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INVARIANTS AND CANONICAL FORMS UNDER FEEDBACK.(U)
NOV 78 P L FALB, W A WOLOVICH AFOSR-77-3182
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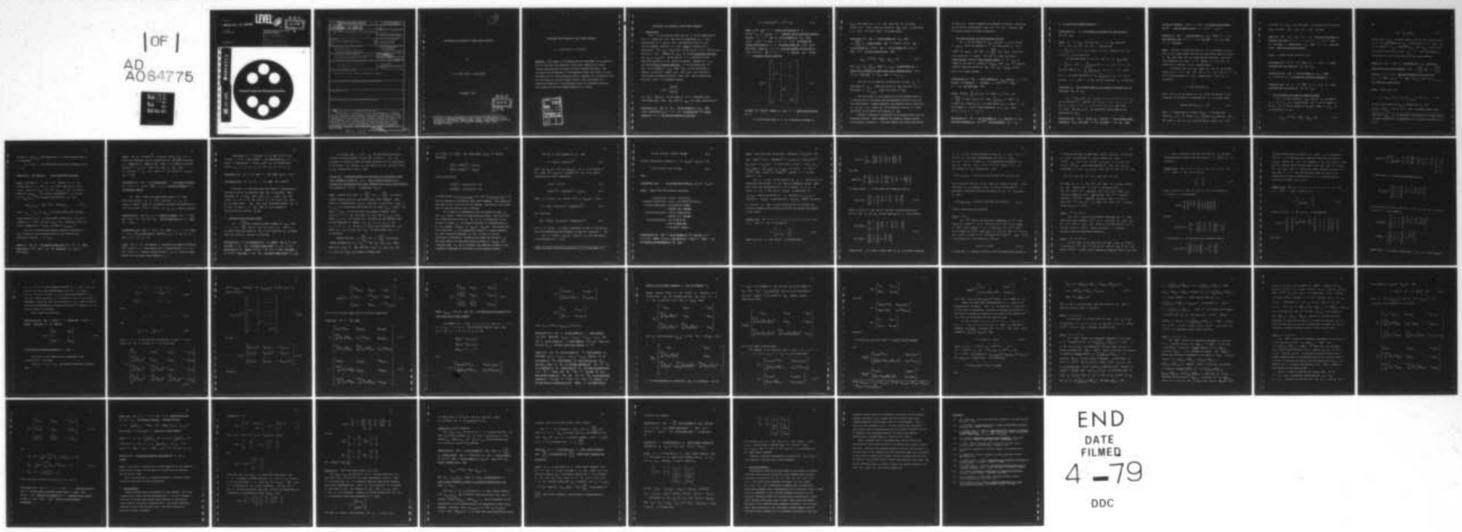

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19 REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: 18 AFOSR-TR-79-0075 · 2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER: 9
4. TITLE (and Subtitle): 6 INVARIANTS AND CANONICAL FORMS UNDER FEEDBACK	5. TYPE OF REPORT & PERIOD COVERED: Interim rept.
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): 10 P.L. Falb and W.A. Wolovich	8. CONTRACT OR GRANT NUMBER(s): 15 AFOSR-77-3182
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Brown University, Division of Applied Mathematics, Providence, Rhode Island 02912	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F 16 2304 A1 17 A1
11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Office of Scientific Research/NM, Bolling AFB, Washington, DC 20332	12. REPORT DATE: 11 November 1978
	13. NUMBER OF PAGES: 48 12 50p.
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): 14 LCDS-78-2	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper is concerned with the development of a complete set of invariants and canonical forms under feedback for linear systems characterized by proper rational transfer matrices. The invariants are determined in the frequency domain and consist of the Kronecker set of controllability indices together with a canonical form for the numerator of the transfer matrix under the action of a stabilizer subgroup of the unidoular group of polynomial matrices. Techniques used are algebro-geometric in nature.

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INVARIANTS AND CANONICAL FORMS UNDER FEEDBACK

by

P. L. Falb+ and W. A. Wolovich+

November, 1978

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INVARIANTS AND CANONICAL FORMS UNDER FEEDBACK

P. L. Falb and W. A. Wolovich

Abstract: This paper is concerned with the development of a complete set of invariants and canonical forms under feedback for linear systems characterized by proper rational transfer matrices. The invariants are determined in the frequency domain and consist of the Kronecker set of controllability indices together with a canonical form for the numerator of the transfer matrix under the action of a stabilizer subgroup of the unimodular group of polynomial matrices. The techniques used are algebro-geometric in nature.

INVARIANTS AND CANONICAL FORMS UNDER FEEDBACK

1. Introduction

Let k be an infinite field and let x be an indeterminate over k. Denote by $k[x]$ the ring of polynomials in x with coefficients in k and by $k(x)$ the quotient field of $k[x]$. Call an element $n(x)/d(x)$ of $k(x)$ proper if degree $n(x) \leq$ degree $d(x)$. Let $M_{p,m} = M_{p,m}(k[x])$ be the set of $p \times m$ matrices with entries in $k[x]$. Elements of $M_{p,m}$ are called polynomial matrices. Let $\Sigma_{p,m} = \Sigma_{p,m}(k(x))$ be the set of $p \times m$ matrices of full rank with entries in $k(x)$ which are proper. Elements of $\Sigma_{p,m}$ are called proper transfer matrices. It is well-known that if $T(x)$ is an element of $\Sigma_{p,m}$, then $T(x)$ can be factored as a product $R_T(x)P_T^{-1}(x)$ where $R_T(x)$ and $P_T(x)$ are relatively right prime polynomial matrices. Thus, $T(x)$ can be identified with the $(m+p) \times m$ polynomial matrix

$$\sigma_T(x) = \begin{bmatrix} R_T(x) \\ P_T(x) \end{bmatrix}.$$

Let $\mathscr{U}_m = \mathscr{U}_m(k[x])$ be the group of $m \times m$ unimodular polynomial matrices. Then, $\mathscr{U}_m$ acts on $\Sigma_{p,m}$ via right multiplication.

Definition 1.1. Let $M = (m_{ij})$ be an element of $M_{q,r}$. Then $\partial_j(M) = \max\{\text{degree } m_{ij} \mid i = 1,\ldots,q\}$ is called the j-th column degree of M. M can thus be written in the form

$$M = \Delta_c(M)\,\mathrm{diag}[x^{\partial_1},\ldots,x^{\partial_r}] + M_1 \qquad (1.2)$$

where $\Delta_c(M)$ is a $q \times r$ matrix with entries in k, $\mathrm{diag}[x^{\partial_1},\ldots,x^{\partial_r}]$ is an $r \times r$ diagonal matrix with main diagonal entries $x^{\partial_1},\ldots,x^{\partial_r}$ and M_1 is an element of $M_{q,r}$ with $\partial_j(M_1) < \partial_j(M) = \partial_j$ for $j = 1,\ldots,r$. $\Delta_c(M)$ is called the column coefficient of M. M is column proper if $\Delta_c(M)$ is of full rank. Thus, if $q = r$, M is column proper if and only if $\Delta_c(M) \in GL(k,r)$. Let $n = \partial_1 + \cdots + \partial_r$ and let $S_M(x)$ be the $n \times r$ polynomial matrix given by

$$S_M(x) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ x & 0 & \cdots & 0 \\ \vdots & \vdots & & \\ x^{\partial_1 - 1} & 0 & \cdots & 0 \\ 0 & 1 & & \\ \cdot & x & & \cdot \\ \cdot & \vdots & & \cdot \\ \cdot & x^{\partial_2 - 1} & & \cdot \\ \cdot & 0 & & \cdot \\ \cdot & \vdots & & \\ 0 & 0 & & x^{\partial_m - 1} \end{bmatrix} \qquad (1.3)$$

so that $M_1 = F_M S_M(x)$ where F_M is a $q \times n$ matrix with entries in k.

It is well-known that if P is a nonsingular element of

$M_{m,m}$, then there is a U in $\mathcal{U}_m$ such that PU is column proper ([1]). Thus, under the action of $\mathcal{U}_m$, $\sigma_T(x)$ is equivalent to a $\sigma_T^*(x)$ for which $P_T^*(x)$ is column proper.

Definition 1.4. Let T be an element of $\Sigma_{p,m}$ with $\sigma_T = \begin{bmatrix} R_T \\ P_T \end{bmatrix}$, P_T column proper. Let n = degree det P_T. Let G be an element of GL(k,m) and F be an element of $M_{n,m}(k)$. Call (F,G) a state feedback pair. Set

$$P_{T_{F,G}} = G^{-1}\{P_T - FS_{P_T}\}, \quad R_{T_{F,G}} = R_T \qquad (1.5)$$

and $T_{F,G} = R_{T_{F,G}} P_{T_{F,G}}^{-1}$. Then $T_1 \in \Sigma_{p,m}$ is equivalent to T *under state feedback if there exist state feedback pairs* (F,G), (F_1,G_1) such that $T_1 = T_{F,G}$ and $T = T_{1\,F_1,G_1}$.

Note that it is implicit in Definition 1.4 that σ_{T_1} is equivalent to $\sigma_{T_{F,G}}$ under the action of $\mathcal{U}_m$ and that σ_T is equivalent to $\sigma_{T_{1\,F_1,G_1}}$ under the action of $\mathcal{U}_m$.

The main result of this paper will be the determination of a complete set of invariants and corresponding canonical form for this equivalence. Loosely speaking, the complete set of invariants is $(R_c, \partial_1, \ldots, \partial_m)$ where R_c is a canonical form for R under the action of an appropriate subgroup of $\mathcal{U}_m$.

Section 2 contains a discussion of the system module and the Kronecker indices. State feedback and properly indexed systems are analyzed in section 3. The main results are stated and proved in section 4. Several examples are examined in section 5 including the so-called "controllable" case ([2], [3], [4]). Finally, some concluding remarks are made in section 6.

2. The System Module and the Kronecker Indices

Let T be an element of $\Sigma_{p,m}$ and let σ_T be an element of $M_{p+m,m}$ which corresponds to T. In other words, σ_T is an element of $M_{p+m,m}$ such that $\sigma_T = \begin{bmatrix} R_T \\ P_T \end{bmatrix}$ with R_T, P_T relatively right prime and $T = R_T P_T^{-1}$. Any such σ_T shall be called a linear system (minimal) with transfer matrix T. If $\Sigma_{p,m}$ is viewed in this way as a subset of $M_{p+m,m}$, then $\Sigma_{p,m}$ is invariant (stable) under the action of $\mathcal{U}_m$. Let $S_{p,m} \subset M_{p+m,m}$ be the set of all linear systems.

Proposition 2.1. *Let σ be an element of $S_{p,m}$ and let $\sigma_j = \sigma_j(x)$ be the j-th column of σ (so that $\sigma_j \in (k[x])^{p+m}$). Then $\sigma_1, \ldots, \sigma_m$ are free over $k[x]$.*

Proof: Suppose $\sum_{j=1}^{m} \psi_j(x)\sigma_j(x) = 0$ where $\psi_j \in k[x]$. Let $\sigma = \begin{bmatrix} R_\sigma \\ P_\sigma \end{bmatrix}$ so that $\det P_\sigma \neq 0$. Then $\sum_{j=1}^{m} \psi_j P_{\sigma_j} = 0$ where P_{σ_j} is the j-th column of P_σ. In other words, $P_\sigma \psi = 0$ where ψ is the element of $(k[x])^m$ with components $\psi_1, \ldots, \psi_m$. Since $\det P_\sigma \neq 0$, $\psi_1 = \cdots = \psi_m = 0$.

Definition 2.2. *Let σ be an element of $S_{p,m}$ and let M_σ be the free submodule of $(k[x])^{p+m}$ with generators $\sigma_1, \ldots, \sigma_m$.*

M_σ is called the system module of σ.

Proposition 2.3. M_σ is a complete invariant for the action of $\mathcal{U}_m$ on $S_{p,m}$.

Proof: If $\sigma,\tau \in S_{p,m}$ and there is a U in $\mathcal{U}_m$ such that $\sigma U = \tau$, then $M_\sigma = M_\tau$. For, $\sigma U = \tau$ implies that $\tau_j = \sum_{j=1}^{m} u_{j\ell}\sigma_\ell$, $j = 1,\ldots,m$, so that $M_\tau \subset M_\sigma$. Similarly, $M_\sigma \subset M_\tau$ and so M_σ is an invariant.

On the other hand, if $M_\sigma = M_\tau$ for σ,τ in $S_{p,m}$, then $\sigma_j = \sum_{j=1}^{m} u_{j\ell}\tau_\ell$ and $\tau_\ell = \sum_{r=1}^{m} v_{\ell r}\sigma_r$ for $j = 1,\ldots,m$, $\ell = 1,\ldots,m$. But this implies $\sigma_j = \sum_{\ell=1}^{m}\sum_{r=1}^{m} u_{j\ell}v_{\ell r}\sigma_r$ for $j = 1,\ldots,m$. Since the σ_j are free generators of M_σ, $\sum_{\ell=1}^{m} u_{j\ell}v_{\ell r} = \delta_{jr}$, i.e. $UV = I$ so that $\tau U = \sigma$ with U in $\mathcal{U}_m$ and the invariant M_σ is complete.

Corollary 2.4. The transfer matrix is a complete invariant for the action of $\mathcal{U}_m$ on $S_{p,m}$.

If $\sigma \in S_{p,m}$, let $\sigma_{ij} = \sigma_{ij}(x)$, $i = 1,\ldots,p + m$, $j = 1,\ldots,m$ be the entries in σ and let $R_\sigma = (\sigma_{ij})$, $i = 1,\ldots,p$, $j = 1,\ldots,m$ and $P_\sigma = (\sigma_{ij})$, $i = p + 1,\ldots,p + m$, $j = 1,\ldots,m$. Note that $T_\sigma = R_\sigma P_\sigma^{-1}$ and that R_σ, P_σ are relatively right prime by the definition of $S_{p,m}$.

Definition 2.5. Let σ be in $S_{p,m}$ and let P be a column proper element of $M_{m,m}$ such that $P = P_\sigma U$ for some U in $\mathcal{U}_m$. Then

<u>the set of integers</u> $\{\partial_1(P),\ldots,\partial_m(P)\}$ <u>is called the Kronecker set of</u> σ <u>and is often written</u> $\partial_\sigma = \{\partial_1(\sigma),\ldots,\partial_m(\sigma)\}$.

<u>Theorem 2.6</u>. <u>Let</u> σ <u>be an element of</u> $S_{p,m}$. <u>Then</u> (i) ∂_σ <u>is well-defined; and</u>, (ii) if $\tau = \sigma U$ <u>for some</u> U <u>in</u> $\mathscr{U}_m$, <u>then</u> $\partial_\tau = \partial_\sigma$ (<u>as sets</u>).

<u>Proof</u>: Clearly, in order to prove (i), it is sufficient to show that if two column proper matrices are equivalent under the action of $\mathscr{U}_m$, then their sets of column degrees coincide. This will also establish (ii) since $\tau = \sigma U$ implies $P_\tau = P_\sigma U$ and $P_\tau V$ column proper implies $P_\sigma(UV)$ column proper.

So suppose that $P_1 = P_2 U$ where P_1 and P_2 are column proper and U is in $\mathscr{U}_m$. Let $\{\partial_j^1 \mid j = 1,\ldots,m\}$, $\{\partial_j^2 \mid j = 1,\ldots,m\}$ be the column degrees of P_1, P_2 respectively. Since P_2 is invertible,

$$U = [\mathrm{Adj}\ P_2]P_1/\det P_2 \tag{2.7}$$

where $\mathrm{Adj}\ P_2$ is the adjoint of P_2. Since the adjoint is the transpose of the matrix of cofactors and P_2 is column proper,

$$\text{degree } [\mathrm{Adj}\ P_2]_{ij} \leq n - \partial_i^2 \tag{2.8}$$

where $n = \text{degree} \det P_2 = \text{degree} \det P_1$. It follows that degree $u_{ij} \leq \partial_j^1 - \partial_i^2$ (since $u_{ij} = \sum_{r=1}^{m} [\mathrm{Adj}\ P_2]_{ir} p_{rj}/\det P_2$). But, for fixed j, not all u_{ij} are zero and so, there is an $i(j)$

such that $\partial^1_j \geq \partial^2_{i(j)}$ and vice-versa. By virtue of the following lemma, the sets $\{\partial^1_1,\ldots,\partial^1_m\}$, $\{\partial^2_1,\ldots,\partial^2_m\}$ coincide.

Lemma 2.9. Let $\{\partial_1,\ldots,\partial_m\}$, $\{\varepsilon_1,\ldots,\varepsilon_m\}$ be sets of nonnegative integers such that (i) $\partial_1 + \ldots + \partial_m = \varepsilon_1 + \ldots + \varepsilon_m = n$, and (ii) for each ∂_j, there is an $\varepsilon_{i(j)}$ with $\partial_j \geq \varepsilon_{i(j)}$ and vice-versa. Then the two sets coincide.

Proof: A simple double induction ([5]).

Corollary 2.10. If $M_\sigma = M_\tau$, then $\partial_\sigma = \partial_\tau$ (i.e., ∂_σ is an invariant for the action of $\mathcal{U}_m$ on $S_{p,m}$).

Definition 2.11. Let σ be an element of $S_{p,m}$. Then n_σ = degree det P_σ is called the (McMillan) degree of σ.

Corollary 2.12. If $M_\sigma = M_\tau$, then $n_\sigma = n_\tau$ (i.e., n_σ is an invariant for the action of $\mathcal{U}_m$ on $S_{p,m}$).

3. State Feedback and Properly Indexed Systems

Let T be an element of $\Sigma_{p,m}$ and let σ_T correspond to T with $P_{\sigma_T} = P_T$ column proper. Let G be an element of $GL(k,m)$ and F be an element of $M_{n,m}(k)$ where $n = n_{\sigma_T}$. As in Definition 1.4, set

$$P_{T_{F,G}} = G^{-1}\{P_T - FS_{P_T}\}, \quad R_{T_{F,G}} = R_T \tag{3.1}$$

and

$$T_{F,G} = R_{T_{F,G}} P^{-1}_{T_{F,G}} . \qquad (3.2)$$

Observe that $P_{T_{F,G}}$ is column proper with the same column degrees as P_T and that degree det $P_{T_{F,G}}$ = degree det $P_T = n$. Thus, $T_{F,G}$ is an element of $\Sigma_{p,m}$. However, $R_{T_{F,G}}$ and $P_{T_{F,G}}$ need not be relatively right prime so that $n_{T_{F,G}} \le n$.+ This corresponds to the potential loss of observability under state feedback.

__Lemma 3.3__. __Let__ T __and__ T_1 __be elements of__ $\Sigma_{p,m}$ __which are equivalent under state feedback__. __Let__ $\sigma = \begin{bmatrix} R_T \\ P_T \end{bmatrix}$ __and__ $\sigma_1 = \begin{bmatrix} R_{T_1} \\ P_{T_1} \end{bmatrix}$ __and let__ (F,G), (F_1,G_1) __be state feedback pairs such that__ $T_{F,G} = T_1$ __and__ $T_{1\,F_1,G_1} = T$. __Then__ $R_T, P_{T_{F,G}}$ __and__ $R_{T_1}, P_{T_1\,F_1,G_1}$ __are relatively right prime__.

__Proof__: Simply note that

$$n_\sigma \geq n_{T_{F,G}} = n_{T_1} = n_{\sigma_1} \geq n_{T_1\,F_1,G_1} = n_T = n_\sigma \qquad (3.4)$$

so that (say) degree det $P_{T_{F,G}}$ = degree det P_{T_1}. But $R_{T_1} P^{-1}_{T_1} = R_T P^{-1}_{T_{F,G}}$ and so, if D were a greatest common right

+Note that $n_{T_{F,G}}$ is, by definition, the degree of a linear system (minimal realization) corresponding to the proper transfer matrix $T_{F,G}$.

divisor of $R_T, P_{T_{F,G}}$, then degree det D = 0 which implies that D is unimodular.

If T and T_1 are equivalent under state feedback, write $T \sim T_1$.

Theorem 3.5. The relation $\sim$ is an equivalence relation.

Proof: Obviously, $T \sim T$ and $T \sim T_1$ implies $T_1 \sim T$. So suppose that $T \sim T_1$ and $T_1 \sim T_2$. Then there exist (F,G), (F_1,G_1) such that $T_{F,G} = T_1$ and $T_{1\,F_1,G_1} = T_2$. In view of Lemma 3.3, $R_T, P_{T_{F,G}}$ is a minimal realization of T_1 and so, $R_T, (P_{T_{F,G}})_{F_1,G_1}$ is a minimal realization of T_2. But

$$(P_{T_{F,G}})_{F_1,G_1} = (GG_1)^{-1}\{P_T - (F+GF_1)S_{P_T}\} \qquad (3.6)$$

since $S_{P_{T_{F,G}}} = S_{P_T}$ as $P_{T_{F,G}}$ is column proper with the same column degrees as P_T. In other words, $(F+GF_1, GG_1)$ is a state feedback pair $(\hat{F},\hat{G})$ for which $T_{\hat{F},\hat{G}} = T_2$. Similarly, there is a state feedback pair $(\hat{F}_2,\hat{G}_2)$ for which $T_{2\,\hat{F}_2,\hat{G}_2} = T$.

The goal of determining complete feedback invariants is, thus, reduced to the characterization of the orbits of the equivalence relation $\sim$ in $S_{p,m}$.

Lemma 3.7. Let P_1, P be column proper with $\partial P_1 = \partial P = \partial$. Then there exist (F,G) and U in $\mathcal{U}_m$ such that $P_1 = P_{F,G}U = G^{-1}\{P - FS_P\}U$.

Proof: Let $P_1 = M\,\mathrm{diag}[x^{\partial_i^1}] + N\,S_{P_1}(x)$. Since $\partial P_1 = \partial P = \partial$, there are elementary row and column matrices E_r^m, E_r^n, E_c^m such that $P_1 = \{ME_r^m \mathrm{diag}[x^{\partial_i}] + NE_r^n S_P(x)\}E_c^m$. Thus, it is enough to consider $M_1 \mathrm{diag}[x^{\partial_i}] + N_1\, S_P(x)$ where $M_1 = ME_r^m$, $N_1 = NE_r^n$. Take $G = \Delta_c(P)M_1^{-1}$, $F = [F_P - GN_1]$ where $P = \Delta_c(P)\mathrm{diag}[x^{\partial_i}] + F_P S_P(x)$ and $U = E_c^m$.

Definition 3.8. Let P be column proper. P is properly indexed if $\partial_1(P) \geq \cdots \geq \partial_m(P)$. Call $\sigma \in S_{p,m}$ properly indexed if P_σ is properly indexed.

Let $\mathcal{O}(T)$ denote the equivalence class of T under $\sim$. Then there exists a T_1 in $\mathcal{O}(T)$ such that $\sigma_1 = \sigma_{T_1}$ is properly indexed. Thus, it is enough to consider the characterization of the sets $\mathcal{O}^*(T_1) = \{T_2 | T_2 \sim T_1, \sigma_{T_2}, \sigma_{T_1}$ properly indexed$\}$.

Definition 3.9. Let $W_\partial = \{P|\ P$ properly indexed, $\partial(P) = \partial\}$ and let $S(W_\partial) = \{U \in \mathcal{U}_m |\ W_\partial U = W_\partial\}$ be the stabilizer of W_∂. Write $\mathcal{U}_\partial = S(W_\partial)$.

Proposition 3.10. Let $\partial = \{\partial_1, \ldots, \partial_m\}$ with $\partial_1 \geq \cdots \geq \partial_m$. Then $U = (u_{ij}) \in \mathcal{U}_\partial$ if and only if degree $u_{ij} \leq \partial_j - \partial_i$ if $\partial_j \geq \partial_i$ and $u_{ij} = 0$ if $\partial_j < \partial_i$.

Proof: Let $P \in W_\partial$ and suppose U satisfies the degree conditions. Then $\partial_i(PU) \leq \partial_i$. Since degree det P = degree det$(PU) = n = \Sigma\partial_i$ and $\partial_i(PU) \leq \partial_i$ implies degree det$(PU) \leq \Sigma\partial_i$, PU must be column proper with the same column degrees as P.

Conversely, if $U \in \mathcal{U}_\partial$ and $P \in W_\partial$, then $PU = \hat{P} \in W_\partial$ so that $U = P^{-1}\hat{P} = [\text{Adj } P]\hat{P}/\det P$. But degree$[\text{Adj } P]_{ij} \leq n - \partial_i$ where n = degree det P. Since $\partial_i(\hat{P}) = \partial_i$, it follows that degree $u_{ij} \leq \partial_j - \partial_i$ if $\partial_j \geq \partial_i$ and $u_{ij} = 0$ if $\partial_j < \partial_i$.

Corollary 3.11. If $P \in W_\partial$ and $U \in \mathcal{U}_\partial$, then $S_{PU}(x) = S_P(x)$.

Corolllary 3.12. If $\partial_1 = \partial_2 \ldots = \partial_m$, then $\mathcal{U}_\partial = GL(k,m)$.

Corollary 3.12 indicates that the prospect of determining a canonical form for the quotient under $\mathcal{U}_\partial$ is favorable. More precisely, if $R \in M_{p,m}$ with $\partial_1(R) \leq \partial_1, \ldots, \partial_m(R) \leq \partial_m$, then $\partial_i(RU) \leq \partial_i$ for U in $\mathcal{U}_\partial$ and if $X_\partial = \{R \mid \partial_i(R) \leq \partial_i\}$, then X_∂ is stable under $\mathcal{U}_\partial$. Call R, R_1 equivalent modulo $\mathcal{U}_\partial$ if $R = R_1U$ for some U in $\mathcal{U}_\partial$. Then it is of interest to characterize the quotient $X_\partial / \mathcal{U}_\partial$.

4. Invariants and Canonical Forms

Let $\sigma = \begin{bmatrix} R_\sigma \\ P_\sigma \end{bmatrix}$ be a properly indexed element of $S_{p,m}$ with $\partial = \partial_\sigma = \{\partial_1, \ldots, \partial_m\}$ and let $X_\partial = \{R \mid \partial_i(R) \leq \partial_i\}$. Then X_∂ is stable under the action of $\mathcal{U}_\partial$ on the right and X_∂ is stable under the action of $GL(k,p)$ on the left.

Definition 4.1. R is equivalent to R_1 modulo $\mathcal{U}_\partial$ if $R = R_1U$ for some U in $\mathcal{U}_\partial$. In such a case, write $R \sim_\partial R_1$. R is equivalent to R_1 modulo $GL(k,p) \times \mathcal{U}_\partial$ if $R = HR_1U$ for some H in $GL(k,p)$ and some U in $\mathcal{U}_\partial$. In such a case, write $R \sim_{\partial,p} R_1$.

It is clear that $\sim_\partial$ and $\sim_{\partial,p}$ are equivalence relations. Consider now the quotients $X_\partial/\mathcal{U}_\partial$ and $X_\partial/GL(k,p) \times \mathcal{U}_\partial$. The existence of canonical forms for these quotients will be established in the sequel. So, suppose, for the moment that such canonical forms $R_c, R_{c,p}$, respectively, exist. Then:

<u>Theorem 4.2.</u> <u>A complete system of invariants for equivalence under state feedback is given by</u> (R_c, ∂) <u>and a complete system of invariants for equivalence under state feedback and output transformations (i.e. action of</u> $GL(k,p)$) <u>is given by</u> $(R_{c,p}, \partial)$.

<u>Proof</u>: Suppose first that $\sigma_1 = \begin{bmatrix} R_1 \\ P_1 \end{bmatrix}$ and $\sigma_2 = \begin{bmatrix} R_2 \\ P_2 \end{bmatrix}$ are properly indexed systems which are equivalent under state feedback. Then, there is a feedback pair (F_1, G_1) such that (i) $R_1, G_1^{-1}\{P_1 - F_1 S_{P_1}\} = P_{1\ F_1,G_1}$ are relatively right prime and (ii) $R_1 P_{1\ F_1,G_1}^{-1} = R_2 P_2^{-1}$. But $G_1^{-1}\{P_1 - F_1 S_{P_1}\}$ is column proper and properly indexed. Thus, there is a U in $\mathcal{U}_m$ such that $R_1 = R_2 U$ and $P_{1\ F_1,G_1} = P_2 U$. Hence, $\partial_{\sigma_1} = \partial_{\sigma_2}$ as ordered sets. Moreover, by the argument used to prove Theorem 2.6, degree $u_{ij} \leq \partial_j - \partial_i$ if $\partial_j \geq \partial_i$ and $u_{ij} = 0$ if $\partial_j < \partial_i$ so that $U \in \mathcal{U}_\partial$ in view of Proposition 3.10. Hence, $R_1 \sim_\partial R_2$ so that $R_{1c} = R_{2c}$. In other words, (R_c, ∂) is an invariant.

Suppose now that $\sigma_1 = \begin{bmatrix} R_1 \\ P_1 \end{bmatrix}$ and $\sigma_2 = \begin{bmatrix} R_2 \\ P_2 \end{bmatrix}$ are properly indexed systems with $\partial_{\sigma_1} = \partial_{\sigma_2} = \partial$ and $R_{1c} = R_{2c} = R_c$. Then, there are U_1, U_2 in $\mathcal{U}_\partial$, such that $R_1 U_1 = R_c$, $R_2 U_2 = R_c$ and $\hat{P}_1 = P_1 U_1$, $P_2 = \hat{P}_2 U_2$ are properly indexed with

$T_1 = R_c\hat{P}_1^{-1}$, $T_2 = R_c\hat{P}_2^{-1}$. But, there exist G_1, G_2 in $GL(k,m)$ such that

$$G_1^{-1}\hat{P}_1 = \mathrm{diag}[x^{\partial_i}] + F_1 S_\partial(x)$$
$$G_2^{-1}\hat{P}_2 = \mathrm{diag}[x^{\partial_i}] + F_2 S_\partial(x)$$

and it follows that

$$G_2[G_1^{-1}\hat{P}_1 + (F_2-F_1)S_\partial(x)] = \hat{P}_2$$
$$G_1[G_2^{-1}\hat{P}_2 + (F_1-F_2)S_\partial(x)] = \hat{P}_1 \; .$$

In other words, $\hat{P}_1 \; G_1(F_1-F_2), G_1G_2^{-1} = \hat{P}_2$ and $\hat{P}_2 \; G_2(F_2-F_1), G_2G^{-1} = \hat{P}_1$ and the systems are equivalent under state feedback. This completes the proof of the first part of the theorem. The proof of the second part is entirely similar and is omitted.

So it remains to demonstrate that the canonical forms R_c and $R_{c,p}$ exist. There are three essential ideas. The first idea is to show that the action of $\mathcal{U}_\partial$ on X_∂ is equivalent to the action of a group Γ_∂ of constant matrices on a representation of X_∂ as a subset $\{(C_R, E_R)\}$ of $M_{p,n}(k) \times M_{p,m}(k)$. The second idea is to show that Γ_∂ is a semidirect product of a normal subgroup N_∂ and a reductive subgroup G_∂ ([7]) so that it will be sufficient to determine a canonical form under the action of N_∂. The third idea is to show that certain columns of (C_R, E_R) are invariant under N_∂ and to "project" the remaining columns on the orthogonal complement of the range of the invariant columns.

Now let R be an element of X_∂. Then

$$R = C_R S_\partial(x) + E_R \operatorname{diag}[x^{\partial_i}] \qquad (4.3)$$

where $C_R \varepsilon M_{p,n}(k)$ and $E_R \varepsilon M_{p,m}(k)$. If U is an element of $\mathcal{U}_\partial$, then $RU = C_R S_\partial(x)U + E_R \operatorname{diag}[x^{\partial_i}]U$ and, as is readily established by direct computation,

$$S_\partial(x)U = V_u S_\partial(x) \qquad (4.4)$$

$$\operatorname{diag}[x^{\partial_i}]U = W_u \operatorname{diag}[x^{\partial_i}] + \theta_u S_\partial(x) \qquad (4.5)$$

where $V_u \varepsilon GL(k,n)$, $W_u \varepsilon GL(k,m)$, and $\theta_u \varepsilon M_{m,n}(k)$.+ Thus,

$$RU = [C_R V_u + E_R \theta_u] S_\partial(x) + E_R W_u \operatorname{diag}[x^{\partial_i}] \qquad (4.6)$$

and, similarly,

$$HRU = H[C_R V_u + E_R \theta_u] S_\partial(x) + HE_R W_u \operatorname{diag}[x^{\partial_i}] \qquad (4.7)$$

for H in $GL(k,p)$. In effect, equations 4.6 and 4.7 provide the basis for determining the quotients $X_\partial / \mathcal{U}_\partial$ and $X_\partial / GL(k,p) \times \mathcal{U}_\partial$.

Let $\Gamma = GL(k,n) \times M_{m,n}(k) \times GL(k,m)$ as a set and define a multiplication in Γ via

+These relations follow from equations 4.27-4.30 and Lemma 4.31.

$$(V,\theta,W)\cdot(\tilde{V},\tilde{\theta},\tilde{W}) = (V\tilde{V},\theta\tilde{V} + W\tilde{\theta},W\tilde{W}). \tag{4.8}$$

Further, define the "action" of Γ on $M_{p,n}(k) \times M_{p,m}(k)$ via

$$(C,E)\cdot(V,\theta,W) = (CV + E\theta,EW). \tag{4.9}$$

Then:

<u>Proposition 4.10</u>. Γ <u>is a group which acts on</u> $M_{p,n}(k) \times M_{p,m}(k)$.

<u>Proof</u>: Simply note the following relations:

$$(I,0,I)(V,\theta,W) = (V,\theta,W) = (V,\theta,W)(I,0,I)$$

$$(V,\theta,W)(V^{-1},-W^{-1}\theta V^{-1},W^{-1}) = (I,0,I) = (V^{-1},-W^{-1}\theta V^{-1},W^{-1})(V,\theta,W)$$

$$(V,\theta,W)[(\tilde{V},\tilde{\theta},\tilde{W})(V,\hat{\theta},\hat{W})] = (V,\theta,W)[(\tilde{V}\hat{V},\tilde{\theta}\hat{V} + \tilde{W}\hat{\theta},\tilde{W}\hat{W})]$$

$$= (V\tilde{V}\hat{V},\theta\tilde{V}\hat{V} + W\tilde{\theta}\hat{V} + W\tilde{W}\hat{\theta},W\tilde{W}\hat{W})$$

$$[(V,\theta,W)(\tilde{V},\tilde{\theta},\tilde{W})](\hat{V},\hat{\theta},\hat{W}) = [(V\tilde{V},\theta\tilde{V} + W\tilde{\theta},W\tilde{W})](\hat{V},\hat{\theta},\hat{W})$$

$$= (V\tilde{V}\hat{V},\theta\tilde{V}\hat{V} + W\,\tilde{}\,\hat{V} + W\tilde{W}\hat{\theta},W\tilde{W}\hat{W})$$

$$[(C,E)(V,\theta,W)](\tilde{V},\tilde{\theta},\tilde{W}) = (CV + E\theta,EW)(\tilde{V},\tilde{\theta},\tilde{W})$$

$$= (CV\tilde{V} + E(\theta\tilde{V}+W\tilde{\theta}),EW\tilde{W})$$

$$= (C,E)[(V\tilde{V},\theta\tilde{V} + W\tilde{\theta},W\tilde{W})] \quad .$$

<u>Proposition 4.11</u>. <u>Let</u> U <u>be an element of</u> $\mathcal{U}_\partial$ <u>and let</u> $\psi(U) = (V_u,\theta_u,W_u)$ <u>where</u> V_u,W_u,θ_u <u>are given by</u> 4.4 <u>and</u> 4.5. <u>Then</u> ψ <u>is an injective homomorphism of</u> $\mathcal{U}_\partial$ <u>into</u> Γ.

Proof: First note that $S_\partial(x)(UU_1) = V_u S_\partial(x) U_1 = V_u V_{u_1} S_\partial(x)$ and that $\text{diag}[x^{\partial_i}](UU_1) = [W_u \text{diag}[x^{\partial_i}] + \theta_u S_\partial(x)]\cdot U_1 = W_u W_{u_1} \text{diag}[x^{\partial_i}] + W_u \theta_{u_1} S_\partial(x) + \theta_u V_{u_1} S_\partial(x)$. In other words, $\psi([UU_1]) = (V_u V_{u_1}, \theta_u V_{u_1} + W_u \theta_{u_1}, W_u W_{u_1}) = (V_u, \theta_u, W_u)(V_{u_1}, \theta_{u_1}, W_{u_1}) = \psi(U)\psi(U_1)$. If $\psi(U) = (I,0,I)$, then $\text{diag}[x^{\partial_i}]U = \text{diag}[x^{\partial_i}]$ implies $U = I$ so that kernel $\psi = \{I\}$.

Let $\Gamma_\partial = \psi(\mathcal{U}_\partial)$ be the image of $\mathcal{U}_\partial$ in Γ. Proposition 4.11 essentially states that Γ_∂ and $\mathcal{U}_\partial$ are isomorphic groups. Moreover, since the representation 4.3 of an element R in X_∂ is unique, $R \sim_\partial R_1$ if and only if (C_R, E_R) is equivalent to (C_{R_1}, E_{R_1}) modulo the action of Γ_∂. Similarly $R \sim_{\partial,p} R_1$ if and only if (C_R, E_R) is equivalent to (C_{R_1}, E_{R_1}) modulo the action of $GL(k,p) \times \Gamma_\partial$. Now, it will be instructive to consider the following examples which serve to motivate the general development of the sequel.

Example 4.12. Let $m = 2$, $n = 3$, $\partial_1 = 2$, $\partial_2 = 1$, $\partial = \{2,1\}$, and $p \geq 1$. Then $U \in \mathcal{U}_\partial$ if and only if

$$U = \begin{bmatrix} a & 0 \\ b + cx & d \end{bmatrix} \qquad (4.13)$$

where $a,b,c,d \in k$ and $ad \neq 0$. It follows that

$$S_{\partial}(x)U = \begin{bmatrix} a & 0 \\ b+cx & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ x & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ b & c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ x & 0 \\ 0 & 1 \end{bmatrix}$$

and that

$$\mathrm{diag}[x^{\partial_i}]U = \begin{bmatrix} x^2 & 0 \\ 0 & x \end{bmatrix} \begin{bmatrix} a & 0 \\ b+cx & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \begin{bmatrix} x^2 & 0 \\ 0 & x \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & b & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ x & 0 \\ 0 & 1 \end{bmatrix} .$$

In other words, Γ_{∂} is the group with elements given by

$$(V_u, \theta_u, W_u) = \left(\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ b & c & d \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & b & 0 \end{bmatrix}, \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \right) \qquad (4.14)$$

where $a,b,c,d \in k$ and $ad \neq 0$ and with multiplication given by 4.8. Let N_{∂} and G_{∂} be the subgroups of Γ_{∂} with elements

$$(V_u, \theta_N, W_N) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & c & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & b & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \right)$$

$$(V_G, 0, W_G) = \left(\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & d \end{bmatrix}, 0, \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \right) \qquad (4.15)$$

respectively. It is easy to check that (i) N_{∂} is a normal subgroup of Γ_∂, (ii) N_∂ is the unipotent radical of Γ_∂ ([7]), (iii) G_∂ acts on N_∂ via inner automorphisms, and (iv) $\Gamma_\partial = N_\partial G_\partial$, $N_\partial \cap G_\partial = (I,0,I)$ so that Γ_∂ is a semi-direct product of N_∂ and G_∂. Now let $(C,E) = (C^1\ C^2\ C^3\ E^1\ E^2)$ where C^i, E^j are the columns of C,E respectively. Then

$$(C,E)(V_u,\theta_u,W_u) = (a(C^1+bC^3)\ a(C^2+cC^3+bE^2)\ dC^3\ a(E^1+cE^2)\ dE^2)$$

and a canonical form for (C,E) under this action is sought. Since the group G_∂ is reductive ([7]), it will be sufficient to determine a canonical form for the action of N_∂. If $(V_N,\theta_N,W_N) \in N_\partial$, then

$$(C,E)(V_N,\theta_N,W_N) = (C^1+bC^3\ C^2+cC^3+bE^2\ C^3\ E^1+cE^2\ E^2). \qquad (4.16)$$

Several cases must be considered.

<u>Case 1</u>: $E^2 \neq 0$

Let $(E^2)^\perp$ denote the orthogonal complement of E^2. Then there is a unique c^* in k such that $E^1 + c^*E^2$ is an element of $(E^2)^\perp$ and a unique b^* in k such that $C^2 + c^*C^3 + b^*E^2$ is an element of $(E^2)^\perp$. Note that $b^* = b^*(C,E)$ and $c^* = c^*(C,E)$. Let $(C^*,E^*) = (C^1+b^*\ C^3\ C^2 + c^*C^3+b^*E^2\ C^3\ E^1+c^*E^2\ E^2)$. Then (C^*,E^*) is equivalent to (C,E) modulo N_∂ and set

$$\phi(C,E) = (C^*,E^*). \qquad (4.17)$$

To show that ϕ defines a canonical form for equivalence modulo N_∂

it will be sufficient to show that $\phi(C,E) = \phi(C_1,E_1)$ if and only if (C,E) is equivalent to (C_1,E_1) modulo N_∂. Since $(C^*,E^*) \sim_{N_\partial} (C,E)$ and $(C_1^*,E_1^*) \sim_{N_\partial} (C_1,E_1)$, it is clear that $\phi(C,E) = \phi(C_1,E_1)$ implies equivalence of (C,E) and (C_1,E_1). Conversely, if $(C,E) \sim_{N_\partial} (C_1,E_1)$, then $E_1^2 = E^2$ and $C_1^3 = C^3$ and

$$C_1^1 = C^1 + bC^3,\ C_1^2 = C^2 + cC^3 + bE^2,\ E_1^1 = E^1 + cE^2$$

for some b,c in k. But $E_1^{1*} = E_1^1 + c_1^*E^2$ is a (unique) element of $(E_2)^\perp$ implies that $c^* = c_1^* + c$ and hence, that $E_1^{1*} = E^1 + c^*E^2 = E^{1*}$. Similarly, $C_1^{2*} = C_1^2 + c_1^*C^3 + b_1^*E^2$ is a (unique) element of $(E^2)^\perp$ implies that $C_1^{2*} = C^2 + (c+c_1^*)C^3 + (b_1^*+b)E^2 = (C^2+c^*C^3) + (b_1^*+b)E^2$ and hence, that $b^* = b_1^* + b$. Thus, $C_1^{2*} = C^{2*}$ and $C_1^{1*} = C_1^1 + b_1^*C^3 = C^1 + b^*C^3 = C^{1*}$. In other words, $\phi(C_1,E_1) = \phi(C,E)$.

<u>Case 2</u>: $E^2 = 0,\ C^3 \neq 0$.

Let $(C^3)^\perp$ denote the orthogonal complement of C^3. Then there are unique elements b^*,c^* of k such that $C^1 + b^*C^3$ and $C^2 + c^*C^3$ are in $(C_3)^\perp$. Let $(C^*,E^*) = (C^1+b^*C^3\ \ C^2+c^*C^3\ \ C^3\ \ E^1\ \ 0)$ and set $\phi(C,E) = (C^*,E^*)$. Just as in Case 1, ϕ defines a canonical form for equivalence modulo N_∂.

<u>Case 3</u>: $E^2 = 0,\ C^3 = 0$.

In this case, it is clear from 4.16 that $(C,E) \sim_{N_\partial} (C^1,E^1)$ if and only if $C^1 = C_1^1,\ C^2 = C_1^2,\ 0 = C^3 = C_1^3,\ E^1 = E_1^1,\ 0 = E^2 = E_1^2$. Hence, $(C^*,E^*) = \phi(C,E) = (C,E)$ defines a canonical form in this case.

Thus, a canonical form for the action of N_∂ has been determined and a fortiori for the action of Γ_∂ (since G_∂ is reductive).

Example 4.18. Let $m = 2$, $n = 4$, $\partial_1 = \partial_2 = 2$, $\partial = \{2,2\}$ and $p \geq 1$. Then $U \in \mathcal{U}_\partial$ if and only if

$$U = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{4.18}$$

where $a,b,c,d \in k$ and $ad - bc \neq 0$ i.e., if and only if $U \in GL(k,2)$. It follows that

$$S_\partial(x)U = \begin{bmatrix} 1 & 0 \\ x & 0 \\ 0 & 1 \\ 0 & x \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ x & 0 \\ 0 & 1 \\ 0 & x \end{bmatrix}$$

and that

$$\mathrm{diag}[x^{\partial_i}]U = \begin{bmatrix} x^2 & 0 \\ 0 & x^2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x^2 & 0 \\ 0 & x^2 \end{bmatrix}.$$

In other words, Γ_∂ is the group with elements given by

$$(V_u, \theta_u, W_u) = \left(\begin{bmatrix} a & 0 & b & 0 \\ 0 & d & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}, 0, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)$$

and with multiplication given by 4.8. Since Γ_∂ is isomorphic to $GL(k,2)$, Γ_∂ is reductive and a canonical form for its action on (C_R, E_R) exists ([6], [7]). For example, if E_R is of rank 2, then E_R^* is a reduced column echelon matrix with $E_R^* = E_R W^*$, W^* unique, and $C_R^* = C_R V^*$, with V^* having the "same" non-zero entries as W^*. Note that if $p > 4$, then the canonical form will exist on an appropriately chosen "stratification" of X_∂ and will <u>not</u> be a continuous canonical form ([8]).

<u>Example 4.20</u>. Let $m = 2$, $n = 4$, $\partial_1 = 3$, $\partial_2 = 1$, $\partial = \{3,1\}$ and $p \geq 1$. Then $U \in \mathcal{U}_\partial$ if and only if

$$U = \begin{bmatrix} a & 0 \\ b+cx+dx^2 & e \end{bmatrix} \tag{4.21}$$

where $a,b,c,d,e \in k$ and $ae \neq 0$. It follows that

$$S_\partial(x)U = \begin{bmatrix} 1 & 0 \\ x & 0 \\ x^2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ b+cx+dx^2 & e \end{bmatrix} = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ b & c & d & e \end{bmatrix} \begin{bmatrix} 1 & 0 \\ x & 0 \\ x^2 & 0 \\ 0 & 1 \end{bmatrix}$$

and that

$$\mathrm{diag}[x^{\partial_i}]U = \begin{bmatrix} x^3 & 0 \\ 0 & x \end{bmatrix}\begin{bmatrix} a & 0 \\ b+cx+dx^2 & e \end{bmatrix} = \begin{bmatrix} a & 0 \\ d & e \end{bmatrix}\begin{bmatrix} x^3 & 0 \\ 0 & x^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & b & c & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ x & 0 \\ x^2 & 0 \\ 0 & 1 \end{bmatrix}.$$

In other words, Γ_∂ is the group with elements given by

$$(V_u, \theta_u, W_u) = \left(\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ b & c & d & e \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & b & c & 0 \end{bmatrix}, \begin{bmatrix} a & 0 \\ d & e \end{bmatrix} \right)$$

and with multiplication given by 4.8. Let N_∂ and G_∂ be the subgroups of Γ_∂ with elements

$$(V_N, \theta_N, W_N) = \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ b & c & d & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & b & c & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \right)$$

$$(V_G, 0, W_G) = \left(\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & e \end{bmatrix}, 0, \begin{bmatrix} a & 0 \\ 0 & e \end{bmatrix} \right)$$

respectively. It is easy to check that (i) N_∂ is a normal subgroup of Γ_∂, (ii) N_∂ is the unipotent radical of Γ_∂ ([7]), (iii) G_∂ acts on N_∂ via inner automorphisms, and (iv) $\Gamma_\partial = N_\partial G_\partial$, $N_\partial \cap G_\partial = (I,0,I)$ so that Γ_∂ is a semi-direct product of N_∂ and G_∂. Since the group G_∂ is reductive ([7]), it is enough to determine a canonical form for the action of N_∂. Such a canonical form can be determined by the same methods used in Example 4.12 (as will be shown in the sequel).

Now, recall the following:

<u>Definition 4.22</u>. <u>Let</u> A <u>be an</u> $n \times m$ <u>matrix and</u> B <u>be a</u> $p \times q$ <u>matrix</u>. <u>Then the</u> $np \times mq$ <u>matrix</u>

$$A \otimes B = \begin{bmatrix} Ab_{11} & \cdots & Ab_{1q} \\ \vdots & & \vdots \\ Ab_{p1} & \cdots & Ab_{pq} \end{bmatrix} \tag{4.23}$$

<u>is called the Kronecker product of</u> A <u>and</u> B.

Note that if the dimensions are compatible, then $(A \otimes B)(C \otimes D) = (AC \otimes BD)$.

Now let $\partial = \{\partial_1,\ldots,\partial_m\}$ be properly indexed and suppose that

$$\partial_1 = \partial_2 \cdot \cdot \cdot \cdot = \partial_{q_1} = \varepsilon_1$$
$$\partial_{q_1+1} = \cdot \quad \cdot \cdot \cdot \cdot \cdot = \partial_{q_1+q_2} = \varepsilon_2$$
$$\vdots \qquad\qquad \vdots \qquad (4.24)$$
$$\partial_{m-q_\ell+1} = \cdot \quad \cdot \cdot \cdot \cdot \cdot = \partial_m = \varepsilon_\ell$$

where

$$\varepsilon_1 > \varepsilon_\ell > \ . \ . \ . > \varepsilon_\ell \geq 1. \qquad (4.25)$$

Then

$$\sum_{i=1}^{\ell} q_i = m, \quad \sum_{i=1}^{\ell} q_i \varepsilon_i = n \qquad (4.26)$$

where $n = \Sigma \partial_j$. It follows from Proposition 3.10 that U is an element of $\mathcal{U}_\partial$ if and only if U is of the form

$$U = \begin{bmatrix} A_{q_1,q_2} & O_{q_1,q_2} & O_{q_1,q_3} & \cdots O_{q_1,q_\ell} \\ \sum_{j=0}^{\varepsilon_1-\varepsilon_2} B^j_{q_2,q_1} x^j & A_{q_2,q_2} & O_{q_2,q_3} & \cdots O_{q_2,q_\ell} \\ \sum_{j=0}^{\varepsilon_1-\varepsilon_3} B^j_{q_3,q_1} x^j & \sum_{j=0}^{\varepsilon_2-\varepsilon_3} B^j_{q_3,q_2} x^j & A_{q_3,q_3} & \cdots O_{q_3,q_\ell} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{j=0}^{\varepsilon_1-\varepsilon_\ell} B^j_{q_\ell,q_1} x^j & \sum_{j=0}^{\varepsilon_2-\varepsilon_\ell} B^j_{q_\ell,q_2} x^j & \sum_{j=0}^{\varepsilon_3-\varepsilon_\ell} B^j_{q_\ell,q_3} x^j & \cdots A_{q_\ell,q_\ell} \end{bmatrix}$$

(4.27)

where $A_{q_i,q_i} \in GL(k,q_i)$. Let $S_{q_i\varepsilon_i,q_i}(x)$ be the $q_i\varepsilon_i \times q_i$ matrix given by

$$S_{q_i\varepsilon_i,q_i}(x) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ x & 0 & \cdots & 0 \\ \vdots & \vdots & & \\ x^{\varepsilon_i-1} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & x^{\varepsilon_i-1} & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & & x^{\varepsilon_i-1} \end{bmatrix} \tag{4.28}$$

so that

$$S_\partial(x) = \begin{bmatrix} S_{q_1\varepsilon_1,q_1}(x) & O_{q_1\varepsilon_1,q_2} & \cdots & O_{q_1\varepsilon_1,q_\ell} \\ O_{q_2\varepsilon_2,q_1} & S_{q_2\varepsilon_2,q_2}(x) & \cdots & O_{q_2\varepsilon_2,q_\ell} \\ \vdots & \vdots & & \\ O_{q_\ell\varepsilon_\ell,q_1} & O_{q_\ell\varepsilon_\ell,q_2} & \cdots & S_{q_\ell\varepsilon_\ell,q_\ell}(x) \end{bmatrix}. \tag{4.29}$$

Moreover,

$$\operatorname{diag}[x^{\partial_i}] = \begin{bmatrix} x^{\varepsilon_1} I_{q_1,q_1} & O_{q_1,q_2} & \cdots & O_{q_1,q} \\ O_{q_2,q_1} & x^{\varepsilon_2} I_{q_2,q_2} & \cdots & O_{q_2,q_\ell} \\ \vdots & \vdots & & \\ O_{q_\ell,q_1} & O_{q_\ell,q_2} & \cdots & x^{\varepsilon_\ell} I_{q_\ell,q_\ell} \end{bmatrix} \tag{4.30}$$

and so the following lemma holds via a direct computation.

<u>Lemma 4.31</u>. <u>If</u> $U \varepsilon \mathscr{U}_\partial$, <u>then</u>

$$V_u = \begin{bmatrix} I_{\varepsilon_1,\varepsilon_1} \otimes A_{q_1,q_1} & O_{q_1\varepsilon_1,q_2\varepsilon_2} & O_{q_1\varepsilon_1,q_\ell\varepsilon_\ell} \\ \sum_{j=0}^{\varepsilon_1-\varepsilon_2} E^j_{\varepsilon_2,\varepsilon_1} \otimes B^j_{q_2,q_1} & I_{\varepsilon_2,\varepsilon_2} \otimes A_{q_2,q_2} & O_{q_2\varepsilon_2,q_\ell\varepsilon_\ell} \\ \vdots & \vdots & \\ \sum_{j=0}^{\varepsilon_1-\varepsilon_\ell} E^j_{\varepsilon_\ell,\varepsilon_1} \otimes B^j_{q_\ell,q_1} & \sum_{j=0}^{\varepsilon_2-\varepsilon_\ell} E^j_{\varepsilon_\ell,\varepsilon_2} \otimes B^j_{q_\ell,q_2} & I_{\varepsilon_2,\varepsilon_\ell} \otimes A_{q_\ell,q_\ell} \end{bmatrix} \tag{4.31}$$

$$\theta_u = \begin{bmatrix} O_{q_1,q_1\varepsilon_1} & O_{q_1,q_2\varepsilon_2} & \cdots & O_{q_1,q_\ell\varepsilon_\ell} \\ \sum_{j=0}^{\varepsilon_1-\varepsilon_2-1} E^j_{1,\varepsilon_1} \otimes B^j_{q_2,q_1} & O_{q_2,q_2\varepsilon_2} & \cdots & O_{q_2,q_\ell\varepsilon_\ell} \\ \vdots & \vdots & & \\ \sum_{j=0}^{\varepsilon_1-\varepsilon_2-1} E^j_{1,\varepsilon_1} \otimes B^j_{q_\ell,q_1} & \sum_{j=0}^{\varepsilon_2-\varepsilon_\ell-1} E^j_{1,\varepsilon_2} \otimes B^j_{q_\ell,q_2} & \cdots & O_{q_\ell,q_\ell\varepsilon_\ell} \end{bmatrix} \tag{4.32}$$

$$W_u = \begin{bmatrix} A_{q_1,q_1} & O_{q_1,q_2} & \cdots & O_{q_1,q_\ell} \\ B^{\varepsilon_1-\varepsilon_2}_{q_2,q_1} & A_{q_2,q_2} & \cdots & O_{q_2,q_\ell} \\ \vdots & \vdots & & \\ B^{\varepsilon_1-\varepsilon_\ell}_{q_\ell,q_1} & B^{\varepsilon_2-\varepsilon_\ell}_{q_\ell,q_2} & \cdots & A_{q_\ell,q_\ell} \end{bmatrix} \qquad (4.34)$$

<u>where</u> $A_{q_i,q_i} \in GL(k,q_i)$ <u>and</u> $E^j_{r,s}$ <u>are matrices of the appropriate rank with unit or zero columns.</u>

For example, if $n = qm + r$, $0 \le r \le m - 1$, and $\partial_1 = \ldots = \partial_r = q + 1$, $\partial_{r+1} = \ldots = \partial_m = q$ (the so-called "generic" case), then $q_1 = r$, $q_2 = m - r$, $\partial_1 = q + 1$, $\partial_2 = q$ and

$$E^0_{q,q+1} = [I_{q,q} \; O_{q,1}]$$

$$E^1_{q,q+1} = [O_{q,1} \; I_{q,q}]$$

$$E^0_{1,q+1} = [0 \; \ldots \; 0 \; 1]$$

and

$$V_u = \begin{bmatrix} I_{q+1,q+1} \otimes A_{r,r} & O_{(q+1)r,q(m-r)} \\ \sum_{j=0}^{1} E^j_{q,q+1} \otimes B^j_{m-r,r} & I_{q,q} \otimes A_{m-r,m-r} \end{bmatrix}$$

$$\theta_u = \begin{bmatrix} O_{r,(q+1)r} & O_{r,q(m-r)} \\ E^0_{1,q+1} \otimes B^0_{m-r,r} & O_{m-r,q(m-r)} \end{bmatrix}$$

$$W_u = \begin{bmatrix} A_{r,r} & O_{r,m-r} \\ B^1_{m-r,r} & A_{m-r,m-r} \end{bmatrix}$$

with $A_{r,r} \in GL(k,r)$, $A_{m-r,m-r} \in GL(k,m-r)$.

Definition 4.35. Let N_∂ be the subset of Γ_∂ with elements (V_N, θ_N, W_N) such that $A_{q_1,q_1} = I_{q_1,q_1}, \ldots, A_{q_\ell,q_\ell} = I_{q_\ell,q_\ell}$ and let G_∂ be the subset of Γ_∂ with elements (V_G, θ_G, W_G) such that all the $B^r_{q_i,q_j}$ are zero (note this implies $\theta_G = 0$).

Lemma 4.36. Let $\mathcal{U}_N$ be the subset of $\mathcal{U}_\partial$ with elements U_N such that $A_{q_1,q_1} = I_{q_1,q_1}, \ldots, A_{q_\ell,q_\ell} = I_{q_\ell,q_\ell}$ and let $\mathcal{U}_G$ be the subset of $\mathcal{U}_\partial$ with elements U_G such that all the $B^r_{q_i,q_j}$ are zero. Then: (i) $\mathcal{U}_N$ is a normal subgroup of $\mathcal{U}_\partial$; (ii) $\mathcal{U}_G$ is a subgroup of $\mathcal{U}_\partial$ which acts on $\mathcal{U}_N$ via inner automorphisms; (iii) $\mathcal{U}_\partial = \mathcal{U}_N \mathcal{U}_G$ and $\mathcal{U}_N \cap \mathcal{U}_G = \{I\}$ so that $\mathcal{U}_\partial$ is a semi-direct product of $\mathcal{U}_N$ and $\mathcal{U}_G$; (iv) $\mathcal{U}_N$ is the unipotent radical of $\mathcal{U}_\partial$; and, (v) $\psi(\mathcal{U}_N) = N_\partial$, $\psi(\mathcal{U}_G) = G_\partial$ where ψ is the map given in Proposition 4.11. Hence, Γ_∂ is a semi-direct

<u>product of its normal subgroup</u> N_∂ <u>and its subgroup</u> G_∂.

<u>Proof</u>: Clearly, $\psi(\mathscr{U}_N) = N_\partial$ and $\psi(\mathscr{U}_G) = G_\partial$. Moreover, it is obvious that $\mathscr{U}_G$ is a subgroup and that $\mathscr{U}_G \cap \mathscr{U}_N = \{I\}$. If $U \in \mathscr{U}_\partial$ is given by (4.27), then $U = U_{N_1} U_G$ where

$$U_{N_1} = \begin{bmatrix} I_{q_1,q_1} & O_{q_1,q_2} & \cdots & O_{q_1,q_\ell} \\ \sum_{j=0}^{\varepsilon_1-\varepsilon_2} B^j_{q_2,q_1} A^{-1}_{q_1,q_1} x^j & I_{q_2,q_2} & \cdots & O_{q_2,q_\ell} \\ \vdots & \vdots & & \\ \sum_{j=0}^{\varepsilon_1-\varepsilon_\ell} B^j_{q_\ell,q_1} A^{-1}_{q_1,q_1} x^j & \sum_{j=0}^{\varepsilon_2-\varepsilon_\ell} B^j_{q_\ell,q_2} A^{-1}_{q_2,q_2} x^j & \cdots & I_{q_\ell,q_\ell} \end{bmatrix}$$

and U_G = block diagonal $[A_{q_i,q_i}]$, so that $\mathscr{U}_\partial = \mathscr{U}_N \mathscr{U}_G$. Since

$$U_N U_{N'} = \begin{bmatrix} I_{q_1,q_1} & O_{q_1,q_2} & \cdots \\ \sum_{j=0}^{\varepsilon_1-\varepsilon_2} (B^j_{q_2,q_1} + B'^j_{q_2,q_1}) x^j & I_{q_2,q_2} & \cdots \\ \sum_{j=0}^{\varepsilon_1-\varepsilon_3} B^j_{q_3,q_1} x^j + \sum_{j=j_1+j_2=0}^{\varepsilon_1-\varepsilon_3} B^{j_1}_{q_3,q_2} B'^{j_2}_{q_2,q_1} x^j & \sum_{j=0}^{\varepsilon_2-\varepsilon_3} (B^j_{q_3,q_2} + B'^j_{q_3,q_2}) x^j & \cdots \\ \vdots & & \vdots \end{bmatrix}$$

it is straightforward to check that $\mathscr{U}_N$ is a subgroup. Now let

$U = U_{N_1} U_G$ be an element of $\mathscr{U}_\partial$ and let U_N be an element of $\mathscr{U}_N$. Then $UU_NU^{-1} = U_{N_1} U_G U_N U_G^{-1} U_{N_1}^{-1}$ and so it will be enough to show that $U_G U_N U_G^{-1}$ is an element of $\mathscr{U}_N$. However, direct computation gives

$$U_G U_N U_G^{-1} = \begin{bmatrix} I_{q_1,q_1} & O_{q_1,q_2} & \cdots & O_{q_1,q_\ell} \\ \sum_{j=0}^{\varepsilon_1-\varepsilon_2} A_{q_2,q_2} B^j_{q_2,q_1} A^{-1}_{q_1,q_1} x^j & I_{q_2,q_2} & \cdots & O_{q_2,q_\ell} \\ \vdots & \vdots & & \\ \sum_{j=0}^{\varepsilon_1-\varepsilon_\ell} A_{q_\ell,q_\ell} B^j_{q_\ell,q_1} A^{-1}_{q_1,q_1} x^j & \sum_{j=0}^{\varepsilon_2-\varepsilon_\ell} A_{q_\ell,q_\ell} B^j_{q_\ell,q_2} A^{-1}_{q_2,q_2} x^j & \cdots & I_{q_\ell,q_\ell} \end{bmatrix}$$

and so the lemma is established.

For example, in the "generic" case $n = qm + r$, $0 \le r \le m - 1$, $\partial_1 = \ldots = \partial_r = q + 1$, $\partial_{r+1} = \ldots = \partial_m = q$, it is clear that

$$V_N = \begin{bmatrix} I_{q+1,q+1} \otimes I_{r,r} & O_{(q+1)r,q(m-r)} \\ \sum_{j=0}^{1} E^j_{q,q+1} \otimes B^j_{m-r,r} & I_{q,q} \otimes I_{m-r,m-r} \end{bmatrix}$$

$$\theta_N = \begin{bmatrix} O_{r,(q+1)r} & O_{r,q(m-r)} \\ E^0_{1,q+1} \otimes B^0_{m-r,r} & O_{m-r,q(m-r)} \end{bmatrix} \quad (4.37)$$

$$W_N = \begin{bmatrix} I_{r,r} & O_{r,m-r} \\ B^1_{m-r,r} & I_{m-r,mr} \end{bmatrix}$$

and that

$$V_G = \begin{bmatrix} I_{q+1,q+1} \otimes A_{r,r} & O_{(q+1)r,q(m-r)} \\ O_{q(m-r),(q+1)r} & I_{q,q} \otimes A_{m-r,m-r} \end{bmatrix}$$

$$W_G = \begin{bmatrix} A_{r,r} & O_{r,m-r} \\ O_{m-r,r} & A_{m-r,m-r} \end{bmatrix}$$

Moreover,

$$(V_G,0,W_G)(V_N,\theta_N,W_N)(V_G^{-1},0,W_G^{-1}) = (V_G V_N V_G^{-1}, W_G \theta_N V_G^{-1}, W_G W_N W_G^{-1})$$

and

$$V_G V_N V_G^{-1} = \begin{bmatrix} I_{q+1,q+1} \otimes I_{r,r} & O_{(q+1)r,q(m-r)} \\ \sum_{j=0}^{1} E^j_{q,q+1} \otimes A_{m-r,m-r} B^j_{m-r,r} A^{-1}_{r,r} & I_{q,q} \otimes I_{m-r,m-r} \end{bmatrix}$$

$$W_G \theta_N V_G^{-1} = \begin{bmatrix} O_{r,(q+1)r} & O_{r,q(m-r)} \\ E^0_{1,q+1} \otimes A_{mr,mr} B^0_{m-r,r} A^{-1}_{r,r} & O_{m-r,q(m-r)} \end{bmatrix}^{\dagger}$$

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†Note $A_{m-r,m-r} = I_{1,1} \otimes A_{m-r,m-r}$ so that $A_{m-r,m-r}\,(E^0_{1,q+1} \otimes B^0_{m-r,r})$ $= E^0_{1,q+1} \otimes A_{m-r,m-r} B^0_{m-r,r}$.

$$W_G W_N W_G^{-1} = \begin{bmatrix} I_{r,r} & O_{r,m-r} \\ A_{m-r,m-r} B^1_{m-r,r} A^{-1}_{r,r} & I_{m-r,m-r} \end{bmatrix}$$

imply that $(V_G, 0, W_G)(V_N, \theta_N, W_N)(V_G^{-1}, 0, W_G^{-1})$ is an element of N_∂.

Lemma 4.36 shows that the properties of Γ_∂ used in the examples hold in general. Since G_∂ is reductive, it again will be sufficient to determine a canonical form under the action of N_∂. The method is entirely analogous to that used in the examples and will first be illustrated for the "generic" case.

So let $n = qm + r$, $0 \leq r \leq m - 1$, $\partial_1 = \ldots = \partial_r = q + 1$, $\partial_{r+1} = \ldots = \partial_m = q$, $\partial_1 = q + 1$, $\partial_2 = q$. Then N_∂ consists of elements (V_N, θ_N, W_N) given by 4.37. If $p \geq 1$ and (C,E) is an element of $M_{p,n}(k) \times M_{p,m}(k)$, then

$$C = (\underline{C}\ \underline{D}),\ E = (\underline{E}\ \underline{F}) \tag{4.38}$$

where $\underline{C} = (C^1 \ldots C^{(q+1)r})$, $\underline{D} = (C^{(q+1)r+1} \ldots C^n) = (D^1 \ldots D^{q(m-r)})$, $\underline{E} = (E^1 \ldots E^r)$, and $\underline{F} = (E^{r+1} \ldots E^m) = (F^1 \ldots F^{m-r})$ are elements of $M_{p,(q+1)r}(k)$, $M_{p,q(m-r)}(k)$, $M_{p,r}(k)$, and $M_{p,m-r}(k)$, respectively. It follows that

$$(C,E)(V_N, \theta_N, W_N) = (CV_N + E\theta_N, EW_N)$$

and

$$CV_N = (\underline{C} + \underline{D}(E^0_{q,q+1} \otimes B^0_{m-r,r} + E^1_{q,q+1} \otimes B^1_{m-r,r})\ \underline{D})$$

$$E_N = (\underline{F}(E^0_{1,q+1} \otimes B^0_{m-r,r})\ O_{p,q(m-r)}) \qquad (4.39)$$

$$EW_N = (\underline{E} + \underline{F}B^1_{m-r,r}\ \underline{F}).$$

Thus, $\underline{D}$ and $\underline{F}$ are invariant under the action of N_∂. Again, there are several cases to consider.

Case 1: $\underline{D} = 0$, $\underline{F} = 0$.

In this case, it is clear from 4.39 that $(C,E) \sim_{N_\partial} (C_1,E_1)$ if and only if $\underline{C} = \underline{C}_1$, $0 = \underline{D} = \underline{D}_1$, $\underline{E} = \underline{E}_1$, and $0 = \underline{F} = \underline{F}_1$. Hence $(C^*,E^*) = \phi(C,E) = (\underline{C},E)$ defines a canonical form.

Case 2: $\underline{D} \neq 0$, $\underline{F} = 0$.

Let $\Re(\underline{D})^\perp$ denote the orthogonal complement of the range of $\underline{D}$. Consider the set $\{\underline{C} + \underline{D}(\sum_{j=0}^{1} E^j_{q,q+1} \otimes B^j_{m-r,r})\} \cap \Re(\underline{D})^\perp$. If $\underline{C} + \underline{D}X$ and $\underline{C} + \underline{D}X_1$ are elements of this set, then $\underline{D}(X-X_1) = 0$ (being an element of $\Re(\underline{D}) \cap \Re(\underline{D})^\perp$). Thus, if the set is non-empty, it contains a unique element $\underline{C} + \underline{D}(\sum_{j=0}^{1} E^j_{q,q+1} \otimes \hat{B}^j_{m-r,r}) = \underline{C}^*$ (caution: $\hat{B}^j_{m-r,r}$ are not unique in general). In this case, set $(C^*,E^*) = \phi(C,E) = ([\underline{C}^*\ \underline{D}], [\underline{E}\ 0])$. Then, $(C^*,E^*) \sim_{N_\partial} (C,E)$ and it is claimed that (C^*,E^*) defines a canonical form. Clearly, if $(C^*_1,E^*_1) = (C^*_1,E^*_1)$, then $(C,E) \sim_{N_\partial} (C_1,E_1)$. On the other hand, if $(C,E) \sim_{N_\partial} (C_1,E_1)$, then $\underline{F} = \underline{F}_1 = 0$ and $\underline{D} = \underline{D}_1$ so that $\underline{E} = \underline{E}_1$ and $\underline{C}_1 = \underline{C} + \underline{D}(\sum_{j=0}^{1} E^j_{q,q+1} \otimes \tilde{B}^j_{m-r,r})$ for some $\tilde{B}^j_{m-r,r}$. But

$\underline{C}_1 + \underline{D}(\sum_{j=0}^{1} E^j_{q,q+1} \otimes \hat{B}^j_{m-r,r}) = \underline{C} + \underline{D}(\sum_{j=0}^{1} E^i_{q,q+1} \otimes B'^j_{m-r,r})$ where

$B'^j_{m-r,r} = \tilde{B}^j_{m-r,r} + B'^j_{m-r,r}$, so that $\underline{C}^*_1 = \underline{C}_1 + \underline{D}(\sum_{j=0}^{1} E^j_{q,q+1} \otimes \hat{B}'^j_{1m-r,r})$ is a unique element of $\Re(\underline{D})^\perp$ implies that $\underline{C}^*_1 = \underline{C} + \underline{D}(\sum_{j=0}^{1} E^j_{q,q+1} \otimes [\tilde{B}^j_{m-r,r} + \hat{B}_{1m-r,r}])$ is an element of $\Re(\underline{D})^\perp \cap \{\underline{C} + \underline{D}(\sum_{j=0}^{1} E^j_{q,q+1} \otimes B^j_{m-r,r})\}$. Since $\underline{C}^*$ is a unique such element, it follows that $\underline{C}^*_1 = \underline{C}^*$ and hence that $(C^*,E^*) = (C^*_1,E^*_1)$.

Finally, if the set $\{\underline{C} + \underline{D}\sum_{j=0}^{1} E^j_{q,q+1} \otimes B^j_{m-r,r})\} \cap \Re(\underline{D})^\perp$ is empty, set $(C^*,E^*) = \phi(C,E) = ([\underline{C}\ \underline{D}],\ [\underline{E}\ 0]) = (C,E)$. It is clear that (C^*,E^*) is a canonical form in this situation.

Case 3: $\underline{D} \neq 0$, $\underline{F} = 0$

Let $\Re(\underline{F})^\perp$ denote the orthogonal complement of the range of $\underline{F}$. Consider the set $\{\underline{E} + \underline{F}B^1_{m-r,r}\} \cap \Re(\underline{F})^\perp$. If this set is non-empty, it contains a unique element $\underline{E}^* = \underline{E} + \underline{F}\hat{B}^1_{m-r,r}$ (caution: $\hat{B}^1_{m-r,r}$ is not necessarily unique). Let $\Re(\underline{D},\underline{F})^\perp$ denote the orthogonal complement of the range of $(\underline{D},\underline{F})$. Consider the set $\{\underline{C} + \underline{D}(E^0_{q,q+1} \otimes B^0_{m-r,r} + E^1_{q,q+1} \otimes \hat{B}^1_{m-r,r}) + \underline{F}(E^0_{1,q+1} \otimes B^0_{m-r,r})\} \cap \Re(\underline{D},\underline{F})^\perp$. Again, if this set is non-empty, it contains a unique element $\underline{C}^* = \underline{C} + \underline{D}(E^0_{q,q+1} \otimes \hat{B}^0_{m-r,r} + E^1_{q,q+1} \otimes \hat{B}^1_{m-r,r}) + \underline{F}(E^0_{1,q+1} \otimes \hat{B}^0_{m-r,r})$. Note that $\underline{C}^*$ is independent of the choice of $\hat{B}^1_{m-r,r}$ for which $\underline{E}^* = \underline{E} + \underline{F}\hat{B}^1_{m-r,r}$ since, if $\underline{C} + \underline{D}X + \underline{F}Y$ and $\underline{C} + \underline{D}X_1 + \underline{F}Y_1$ are in $\Re(\underline{D},\underline{F})^\perp$, then

$\underline{D}(X-X_1) + \underline{F}(Y-Y_1)$ is an element of $\mathfrak{R}(\underline{D},\underline{F}) \cap \mathfrak{R}(\underline{D},\underline{F})^{\perp}$ and so, $\underline{D}(X-X_1) + \underline{F}(Y-Y_1) = 0$. So set $(C^*,E^*) = \phi(C,E) = ([\underline{C}^*\ \underline{D}],\ [\underline{E}^*\ \underline{F}])$. Then $(C^*,E^*) \sim_{N_\partial} (C,E)$ and it is claimed that (C^*,E^*) defines a canonical form. Clearly, if $(C^*,E^*) = (C_1^*,E_1^*)$, then $(C,E) \sim_{N_\partial} (C_1,E_1)$. Conversely, if $(C,E) \sim_{N_\partial} (C_1,E_1)$, then $\underline{F} = \underline{F}_1$ and $\underline{D} = \underline{D}_1$. Since $\underline{E}_1 = \underline{E} + \underline{F}\tilde{B}^1_{m-r,r}$, $\underline{E}_1 + \underline{F}B^1_{m-r,r} = \underline{E} + \underline{F}(\tilde{B}^1_{m-r,r} + B^1_{m-r,r})$ and it follows that $\underline{E}_1^* = \underline{E}^*$ and that $\hat{B}^1_{1m-r,r}$ can be taken so that $\hat{B}^1_{1m-r,r} + B^1_{m-r,r} = \hat{B}^1_{m-r,r}$. Arguing in a manner entirely analogous to that used in Case 2, it is easy to show that $\underline{C}_1^* = \underline{C}^*$. The situation when the various intersections are empty can also be treated in a manner entirely similar to that used in Case 2.

<u>Case 4</u>: $\underline{D} = 0$, $\underline{F} \neq 0$.

Let $\mathfrak{R}(F)^{\perp}$ denote the orthogonal complement of the range of $\underline{F}$ and consider the sets $\{\underline{E} + \underline{F}B^1_{m-r,r}\} \cap \mathfrak{R}(\underline{F})^{\perp}$ and $\{\underline{C} + \underline{F}(E^0_{1,q+1} \otimes B^0_{m-r,r})\} \cap \mathfrak{R}(\underline{F})^{\perp}$. If these sets are non-empty, they contain unique elements $\underline{E}^*,\underline{C}^*$ respectively and $(C^*,E^*) = \phi(C,E) = ([\underline{C}^*\ 0],\ [\underline{E}^*\ \underline{F}])$ is the desired canonical form (as is readily demonstrated via the methods used in the previous cases). If either of the sets are empty, then the appropriate $B^j_{m-r,r}$ is taken to be 0 to obtain the canonical form. Thus, a canonical form exists for the generic case.

Now it is time to consider the general case. So let $\partial = \{\partial_1,\ldots,\partial_m\}$ be properly indexed and let $q_1,\ldots,q_\ell$, $\varepsilon_1,\ldots,\varepsilon_\ell$ be given by 4.24 and 4.25. Let $p \geq 1$ and let (C,E)

be an element of $M_{p,n}(k) \times M_{p,m}(k)$. Then

$$C = (\underline{C}^1, \underline{C}^2, \ldots \underline{C}^\ell), \quad E = (\underline{E}^1, \underline{E}^2, \ldots \underline{E}^\ell) \tag{4.40}$$

where $\underline{C}^i$ is in $M_{p,q_i\varepsilon_i}(k)$ and $\underline{E}^i$ is in $M_{p,q_i}(k)$. Let (V_N, θ_N, W_N) be an element of N_∂ so that

$$V_N = \begin{bmatrix} I_{\varepsilon_1,\varepsilon_1} \otimes I_{q_1,q_1} & O_{q,\varepsilon_1,q_2\varepsilon_2} & \cdots & O_{q_1\varepsilon_1,q_\ell\varepsilon_\ell} \\ \sum_{j=0}^{\varepsilon_1-\varepsilon_2} E^j_{\varepsilon_2,\varepsilon_1} \otimes B^j_{q_2,q_1} & I_{\varepsilon_2,\varepsilon_2} \otimes I_{q_2,q_2} & \cdots & O_{q_2\varepsilon_2,q_\ell\varepsilon_\ell} \\ \vdots & & & \vdots \\ \sum_{j=0}^{\varepsilon_1-\varepsilon_\ell} E^j_{\varepsilon_\ell,\varepsilon_1} \otimes B^j_{q_\ell,q_1} & \sum_{j=0}^{\varepsilon_2-\varepsilon_\ell} E^j_{\varepsilon_\ell,\varepsilon_2} \otimes B^j_{q_\ell,q_2} & \cdots & I_{\varepsilon_\ell,\varepsilon_\ell} \otimes I_{q_\ell,q_\ell} \end{bmatrix}$$

$$\theta_N = \begin{bmatrix} O_{q_1,q_1\varepsilon_1} & O_{q_1,q_2\varepsilon_2} & \cdots & O_{q_1,q_\ell\varepsilon_\ell} \\ \sum_{j=0}^{\varepsilon_1-\varepsilon_2-1} E^j_{1,\varepsilon_1} \otimes B^j_{q_2,q_1} & O_{q_2,q_2\varepsilon_2} & \cdots & O_{q_2,q_\ell\varepsilon_\ell} \\ \vdots & & & \vdots \\ \sum_{j=0}^{\varepsilon_1-\varepsilon_\ell-1} E^j_{1,\varepsilon_1} \otimes B^j_{q_\ell,q_1} & \sum_{j=0}^{\varepsilon_2-\varepsilon_\ell-1} E^j_{1,\varepsilon_2} \otimes B^j_{q_\ell,q_2} & \cdots & O_{q_\ell,q_\ell\varepsilon_\ell} \end{bmatrix}$$

$$W_N = \begin{bmatrix} I_{q_1,q_1} & O_{q_1,q_2} & \cdots & O_{q_1,q_\ell} \\ B^{\varepsilon_\ell-\varepsilon_2}_{q_2,q_1} & I_{q_2,q_2} & \cdots & O_{q_2,q_\ell} \\ \vdots & \vdots & & \\ B^{\varepsilon_1-\varepsilon_\ell}_{q_\ell,q_1} & B^{\varepsilon_2-\varepsilon_\ell}_{q_\ell,q_2} & \cdots & I_{q_\ell,q_\ell} \end{bmatrix} \tag{4.41}$$

and

$$CV_N = (\underline{C}^1 + \sum_{i=2}^{\ell} \underline{C}^i (\sum_{j=0}^{\varepsilon_1-\varepsilon_i} E^j_{\varepsilon_1,\varepsilon_1} \otimes B^j_{q_i,q_1}), \ldots, \underline{C}^\ell)$$

$$E\theta_N = (\sum_{i=2}^{\ell} \underline{E}^i (\sum_{j=0}^{\varepsilon_1-\varepsilon_i-1} E^j_{1,\varepsilon_1} \otimes B^j_{q_i,q_1}), \ldots, \underline{O}) \tag{4.42}$$

$$EW_N = (E^1 + \sum_{i=2}^{\ell} \underline{E}^i B^{\varepsilon_1-\varepsilon_i}_{q_i,q_1}, \ldots, \underline{E}^\ell)$$

These equations determine the action of N_∂ on X_∂.

Definition 4.43. Let A_i, $i = 1,\ldots s$ be $p \times r_i$ constant matrices. Let k^p be the space of column vectors with p rows. Then $\mathfrak{R}(\underline{A}_1,\ldots,\underline{A}_s)$ denotes the subspace of k^p spanned by the columns of the A_i and $\mathfrak{R}(\underline{A}_1,\ldots,\underline{A}_s)^\perp$ denotes its orthogonal complement.

Lemma 4.44. Let A_i, $i = 1,\ldots,s$ be $p \times r_i$ constant matrices. Let $d_2,\ldots,d_s$ be positive integers. Consider the set

$$Q = \{\underline{A}_1 + \sum_{j=2}^{s} \underline{A}_j B^j_{r_j,d_j}\} \cap \Re(\underline{A}_2,\ldots\underline{A}_s)^{\perp} \quad \text{where} \quad B^j_{r_j,d_j} \in M_{r_j,d_j}(k).$$

Then either Q is empty or Q contains a unique element.

Proof: Let $X_1 = \underline{A}_1 + \sum_{j=2}^{s} \underline{A}_j B^j_{r_j d_j}$ and $X_2 = \underline{A}_1 + \sum_{j=2}^{s} \underline{A}_j B'^j_{r_j,d_j}$ be elements of Q. Then $X_1 - X_2 = \sum_{j=2}^{s} \underline{A}_j (B^j_{r_j,d_j} - B'_{r_j,d_j})$ is an element of $\Re(\underline{A}_2,\ldots,\underline{A}_s)^{\perp} \cap \Re(\underline{A}_2,\ldots,\underline{A}_s) = \{0\}$ so that $X_1 = X_2$.

Theorem 4.45. A canonical form for the action of N_∂ on X_∂ exists.

Proof: The proof is essentially a tedious exercise in the repeated application of Lemma 4.44 and should be clear from the examples and the generic case.

Thus, the existence of a complete system of invariants under feedback equivalence has been established.

5. Some Examples

Several examples shall be examined in this section. The first illustrates the fact that the Kronecker set ∂ is not a complete invariant for either equivalence under feedback or equivalence under feedback and output transformations. The second contains a treatment of the "controllable" case. The third involves an analysis of output feedback.

Example 5.1. Let

$$T(x) = \begin{bmatrix} 1/x^2 & (x+1)/x \\ 0 & 1/x \end{bmatrix}, \quad T_1(x) = \begin{bmatrix} (1-x)/x^2 & 0 \\ 0 & 1/x \end{bmatrix}$$

Then $T(x) = R(x)P^{-1}(x)$, $T_1(x) = R_1(x)P_1^{-1}(x)$ where

$$R(x) = \begin{bmatrix} 1 & x+1 \\ 0 & 1 \end{bmatrix}, \quad R_1(x) = \begin{bmatrix} 1-x & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$P(x) = P_1(x) = \begin{bmatrix} x^2 & 0 \\ 0 & x \end{bmatrix}$$

Note that R,P and R_1,P_1 are relatively right prime, that $P = P_1$ is properly indexed, and that $\partial_T = \{2,1\} = \partial_{T_1}$. However, R is not equivalent to R_1 under $\mathcal{U}_\partial$ (or $GL(k,2) \times \mathcal{U}_\partial$) since R is unimodular but R_1 is not unimodular. The fact that R and R_1 are not equivalent under $\mathcal{U}_\partial$ can also be established via examination of the canonical forms R_c, R_{1c}. For,

$$R(x) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ x & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x^2 & 0 \\ 0 & x \end{bmatrix}$$

$$R_1(x) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ x & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x^2 & 0 \\ 0 & x \end{bmatrix}$$

so that

$$C_R^* = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_R^* = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$C_{R_1}^* = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{R_1}^* = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and $(C_R^*, E_R^*) \neq (C_{R_1}^*, E_{R_1}^*)$.

<u>Example 5.2</u>. "The Controllable Case" ([2], [3]).

Let $\Sigma_{n,m}^c \subset \Sigma_{n,m}$ be the set of $n \times m$ transfer matrices $T(x)$ such that $T(x) = I(xI-A)^{-1}B$ for some controllable (A,B,I). Then it is claimed that ∂_T is a complete invariant under state feedback and output transformations. Since ∂_T is an invariant, it is enough to show that if T, T_1 are elements of $\Sigma_{n,m}^c$ with $\partial_T = \partial_{T_1}$, then T and T_1 are equivalent. However, as is well-known ([2], [3], [4]), T is equivalent under state feedback to $\hat{T}$ where

$$\sigma_{\hat{T}} = \begin{bmatrix} Q^{-1} S_\partial(x) \\ \operatorname{diag}[x^{\partial_i}] \end{bmatrix}$$

for some $Q \in GL(k,n)$ and similarly, for T_1. In other words, In other words, $\hat{R} = Q^{-1}S_\partial(x)$ and $\hat{R}_1 = Q_1^{-1}S_\partial(x)$. Hence, $\hat{R} = (Q^{-1}Q_1)\hat{R}_1$ and $\hat{T}$ is equivalent to $\hat{T}_1$.

Example 5.3 "Output Feedback"

Let $\Sigma^s_{p,m} \subset \Sigma_{p,m}$ be the set of $p \times m$ transfer matrices $T(x)$ which are strictly proper i.e., if $T(x) = (n_{ij}(x)/d_{ij}(x))$, then degree n_{ij} < degree d_{ij}. Let $S^s_{p,m} \subset S_{p,m}$ be the corresponding set of strictly proper linear systems.

Definition 5.4. Let T be an element of $\Sigma^s_{p,m}$ with $\sigma_T = \begin{bmatrix} R_T \\ P_T \end{bmatrix}$, P_T column proper. Let n = degree det P_T. Let G be an element of $GL(k,m)$ and H be an element of $M_{m,p}(k)$. Call (H,G) an output feedback pair. Set

$$P_{T_{H,G}} = G^{-1}\{P_T - HR_T\},\ R_{T_{H,G}} = R_T \tag{5.5}$$

and $T_{H,G} = R_{T_{H,G}} P^{-1}_{T_{H,G}}$. Then $T_1 \in \Sigma^s_{p,m}$ is equivalent to T under output feedback if there is an output feedback pair such that $T_1 = T_{H,G}$.

Note that if T_1 is equivalent to T under output feedback, then $P_{T_{H,G}}, R_{T_{H,G}}$ are relatively right prime since $AR_T + BP_T = I$ implies $(A+BGG^{-1}H)R_{T_{H,G}} + (BG)P_{T_{H,G}} = I$. This corresponds to the preservation of both controllability and observability under output feedback. Moreover, since $T_{1(-H,G^{-1})} = T$ and $R_T = C_{R_T}S_\partial(x)$ so that $HR_T = (HC_{R_T})S_\partial(x)$, it is clear that equivalence under output feedback, implies equivalence under state feedback.

Now, if T is an element of $\Sigma^{s}_{p,m}$ with $\sigma_T = \begin{bmatrix} R_T \\ P_T \end{bmatrix}$, then there is a U in $\mathscr{U}_{\partial_T}$ such that $R_T U = R_c$, the canonical form under $\mathscr{U}_{\partial_T}$, and $P_T U = P_c$ is properly indexed. Since $T = R_T P_T^{-1} = R_c P_c^{-1}$, it may be assumed that $\sigma = \sigma_T = \begin{bmatrix} R_c \\ P_c \end{bmatrix}$.

<u>Lemma 5.6.</u> <u>If</u> T <u>is equivalent to</u> T_1 <u>under output feedback,</u> <u>then</u> $\begin{bmatrix} R_c \\ P_c \end{bmatrix}$ <u>is equivalent to</u> $\begin{bmatrix} R_{1c} \\ P_{1c} \end{bmatrix}$ <u>under output feedback and conversely</u>.

<u>Proof</u>: If T is equivalent to T_1 under output feedback, then $R_1 = R \cdot U$ and $P_1 = P_{H,G} U$ for some U in $\mathscr{U}_m$. Since P_1, P are properly indexed, $P_{H,G}$ is properly indexed and U is an element of $\mathscr{U}_\partial$. But $R_{1c} = R_1 V_1$ with V_1 in $\mathscr{U}_\partial$ and $R = R_c W$ with W in $\mathscr{U}_\partial$ together imply that $R_{1c} = R_c(WUV_1)$ and $P_{1c} = P_{H,G} UV_1 = G^{-1}\{P_c W - HR_c W\}UV_1 = P_{cH,G}(WUV_1)$. Thus, $\begin{bmatrix} R_c \\ P_c \end{bmatrix}$ is equivalent to $\begin{bmatrix} R_{1c} \\ P_{1c} \end{bmatrix}$ under output feedback. The converse is demonstrated by

reversing the argument.

<u>Definition 5.7.</u> <u>Let</u> $\sigma = \begin{bmatrix} R \\ P \end{bmatrix}$ <u>be an element of</u> $S^s_{p,m}$ <u>and let</u> $r_1, \ldots, r_p, p_1, \ldots, p_m$ <u>denote the rows of</u> σ. <u>Then</u> $\Re_k(\sigma) = \text{span}_k[r_1, \ldots, r_p, p_1, \ldots, p_m]$ <u>is the span over</u> k <u>of the rows</u> <u>of</u> σ.

<u>Theorem 5.8.</u> T <u>is equivalent to</u> T_1 <u>under output feedback if and only if</u> $\partial_T = \partial_{T_1}$, $R_c = R_{1c}$ <u>and</u> $\Re_k(\sigma_T) = \Re_k(\sigma_{T_1})$.

<u>Proof</u>: If T is equivalent to T_1 under output feedback, then T is equivalent to T_1 under state feedback and so, $\partial_T = \partial_{T_1}$ and $R_c = R_{1c}$. Moreover, in view of Lemma 5.6,

$$\begin{bmatrix} I & 0 \\ -G^{-1}H & G^{-1} \end{bmatrix} \begin{bmatrix} r_{c1} \\ \vdots \\ r_{cp} \\ p_{c1} \\ \vdots \\ p_{cm} \end{bmatrix} = \begin{bmatrix} r_{1c1} \\ \vdots \\ r_{1cp} \\ p_{1c1} \\ \vdots \\ p_{1cm} \end{bmatrix}$$

so that $[r_{1c1} \cdots r_{1cp}\ p_{1c1} \cdots p_{1cm}] \subset \Re_k(\sigma_T)$. Similarly, $[r_{c1} \cdots r_{cp}\ p_{c1} \cdots p_{cm}] \subset \Re_k(\sigma_{T_1})$ and so, $\Re_k(\sigma_T) = \Re_k(\sigma_{T_1})$.

Conversely, if $\partial_T = \partial_{T_1}$, $R_c = R_{1c}$ and $\Re_k(\sigma_T) = \Re_k(\sigma_{T_1})$, then $r_{1c1} = r_{c1}, \ldots, r_{1cp} = r_{cp}$ and $[r_{1c1} \ldots r_{1cp}\ p_{1c1} \cdots p_{1cp}] \subset \Re_k(\sigma_T)$. It follows that

$$\begin{bmatrix} I & 0 \\ N & M \end{bmatrix} \begin{bmatrix} r_{c1} \\ \vdots \\ r_{cp} \\ p_{c1} \\ \vdots \\ p_{cm} \end{bmatrix} = \begin{bmatrix} r_{c1} \\ \vdots \\ r_{cp} \\ p_{1c1} \\ \vdots \\ p_{1cm} \end{bmatrix}$$

for suitable N,M i.e. $NR_c + MP_c = P_{1c}$. But $\partial_i(R_c) < \partial_i(P_c)$, P_c, P_{1c} column proper, together imply $M \varepsilon GL(k,m)$. Thus, $P_{1c} = P_{cH,G}$ with $H = -GN$, $G = M^{-1}$ and so, T is equivalent to T_1 under output feedback.

Theorem 5.8 may be interpreted as stating that $(R_c, \partial, \mathfrak{R}_k(\sigma))$ is a complete invariant under output feedback.

6. Concluding Remarks

Considerable research has been done on the problem of finding invariants and canonical forms for linear systems under various equivalence relations (e.g. [2], [3], [4], [5], [9], [10], [11], [12].) For controllable systems, Brunovsky ([2]) and others ([3], [4], [11], [12]) determined a complete set of invariants under state feedback and a corresponding set of canonical forms. Kalman ([4]) and Rosenbrock ([11]) related feedback invariants to the classical Kronecker theory of singular pencils of matrices. Morse ([10]) studied invariants under a rather large group and Wonham and Morse ([12]) examined state feedback invariants. In a pivotal paper, Wang and Davison ([9]) developed a sound complete set of invariants under feedback with a reasonable indication of the true algebraic group acting on an algebraic set nature of the problem. Hermann and Martin ([3]) treated the controllable case using algebro-geometric methods and a result of Grothendieck. More or less with the exception of [3], all the results were developed in state space form for systems with strictly proper transfer matrices. In addition, the techniques used do not seem to be readily generalized to systems where k need not be a field.

Here a complete set of invariants and canonical forms are determined in the frequency domain for systems with proper transfer matrices. Moreover, the algebro-geometric nature of the problem is evident (see [8] for example) and the techniques used can be extended to the case of systems over integral domains without any difficulty. In addition, the methods used to obtain the canonical form under $\mathcal{U}_\partial$ can be employed to prove a "moduli" result for general groups of the form $N_\partial G_\partial$ where N_∂ has certain properties.

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