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NUCLEAR BURST INDUCED SHOCK WAVE MODELING OF ENERGETIC ELECTRON--ETC(U)

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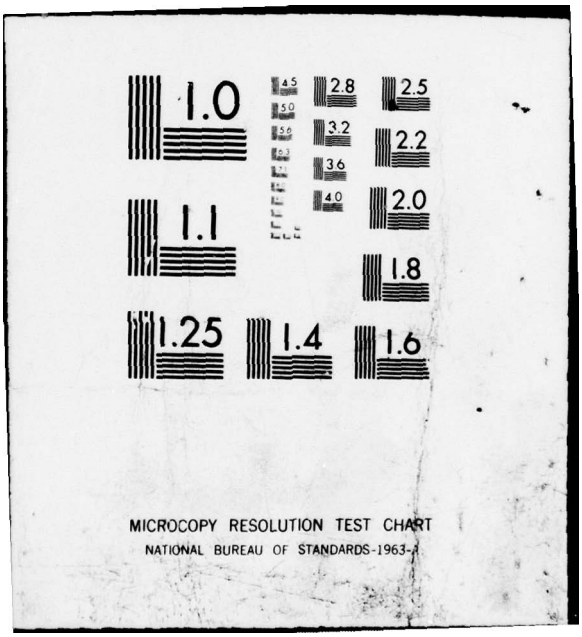
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(6) NUCLEAR BURST INDUCED SHOCK WAVE MODELING OF ENERGETIC ELECTRON INJECTION INTO THE MAGNETOSPHERE: Eigenmode description of the hot ion cyclotron beam-whistler mode instability in parallel shock waves

(10) Kenneth I. Golden,
John W. Cipolla, Jr.
Michael B. Silevitch

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Departments of Electrical and Mechanical Engineering
Northeastern University
Boston, Massachusetts 02115

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The local linear dispersion relation for the hot ion cyclotron beam--whistler mode instability near the leading edge of parallel shock waves is generalized into a WKB eigenvalue problem. Its solution reveals the existence of stationary eigen growth modes capable of generating turbulence in the leading edge region.

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PREFACE

This paper will appear in the Physics of Fluids. Dr. Liu Chen (Plasma Physics Laboratory, Princeton University) has been a principal participant in the research and is cited as a co-author.

SUMMARY

The local linear dispersion relation for the hot ion cylotron beam-whistler mode instability near the leading edge of parallel shock waves is generalized into a WKB eigenvalue problem. Its solution reveals the existence of stationary eigen growth modes capable of generating turbulence in the leading edge region.

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I. INTRODUCTION

It has been theoretically demonstrated that the low frequency ion cyclotron beam-whistler plasma instability is a principal source of turbulence in parallel shock waves* in collisionless plasmas (Refs. 1,2). The local analyses of Refs. 1a,b point to the existence of unstable modes near the leading edge of the shock which propagate at the velocity of the leading edge for shock Alfvén Mach numbers M_A greater than some critical value M*. Viewed more conveniently in the rest frame of the shock front, this suggests that such stationary** modes have ample time to grow to amplitudes sufficiently large to scatter the incoming (unshocked) ions thereby providing the key mechanism for turbulent dissipation in the leading edge region. These stationary modes were determined from the $\underset{\sim}{k}||\underset{\sim}{B}_0$ linear dispersion relation at points throughout the shock layer under the assumption that the local ion distribution function is a Mott-Smith superposition of interpenetrating unshocked and shocked ion flows, each flow being characterized by its own density, average velocity, and temperature; the electrons, unlike the ions, were modeled as being warm fluid. The value of M* depends on whether one adopts a hydrodynamical or kinetic description of the shocked ions. In Ref. 1a where they are modeled (for the sake of mathematical simplicity) as a cold monoenergetic ion cyclotron beam driving unstable the leading edge whistlers, M* = 5.5; in Ref. 1b, where they are more realistically considered to be thermalized, M* = 2.77.

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1. a. Golden, K.I., L.M. Linson and S.A. Mani, Phys. Fluids 16, 2319 (1973).

   b. Cipolla, J.W., K.I. Golden and M.B. Silevitch, Phys. Fluids 20, 282 (1977).

2. Chen, Liu and K.I. Golden, Phys. Lett. 58A, 462 (1976).

* Shock waves which propagate along the constant applied magnetic field $\underset{\sim}{B}_0$.

** In the sequel, modes having zero group velocity in the rest frame of the shock leading edge are referred to as stationary modes.

Recently, two of the authors substantially refined the previous local hydrodynamic theory (Ref. 1a) by generalizing its $\underset{\sim}{k}||\underset{\sim}{B}_o$ linear dispersion relation into a WKB type eigenvalue formulation more appropriate for a shock layer description of ion cyclotron beam-whistler mode interactions near the leading edge (Ref. 2). All of the key elements of the local theory are retained in this WKB formulation. The most significant difference between the local (Ref. 1a) and WKB (Ref. 2) hydrodynamic theories is that in the former there is one and only one frequency and growth rate corresponding to a given wave number k_o while the latter features a broad spectrum of eigen frequencies and eigen growth rates corresponding to k_o.

The purpose of this paper is to reformulate and solve the WKB eigenvalue problem under the assumption that the shocked ions are more realistically treated as hot Maxwellian particles. This we do in Section II, III and IV. In Section V we discuss the results and draw our conclusions.

II. WKB DISPERSION RELATION

In this Section we present a derivation of the $\underset{\sim}{k}||\underset{\sim}{B}_o$ WKB type dispersion relation for ion cyclotron beam-whistler mode interactions near the leading edge of the parallel shock layer.

We work in the rest frame of the leading edge (located at $z = 0$; z increases across the layer). In this frame, unshocked plasma enters the layer from the upstream ($z < 0$) region with constant velocity V_u and shocked plasma emerges from the layer and flows through the downstream region with constant velocity $V_d < V_u$. Adopting the Mott-Smith description for the ion velocity distribution function,

$$f_i(z,\underset{\sim}{v}) = n_u(z)\delta(\underset{\sim}{v} - \underset{\sim}{e}_z V_u) + n_d(z)(\pi C_d^2)^{-3/2}\exp[-(\underset{\sim}{v} - \underset{\sim}{e}_z V_d)^2/C_d^2], \qquad (1)$$

$$C_d^2 = 2\kappa T_d/m_i,$$

we assume that the shock layer is a superposition of interpenetrating upstream cold* and downstream hot ion flows, each flow being characterized by its own density $n_s(z)$, average velocity V_s, and temperature T_s (s=u,d)**. The mean motion of the warm background electrons is dictated by the requirements of local charge and current neutrality:

$$n_u(z) + n_d(z) = n_e(z), \tag{2}$$

$$n_u(z)V_u + n_d(z)V_d = n_e(z)V_e(z), \tag{3}$$

compatible with the equation of continuity. Or equivalently,

$$V_e(\eta) = (1 - \eta)V_u + \eta V_d, \tag{4}$$

where $\eta(z) = n_d(z)/[n_u(z) + n_d(z)]$ is the beam strength parameter. Note that just inside the leading edge ($0 \lesssim z << L$, L = typical scale length of shock), $\eta << 1$ since $n_d(z) << n_u(z)$ there.

We assume the following WKB form for the perturbed transverse (perpendicular to $\underset{\sim}{e}_z$) electric field vector:

$$\underset{\sim}{E}_\perp(z,t) = \tilde{\underset{\sim}{E}}_\perp(z)\exp[i(k_o z - \omega_s t)], \tag{5}$$

where: $\omega_s = \omega_o + i\gamma$, $|\gamma| << |\omega_o|$,

$$|k_o| >> |\tilde{E}'_\alpha(z)|/|\tilde{E}_\alpha(z)| >> L^{-1}, \ (\alpha = x,y) \tag{6}$$

In our previous local theory (Ref. 1b), it was found that $|\gamma/\omega_o| \sim o(\eta)$. We shall retain this principal feature in the present WKB formulation. In terms of a smallness parameter, $\varepsilon << 1$, the appropriate ordering is then taken to be

$$|k_o|^{-1}\ |\tilde{E}'_\alpha(z)|/|\tilde{E}_\alpha(z)| \sim o(\varepsilon), \quad |k_o L|^{-1} \sim o(\varepsilon^2) \tag{7a,b}$$

$$\eta \sim o(\varepsilon^2). \tag{8}$$

---

* Aside from the $T_u = 0$ approximation in Eq. (1), made for the sake of mathematical expediency, it is, nevertheless understood that there is a small but sufficient amount of upstream warmth to preclude the possibility of switch-on phenomena. The effects of upstream warmth are discussed in Ref. 1b.

** Note that the upstream ion density $n_u(z)$ decreases from its upstream value $n_{u\infty}$ at the leading edge to zero at the trailing edge, while $n_d(z)$ has the opposite dependence on z.

In the sequel, we shall keep terms up to and including $o(\varepsilon^2)$ smallness. For the very low frequencies of interest in this paper, vacuum displacement currents are negligibly small, so that the appropriate Maxwell equation is

$$\frac{\partial^2 \underset{\sim}{E}_\perp(z,t)}{\partial z^2} \simeq \frac{4\pi}{c^2} \frac{\partial \underset{\sim}{j}_\perp(z,t)}{\partial t}, \tag{9}$$

where $\underset{\sim}{j}_\perp(z,t) = en_e(z)[(1 - \eta)\underset{\sim}{V}_{u\perp}(z,t) + \eta \underset{\sim}{V}_{d\perp}(z,t) - \underset{\sim}{V}_{e\perp}(z,t)]$ (10)

is the transverse current due to the perturbed motions of the plasma particles. Upon combining Eqs. (5), (9), and (10), one obtains the following Maxwell's equation for the envelope $\tilde{E}_\alpha(z)$, $(\alpha = x,y)$:

$$(ik_o + \frac{d}{dz})^2 \tilde{E}_\alpha(z) + \frac{4\pi i e \omega_s n_e(z)}{c^2} [(1 - \eta)\tilde{V}_{u\alpha}(z) + \eta \tilde{V}_{d\alpha}(z) - \tilde{V}_{e\alpha}(z)] = 0. \tag{11}$$

We next calculate from appropriate equations of motion the perturbed transverse velocity envelopes in the second ℓ.h.s. term of (11).

A. Upstreaming Cold Ions. Since the upstream ions are cold, the momentum equations

$$(\omega_s - k_o V_u + iV_u \frac{d}{dz})\tilde{V}_{ux}(z) = (ie/m_i)[\tilde{E}_x(z) - (1/c)V_u \tilde{B}_y(z) + (1/c)\tilde{V}_{uy}(z)B_o], \tag{12}$$

$$(\omega_s - k_o V_u + iV_u \frac{d}{dz})\tilde{V}_{uy}(z) = (ie/m_i)[\tilde{E}_y(z) + (1/c)V_u \tilde{B}_x(z) - (1/c)\tilde{V}_{ux}(z)B_o], \tag{13}$$

and Maxwell's equations

$$\tilde{B}_y(z) = -(ic/\omega_s)(ik_o + \frac{d}{dz})\tilde{E}_x(z), \tag{14}$$

$$\tilde{B}_x(z) = (ic/\omega_s)(ik_o + \frac{d}{dz})\tilde{E}_y(z), \tag{15}$$

are most suitable for the calculation of the perturbed ion velocity components in terms of the perturbed electric field components. Upon combining these equations and taking account of the fact that $\tilde{E}_y = i\tilde{E}_x$ for the right circularly polarized whistler modes under consideration here, one obtains

$$(\omega_s - k_o V_u + \Omega_i + iV_u \frac{d}{dz})\tilde{V}_{ux}(z) \simeq (ie/m_i\omega_s)(\omega_s - k_o V_u + iV_u \frac{d}{dz})\tilde{E}_x(z), \qquad (16)$$

where $\Omega_i = eB_o/cm_i$. The solution to Eq. (16) is, to $o(\varepsilon^2)$,

$$\tilde{V}_{ux}(z) \simeq (ie/m_i\omega_s)\frac{\omega_s - k_o V_u}{\omega_s - k_o V_u + \Omega_i}\left[1 + \frac{i\Omega_i V_u}{(\omega_s - k_o V_u)(\omega_s - k_o V_u + \Omega_i)}\frac{d}{dz} + \right. \qquad (17)$$

$$\left. + \frac{\Omega_i V_u^2}{(\omega_s - k_o V_u)(\omega_s - k_o V_u + \Omega_i)^2}\frac{d^2}{dz^2}\right]\tilde{E}_x(z).$$

B. Downstreaming Hot Ions. Since near the leading edge the contribution of the downstreaming hot ion beam to the transverse current perturbation is at most $o(\varepsilon^2)$ smallness compared with the other contributions (see Eq. (10)), we need only calculate $V_{dx}(z)$ to O(1). This is routinely done from linearized Vlasov theory and one readily obtains

$$\tilde{V}_{dx}(z) = (ie/m_i\omega_s)\left[\frac{1}{\sqrt{\pi}\, C_d}\int_{-\infty}^{\infty} du\, e^{-u^2/C_d^2}\frac{\omega_s - k_o V_d}{\omega_s - k_o V_d + \Omega_i - k_o u}\right]\tilde{E}_x(z). \qquad (18)$$

C. Electron Fluid. For the low frequencies of interest in this paper, we assume that $|\omega_s|$, $|k_o V_e(z)| << |\Omega_e|$. The appropriate description of the perturbed electron fluid motion is then given by the equations:

$$\tilde{E}_x(z) - (1/c)V_e(z)\tilde{B}_y(z) + (1/c)\tilde{V}_{ey}(z)B_o \simeq 0, \qquad (19)$$

$$\tilde{E}_y(z) + (1/c)V_e(z)\tilde{B}_x(z) - (1/c)\tilde{V}_{ex}(z)B_o \simeq 0. \qquad (20)$$

Upon combining Eqs. (15) and (20), one readily obtains the desired equation

$$\tilde{V}_{ex}(z) \simeq (c/B_o)\left[1 + \frac{iV_e(z)}{\omega_s}(ik_o + \frac{d}{dz})\right]\tilde{E}_y(z) =$$

$$= i(c/B_o\omega_s)[\omega_s - k_o V_e(z) + iV_e(z)\frac{d}{dz}]\tilde{E}_x(z). \qquad (21)$$

It is now a straightforward matter to assemble the WKB dispersion equation from Eqs. (11), (17), (18), and (21). Taking account of Eq. (4) and noting that

$$\frac{4\pi[n_u(z) + n_d(z)]e^2}{m_i c^2} = \frac{\Omega_i^2}{C_A^2(z)},$$

where the local Alfvén speed

$$C_A(z) = \begin{cases} C_A(0)[1 - \eta(1 - V_d/V_u)]^{1/2}, & z > 0, \\ C_A(0), & z < 0, \end{cases}$$

one ultimately obtains to $O(\varepsilon^2)$:

$$D(\omega_o, k_o - i\frac{d}{dz}, \eta)\tilde{E}_x(z) = 0, \tag{22}$$

where:

$$D(\omega_o, k_o - i\frac{d}{dz}, \eta) = D_r(\omega_o, k_o - i\frac{d}{dz}, \eta) + iD_i(\omega_o, k_o, \eta), \tag{23}$$

$$D_r(\omega_o, k_o - i\frac{d}{dz}, \eta) = D_r(\omega_o, k_o, o) + \frac{\partial D_r(\omega_o, k_o, o)}{\partial k_o}(-i\frac{d}{dz}) + \tag{24}$$

$$+ \frac{1}{2}\frac{\partial^2 D_r(\omega_o, k_o, o)}{\partial k_o^2}(-i\frac{d}{dz})^2 + \frac{1}{2}\eta\left(\frac{\partial^2 D_r(\omega_o, k_o, \eta)}{\partial\sqrt{\eta}^2}\right)_{\sqrt{\eta}=0},$$

$$D_r(\omega_o, k_o, o) = 1 - \frac{\omega_o}{\Omega_i} + \frac{k_o V_u}{\Omega_i} + \frac{k_o^2 C_A^2(o)}{\Omega_i^2} - \frac{\Omega_i}{\omega_o - k_o V_u + \Omega_i}, \tag{25}$$

$$D_r(\omega_o, k_o, \eta) = D_r(\omega_o, k_o, o) + \eta\left[\frac{k_o(V_d - V_u)}{\Omega_i} + \frac{k_o^2 C_A^2(o)}{\Omega_i^2}\left(\frac{V_d}{V_u} - 1\right)\right. \tag{26}$$

$$\left. + \frac{\Omega_i}{\omega_o - k_o V_u + \Omega_i} - \frac{1}{\sqrt{\pi}\, C_d} P \int_{-\infty}^{\infty} du\, e^{-u^2/C_d} \frac{(\Omega_i - k_o u)}{\omega_o - k_o V_d + \Omega_i - k_o u}\right],$$

$$D_i(\omega_o,k_o,\eta) = -\frac{\gamma\Omega_i}{\tilde{\Omega}^2} + \frac{\sqrt{\pi}\,\eta}{|k_o|C_d}(k_oV_d - \omega_o)\exp\left[-\frac{(\omega_o - k_oV_d + \Omega_i)^2}{k_o^2C_d^2}\right], \tag{27}$$

$$\tilde{\Omega}^{-2} = \Omega_i^{-2} - (\omega_o - k_oV_u + \Omega_i)^{-2} > 0. \tag{28}$$

It is now evident that Eq. (22) is the WKB generalization of the algebraic linear dispersion relation

$$D(\omega_o,k_o,\eta) = 0 \tag{29}$$

for the local theory (Ref. 1b).

The analysis of Ref. 1b revealed that for $M_A \gtrsim 2.77$, stationary unstable whistler waves exist near the leading edge and are driven unstable by their interaction with the downstream hot ion cyclotron drift mode only if $k_oV_d - \omega_o > 0$. These stationary modes are the ones which evidently have ample time to grow to sufficiently large amplitude to irreversibly scatter the upstream ions. One can approximately incorporate this same stationarity of leading edge whistlers into the present WKB formulation by supposing that (ω_o,k_o) satisfy the cold plasma whistler mode dispersion relation,

$$D_r(\omega_o,k_o,0) = 0 \tag{30}$$

at $z = 0$ and the zero group velocity condition there equivalent to

$$\frac{\partial D_r(\omega_o,k_o,0)}{\partial\kappa_o} = 0. \tag{31}$$

We see now that $D_r(\omega_o,k_o - id/dz,\eta)$ is a real operator of $O(\varepsilon^2)$. In assuming $\gamma \sim O(\eta) \sim O(\varepsilon^2)$ (as suggested by local theory and Eq. (7)), the pure imaginary part $D_i(\omega_o,k_o,\eta)$ and, consequently, $D(\omega_o,k_o - id/dz,\eta)$ must also be $O(\varepsilon^2)$. Note from Eq. (31) that, while modes at the leading edge ($z = 0$) are exactly stationary, those nearby in the region $0 < z << L$ will have group velocities which are, however, negligibly small, i.e. $V_{gn} = \partial\omega_{on}/\partial k_o \sim O(\varepsilon^2V_u)$, $n = 0,1,2,\ldots$. This is discussed at greater length in Section V.

III. FORMULATION OF THE EIGENVALUE PROBLEM

We turn next to the formulation of the eigenvalue problem. Near the leading edge, $\eta(z) \simeq n_d(z)/n_u(0)$ and we assume its profile to be:

$$\eta(z) = \begin{cases} 0, & z \leq 0 \\ (z/L)^2, & z \geq 0 \text{ and } z << L. \end{cases} \tag{32a,b}$$

For convenience, we adopt the notation:

$$\tilde{E}_x(z) = \begin{cases} \tilde{E}_-(z), & z < 0 \\ \tilde{E}_+(z), & z > 0. \end{cases}$$

$z < 0$. In this region, we have from Eqs. (22) to (28), (30), (31) and (32a) that

$$\left(\frac{1}{K^2} \frac{d^2}{dz^2} + \frac{i\gamma\Omega_i}{\tilde{\Omega}^2} \right) \tilde{E}_-(z) = 0, \tag{33}$$

$$\text{where } \frac{1}{K^2} = \frac{1}{2} \frac{\partial^2 D_r(\omega_o, k_o, o)}{\partial k_o^2} = \frac{C_A^2(0)}{\Omega_i^2} - \frac{\Omega_i V_u^2}{(\omega_o - k_o V_u + \Omega_i)^3} \cdot \tag{34}$$

From Eq. (30) and the zero group velocity condition (31) written equivalently as

$$M_A = V_u/C_A(0) = -x_o + (1 + x_o^2/4)^{1/2} + (x_o^2/4)(1 + x_o^2/4)^{-1/2}, \tag{35}$$

$$x_o = k_o C_A(0)/\Omega_i,$$

it can be shown that $K^2 > 0$. (36)

$z > 0$. For this region, we have from (26) and (32b) that

$$\frac{1}{2} \eta \frac{\partial^2 D_r(\omega_o, k_o, \eta)}{\partial \sqrt{\eta}^2} = (z/L)^2 \left[\frac{k_o(V_d - V_u)}{\Omega_i} + \frac{k_o^2 C_A^2(0)}{\Omega_i^2} \left(\frac{V_d}{V_u} - 1 \right) + \right.$$

$$\left. + \frac{\Omega_i}{\omega_o - k_o V_u + \Omega_i} - \frac{1}{\sqrt{\pi} C_d} P \int_{-\infty}^{\infty} du\, e^{-u^2/C_d^2} \frac{(\Omega_i - k_o u)}{(\omega_o - k_o V_d + \Omega_i) - k_o u} \right]. \tag{37}$$

Then from Eqs. (22) to (25), (27), (28), (30), (31) and (37), one obtains the desired equation

$$\left(\frac{1}{K^2} \frac{d^2}{dz^2} + \frac{i\gamma\Omega_i}{\tilde{\Omega}^2} - \frac{Rz^2}{L^2} \right) \tilde{E}_+(z) = 0, \tag{38}$$

where: $R = R_r + iR_i$, (39)

$$R_r = \frac{1}{2} \frac{\partial^2 D_r(\omega_o, k_o, \eta)}{\partial \sqrt{\eta}^2} = \tag{40}$$

$$= M_A^{-1}(1 + x_o^2/4)^{-1/2} \left[1 + \frac{3|x_o|(1 + x_o^2/2)(M_A^2 - 1)}{4M_A} \right]$$

$$- \frac{1}{\sqrt{\pi}\, C_d} P \int_{-\infty}^{\infty} du e^{-u^2/C_d^2} \frac{(\Omega_i - k_o u)}{(\omega_o - k_o V_d + \Omega_i) - k_o u},$$

$$R_i = \frac{\sqrt{\pi}}{|k_o| C_d} (k_o V_d - \omega_o) \exp \left[- \frac{(\omega_o - k_o V_d + \Omega_i)^2}{k_o^2 C_d^2} \right]. \tag{41}$$

We note that Eq. (40) is expressed mostly in terms of the more convenient shock Alfvén Mach number M_A. This equation is easily derived from Eq. (37) by use of the gasdynamic Rankine-Hugoniot shock relation*

$$\frac{V_d}{V_u} = \frac{M_A^2 + 3}{4M_A^2} \tag{42}$$

The specification of the eigenvalue problem is now completed when, aside from the usual required boundedness of $\tilde{E}_-(z \to -\infty)$ and $\tilde{E}_+(z \to +\infty)$, one demands that $\tilde{E}(z)$ and its first derivative be continuous at $z = 0$, i.e.

$$\tilde{E}_-(0) = \tilde{E}_+(0), \tag{43}$$

$$\tilde{E}_-'(0) = \tilde{E}_+'(0). \tag{44}$$

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* For the derivation of Eq. (43) we have assumed $C_A^2(0) = 5\kappa T_u/3m_i$ to avoid a switch-on occurrence.

IV. SOLUTION TO THE EIGENVALUE PROBLEM

$z < 0$. If one makes the ansatz

$$|\arg(\gamma^{1/2} e^{-i\pi/4})| < \pi/2, \tag{45}$$

one then has for $z < 0$,

$$\tilde{E}_-(z) = A_- \exp(K_- z), \tag{46}$$

where

$$K_- = (K^2 \Omega_i \gamma e^{-i\pi/2} / \tilde{\Omega}^2)^{1/2} \tag{47}$$

The other solution is discarded due to its divergence at $z = -\infty$. The matching conditions (43) and (44) at $z = 0$ therefore become

$$\tilde{E}_+(0) = A_-, \tag{48}$$

$$\tilde{E}_+'(0) = K_- A_-. \tag{49}$$

$z > 0$. For $z > 0$, Eq. (38) can be reduced to the standard equation for parabolic cylinder functions (Ref. 3),

$$[(d^2/d\xi^2) - (\tfrac{1}{4}\xi^2 + a^2)]\tilde{E}_+(\xi) = 0. \tag{50}$$

Here $\xi = z/\alpha$,

$$\alpha = i\left(\frac{L^2}{4RK^2}\right)^{1/4} \tag{51}$$

and $$a = \frac{i\gamma\Omega_i |K| L}{2\tilde{\Omega}^2 \sqrt{R}} . \tag{52}$$

The solution, well behaved at $z = \infty$, is then

$$\tilde{E}_+(\xi) = A_+ V(a,\xi). \tag{53}$$

From the matching conditions (48) and (49) at $z = 0$, one obtains the following conditions for the eigenvalues:

$$\gamma_n = \frac{2ia_n \tilde{\Omega}^2 R^{1/2}}{\Omega_i |K| L} \qquad (n = 0,1,2,\ldots) \tag{54}$$

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3. M. Abramowitz and I.A. Segun, Handbook of Mathematical Functions, (Dover, New York, 1968), p.686.

and a_n is the nth root of the equation

$$\sqrt{a/2} = \frac{\sin[\pi(1/4 - a/2)]}{\sin[\pi(3/4 - a/2)]} \frac{\Gamma(3/4 - a/2)}{\Gamma(1/4 - a/2)} . \tag{55}$$

By writing Eq. (55) in the form,

$$\sqrt{a/2} = \frac{\sin^2[\pi(1/4 - a/2)]}{\sin^2[\pi(3/4 - a/2)]} \frac{\Gamma(3/4 + a/2)}{\Gamma(1/4 + a/2)} ,$$

one can see that all its <u>real</u> roots must be positive.* Numerical computations bear this out: We obtain $a_0 \simeq 0.09$, $a_1 \simeq 1.004$, $a_2 \simeq 2$,....

For those a_n's which are real, Eq. (54) now reads

$$\gamma_n = \frac{-2i|a_n|\tilde{\Omega}^2 R^{1/2}}{\Omega_i |K| L} , \tag{56}$$

or equivalently,

$$|\gamma_n| = \frac{2|a_n|\tilde{\Omega}^2 (R_r^2 + R_i^2)^{1/4}}{\Omega_i |K| L} , \tag{57a}$$

$$\arg \gamma_n = -\frac{\pi}{2} + \frac{1}{2} \tan^{-1}\left(\frac{R_i}{R_r}\right). \tag{57b}$$

Evidently, for instability (which corresponds to Re $\gamma_n > 0$), it is necessary that**

$$R_i > 0, \tag{58}$$

or $-(\pi/2) < \arg \gamma_n < 0$.

We note that γ_n satisfies the ansatz Eq. (45). Eqs. (41) and (58) reveal that there can be instability only if $k_o V_d - \omega_o > 0$, consistent with the instability criterion of our local theory (Ref. 1b). Since $|a_n|$, $|(K_-)_n|$ and the turning point $|z_{tn}|^2 = |4a_n\alpha^2|$ increase with γ_n, our results indicate that modes with higher growth rates are peaked deeper inside the shock front and extend further

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* We believe that all the roots of Eq. (55) are real, but this has yet to be proved.

** Stable modes correspond to $R_i < 0$ or $-\pi < \arg R < 0$.

into the shock. This is physically expected since the instability is driven by finite η and η increases with z. The validity of the approximations made here requires $|k_o| >> |(K_-)_n|$ and $|z_{tn}| << L$ thereby giving an upper bound to the acceptable values of γ_n. Keeping this in mind, Eq. (57a) and our ordering scheme given by Eqs. (7) and (8) guarantee that

$|\gamma_n/\Omega_i| \sim O[(|K|L)^{-1}] \sim O[(|k_o|L)^{-1}] \sim O(\epsilon^2) \sim O(\eta)$, again consistent with the local theory.

V. DISCUSSION AND CONCLUSIONS

Perhaps the most suitable way to enumerate the similarities and differences between the WKB and local (Ref. 1b) theories is to cite the relevant frequency and growth rate formulas.

WKB theory

$$\omega_{on}(k_o,\eta) = \omega_o(k_o) + \frac{2|a_n|}{|K|L}\frac{\tilde{\Omega}^2}{\Omega_i}|R|^{1/2}\cos(\tfrac{1}{2}\tan^{-1}\frac{R_i}{R_r}), \tag{60}$$

$$\gamma_n(k_o,\eta) = \frac{2|a_n|}{|K|L}\frac{\tilde{\Omega}^2}{\Omega_i}|R|^{1/2}\sin(\tfrac{1}{2}\tan^{-1}\frac{R_i}{R_r}), \quad (n = 0,1,2,\ldots) \tag{61}$$

LOCAL theory

$$\omega_o(k_o,\eta) = \omega_o(k_o) + \eta\frac{\tilde{\Omega}^2}{\Omega_i}R_r, \tag{62}$$

$$\gamma(k_o,\eta) = \eta\frac{\tilde{\Omega}^2}{\Omega_i}R_i, \tag{63}$$

where

$$\omega_o(k_o) = k_oV_u + \frac{k_o^2C_A^2(0)}{\Omega_i} - k_oC_A(0)\left(1 + \frac{k_o^2C_A^2(0)}{4\Omega_i^2}\right)^{1/2} \tag{64}$$

is the whistler mode frequency at the leading edge (see Eq. (25)).

We have already pointed out that in both theories:

(i) $0 < \arg R < \pi$ or R_i must be positive for instability (stable modes correspond to $-\pi < \arg R < 0$ or $R_i < 0$);

(ii) growth rates are $O(\varepsilon^2)$ in smallness.

Furthermore, in the vicinity of the leading edge ($0 < z << L$), both theories indicate a real frequency shift (from its $z = 0$ value) of $O(\varepsilon^2)$; in the WKB case, this shift is clearly seen to be positive.

Noting that Eq. (31) is equivalent to the condition $\partial\omega_o(k_o)/\partial k_o = 0$, the eigen group velocities calculated from Eq. (60) are negligibly small, i.e.,

$$V_{gn} = \partial\omega_o(k_o,\eta)/\partial k_o = \frac{\partial}{\partial k_o}\left[\frac{2|a_n|}{|K|L}\frac{\tilde{\Omega}^2}{\Omega_i}|R|^{1/2}\cos\left(\frac{1}{2}\tan^{-1}\frac{R_i}{R_r}\right)\right] \sim O(\varepsilon^2 V_u), \qquad (65)$$

$(n = 0,1,2,\ldots)$

Eqs. (61) and (65) permit us to estimate the distance Δ required for one e-fold growth of the very slowly rightward propagating eigen whistlers. One obtains

$$\Delta_n = \gamma_n^{-1} V_{gn} \sim O(V_u/\Omega_i) \sim O(k_0^{-1}) = O(\lambda_o) \sim O(\varepsilon z) \sim O(\varepsilon^2 L). \qquad (66)$$

Suppose that $\varepsilon \tilde{>} 0.1$ (corresponding to beam strengths $\eta \sim .01$). By the time the mode has moved a distance $z \tilde{<} |z_{tn}|$ through the leading edge region, it will have undergone 10 e-folding growths in its amplitude and should therefore be capable of scattering incoming ions, especially since it remains in the leading edge region for a time long compared with the residence time of an incoming particle ($V_{gn}/V_u \sim .01$). Thus in our WKB theory, the eigen growth modes can be considered to be approximately stationary, an essential condition for the generation of whistler turbulence.

Finally, conditions (30), (37) and (58a) indicate that the stationary ion cyclotron beam-whistler eigenmode instability is operative near the leading edge for $M_A \tilde{>} 2.77$.

VI. RECOMMENDATION

We recommend that this WKB theory be re-formulated for the more realistic shock layer model where the unshocked plasma is warm. Such a reformulation should be a relatively easy task.