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J. Barkley Rosser

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ABSTRACT

Biharmonic functions are much used in the theory of elastic solids. A problem of long standing has been to produce a formula involving only the boundary conditions which gives a bound for the value of a biharmonic function in the interior of a region. Using results of Miranda, this problem is solved if the region is a circle or rectangle.

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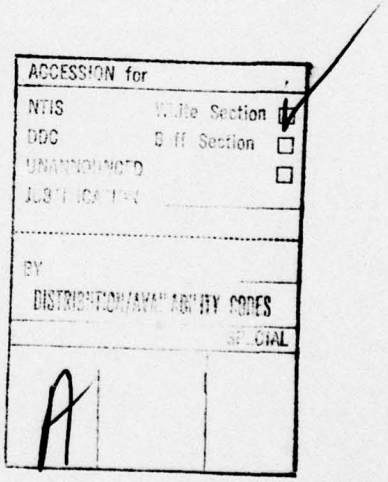
SIGNIFICANCE AND EXPLANATION

Biharmonic functions are much used in the theory of elasticity. A majorization formula gives an upper bound for $|u|$ in terms of the boundary conditions of u, where u is a biharmonic function. Such a formula would be useful in estimating the size of u, providing one has explicit constants in the formula. Apparently the best results to date (Miranda, 1948) had shown that a formula exists for each region, but gave no way to calculate the constants appearing in the formula for any region. In this report are supplied explicit constants for the two cases of a circle and a rectangle.



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MAJORIZATION FORMULAS FOR A BIHARMONIC FUNCTION OF TWO VARIABLES

J. Barkley Rosser

1. BACKGROUND. We propose to extend majorization formulas given by Miranda in [2]. He was considering a biharmonic function $u(x,y)$; that is,

$$\nabla^2\nabla^2 u = 0 \tag{1.1}$$

in a region T. He allowed T to be multiply connected, but required the curvature of the boundary (or boundaries) to be continuous around the boundary, FT, and the tangent to be continuously turning. On the boundary, values of u were specified,

$$u = f \ , \tag{1.2}$$

and values of the inward normal derivative were specified,

$$\frac{du}{d\nu} = g \ . \tag{1.3}$$

He considered f and g as functions of the arc length around the boundary. He also assumed that u and its first partial derivatives are continuous in T, up to and including (one-sided) continuity at the boundary.

His Theorem II says that if $f \equiv 0$, then in T

$$|u(x,y)| \leq \sqrt{2\phi(x,y)}\ \max|g| \ , \tag{1.4}$$

where the maximum of $|g|$ is taken over the boundary and $\phi(x,y)$ is the function which satisfies

$$\nabla^2\phi = -1 \tag{1.5}$$

inside T and has $\phi \equiv 0$ on FT.

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Upon relaxing the requirement that u be zero on the boundary, Miranda also had a majorization formula. He assumed continuity of f' and g, and concluded that there are nonnegative constants K_1 and K_2, depending solely on the region T, such that in T one has

$$|u(x,y)| \leq K_1\delta[\max|g| + \max|f'|] + (1 + K_2\delta)\max|f| , \tag{1.6}$$

where δ is the (minimum) distance from (x,y) to FT. See his Theorem VI.

This has the shortcoming that no clue is available as to what might be the sizes of K_1 and K_2.

Professor L. Collatz has pointed out in conversation that, if one is given boundary conditions for u, it may be possible to contrive a specifically given $\bar{u}$ whose boundary conditions are not greatly different from those of u. Then, if one had specific constants in (1.6), one could bound the difference between u and $\bar{u}$ by means of (1.6). We undertake to find in two cases majorization formulas involving specific constants that can be used as Professor Collatz suggests.

We will consider the two cases where T is a circle and where T is a rectangle, and will supply majorization formulas with specific constants; for the rectangle we have to assume in addition that f'' is of bounded variation, and will find $\max|f''|$ appearing in the corresponding majorization formula. We also need a slight additional hypothesis on u at the four corners of the rectangle.

In Section 2, we collect some auxiliary formulas. In Sections 3 and 4 respectively, we treat the circle and rectangle. The relevant majorization formulas are given near the ends of the two sections in Theorems 1 and 2. In Section 5 we discuss the possibility of weakening the hypotheses of Theorem 2.

2. AUXILIARY FORMULAS. We collect here various pieces of information that will be useful in subsequent sections.

Let $z = re^{i\theta}$ be a complex variable. If $|z| < 1$, then

$$\frac{1}{1 - z} = 1 + z + z^2 + \cdots . \tag{2.1}$$

Integrating gives

$$\ell n(1 - z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \cdots . \tag{2.2}$$

Taking real and imaginary parts gives

$$\frac{1}{2}\ell n(1 - 2r\cos\theta + r^2) = -\sum_{n=1}^{\infty} \frac{r^n\cos n\theta}{n} , \tag{2.3}$$

$$\arctan\frac{r\sin\theta}{1 - r\cos\theta} = \sum_{n=1}^{\infty} \frac{r^n\sin n\theta}{n} . \tag{2.4}$$

Taking $r = 1$ in (2.3) gives

$$\frac{1}{2}\ln 2 + \frac{1}{2}\ln(1 - \cos\theta) = -\sum_{n=1}^{\infty} \frac{\cos n\theta}{n} \quad . \tag{2.5}$$

Then

$$\int_0^{\frac{\pi}{2}} \ln(1 - \cos\theta)\,d\theta = -\frac{\pi}{2}\ln 2 - 2C \ , \tag{2.6}$$

where C is Catalan's constant,

$$C = \sum_{k=0}^{\infty} (-1)^k (2k + 1)^{-2} \ . \tag{2.7}$$

Properties of C are discussed on p. 807 of Abramowitz and Stegun [1], and a value to 18 decimals is given on p. 812. Rounded off to 5 decimals, it is

$$C \cong 0.91597. \tag{2.8}$$

From (2.5), we get

$$\int_{\frac{\pi}{2}}^{\pi} \ln(1 - \cos\theta)\,d\theta = 2C - \frac{\pi}{2}\ln 2 \ . \tag{2.9}$$

Taking $r = 1$ in (2.4) gives

$$\frac{\pi}{2} - \frac{\theta}{2} = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n} \qquad (0 < \theta < 2\pi) \ . \tag{2.10}$$

Taking $r = -1$ in (2.4) gives

$$\frac{\theta}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin n\theta}{n} \qquad (-\pi < \theta < \pi) \ . \tag{2.11}$$

By contour integration we can show that for $0 \leq r < R$

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{\ell^2}\,dt = 1 \tag{2.12}$$

$$\frac{(R^2 - r^2)^2}{2\pi R} \int_0^{2\pi} \frac{R - r\cos(t - \theta)}{\ell^4}\,dt = 1 \ , \tag{2.13}$$

where

$$\ell^2 = R^2 - 2Rr\cos(t - \theta) + r^2 \ . \tag{2.14}$$

Indeed (2.12) is worked out on pp. 112-113 of Whittaker and Watson [3] as an illustrative example of contour integration.

3. FORMULA FOR A CIRCLE. Let us take Miranda's region T to be a circle. Let u satisfy (1.1), (1.2), and (1.3). Let the circle have radius R. Choose polar coordinates r and θ, with the origin at the center of the circle. Then, by a formula of Lauricella [4], we have

$$\begin{aligned} u(r,\theta) = {} & \frac{1}{2\pi}\int_0^{2\pi} f(t)\,\frac{R^2 - r^2}{\ell^2}\,dt \\ & - \frac{r(R^2 - r^2)}{2\pi R}\int_0^{2\pi} f'(t)\,\frac{\sin(t - \theta)}{\ell^2}\,dt \\ & + \frac{R^2 - r^2}{4\pi R}\int_0^{2\pi} g(t)\,\frac{R^2 - r^2}{\ell^2}\,dt \ , \end{aligned} \tag{3.1}$$

where ℓ is the distance from the point (r,θ) to the point (R,t); that is, ℓ^2 is given by (2.14).

This formula is usually cited with a minus before the last term on the right, because Lauricella was taking $g(\theta)$ to be $\partial u/\partial r$, whereas we are taking $g(\theta)$ to be the inward normal derivative. Integration by parts in the second term on the right of (3.1) reduces (3.1) to

$$u(r,\theta) = \frac{(R^2 - r^2)^2}{2\pi R}\int_0^{2\pi} f(t)\,\frac{R - r\cos(t - \theta)}{\ell^4}\,dt + \frac{R^2 - r^2}{4\pi R}\int_0^{2\pi} g(t)\,\frac{R^2 - r^2}{\ell^2}\,dt \ . \tag{3.2}$$

This formula must be used with some caution. If, at $\theta = \theta_0$, $f'(\theta)$ has a jump discontinuity and $g(\theta)$ is continuous, then the limit of the inward normal derivative as one approaches the circumference along the ray $\theta = \theta_0$ is

$$g(\theta_0) + \frac{f'(\theta_0+) - f'(\theta_0-)}{\pi} \ . \tag{3.3}$$

This and other idiosyncracies of (3.2) are discussed in Picone [5]; see especially pp. 216-217 and pp. 257-261.

If we make the same assumptions that Miranda made to derive (1.4) and (1.6), namely that u and its first partial derivatives are continuous in T up to and including the circumference, that will assure us that f' and g are continuous. Then (3.2) defines the one and only u satisfying (1.1), (1.2), and (1.3) for the circle.

As the integrands in (2.12) and (2.13) are positive, one can immediately conclude from (3.2) by the mean value theorem that, for some $\tau(r,\theta)$ and $\sigma(r,\theta)$,

$$u = f(\tau(r,\theta)) + \frac{R^2 - r^2}{2R} g(\sigma(r,\theta)) \quad . \tag{3.4}$$

As a consequence, if $g(\sigma)$ is nonpositive for all σ, we have

$$u \leq \max f \quad . \tag{3.5}$$

Similarly, if $g(\sigma)$ is nonnegative for all σ, one has

$$u \geq \min f \quad . \tag{3.6}$$

In Picone [5], at the bottom of p. 216, these surprising conclusions are attributed to Miranda, who apparently used essentially the same proof.

From (3.4), we immediately get our majorization formula.

Theorem 1. Let u be a biharmonic function satisfying (1.1), (1.2), and (1.3) in a circle of radius R, such that u and its first partial derivatives are continuous up to and including the circumference. Then at a point r units from the center,

$$|u| \leq \max|f| + \frac{R^2 - r^2}{2R} \max|g| \quad . \tag{3.7}$$

Clearly this is much superior to Miranda's Theorem VI (see (1.6)). Also, if $f \equiv 0$ it is appreciably better than Miranda's Theorem II (see (1.4)). Indeed, for the region under consideration, the ϕ of Miranda's Theorem II is

$$\phi = (R^2 - r^2)/4 \quad . \tag{3.8}$$

This ϕ is biharmonic. If we put it for u in (3.7), we find that it itself is given as an upper bound for itself. Clearly this is the best possible. However, if we use this ϕ for u in Miranda's Theorem II, namely (1.4), a bound for u at the center of the circle is given which is $\sqrt{2}$ times as great as the actual value of u at that point.

4. FORMULA FOR A RECTANGLE. If T is a simply connected region more irregular than a circle, one may map it conformally into a circle. If the region has a smooth enough boundary, the conformality may extend out to the boundary, so that normal derivatives go into normal derivatives. If one has a formula for the conformal mapping, it may be tractable enough that one can calculate factors of proportionality for the various derivatives. Thus one can sometimes convert (3.7) into a majorization formula for the more general region.

When one maps a rectangle into a circle, conformality certainly does not extend out to the boundary at the four corners. In any case the formula for the transformation is much too complicated to be of much use. So we give a separate treatment for the case that T is a rectangle.

Let a and b be the lengths of the sides of the rectangle. Choose coordinates so that one corner of the rectangle is at the origin, and the rectangle extends a units along the positive x-axis, and b units along the positive y-axis.

Let u satisfy (1.1), (1.2), and (1.3). We assume that u and its first partial derivatives are continuous up to and including the perimeter. Also, on each side we assume that f'' is of bounded variation in the closed interval consisting of that side.

At the corners, strange things can happen. For one thing, a normal derivative is not defined at a corner, nor do the normal derivatives along the two sides have to have the same limits as one approaches a corner. However, continuity of $\partial u/\partial x$ as one goes to the corner means that the limit of a normal derivative on a vertical side as one approaches an upper corner must be either f' or $-f'$ along the top at the corner. Whether it is f' or $-f'$ depends on which direction is taken as increasing arc length, and will be different at different corners. We assume further:

Boundedness Hypothesis. For each corner there is a neighborhood of that corner within which $\nabla^2 u$ is bounded.

As an illustration of a need for a boundedness hypothesis, we cite the following example. Consider the first quadrant of a circle of radius unity with center at the origin. If a function is harmonic inside this region and is zero around the boundary, it must be identically zero by the maximum principle. But note the harmonic function

$$v = r^2 \sin 2\theta - \frac{\sin 2\theta}{r^2} \quad . \tag{4.1}$$

It is zero around the boundary, except for an indeterminacy at the origin, where $r = 0$. However, v is certainly not identically zero. It is because of the unboundedness at the origin that the usual maximum principle for harmonic functions fails. The reason we invoke our Boundedness Hypothesis is to avert a similar difficulty.

As we said, we let u satisfy (1.1), (1.2), and (1.3). Choose $\bar{u}$ to be the harmonic function inside the rectangle such that on the perimeter

$$\bar{u} = f \quad . \tag{4.2}$$

As $\bar{u}$ is harmonic, it is biharmonic. So $u - \bar{u}$ is biharmonic, and is zero on the perimeter. Applying Miranda's Theorem II (see (1.4)), we get

$$|u - \bar{u}| \leq \sqrt{2\phi}\, \max|g - \bar{g}| \quad , \tag{4.3}$$

where $\bar{g}$ is the value of the inward normal derivative for $\bar{u}$, and ϕ is as in (1.4).

We will later show that $\bar{u}$ and its first partial derivatives are continuous up to and including the perimeter. So $u - \bar{u}$ satisfies those of Miranda's hypotheses. However, Miranda also assumed that his region had a boundary with continuous curvature and continuously turning tangent. Lacking these, we proceed as follows.

On p. 99 of Miranda [2] is proved the Lemma that $u\nabla^2 u$ is continuous in the region, and approaches zero on the boundary, if $f \equiv 0$. One can easily modify Miranda's proof to show that if some segment of the boundary has continuous curvature and a continuously turning tangent, then within a closed portion of that segment $u\nabla^2 u$ approaches zero uniformly as one approaches the boundary. So in the interior of each side of the rectangle, one has $(u - \bar{u})\nabla^2(u - \bar{u})$ approaching zero as one approaches the perimeter. In the neighborhood of a corner, $\nabla^2 u$ is bounded. But $\bar{u}$ is harmonic, so that $\nabla^2\bar{u} = 0$. Hence $\nabla^2(u - \bar{u})$ is bounded. But $u - \bar{u}$ approaches zero, continuously. So we conclude that $(u - \bar{u})\nabla^2(u - \bar{u})$ approaches zero as one approaches a corner.

So Miranda's Lemma holds for the rectangle, and the rest of the proof proceeds just as in Miranda [2].

For the ϕ of (4.3), we may start with

$$\phi^* = \frac{1}{2} x(a - x) \quad . \tag{4.4}$$

This is zero on the two vertical sides of the rectangle. Now add to ϕ^* a harmonic function ϕ^{**} which is zero on the two vertical sides, and equal to

$$-\frac{1}{2} x(a - x)$$

on the top and bottom. Then $\phi = \phi^* + \phi^{**}$. By the principle of the maximum, ϕ^{**} will be everywhere nonpositive. So the ϕ of (4.3) will be bounded above by (4.4). It will also be nonnegative, because it is zero on the perimeter. By a similar argument, ϕ is bounded above by

$$\phi^{***} = \frac{1}{2} y(b - y) \quad .$$

We have

$$|u| \leq |u - \bar{u}| + |\bar{u}| \quad .$$

But, by the principle of the maximum,

$$|\bar{u}| \leq \max|f| \quad .$$

So, by (4.3) we get

$$|u| \leq \max|f| + \sqrt{2\phi}\ \{\max|g| + \max|\bar{g}|\} \quad . \tag{4.5}$$

So we wish to find $\max|\bar{g}|$.

Although f' will make random jumps at the corners, f will be continuous at each corner, since u is to be continuous up to and including the perimeter.

In the sequel, we will use superscripts T, B, L, and R to signify the top, bottom, left, and right sides of the rectangle. We will express $\bar{u}$ as

(4.6) $$\bar{u} = \sum + \sum^T + \sum^B + \sum^L + \sum^R .$$

In this we choose $\sum$ a polynomial

(4.7) $$\sum = A + Bx + Cy + Dxy ,$$

with A, B, C, and D chosen so that $\sum$ has the same value as $\bar{u}$ at each of the four corners of the rectangle. To accomplish this, we set

(4.8) $$A = f(0,0) ,$$

(4.9) $$B = \frac{f(a,0) - f(0,0)}{a} ,$$

(4.10) $$C = \frac{f(0,b) - f(0,0)}{b} ,$$

(4.11) $$D = \frac{f(a,b) - f(a,0) - f(0,b) + f(0,0)}{ab} ;$$

recall that $f(x,y)$ is the value assumed by $\bar{u}$ around the perimeter of the rectangle.

We will first determine the inward normal for $\bar{u}$ along the top of the rectangle. A similar analysis will apply to each of the other three sides.

We have

(4.12) $$-\frac{\partial}{\partial y}\sum = -C - Dx .$$

At $x = 0$, this is

(4.13) $$-\frac{f(0,b) - f(0,0)}{b}$$

and at $x = a$, this is

(4.14) $$-\frac{f(a,b) - f(a,0)}{b} .$$

As the right side of (4.12) is linear, its extreme values must be (4.13) and (4.14). However, (4.13) is bounded below and above by the minimum and maximum of f', evaluated for $x = 0$. (Here we are considering arc length as increasing counterclockwise around the rectangle, so that f' evaluated for $x = 0$ is

$$-\frac{\partial}{\partial y} f(0,y) .)$$

Similarly, (4.14) is bounded below and above by the minimum and maximum of $-f'$, evaluated for $x = a$.

We take

$$F = \max|f| \,, \tag{4.15}$$

$$F' = \max|f'| \,, \tag{4.16}$$

$$F'' = \max|f''| \,, \tag{4.17}$$

taken around the perimeter of the rectangle. Then we have just shown that

$$\left| -\frac{\partial}{\partial y} \sum \right| \leq F' \,. \tag{4.18}$$

We take

$$\sum^T = \sum_{j=1}^{\infty} A_j^T \frac{\sinh \frac{j\pi y}{a}}{\sinh \frac{j\pi b}{a}} \sin \frac{j\pi x}{a} \,, \tag{4.19}$$

$$\sum^B = \sum_{j=1}^{\infty} A_j^B \frac{\sinh \frac{j\pi (b - y)}{a}}{\sinh \frac{j\pi b}{a}} \sin \frac{j\pi x}{a} \,, \tag{4.20}$$

$$\sum^L = \sum_{j=1}^{\infty} A_j^L \frac{\sinh \frac{j\pi (a - x)}{b}}{\sinh \frac{j\pi a}{b}} \sin \frac{j\pi y}{b} \,, \tag{4.21}$$

$$\sum^R = \sum_{j=1}^{\infty} A_j^R \frac{\sinh \frac{j\pi x}{b}}{\sinh \frac{j\pi a}{b}} \sin \frac{j\pi y}{b} \,, \tag{4.22}$$

where

$$A_j^T = \frac{2}{a} \int_0^a \sin \frac{j\pi x}{a} \left\{ f(x,b) - f(0,b) - x \frac{f(a,b) - f(0,b)}{a} \right\} dx \,, \tag{4.23}$$

$$A_j^B = \frac{2}{a} \int_0^a \sin \frac{j\pi x}{a} \left\{ f(x,0) - f(0,0) - x \frac{f(a,0) - f(0,0)}{a} \right\} dx \,, \tag{4.24}$$

$$A_j^L = \frac{2}{b} \int_0^b \sin \frac{j\pi y}{b} \left\{ f(0,y) - f(0,0) - y \frac{f(0,b) - f(0,0)}{b} \right\} dy \,, \tag{4.25}$$

$$A_j^R = \frac{2}{b} \int_0^b \sin \frac{j\pi y}{b} \left\{ f(a,y) - f(a,0) - y \frac{f(a,b) - f(a,0)}{b} \right\} dy \,. \tag{4.26}$$

We note that $\sum^T$ is zero on the sides and bottom of the rectangle, and on the top it equals

$$f(x,b) - f(0,b) - x\,\frac{f(a,b) - f(0,b)}{a}\,,$$

which is the value of

$$\bar{u} - \sum$$

along the top of the rectangle. Similar considerations apply to $\sum^B$, $\sum^L$, and $\sum^R$ with respect to other sides of the rectangle. So

$$\sum^T + \sum^B + \sum^L + \sum^R$$

takes the same values as

$$\bar{u} - \sum$$

around the perimeter of the rectangle. So

$$\sum + \sum^T + \sum^B + \sum^L + \sum^R \tag{4.27}$$

takes the same values as $\bar{u}$ around the perimeter of the rectangle. But (4.27) is harmonic. By the uniqueness of the solution of a harmonic equation, we conclude that (4.6) holds.

We can integrate by parts in (4.23), (4.24), (4.25), and (4.26) to get

$$A_j^T = -\frac{2a}{j^2\pi^2}\int_0^a f''(x,b)\sin\frac{j\pi x}{a}\,dx\,, \tag{4.28}$$

$$A_j^B = -\frac{2a}{j^2\pi^2}\int_0^a f''(x,0)\sin\frac{j\pi x}{a}\,dx\,, \tag{4.29}$$

$$A_j^L = -\frac{2b}{j^2\pi^2}\int_0^b f''(0,y)\sin\frac{j\pi y}{b}\,dy\,, \tag{4.30}$$

$$A_j^R = -\frac{2b}{j^2\pi^2}\int_0^b f''(a,y)\sin\frac{j\pi y}{b}\,dy\,, \tag{4.31}$$

where in (4.28) and (4.29) the double primes indicate the second partial derivatives with respect to x, and in (4.30) and (4.31) the double primes indicate the second partial derivatives with respect to y. Be it recalled that these second derivatives were assumed to be of bounded variation. By the result on p. 172 of Whittaker and Watson [3], the integrals on the right sides of (4.28), (4.29), (4.30), and (4.31) each decrease of the order of j^{-1}. So each of $|A_j^T|$, $|A_j^B|$, $|A_j^L|$, and $|A_j^R|$ goes to zero of the order of j^{-3} as j goes to infinity.

Thus the various series on the right of (4.19), (4.20), (4.21), and (4.22) converge absolutely and uniformly everywhere in the rectangle, including the perimeter, since

$$\left| \frac{\sinh \frac{j\pi y}{a}}{\sinh \frac{j\pi b}{a}} \sin \frac{j\pi x}{a} \right| \leq 1 ,$$

etc. This assures the continuity of $\sum^T$, $\sum^B$, $\sum^L$, and $\sum^R$. If we take $\partial/\partial x$ or $\partial/\partial y$ of $\sum^T$, $\sum^B$, $\sum^L$, or $\sum^R$, we will multiply terms by a constant times j, and replace some sines by cosines, or some sinh's by cosh's; the latter will not make an appreciable difference for large j. After multiplication by j, the coefficients will still go to zero of the order of j^{-2}. This will still assure absolute and uniform convergence everywhere in the rectangle, including the perimeter, so that the partial derivatives will be continuous. Needless to say, $\sum$ and its first partial derivatives are continuous. So, by (4.6), $\bar{u}$ and its first partial derivatives are continuous up to and including the perimeter.

Along the top of the rectangle, the inward normal for $\sum^T$ is

$$(4.32) \qquad -\frac{\partial}{\partial y} \sum^T = -\sum_{j=1}^{\infty} \frac{j\pi}{a} A_j^T \frac{\cosh \frac{j\pi b}{a}}{\sinh \frac{j\pi b}{a}} \sin \frac{j\pi x}{a} .$$

We split the right side of (4.32) into

$$\sum_1 + \sum_2 ,$$

where

$$(4.33) \qquad \sum_1 = -\sum_{j=1}^{\infty} \frac{j\pi}{a} A_j^T \sin \frac{j\pi x}{a} ,$$

$$(4.34) \qquad \sum_2 = -\sum_{j=1}^{\infty} \frac{j\pi}{a} \frac{2A_j^T}{\exp\left(\frac{2j\pi b}{a}\right) - 1} \sin \frac{j\pi x}{a} .$$

Let $0 \leq r \leq 1$. Then, since $|jA_j^T|$ goes to zero of the order of j^{-2}, we conclude

$$\sum_1 = \lim_{r \to 1} \sum_r ,$$

where

$$\sum_r = -\sum_{j=1}^{\infty} \frac{j\pi r^j}{a} A_j^T \sin \frac{j\pi x}{a} .$$

By (4.28), we have for $0 \le r < 1$

$$\sum_r = \frac{2}{\pi}\int_0^a f''(t,b)\left\{\sum_{j=1}^{\infty} \frac{r^j \sin\frac{j\pi t}{a}\sin\frac{j\pi x}{a}}{j}\right\}dt .$$

This gives

$$\sum_r = \frac{1}{\pi}\int_0^a f''(t,b)\left\{\sum_{j=1}^{\infty} \frac{r^j \cos\frac{\pi j(t-x)}{a}}{j} - \sum_{j=1}^{\infty} \frac{r^j \cos\frac{\pi j(t+x)}{a}}{j}\right\}dt .$$

By (2.3), we get

$$\sum_r = \frac{1}{2\pi}\int_0^a f''(t,b)\left\{\ell n\left(1 - 2r\cos\frac{\pi(t+x)}{a} + r^2\right)\right.$$

$$\left. - \ell n\left(1 - 2r\cos\frac{\pi(t-x)}{a} + r^2\right)\right\}dt .$$

As $f''(t,b)$ is of bounded variation, we easily justify taking the limit as $r \to 1$. This gives

$$\sum_1 = \frac{1}{2\pi}\int_0^a f''(t,b)\left\{\ell n\left(1 - \cos\frac{\pi(t+x)}{a}\right) - \ell n\left(1 - \cos\frac{\pi(t-x)}{a}\right)\right\}dt . \tag{4.35}$$

We have

$$\left|\frac{1}{2\pi}\int_0^a f''(t,b)\,\ell n\left(1 - \cos\frac{\pi(t+x)}{a}\right)dt\right|$$

$$= \left|\frac{a}{2\pi^2}\int_{\pi x/a}^{\pi+(\pi x/a)} f''\left(\frac{as}{\pi} - x, b\right)\ell n(1 - \cos s)\,ds\right|$$

$$\le \frac{F''a}{2\pi^2}\int_{\pi x/a}^{\pi+(\pi x/a)} \left|\ell n(1 - \cos s)\right| ds .$$

We have also

$$\left| \frac{-1}{2\pi} \int_0^a f''(t,b) \ell n\left(1 - \cos \frac{\pi(t - x)}{a}\right) dt \right|$$

$$= \left| \frac{-a}{2\pi^2} \int_{-\pi x/a}^{\pi-(\pi x/a)} f''\left(\frac{as}{\pi} + x, b\right) \ell n(1 - \cos s) ds \right|$$

$$\leq \frac{F''a}{2\pi^2} \int_{-\pi x/a}^{\pi-(\pi x/a)} |\ell n(1 - \cos s)| ds$$

$$= \frac{F''a}{2\pi^2} \int_{\pi+(\pi x/a)}^{2\pi+(\pi x/a)} |\ell n(1 - \cos s)| ds \ .$$

Adding these gives

$$|\Sigma_1| \leq \frac{F''a}{2\pi^2} \int_0^{2\pi} |\ell n(1 - \cos s)| ds \ .$$

By (2.6) and (2.9), we have

$$|\Sigma_1| \leq \frac{4CF''a}{\pi^2} \ . \tag{4.36}$$

We turn to Σ_2. We have, of course,

$$0 \leq \frac{\frac{2j\pi b}{a}}{\exp\left(\frac{2j\pi b}{a}\right) - 1} \leq 1 \ .$$

So, by (4.34)

$$|\Sigma_2| \leq \frac{1}{b} \sum_{j=1}^{\infty} |A_j^T| \ .$$

Then, by (4.28),

$$|\Sigma_2| \leq \frac{2a}{\pi^2 b} \sum_{j=1}^{\infty} \frac{1}{j^2} \int_0^a |f''(x,b)| dx \ .$$

So

$$|\Sigma_2| \leq \frac{a^2 F''}{3b} \ .$$

Combining with (4.36) gives

$$\left| - \frac{\partial}{\partial y} \sum^{T} \right| \leq \left\{ \frac{4Ca}{\pi^2} + \frac{a^2}{3b} \right\} F'' \quad . \tag{4.37}$$

Along the top of the rectangle, the inward normal derivative for $\sum^{B}$ is

$$- \frac{\partial}{\partial y} \sum^{B} = \sum_{j=1}^{\infty} \frac{j\pi}{a} \frac{A_j^B}{\sinh \frac{j\pi b}{a}} \sin \frac{j\pi x}{a} \quad . \tag{4.38}$$

We have, of course,

$$0 \leq \frac{\frac{j\pi b}{a}}{\sinh \frac{j\pi b}{a}} \leq 1 \quad .$$

So

$$\left| - \frac{\partial}{\partial y} \sum^{B} \right| \leq \frac{1}{b} \sum_{j=1}^{\infty} |A_j^B| \quad .$$

Then, by (4.29)

$$\left| - \frac{\partial}{\partial y} \sum^{B} \right| \leq \frac{2a}{\pi^2 b} \sum_{j=1}^{\infty} \frac{1}{j^2} \int_0^a |f''(x,0)| dx \quad .$$

So

$$\left| - \frac{\partial}{\partial y} \sum^{B} \right| \leq \frac{a^2}{3b} F'' \quad . \tag{4.39}$$

Along the top of the rectangle, the inward normal derivative for $\sum^{L}$ is

$$- \frac{\partial}{\partial y} \sum^{L} = - \sum_{j=1}^{\infty} \frac{j\pi}{b} A_j^L \frac{\sinh \frac{j\pi(a - x)}{b}}{\sinh \frac{j\pi a}{b}} (-1)^j \quad . \tag{4.40}$$

By (4.30), this gives for $0 < x \leq a$

$$- \frac{\partial}{\partial y} \sum^{L} = \frac{2}{\pi} \int_0^b f''(0,y) \left\{ \sum_{j=1}^{\infty} \frac{\sinh \frac{j\pi(a - x)}{b}}{\sinh \frac{j\pi a}{b}} \frac{(-1)^j \sin \frac{j\pi y}{b}}{j} \right\} dy \quad . \tag{4.41}$$

Temporarily denote the material in the curly brackets by $\sum^*$. By (2.11), it is just the value at (x,y) of the harmonic function in the rectangle which equals $-\pi y/2b$ on the left side and is zero on the other three sides. So, by the principle of the maximum, $\sum^*$ is nonpositive. However

$$\sum^* + \frac{\pi y(a - x)}{2ab}$$

is a harmonic function which is nonnegative on the perimeter. So by the principle of the minimum,

$$\sum^* + \frac{\pi y(a - x)}{2ab}$$

is nonnegative. So we conclude that

$$- \frac{\pi y(a - x)}{2ab} \leq \sum^* \leq 0 .$$

Using this in (4.41) gives

$$\left| - \frac{\partial}{\partial y} \sum^L \right| \leq \frac{a - x}{ab} \int_0^b |f''(0,y)| y dy \leq \frac{b(a - x)F''}{2a} . \tag{4.42}$$

Our proof of this required the assumption $0 < x \leq a$ to insure convergence rapid enough to permit interchange of the order of summation and integration in going from (4.40) to (4.41). But

$$\frac{\partial}{\partial y} \sum^L$$

is continuous for $0 \leq x \leq a$. So (4.42) must hold also for $x = 0$.

A similar argument will give

$$\left| - \frac{\partial}{\partial y} \sum^R \right| \leq \frac{bxF''}{2a} . \tag{4.43}$$

So along the top of the rectangle, we have the inward normal derivative bounded as follows

$$\left| - \frac{\partial}{\partial y} \bar{u} \right| \leq F' + \left\{ \frac{4Ca}{\pi^2} + \frac{2a^2}{3b} + \frac{b}{2} \right\} F'' . \tag{4.44}$$

Along the bottom, one gets the same bound. On each side, one gets the same bound, except with a and b interchanged.

So, for a bound on our normal derivative, we have to use whichever is larger of

$$\frac{4Ca}{\pi^2} + \frac{2a^2}{3b} + \frac{b}{2} \tag{4.45}$$

and

$$\frac{4Cb}{\pi^2} + \frac{2b^2}{3a} + \frac{a}{2} \,. \tag{4.46}$$

If we subtract (4.46) from (4.45), the difference is seen to be

$$\frac{(a - b)}{6ab}\left\{4a^2 + ab + 4b^2 + \frac{24Cab}{\pi^2}\right\} \,.$$

Hence, we see that if $a \geq b$, then (4.45) is greater than or equal to (4.46). So we get the following result:

Theorem 2. Let u be a biharmonic function satisfying (1.1), (1.2), and (1.3) in a rectangle having sides of lengths a and b, where $a \geq b$. Let u and its first partial derivatives be continuous up to and including the perimeter. On each side, let f'' be of bounded variation in the closed interval consisting of that side. At each corner, let there be a neighborhood of the corner within which $\nabla^2 u$ is bounded. Then

$$|u| \leq \max|f| + \sqrt{2\phi}\left\{\max|g| + \max|f'| + \left[\frac{4Ca}{\pi^2} + \frac{2a^2}{3b} + \frac{b}{2}\right]\max|f''|\right\} \,, \tag{4.47}$$

where the maxima are taken over the perimeter, and $\phi(x,y)$ is the function which satisfies

$$\nabla^2\phi = -1 \tag{4.48}$$

inside the rectangle and has $\phi \equiv 0$ on the perimeter, and C is given by (2.7).

NOTE. If the rectangle is oriented with one corner at the origin, one side of length a along the positive x-axis, and another side of length b along the positive y-axis, then ϕ will be defined as in the paragraph beginning just before (4.4). So ϕ is nonnegative and is bounded by

$$\frac{1}{2}\max\{x(a - x),\ y(b - y)\} \,.$$

5. POSSIBLE WEAKENING OF HYPOTHESES. In Thm. 2 we assume continuity of both first derivatives up to and including the perimeter. Could we relax this assumption just at the four corners?

To get some feeling for this, consider the unit square, ABCD (see Fig. 1), situated in the first quadrant with A at the origin. Let us require that u be zero on the perimeter, and ask for an inward normal of -1 along AB and DC and of $+1$ along AD and BC.

Lift the figure out of the plane, and flip it about the diagonal AC; AB goes up and over to the positive of AD, while AD goes down and under to the position of AB. We now have the same boundary conditions that we started with. So, if they determine u uniquely, we must have the same values of u

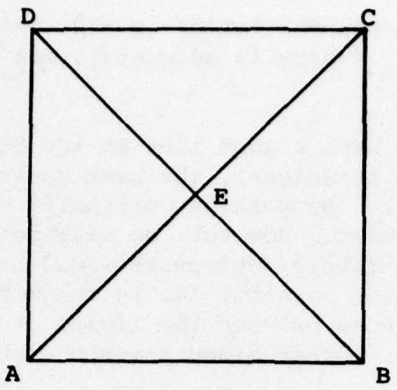

Figure 1

as before. However, since the figure has been flipped upside down, the values along AC have been changed to their negatives. As they come out the same as before, they must be zero. Similarly, we conclude that u is zero along BD.

As we have an inward normal of -1 along AB, the values of u next to AB must be negative. Likely u is negative all inside the triangle AEB, sloping down from AB and up to AE and EB. However, even if this is not the case, consider what happens if one starts vertically from AB, at a distance x from A with $x < 1/2$. One starts off with a slope of -1, which certainly takes one to negative values of u. But by the time one gets up to AE, u has got back up to zero. So one must encounter some place of positive slope. So the slope has gone from -1 to a positive value in a distance less than x. So, someplace along the way $\partial^2 u/\partial y^2$ must be at least $1/x$.

To find out what $\nabla^2 u$ is doing, we have also to get an idea of the behavior of $\partial^2 u/\partial x^2$. Let us go along parallel to AB, and close to it, from AE to BE. We start with $u = 0$ at AE and finish with $u = 0$ at BE. If we are close enough to AB, u will be mostly negative in between. This indicates that $\partial^2 u/\partial x^2$ will tend to be positive.

Thus it appears that as we approach A in the triangle AEB, we will encounter points where $\nabla^2 u$ is greater than $1/x$. Not only are we violating our Boundedness Hypothesis, but it appears possible that $u\nabla^2 u$ is not approaching zero as we get close to A. So Miranda's Lemma is likely failing. This voids our proof of Theorem 2.

This suggests that if we admit discontinuity of first derivatives at a corner, we may entail a violation of our Boundedness Criterion.

If we retain continuity of both first derivatives at the corners, do we really need the Boundedness Hypothesis? The key result is (4.3). As $u - \bar{u}$ is zero along the perimeter, the derivative along the perimeter must also be zero. Given continuity at the corner, $u - \bar{u}$ and both its first derivatives must approach zero continuously as one approaches a corner. This does not

seem to leave much latitude for misbehavior of $u - \bar{u}$. We will conjecture that this suffices to give (4.3) (which is adequate), but do not see at this time how to prove it.

Actually, though we do not have a good idea of the behavior of the u of Fig. 1 (assuming it exists, and is unique), the best guess we can make indicates that it actually satisfies (1.4). So perhaps continuity of first derivatives at the corners is not really needed. However, we will not venture to conjecture this. In view of (3.3), a more likely conjecture would be that one could prove something like (1.4), or its parallel (4.3), but with an extra term on the right involving the differences between the limits at a corner of a first derivative as one approaches the corner along the two sides.

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Biharmonic functions are much used in the theory of elastic solids. A problem of long standing has been to produce a formula involving only the boundary conditions which gives a bound for the value of a biharmonic function in the interior of a region. Using results of Miranda, this problem is solved if the region is a circle or rectangle.

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