $\bar{nx} < t$ . This result is, of course, contained in Tate (1959) and was rediscovered by Pugh (1963) again using the Rao-Blackwell theorem.

For the exponential distribution where the observations are the order statistics  $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(r)}$ , the UMVUE of the tail probability  $R(t) = P\{X \geq t\}$  is

$$(4.18) \quad \left(1 - \frac{t}{T_r}\right)^{r-1},$$

where  $T_r = X_{(1)} + \dots + X_{(r)} + (u-r) X_{(r)}$ . For the truncated two parameter exponential distribution

$$f(x; \theta) = \frac{1}{\theta} e^{-(x-\mu)/\theta}, \quad \mu > 0,$$

the UMVUE of  $R(t)$  is

$$(4.19) \quad \left(\frac{n-1}{n}\right) \left(1 - \frac{t - X_{(1)}}{T_r - nX_{(1)}}\right)^{r-2}.$$

The expression (4.18) is obtained by Basu (1964); the expression (4.19) is obtained by Basu (1964) and by Laurent (1963) for  $r = n$ .

For the Weibull distribution

$$f(x; \theta) = \frac{p}{\theta} x^{p-1} \exp\left[-\frac{x^p}{\theta}\right],$$

letting  $y = x^p$  and noting that  $y$  has an exponential distribution, a direct application of the results for the exponential distribution yields results for the Weibull distribution.

If individual elements are tested until a prescribed number,  $r^*$ , of failures occurs, then the UMVUE of  $R$  is  $(1 - t/T)^{r^*-1}$  for  $t \leq T$ , where  $T$  is the total observed life on all the elements tested. For  $k$  elements in series,  $(1 - kt/T)^{r^*-1}$  for  $t \leq T/k$  is the UMVUE of  $R$ , and for  $k$  elements in parallel it is

$$\sum_{j=1}^k (-1)^{j-1} \binom{k}{j} \left[ \psi(1-j \frac{t}{T}) \right]^{r^*-1}.$$

If a system contains  $k$  elements with redundancy in which one element is in service with  $(k-1)$  replacements, then the system fails only if the sum of all  $k$  lives is less than  $t$ . Now the UMVUE of  $R(t)$  is

$$\sum_{j=0}^{k-1} \binom{r^*-1}{j} (1 - \frac{t}{T})^{r^*-j-1} (\frac{t}{T})^j = I_{1-\frac{t}{T}}(r^*-k, k).$$

An alternative testing procedure is to test individual elements until a preassigned total life  $T$  has passed and we observe the number  $r$ , of failures. Since  $r$  has a Poisson distribution with parameter  $T_0/\theta$ , we may then use results on Poisson probabilities given by Barton (1961) using the Rao-Blackwell theorem, and by Patil (1963), who equates coefficients in a power series.

The UMVUE of  $R(t)$  is then  $\psi[(1-t/T^*)]^r$ . For  $k$  elements in series the UMVUE is  $\psi[(1-kt/T^*)]^r$ ; for  $k$  elements in parallel the UMVUE is



$$\sum_{j=1}^k (-1)^{j-1} \binom{k}{j} \psi \left[ \left(1 - \frac{jt}{T^*}\right)^r \right],$$

and for the redundancy case with  $k$  elements the UMVUE is

$$\sum_{j=0}^{k-1} \binom{r}{j} \psi \left[ \left(1 - \frac{t}{T^*}\right)^{r-j} \left(\frac{t}{T^*}\right)^j \right] = I_{1 - \frac{t}{T^*}}(r-k+1, k),$$

where  $I_x(a, b) = 0$  for  $x \leq 0$ .

The truncated exponential distribution

$$f(x; \theta) = \frac{1}{\theta} \frac{e^{-x/\theta}}{(1 - e^{-x_0/\theta})}, \quad 0 \leq x \leq x_0,$$

has been used in a number of applications, e.g., in fitting rainfall data and irrigation studies. Other examples occur in life testing problems for which the survival probability is of interest. Holla (1967) obtains the UMVUE of the density function and the survival probability. This paper contains an error as noted by Johnson (1968), which also apparently has a lacuna. Sathe and Varde (1969) obtain the UMVUE of the reliability  $R(t) = P\{t \leq X \leq x_0\}$  as

$$1 - \frac{\sum_{j=0}^{m_1} (-1)^j \binom{n-1}{j} \left(1 - \frac{jx_0}{s}\right)^{n-1} - \sum_{j=0}^{m_2} (-1)^j \binom{n-1}{j} 1 - \left(\frac{jx_0+t}{s}\right)^{n-1}}{\sum_{j=0}^{m_1} (-1)^j \binom{n}{j} \left(1 - \frac{kx_0}{s}\right)^{n-1}},$$

where  $m_1 = [s/x_0]$  is the largest integer  $\leq s/x_0$  and  $m_2 = [(s-t)/x_0]$  is the largest integer  $\leq (s-t)/x_0$ .

#### 4.5 Other Methods of Estimation: Bayes Estimates

The development of Bayes estimators of density functions has not been as extensive as that for MLE or UMVUE. However, some results have been obtained.

For the exponential density and reliability function

$$f(x;\theta) = \frac{e^{-x/\theta}}{\theta}, \quad R(t;\theta) = e^{-t/\theta},$$

Bhattacharya (1967) obtains Bayes estimators for three priors:

$$(i) \text{ uniform: } g(\theta;\alpha,\beta) = \frac{1}{\beta-\alpha}, \quad \alpha < \theta < \beta$$

$$(ii) \text{ inverted gamma: } g(\theta;\mu) = \frac{(\mu/\theta)^{\nu+1}}{\mu \Gamma(\nu)} e^{-\mu/\theta},$$

$$(iii) \text{ exponential: } g(\theta;\lambda) = \frac{1}{\lambda} e^{-\theta/\lambda}.$$

We observe the order statistics  $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(r)}$  from the exponential distribution  $f(x;\theta)$ . Define

$$T_r = X_{(1)} + \dots + X_{(r)} + (n-r) X_{(r)}.$$

The Bayes estimators,  $R_B(t)$ , of  $R(t)$  are:

$$(i) \quad R_B(t) = \frac{I_{a_1}(r-1) - I_{b_1}(r-1)}{I_{a_2}(r-1) - I_{b_2}(r-1)} \frac{1}{[1 + (t/T_r)]^{r-1}},$$

where

$$a_1 = (T_r + t)/\alpha, \quad b_1 = (T_r + t)/\beta,$$

$$a_2 = T_r/\alpha, \quad b_2 = T_r/\beta,$$

and  $I_x(n)$  is the incomplete gamma function.

$$(ii) \quad R_B(t) = \frac{1}{\left(1 + \frac{t}{T_r + \mu}\right)^{r+\nu}}.$$

$$(iii) \quad R_B(t) = \frac{K_{r-1}\left(2\sqrt{(T_r+t)/\lambda}\right)}{K_{r-1}\left(2\sqrt{T_r/\lambda}\right)} \frac{1}{\left(1 + \frac{t}{T_r}\right)^{(r-1)/2}},$$

where  $K_\nu(z)$  is the modified Beisel function of the third kind of order  $\nu$ .

Instead of observing time to failure, we may count the number of failures and apply the theory for Poisson probabilities. Now we let  $\delta = 1/\theta$  and assume priors on  $\delta$ :



$$(i) \text{ uniform: } g(\delta; \alpha, \beta) \frac{1}{\beta - \alpha}, \quad \alpha < \delta < \beta,$$

$$(ii) \text{ exponential: } g(\delta; \lambda) = \lambda e^{-\lambda \delta}.$$

Then the Bayes estimators of the reliability function are

$$(i) \quad R_B = \frac{I_a^*(k + \frac{t}{T}, n - k + 1) - I_b^*(k + \frac{t}{T}, n - k + 1)}{I_a^*(k, n - k - 1) - I_b^*(k, n - k + 1)},$$

where  $a = \exp - \alpha^T$ ,  $b = \exp - \beta^T$ , and

$$I_x^*(p, q) = \int_0^x u^{p-1} (1-u)^{q-1} du$$

is the nonnormalized incomplete beta function;

$$(ii) \quad R_B = \frac{B(k + \frac{\lambda + t}{T}, n - k + 1)}{B(k + \frac{\lambda}{T}, n - k + 1)}.$$

If we let  $\alpha \rightarrow 0$ ,  $\beta \rightarrow \infty$  we obtain the Bayes estimators for the improper prior. In case (i):

$$R_B = \frac{1}{(1 + \frac{t}{T})^{r-1}},$$

and in case (ii):

$$R_B = \prod_{r=0}^{n-k} \frac{r+k}{r+k+\frac{t}{T}}.$$



Bayes estimators for the parameter  $\theta$  and  $\theta^{-1}$  in the exponential distribution is obtained by Bhattacharya (1967) and by El-Sayyed (1967) respectively. The two-parameter exponential is studied by Varde (1969).

#### 4.6. Discrete Versions.

Results for discrete distributions have appeared in a number of contexts. For example, Girshick, Mosteller and Savage (1948) obtain unbiased estimators of binomial probabilities based on sequential sampling. Blackwell's 1947 paper showed that conditional expectations were central to this procedure. Since then a number of specific results have appeared.

Barton (1961) shows that if  $X_1, \dots, X_n$  are independent binomial random variables  $b(p, N)$ , then

$$\frac{\binom{\Sigma X}{r} \binom{Nn - \Sigma X}{N-r}}{\binom{Nn}{N}}$$

is the UMVUE of  $\binom{N}{r} p^r (1-p)^{N-r}$  for all integral  $r$ . For the Poisson distribution,

$$\frac{\frac{(n\bar{X})^{(r)}}{r!n^r}}{(1 - \frac{1}{n})^{n\bar{X}}}$$

is the UMVUE of  $e^{-\lambda} \lambda^r / r!$ .

In many examples for discrete distributions, the problem may be posed in terms of equating coefficients in two power series. This idea has reappeared throughout the literature. It is exploited by

Patil (1963) who considers a general class of distributions, called generalized power series distributions of the form

$$p(x;\theta) = a(x) b(\theta) \theta^x, \quad x = 0, 1, \dots$$

Some of the results obtained, in addition to the binomial and Poisson distributions, are:

<u>Estimator</u>	<u>Family</u>	<u>Description</u>
$\binom{z}{k} \frac{s_{n-z}^{n-1}}{n s_z^n}$	$\frac{e^{-\theta} \theta^k}{k! (1-e^{-\theta})}$	Poisson distribution truncated at zero
$\frac{\binom{m}{k} \binom{mn-m}{z-k}}{\binom{mn}{z}}$	$\binom{m}{k} \left(\frac{\theta}{1-\theta}\right)^k \left(\frac{1}{1+\theta}\right)^{m-k}$	binomial
$\frac{\binom{m+k-1}{k} \binom{mn-m+z-k-1}{z-k}}{\binom{mn+z-1}{z}}$	$\binom{m+k-1}{k} \theta^k (1-\theta)^m$	negative binomial
$\frac{z!}{nk(z-k)!} \frac{s_{z-k}^{n-1}}{s_z^n}$	$\frac{\theta^k}{k} \log \frac{1}{1-\theta}$	log series,

where  $s_z^n$  is the Stirling number of the second kind with arguments  $n$  and  $z$ .

#### 4.7. Confidence Intervals for Reliability Functions.

The conversion of estimators to confidence bounds for reliability functions is by no means a straightforward procedure. There have been a number of papers in this area and we mention only a few contributions since this topic is slightly peripheral to the main development.

El-Mawaziny and Buehler (1967) obtain confidence intervals for the reliability of a series system where the  $i^{\text{th}}$  component has an exponential distribution with parameter  $\theta_j$ . The methods are via conditioning, MLE and Bayes estimators.

#### 4.8. Estimation of Special Functions.

Many functions of the parameters of a distribution have been studied, e.g., the moments of a distribution.

Kordonskii and Rozenblit (1976) consider the problem of estimating the characteristic function and a  $k^{\text{th}}$  degree polynomial in the moments for the normal, gamma, Weibull, shifted exponential, uniform, binomial, negative binomial and Poisson distribution. To illustrate these results, the UMVUE of the characteristic function for the normal distribution  $\mathcal{N}(\mu, \sigma^2)$ ,

$$\varphi(\xi) = E e^{i\xi x} = e^{i\xi\mu - \frac{1}{2}\xi^2\sigma^2},$$

is

$$\hat{\varphi}(\xi) = \Gamma\left(\frac{n-1}{2}\right) \left(\frac{4n}{\xi^2 s^2 (n-1)}\right)^{(n-3)/4} e^{i\xi \bar{x}} J_{(n-3)/2}\left(\xi \sqrt{s^2 (n-1)/n}\right),$$



where  $s^2 = \sum (x_i - \bar{x})^2$  and  $J_a(z)$  is the Bessel function of the first kind.

The entropy and Kullback-Leibler information numbers have generally been estimated by maximum likelihood (see e.g., Miller and Madow (1954)). Ghurye and Olkin (1969) provide UMVUE for the normal family. For example, if  $X \sim \mathcal{N}(\mu, \Sigma)$  then

$$E \log f(X; \mu, \Sigma) = -\frac{p}{2}(1 + \log 2\pi) - \frac{1}{2} \log |\Sigma|,$$

so that we need to estimate  $\log |\Sigma|$ . The result is that if  $\mathcal{L}(S) = W(p, m; \Sigma)$ , then

$$E \log |S| = \log |\Sigma| + p \log 2 + \sum_{k=1}^p \frac{d \log \Gamma(\alpha)}{d\alpha} \Big|_{\alpha=(m-1+1)/2}.$$

The last term may be obtained from tables of the digamma function.

##### 5. Comparisons of Alternative Methods of Estimation.

Curiously there have been few studies of the efficiency of alternative estimators. The most extensive study is that of Zacks and Even (1966a, 1966b).

Glasser (1962) notes that

$$\hat{p} \equiv \binom{x}{k} \left(\frac{1}{n}\right)^k \left(1 - \frac{1}{n}\right)^{x-k}$$

is a UMVUE for  $p(k; \lambda) \equiv \lambda^k e^{-k\lambda} / k!$  and then obtains

$\text{Var}(\hat{p})$  and the UMVUE of  $V(\hat{p})$ .

A similar result is obtained for the exponential distribution  
 $p \equiv P\{X > t\} = e^{-t\lambda}$ . The estimator is

$$\hat{p} = \left(1 - \frac{t}{n}\right)^x, \quad \text{Var}(\hat{p}) = e^{-t\lambda} (e^{\lambda t^2/n} - 1),$$

and the UMVUE of  $V(\hat{p})$  is

$$\left(1 - \frac{t}{n}\right)^{2x} - \left(1 - \frac{2t}{n}\right)^x.$$

Zacks and Even (1966) proceed as follows. For several families they consider the UMVUE and MLE of the reliability. We denote these by  $R_{MVU}$  and  $R_{MLE}$ . They obtain an exact or approximate expression for the  $\text{Var}(R_{MVU})$ ,  $\text{Var}(R_{MLE})$ , and the Cramér-Rao lower bound. The efficiency of either estimator is then a ratio of the mean squared error to the lower bound. Finally, the ratio of MSE of the two estimators is also plotted.

The qualitative results that prevail are as follows:

Exponential distribution,  $R(t;\theta) = e^{-t/\theta}$ .

MLE better for moderate values of  $t/\theta$ ,

UMVUE better for small and large values of  $t/\theta$ .

Normal distribution,  $R(t;\theta) = \Phi\left(\frac{t-\mu}{\sigma}\right)$  .

MLE better in central range of  $(t-\mu)/\sigma$  ,

UMVUE better in tails.

Poisson distribution,  $R(t;\theta) = P\{X=0\} = e^{-\lambda}$  .

MLE better for small values of  $t\theta/n$

UMVUE better for large values of  $t\theta/n$  .



## References

1. Barton, D. E. (1961), Unbiased Estimation of a Set of Probabilities, Biometrika 48, 227-229.
2. Basu, A. P. (1964), Estimates of Reliability for Some Distributions Useful in Life Testing, Technometrics, 6, 215-219.
3. Bhattacharya, S. K. (1967), Bayesian Approach to Life Testing and Reliability Estimation, J. Amer. Statist. Assoc. 62, 48-62.
4. Bickel, P. J. and E. L. Lehmann (1969), Unbiased Estimation in Convex Families, Ann. Math. Statist., 40, 1523-1536.
5. Church, J. D. and B. Harris (1970), The Estimation of Reliability from Stress-strength Relationships, Technometrics 12, 49-54.
6. Conolly, B. W. (1955), Unbiased Premiums for Stop-loss Reinsurance, Skand.Aktuarietidskrift 38, 127-134.
7. Diveev, R. Kh. (1964), Unbiased Estimates for Some Classes of Distributions, in collection: Theory of Probability and Mathematical Statistics, 1, 68-76, Izd-vo "Nauka", Tashkent. (Russian).
8. Downton, F. (1973), The Estimation of  $\Pr(Y < X)$  in the Normal Case, Technometrics, 15, 551-558.
9. Eaton, M. L. and C. N. Morris (1970), The Application of Invariance to Unbiased Estimation, Ann. Math. Statist. 41, 1708-1716.
10. Ellison, B. E. (1964), Two Theorems for Inferences About the Normal Distribution with Applications in Acceptance Sampling, J. Amer. Statist. Assoc., 59, 84-96.
11. El Mawaziny, A. H. and R. J. Buehler (1967), Confidence Limits for the Reliability of Series Systems, J. Amer. Statist. Assoc., 62, 1452-1459.
12. El-Sayyed, G. M. (1967), Estimation of the Parameter at an Exponential Distribution, J. Roy Statist. Soc., B, 29, 525-531.
13. Folks, J. L., D. A. Pierce and C. Stewart (1965), Estimating the Fraction of Acceptable Product, Technometrics, 7, 43-50.
14. Ghurye, S. G. and I. Olkin (1969), Unbiased Estimation of Some Multivariate Probability Densities and Related Functions, Ann. Math. Statist., 40, 1261-1271.

15. Glasser, G. J. (1962), Minimum Variance Unbiased Estimations for Poisson Probabilities, Technometrics, 4, 409-418.
16. Gray, H. L., T. A. Watkins and W. R. Schucany (1973), On the Jack-knife Statistic and its Relation to UMVU Estimators in the Normal Case, Comm. in Statist., 2, 285-320.
17. Halperin, M. (1952), Maximum Likelihood Estimation in Truncated Samples, Ann. Math. Statist., 23, 226-238.
18. Healy, W. C., Jr. (1956), Unbiased Estimation of the Normal Distribution Function, (abstract), Ann. Math. Statist., 27, 873.
19. Holla, M. S. (1967), Reliability Estimation of the Truncated Exponential Model, Technometrics, 9, 332-335.
20. Johnson, N. L. (1968), Letter to the Editor, Technometrics, 10, 429.
21. Kolmogorov, A. N. (1950), Unbiased Estimates, Izv. Akad. Nauk SSSR, Ser. Mat. 14, 303-326. Amer. Math. Soc. Trans. Series 1, Vol. 11, (1962) 144-170.
22. Kordonskiĭ, H. B. and P. Ya Rozenblit (1976), Unbiased Estimation of Moment Polynomials, Teor. Veroyatnost. i Primenen 21, 203-209. (English Translation: Theor. Probability Appl. 21, 199-204.)
23. Laurent, A. G. (1963), Conditional Distribution of Order Statistics and Distribution of the Reduced  $i$ th Order Statistic of the Exponential Model, Ann. Math. Statist., 34, 652-657.
24. Lentner, M. M. and R. J. Buehler (1963), Some Inferences About Gamma Parameters with an Application to a Reliability Problem, J. Amer. Statist. Assoc., 58, 670-677.
25. Lieberman, G. J. and G. J. Resnikoff (1955), Sampling Plans for Inspection by Variables, J. Amer. Statist. Assoc., 50, 457-516.
26. Likes, J. (1972), Estimation of Probabilities,  $\Pr(a < x < b)$  for a Pareto Distribution, Statist. H., 13, 291-297.
27. Lipow, M. and R. L. Eidemiller (1964), Application of the Bivariate Normal Distribution to a Stress vs Strength Problem in Reliability Analysis, Technometrics, 6, 325-328.
28. Lumel'skiĭ, Ya.P. (1969), Unbiased Estimates for Shorter Plans for Statistical Control of the Sequential Analysis Type, Proceedings of the First Symposium on Statistical Problems in Technical Cybernetics, Vol. 3, "Adaptive Systems, Large Systems" Izd-vo "Nauka", Moscow (Russian).

29. Lumel'skii, Ya.P. and P. N. Sapozhnikov (1969), Unbiased Estimation of Density Functions, Theor. Probability Appl., 14, 357-364.
30. Madansky, A. (1965), Approximate Confidence Limits for the Reliability of Series and Parallel Systems, Technometrics, 7, 495-503.
31. Neyman, J. and E. L. Scott (1960), Correction for Bias Introduced by a Transformation of Variables, Ann. Math. Statist., 31, 643-655.
32. Patil, G. P. (1963), Minimum Variance Unbiased Estimation and Certain Problems of Additive Number Theory, Ann. Math. Statist., 34, 1050-1056.
33. Patil, G. P. and J. K. Wani (1966), Minimum Variance Unbiased Estimation of the Distribution Function Admitting a Sufficient Statistic., Ann. Inst. Statist. Math., 18, 39-47.
34. Pugh, E. L. (1963), The Best Estimate of Reliability in the Exponential Case, Operations Research, 11, 57-61.
35. Roy, J. and S. Mitra (1957), Unbiased Minimum Variance Unbiased Estimation on a Class of Discrete Distribution, Sankhya, 18, 371-378.
36. Rutemiller, H. C. (1966), Point Estimation of Reliability of a System Comprised of k Elements from the same Exponential Distribution, J. Amer. Statist. Assoc., (61), 1029-1032.
37. Sapozhnikov, P. N. (1968), Unbiased Estimates of Distribution Laws of Integer-valued Random Variables, Perm. Gos. Ped. Inst. U čen. Zap., 61, 82-90.
38. Sathe, Y. S. and S. D. Varde (1969a), On Minimum Variance Unbiased Estimation of Reliability, Ann. Math. Statist., 40, 710-714.
39. Sathe, Y. S. and S. D. Varde (1969b), Minimum Variance Unbiased Estimation of Reliability for the Truncated Exponential Distribution, Technometrics, 11, 609-612.
40. Schmetterer, L. (1960), On a Problem of J. Neyman and E. Scott, Ann. Math. Statist., 31, 656-661.
41. Seheult, A. H. and C. P. Quesenberry, On Unbiased Estimation of Density Functions, Ann. Math. Statist., 42, 1434-1438.
42. Tate, R. F. (1959), Unbiased Estimation: Functions of Location and Scale Parameters, Ann. Math. Statist., 30, 341-366.



43. Tracy, D. S. and B. C. Gupta (1971), A New System of Unbiased Estimates for Moments and Their Products, Bull. Intern. Statist. Inst., 44, 387-391.
44. Vajda, S. (1951), Analytical Studies in Stop-loss Reinsurance, Skand. Aktuarietidskrift, 34, 158-175.
45. Vajda, S. (1955), Analytical Studies in Stop-loss Reinsurance II, Skand. Aktuarietidskrift, 38, 180-191.
46. Varde, S. D. (1969), Life Testing and Reliability Estimation for the Two Parameter Exponential Distribution, J. Amer. Statist. Assoc., 64, 621-631.
47. Washio, Y., H. Morimoto and N. Ikeda (1956), Unbiased Estimation Based on Sufficient Statistics, Bull. Math. Statist., 6, 69-94.
48. Watkins, T. A. and D. M. Kern (1973), On UMVU Estimators Related to the Multivariate Normal Distribution, Comm. in Statist., 2, 321-326.
49. Woodward, W. A. and H. L. Gray (1975), Minimum Variance Unbiased Estimation in the Gamma Distribution, Comm. in Statist., 4, 907-922.
50. Zacks, S. and M. Even (1966a), The Efficiencies in Small Samples of the Maximum Likelihood and Best Unbiased Estimations of Reliability Functions, J. Amer. Statist. Assoc., 61, 1033-1051.
51. Zacks, S. and M. Even (1966b), Minimum Variance Unbiased and Maximum Likelihood Estimators of Reliability Functions for Systems in Series and in Parallel, J. Amer. Statist. Assoc., 61, 1952-1062.

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