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UNIQUENESS OF SOLUTIONS OF THE INITIAL-VALUE PROBLEM FOR $u_t - \Delta\varphi(u) = 0$

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(6) UNIQUENESS OF SOLUTIONS OF THE INITIAL-VALUE PROBLEM
FOR $u_t^{\eta} - \Delta\varphi(u) = 0$

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ABSTRACT

Equations of the form $u_t - \Delta\varphi(u) = 0$ arise in mathematical models of many physical situations. The uniqueness of solutions of the associated initial-value problem in $\mathbb{R}^N$ is considered in this paper. It is shown that bounded weak solutions (i.e., solutions in the sense of distributions) which further satisfy an integrability condition are unique.

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SIGNIFICANCE AND EXPLANATION

In this paper it is shown that the solutions of initial-value problems for a class of quasilinear parabolic equations of broad interest in applications are uniquely identified by rather mild conditions. It is desirable to know very weak conditions which imply uniqueness of solutions of problems of this sort for a variety of reasons. One such reason is that if an approximation process is given to approximate the solution, one need only check that these weak conditions are satisfied by limits of the approximations in order to establish convergence of the process to the correct solution.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

UNIQUENESS OF SOLUTIONS OF THE INITIAL-VALUE PROBLEM FOR $u_t - \Delta\varphi(u) = 0$

Haim Brézis and Michael G. Crandall

Introduction:

This paper is concerned with the uniqueness of solutions of the initial value problem

$$(1)\quad \begin{cases} \text{(i)} & u_t - \Delta\varphi(u) = 0, \quad 0 < t < T, \quad x \in \mathbb{R}^N, \\ \text{(ii)} & u(0,x) = u_0(x), \quad x \in \mathbb{R}^N, \end{cases}$$

where

$$(2)\qquad \varphi : \mathbb{R} \to \mathbb{R} \text{ is nondecreasing, continuous and } \varphi(0) = 0 .$$

Equations of this sort arise in many applications. These include heat flow in materials with a temperature dependent conductivity, flow in a porous medium, the Stefan problem, biological models, etc.

The main result is formulated below. We have set $Q = (0,T) \times \mathbb{R}^N$ and the expression "in $\mathcal{D}'(Q)$" means in the sense of distributions on Q.

Theorem 1. Let (2) hold and $u, \hat{u}$ satisfy

$$(3)\qquad u, \hat{u} \in L^\infty(Q) ,$$

$$(4)\qquad u_t - \Delta\varphi(u) = \hat{u}_t - \Delta\varphi(\hat{u}) \text{ in } \mathcal{D}'(Q) ,$$

$$(5)\qquad u - \hat{u} \in L^1(Q) ,$$

and

$$(6)\qquad \underset{t \downarrow 0}{\text{essential limit}} \int_{\mathbb{R}^N} |u(t,x) - \hat{u}(t,x)|\,dx = 0 .$$

Then $u = \hat{u}$ a.e. on Q.

Theorem 1 implies that bounded solutions u of (1)(i) in the sense of distributions which further satisfy $w(t,x) = u(t,x) - u_0(x) \in L^1(Q)$ and $w(t,\cdot) \to 0$ in $L^1(\mathbb{R}^N)$ as $t \downarrow 0$ are

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unique. Among the earlier uniqueness results we mention the works of Sabinina [11] and Vol'pert and Hudjaev [14]. Sabinina announces a theorem which can be proved by the method exposed in [10] while Vol'pert and Hudjaev consider a broad class of equations (including first order ones) and use the relatively deep theory of BV spaces both in the formulation of their results and the proofs. In any case, as applied to (1), these results assume significantly more regularity of φ than mere continuity as well as conditions on grad $\varphi(u)$ which we do not impose. Here "grad" denotes the gradient with respect to $(x_1,\dots,x_N)$. On the other hand, given (3), our conditions (5) and (6) are somewhat more stringent than those of [11]. This will be rectified in the remarks ending Section 1. Other works concerning the uniqueness question for (1) and variants of it are, for the most part, concerned with one space variable. See, for example, Gilding and Peletier [6], Kalašnikov [7], Kamin [8], and Kershner [9].

In some circumstances of interest we can weaken (3) (which corresponds to $u_0 \in L^\infty(\mathbb{R}^N)$ in (1)). In particular, we have:

<u>Theorem 2</u>. Let $\alpha > \max((N-2)/N, 0)$ and $\varphi(r) = |r|^\alpha \operatorname{sign} r$. Then for each $u_0 \in L^1(\mathbb{R}^N)$ there is exactly one function u satisfying

$$u \in C([0,\infty):L^1(\mathbb{R}^N)) \cap L^\infty([a,\infty) \times \mathbb{R}^N) \quad \text{for every } a > 0 , \tag{7}$$

$$u_t - \Delta\varphi(u) = 0 \quad \text{in} \quad \mathcal{D}'((0,\infty) \times \mathbb{R}^N) , \tag{8}$$

and

$$u(0,\cdot) = u_0(\cdot) . \tag{9}$$

Theorem 1 is proved in Section 1 and Theorem 2 is proved in Section 2. Both sections include remarks concerning variations of these results.

Section 1. The Proof of Theorem 1.

Let $u, \hat{u}$ be as in Theorem 1. Then the functions $z = u - \hat{u}$, $h = \varphi(u) - \varphi(\hat{u})$ satisfy the conditions of the following lemma, which therefore implies Theorem 1.

Proposition 1. Let $z \in L^1(Q) \cap L^\infty(Q)$ and $h \in L^\infty(Q)$. Let

$$z_t - \Delta h = 0 \quad \text{in } \mathcal{D}'(Q) \quad , \tag{1.1}$$

$$zh \geq 0 \quad \text{a.e. in } Q \quad , \tag{1.2}$$

$$\text{meas}\{(t,x) \in Q : |h(t,x)| > \xi\} < \infty \quad \text{for each } \xi > 0 \quad , \tag{1.3}$$

where $\text{meas}\, A$ is the Lebesgue measure of A, and

$$\underset{t \downarrow 0}{\text{essential limit}} \int_{\mathbb{R}^N} |z(t,x)|\,dx = 0 \quad . \tag{1.4}$$

Then $z = 0$ a.e. on Q.

It is obvious that $z = u - \hat{u}$ and $h = \varphi(u) - \varphi(\hat{u})$ satisfy the conditions of Lemma 1, except perhaps for (1.3). In order to verify (1.3) observe that since $u, \hat{u} \in L^\infty(Q)$ and φ is continuous, for each $\xi > 0$ there is a $\delta > 0$ such that $|\varphi(u(t,x)) - \varphi(\hat{u}(t,x))| > \xi$ implies $|u(t,x) - \hat{u}(t,x)| > \delta$. But $u - \hat{u} \in L^1(Q)$ implies $\text{meas}\{(t,x) : |u(t,x) - \hat{u}(t,x)| > \delta\} < \infty$ and so (1.3) holds.

Proof of Proposition 1. It is well-known (e.g. [3], [12]) that for each $\varepsilon > 0$ and $g \in L^p(\mathbb{R}^N)$, $1 \leq p \leq \infty$, the problem

$$\varepsilon v_\varepsilon - \Delta v_\varepsilon = g \quad \text{in } \mathcal{D}'(\mathbb{R}^N) \tag{1.5}$$

has a unique solution $v_\varepsilon \in L^p(\mathbb{R}^N)$. Defining B_ε by $B_\varepsilon g = v_\varepsilon$ one also has the estimate

$$\varepsilon \|B_\varepsilon g\|_p \leq \|g\|_p \tag{1.6}$$

where $\| \; \|_p$ will denote either the norm of $L^p(\mathbb{R}^N)$ or the norm of $L^p(Q)$ depending on the context. Because of (1.6), B_ε defines mappings $B_\varepsilon : L^p(Q) \to L^p(Q)$ for $1 \leq p \leq \infty$ and (1.6) holds equally for $g \in L^p(\mathbb{R}^N)$ and $g \in L^p(Q)$. Under the assumptions of Proposition 1 we have

$$\int_0^T \int_{\mathbb{R}^N} (z\psi_t + h\Delta\psi)\,dx\,dt = 0 \quad \text{for } \psi \in \mathcal{D}(Q) \tag{1.7}$$

where $\mathcal{D}(Q)$ is the space of C^∞ functions with compact support in Q. Fixing $\gamma \in \mathcal{D}(Q)$ we wish to set $\psi = B_\varepsilon\gamma$ in (1.7). Clearly $B_\varepsilon\gamma \in C^\infty(Q)$ (since B_ε commutes with differentiations) and $(B_\varepsilon\gamma)(t,x) = 0$ for t near 0 or T. Moreover, since $z, h \in L^\infty(Q)$, (1.7) clearly continues to hold for $\psi \in C^\infty(Q) \cap L^1(Q)$ with $\psi(t,x) = 0$ for t near 0 and T provided that ψ_t, $\Delta\psi$ and $|\text{grad } \psi| \in L^1(Q)$. $B_\varepsilon\gamma$ has these properties. Moreover, $\Delta B_\varepsilon\gamma = \varepsilon B_\varepsilon\gamma - \gamma$ and $(B_\varepsilon\gamma)_t = B_\varepsilon(\gamma_t)$. Thus

$$(1.8)\qquad \int_0^T \int_{\mathbb{R}^N} (zB_\varepsilon(\gamma_t) + h(\varepsilon B_\varepsilon\gamma - \gamma))\,dxdt =$$

$$\int_0^T \int_{\mathbb{R}^N} ((B_\varepsilon z)(\gamma_t) + (\varepsilon B_\varepsilon h - h)\gamma)\,dxdt = 0 \quad \text{for } \gamma \in \mathcal{D}(Q) \quad ,$$

where the first equality is due to the obvious symmetry of B_ε and the absolute convergence of all integrals involved. Thus

$$(1.9)\qquad (B_\varepsilon z)_t = \varepsilon B_\varepsilon h - h \quad \text{in } \mathcal{D}'(Q) \quad .$$

For notational convenience we denote $z(t,\cdot)$ by $z(t)$ and $\int_{\mathbb{R}^N} p(x)q(x)dx$ by (p, q) when $pq \in L^1(\mathbb{R}^N)$. Since $z, B_\varepsilon z \in L^1(Q) \cap L^\infty(Q)$

$$g_\varepsilon(t) = (B_\varepsilon z(t), z(t))$$

is defined for almost all $t \in [0,T]$. Assume we can demonstrate that

$$(1.10)\qquad \lim_{\varepsilon \downarrow 0} g_\varepsilon(t) = \lim_{\varepsilon \downarrow 0}(B_\varepsilon z(t), z(t)) = 0 \quad \text{a.e. } t \in [0,T] \quad .$$

It will follow that $z(t) = 0$ a.e.. Indeed, if $w \in L^2(\mathbb{R}^N)$ then $\varepsilon B_\varepsilon w - \Delta B_\varepsilon w = w$ and so

$$(B_\varepsilon w, w) = (B_\varepsilon w, \varepsilon B_\varepsilon w - \Delta B_\varepsilon w) = \varepsilon\|B_\varepsilon w\|_2^2 + \|\,|\text{grad } B_\varepsilon w|\,\|_2^2 \quad .$$

Thus $(B_\varepsilon w, w) \to 0$ as $\varepsilon \to 0$ implies $\varepsilon B_\varepsilon w \to 0$ in $L^2(\mathbb{R}^N)$ and $\Delta B_\varepsilon w = \text{div}(\text{grad } B_\varepsilon w) \to 0$ in $\mathcal{D}'(\mathbb{R}^N)$ (since $\text{grad } B_\varepsilon w \to 0$ in $L^2(\mathbb{R}^N)^N$). Therefore $\varepsilon B_\varepsilon w - \Delta B_\varepsilon w = w \to 0$ in $\mathcal{D}'(\mathbb{R}^N)$ and $w = 0$ a.e. In this way Proposition 1 will follow if we can verify (1.11). This will involve two main steps. From (1.9) and the various properties of h and z we will deduce that g_ε is absolutely continuous (upon correction on a set of measure zero) and

$$(1.11)\qquad g'_\varepsilon(t) = 2(\varepsilon B_\varepsilon h(t) - h(t), z(t)) \quad \text{a.e. } t \in [0,T]$$

where $h(t)$ abbreviates $h(t,\cdot)$. We assume these facts for the moment and show how to complete the proof of (1.11). From $z \in L^\infty(Q)$, (1.4) and (1.6) it follows that

(1.12) $$g_\varepsilon(0+) = \underset{t \downarrow 0}{\text{essential limit}}(B_\varepsilon z(t), z(t)) = 0 \quad .$$

This equality together with (1.2), (1.12) and the symmetry of B_ε imply

(1.13) $$g_\varepsilon(t) \leq 2 \int_0^t (\varepsilon B_\varepsilon h(s), z(s))ds = 2 \int_0^t (h(s), \varepsilon B_\varepsilon z(s))ds \quad .$$

Now $|(\varepsilon B_\varepsilon h(s), z(s))| \leq \|\varepsilon B_\varepsilon h\|_\infty \|z(s)\|_1 \leq \|h\|_\infty \|z(s)\|_1$ by (1.6). Since $s \to \|z(s)\|_1 \in L^1(0,T)$, (1.10) will follow from the dominated convergence theorem and (1.13) if

$$\lim_{\varepsilon \downarrow 0} (h(s), \varepsilon B_\varepsilon z(s)) = 0 \text{ a.e. } s \in [0,T] \quad .$$

In view of our various assumptions, this last equality follows from:

<u>Lemma 1</u>. Let $p \in L^\infty(\mathbb{R}^N)$ and $\operatorname{meas}\{x \in \mathbb{R}^N : |p(x)| > \xi\} < \infty$ for $\xi > 0$. Let $q \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then $\lim_{\varepsilon \downarrow 0} (p, \varepsilon B_\varepsilon q) = 0$.

<u>Proof of Lemma 1</u>. We have

(1.14) $$\begin{aligned} \left|\int \varepsilon p B_\varepsilon q dx\right| &\leq \left|\int_{\{x:|p(x)| > \xi\}} \varepsilon p B_\varepsilon q dx\right| + \xi \int |\varepsilon B_\varepsilon q| dx \\ &\leq \operatorname{meas}\{x : |p(x)| > \xi\} \|p\|_\infty \|\varepsilon B_\varepsilon q\|_\infty + \xi \|\varepsilon B_\varepsilon q\|_1 \\ &\leq \operatorname{meas}\{x : |p(x)| > \xi\} \|p\|_\infty \|\varepsilon B_\varepsilon q\|_\infty + \xi \|q\|_1 \quad . \end{aligned}$$

To proceed, we verify that

(1.15) $$\lim_{\varepsilon \downarrow 0} \|\varepsilon B_\varepsilon q\|_\infty = 0 \quad .$$

In fact, scaling arguments show that

(1.16) $$(\varepsilon B_\varepsilon q)(x) = \varepsilon^{N/2} \int k(\sqrt{\varepsilon}(x-y)) q(y) dy$$

where k is the kernel associated with B_1:

(1.17) $$B_1 q(x) = \int k(x-y) q(y) dy \quad .$$

The properties of this kernel we use below can be obtained from [12]. Simple estimations now yield

$$|\varepsilon B_\varepsilon q(x)| \le \varepsilon^{N/2} C(r)\|q\|_1 + \varepsilon^{N/2}\|q\|_\infty \int_{\{\sqrt{\varepsilon}|x-y| \le r\}} k(\sqrt{\varepsilon}(x-y))dy$$

for $r > 0$, where $C(r) = \sup\{k(x) : |x| \ge r\}$. Since $C(r) < \infty$ for $r > 0$, this last estimate shows that

$$\limsup_{\varepsilon \downarrow 0} \|\varepsilon B_\varepsilon q\|_\infty \le \|q\|_\infty \int_{\{|y| \le r\}} k(y)dy$$

for $r > 0$. But $k \in L^1(\{x: |x| \le 1\})$ and the right hand side above therefore tends to zero as $r \downarrow 0$, establishing (1.15). Returning to (1.14), we find now that

$$\limsup_{\varepsilon \downarrow 0} \left|\int \varepsilon p B_\varepsilon q dx\right| \le \xi\|q\|_1 \quad \text{for} \quad \xi > 0$$

and Lemma 1 follows on letting ξ tend to 0. ∎

It remains to verify the absolute continuity of g_ε and (1.11). For notational simplicity we set $\varepsilon = 1$ and write B, g instead of B_1, g_1. Let

$$z_\delta(t, x) = (\rho_\delta * \bar{z})(t, x) = \int_{-\infty}^{\infty} \rho_\delta(t-s)\bar{z}(s, x)ds$$

where $\bar{z} = z$ on Q, $\bar{z} = 0$ outside Q and ρ_δ is a standard family of mollifiers in t with ρ_δ supported in $[-\delta, \delta]$. It is clear that z_δ, Bz_δ and $(Bz_\delta(t), z_\delta(t))$ are smooth in t and we have

$$\frac{d}{dt}(Bz_\delta, z_\delta) = 2\left(\frac{\partial}{\partial t}Bz_\delta, z_\delta\right) = 2\left(Bz_\delta, \frac{\partial}{\partial t}z_\delta\right) \quad \text{on} \quad \mathbb{R} \,. \tag{1.18}$$

Next we claim that almost everywhere on $(\delta, T-\delta) \times \mathbb{R}^N$

$$\frac{\partial}{\partial t}Bz_\delta = \rho_\delta * (B\bar{h} - \bar{h}) \tag{1.19}$$

where $\bar{h} = h$ on Q and $\bar{h} = 0$ outside Q. Indeed if $\gamma \in \mathcal{D}((\delta, T-\delta) \times \mathbb{R}^N)$ and $\check{\rho}_\delta(s) = \rho_\delta(-s)$ we find

$$\int_{\mathbb{R}}\int_{\mathbb{R}^N}\left(\frac{\partial}{\partial t}Bz_\delta\right)\gamma\,dxdt = \int_{\mathbb{R}}\int_{\mathbb{R}^N}(-B\bar{z})\frac{\partial}{\partial t}(\check{\rho}_\delta*\gamma)\,dxdt$$

$$= \int_Q (Bh - h)(\check{\rho}_\delta*\gamma)\,dxdt = \int_{\mathbb{R}}\int_{\mathbb{R}^N}(\rho_\delta*(B\bar{h} - \bar{h}))\gamma\,dxdt$$

by (1.9) and the fact $\check{\rho}_\delta*\gamma \in D(Q)$. Using (1.18) and (1.19) we see that for $\zeta \in D(0,T)$ and sufficiently small δ,

$$-\int_0^T (Bz_\delta(s), z_\delta(s))\zeta'(s)ds = 2\int_0^T (\rho_\delta*(B\bar{h} - \bar{h})(s), z_\delta(s))\zeta(s)ds \quad .$$

Since $z_\delta \to z$ in $L^1(Q)$ and $\|z_\delta\|_\infty \le \|z\|_\infty$ it follows easily that

$$-\int_0^T g(s)\zeta'(s)ds = 2\int_0^T ((B\bar{h} - \bar{h})(s), z(s))\zeta(s)ds \quad .$$

The last result shows that g is absolutely continuous and $g'(t) = 2(Bh(t) - h(t), z(t))$ a.e. The proof of Proposition 1, and hence Theorem 1, is complete. ∎

Remarks on variations:

(1.20) The inhomogeneous equation: The way we have formulated Theorem 1 it is directly applicable to the generalization $u_t - \Delta\varphi(u) = f(t, x)$ of (1)(i).

(1.21) Discontinuous φ: The continuity or even the single-valuedness of φ was used only to establish that $h = \varphi(u) - \varphi(\hat{u})$ satisfied (1.3). This can be arrived at in other ways. For example, if $u, \hat{u} \in L^p(Q)$ for some p, $1 \le p < \infty$, and φ is continuous at 0 we have (1.3) satisfied.

(1.22) Assumption of the initial-value: We have assumed that the initial condition is satisfied in the strong form (1) (ii) which corresponds to (1.4). This was to simplify the presentation and is justified by the existence theory which we have in mind (see Section 2) which provides solutions satisfying (1)(ii). However, it is quite interesting to weaken (1.4) to the requirement

(1.23) $$\int_0^T\int_{\mathbb{R}^N}(z\psi_t + h\Delta\psi)\,dxdt = 0 \;\forall\psi \in C_0^\infty([0,T) \times \mathbb{R}^N) \quad ,$$

where $C_0^\infty([0,T) \times \mathbb{R}^N)$ means the C^∞ functions vanishing for t near T and large $|x|$, especially in view of the existing literature. In fact, the entire proof remains intact under this change of hypotheses except for the verification of (1.12). We briefly describe how to verify (1.12) under the assumption (1.23).

First, if $\psi \in C_0^\infty([0,T) \times \mathbb{R}^N)$, $0 \le a < b < T$ and

$$g(t) = \begin{cases} 1 & \text{if } b < t \\ \frac{1}{b-a}(t-a) & \text{if } a < t < b \\ 0 & \text{if } t < a \end{cases}$$

we can approximate $g(t)$ by smooth functions and establish that

$$\int_0^T \int_{\mathbb{R}^N} (z(g'\psi + g\psi_t) + hg\Delta\psi))dxdt = 0 = \int_0^T \int_{\mathbb{R}^N} (z\psi_t + h\Delta\psi)dxdt \quad .$$

As $a, b \to 0$, this implies that

$$\lim_{\substack{b,a\to 0+ \\ b>a}} \frac{1}{b-a} \int_a^b \int_{\mathbb{R}^N} z(t,x)\psi(t,x)dxdt = 0 \tag{1.24}$$

for $\psi \in C_0^\infty([0,T):\mathbb{R}^N)$. Taking ψ independent of t (as we may clearly do) and recalling $z \in L^\infty(Q)$, we deduce that

$$\lim_{\substack{b,a\to 0+ \\ b>a}} \frac{1}{b-a} \int_a^b \int_{\mathbb{R}^N} z(t,x)\psi(x)dxdt = 0 \ \forall\psi \in L^1(\mathbb{R}^N) \quad . \tag{1.25}$$

From (1.9) it follows that

$$B_\varepsilon z(t,x) - B_\varepsilon z(s,x) = \int_s^t (\varepsilon B_\varepsilon h(\tau,x) - h(\tau,x))d\tau$$

for almost all $(t, s, x) \in (0,T) \times (0,T) \times \mathbb{R}^N$. Multiplying this result by $\psi \in C_0^\infty(\mathbb{R}^N)$, integrating over $x \in \mathbb{R}^N$ and then averaging over s, $a < s < b$, produces

$$\int_{\mathbb{R}^N} B_\varepsilon z(t,x)\psi(x)dx - \frac{1}{b-a}\int_a^b \int_{\mathbb{R}^N} z(s,x)B_\varepsilon\psi(x)dx\,ds$$

$$= \frac{1}{b-a}\int_a^b \int_{\mathbb{R}^N} \int_s^t (\varepsilon B_\varepsilon h(\tau,x) - h(\tau,x))d\tau\psi(x)dxds \quad .$$

Using (1.25) we can pass to the limit above as $b, a \to 0+$ to obtain

$$\int_{\mathbb{R}^N} (B_\varepsilon z)(t,x)\psi(x)dx = \int_0^t \int_{\mathbb{R}^N} (\varepsilon B_\varepsilon h(\tau,x) - h(\tau,x))\psi(x)dxd\tau \quad .$$

This relation holds for all $\psi \in L^1(\mathbb{R}^N)$ for almost all t. Hence

$$B_\varepsilon z(t,x) = \int_0^t (\varepsilon B_\varepsilon h(\tau,x) - h(\tau,x))dx$$

for almost all $(t,x) \in [0,T) \times \mathbb{R}^N$ and $\|B_\varepsilon z(t)\|_\infty \le t\, 2\|h\|_\infty$. Finally, $|(B_\varepsilon z(t), z(t)|$ $\le 2t\|h\|_\infty\|z(t)\|_1$, so $\|z(t)\|_1 \in L^1(0,T)$ and the existence of $g_\varepsilon(0+)$ imply (1.12).

(1.26) Other integrability conditions. We note that Proposition 1 remains valid if $z \in L^1(Q) \cap L^\infty(Q)$ and (1.3) are replaced by $z, h \in L^2(Q)$ and (1.4) is replaced by (1.23). The proof of this assertion consists of mild adaptations of the above arguments (several points being easier). Recalling the relationships of Proposition 1, Theorem 1 and (1), this proves uniqueness of weak solutions of (1) which satisfy $u \in L^\infty(Q)$ and $u - u_0 \in L^2(Q)$ if φ is locally Lipschitzian. Indeed, if $u, \hat{u}$ are two such solutions and $h = \varphi(u) - \varphi(\hat{u})$, then $|h| \le C|u-\hat{u}| \in L^2(Q)$ for some constant C. This result strongly generalizes the uniqueness assertion of [11].

Section 2. The proof of Theorem 2.

The abstract theory of evolution equations governed by accretive operators (see, e.g., [1], [5]) in conjunction with [3] provides a great deal of information concerning the solution of (1). The basic idea is that for each $g \in L^1(\mathbb{R}^N)$ the problem

$$v - \Delta\varphi(v) = g \quad \text{in } \mathcal{D}'(\mathbb{R}^N) \tag{2.1}$$

has a solution $v \in L^1(\mathbb{R}^N)$ which is unique within a suitable class (see [3]). To reduce (2.1) to the problem studied in [3], put $u = \varphi(v)$, $\beta = \varphi^{-1}$ and rewrite (2.1) as $\beta(u) - \Delta u \ni g$. The mapping A defined by $A:v \to g - v$ when $g \in L^1(\mathbb{R}^N)$ and v is the unique solution of (2.1) is m-accretive in $L^1(\mathbb{R}^N)$ and $\overline{D(A)} = L^1(\mathbb{R}^N)$. Thus (1) has a solution in the sense of the abstract theory if $u_0 \in L^1(\mathbb{R}^N)$ (see [1], [5]). Let $u(t,\cdot) = S(t)u_0$ denote this solution; in particular $u \in C([0,\infty); L^1(\mathbb{R}^N))$. Under the assumptions of Theorem 1 it is easy to see that if also $u_0 \in L^\infty(\mathbb{R}^N)$ then $u \in L^\infty([0,\infty) \times \mathbb{R}^N)$ and $u_t - \Delta\varphi(u) = 0$ in $\mathcal{D}'((0,\infty) \times \mathbb{R}^N)$. Thus the existence theory complements the uniqueness theory. With some further restrictions on φ (see [2],[13] for precise conditions) which are satisfied in the special case of Theorem 2, we have $u \in L^\infty([a,\infty) \times \mathbb{R}^N)$ for $a > 0$ if only $u_0 \in L^1(\mathbb{R}^N)$. Thus the <u>existence</u> claim of Theorem 2 is clear. We now prove the <u>uniqueness</u>. Assume u is any solution of (7), (8), (9). Then for $h > 0$ the functions $u(t + h,\cdot)$ and $S(t)u(h)$ are two solutions of (1)(i) with the same initial value $u(h,\cdot)$. It follows from Theorem 1 that $S(t)u(h) = u(t + h,\cdot)$ for $t \geq 0$. As $h \to 0$ we see that $S(t)u_0 = u(t,\cdot)$ and the uniqueness is proved. ∎

To illustrate the use of the existence theory in extending uniqueness results in a slightly more complex way (but by no means the most complex), we indicate the proof of one more result.

<u>Theorem 3</u>. Let $T > 0$, (2) hold and $p:\mathbb{R} \to \mathbb{R}$ be Lipschitz continuous with $p(0) = 0$. Let $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then there is exactly one function u satisfying

$$u \in C([0,T]:L^1(\mathbb{R}^N)) \cap L^\infty([0,T]:\mathbb{R}^N) \quad , \tag{2.2}$$

$$u(0,\cdot) = u_0(\cdot) \quad , \tag{2.3}$$

and

$$u_t - \Delta\varphi(u) + p(u) = 0 \quad \text{in } \mathcal{D}'(Q) \quad . \tag{2.4}$$

Proof of Theorem 3. Assume u and $\hat{u}$ satisfy (2.2) - (2.4). Let $f(t,x) = p(u(t,x))$, so $f \in L^1(Q) \cap L^\infty(Q)$ (by the restrictions on p). The theory mentioned above guarantees the existence of a $v \in L^\infty(Q) \cap C([0,T]:L^1(\mathbb{R}^N))$ such that $v_t - \Delta\varphi(v) + f = 0$ in $\mathcal{D}'(Q)$ and $v(0,x) = u_0(x)$. Theorem 1 implies (see Remark (1.20)) that $v \equiv u$. Similarly we can construct $\hat{v}$ from $\hat{u}$ and $\hat{v} \equiv \hat{u}$. But the existence theory which provided v and $\hat{v}$ also implies that if $w(t) = v(t,\cdot) - \hat{v}(t,\cdot) = u(t,\cdot) - \hat{u}(t,\cdot)$ then

$$\|w(t)\|_1 \le \|w(0)\| + \int_0^t \|f(\tau,\cdot) - \hat{f}(\tau,\cdot)\|_1 d\tau$$

$$\le 0 + K\int_0^t \|u(\tau,\cdot) - \hat{u}(\tau,\cdot)\|_1 d\tau = K\int_0^t \|w(\tau)\|_1 d\tau$$

where K is a Lipschitz constant for p. Thus $w = 0$ and uniqueness is proved. Existence follows from the considerations mentioned above. ∎

Remark. A result comparable to Theorem 3 in bounded domains has been obtained in [4] (Proposition 5.2).

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Equations of the form $u_t - \Delta\varphi(u) = 0$ arise in mathematical models of many physical situations. The uniqueness of solutions of the associated initial-value problem in $\mathbb{R}^N$ is considered in this paper. It is shown that bounded weak solutions (i.e., solutions in the sense of distribution) which further satisfy an integrability condition are unique.

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