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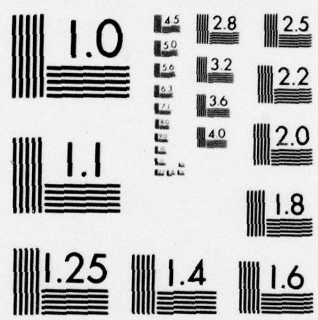

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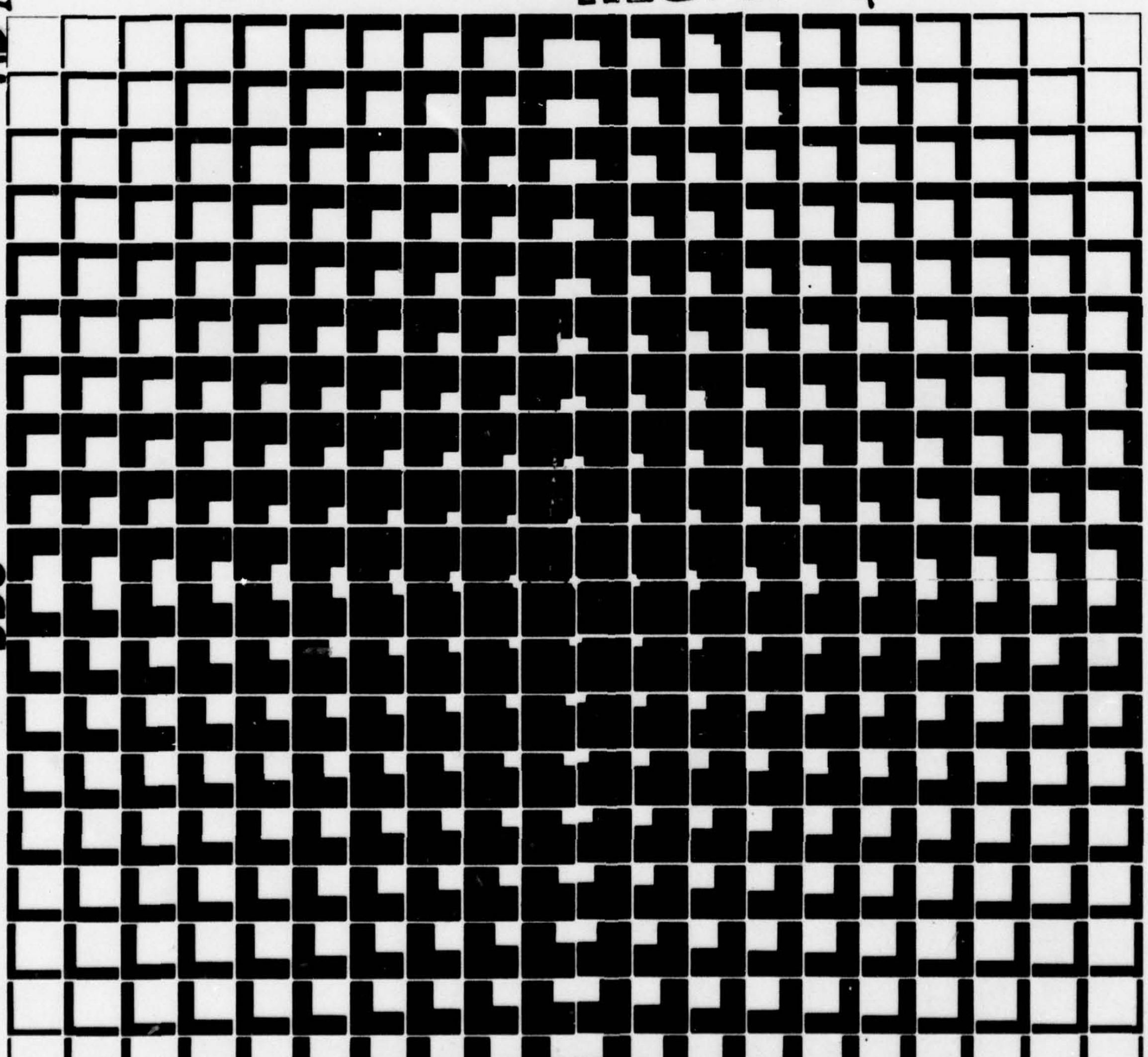
RESEARCH PAPER
FPL 322
1978

Forest Products Laboratory
Forest Service
U.S. Department of Agriculture
Madison, Wis.



Abstract

A unitized stack of containers in transit is susceptible to dynamic overloading due to vibrations in the transporting vehicle. The boxes' compressive stiffnesses interact with the content masses to amplify or attenuate the vehicle motions through the height of the column. Modeling a unitized load as a multiple-degree-of-freedom vibration system provides for identifying its sensitivity to the frequencies inherent to the transportation environment. This report presents the theoretical analysis of the analog that represents a stack of containers and an example that carries the mathematics through a package design problem. To supplement the manual computations which are too time-consuming for practical packaging design, a computer program--not included herein--is discussed. This program plots the transmissibility in each container over a range in frequencies, including the damaging resonants. An example using the program shows employment of the generated plot for unitized package designing.

Abbreviations Used

cycles/s = cycles per second

G = gravities

in./s^2 = inches per second squared

lbf = pounds-force

lbf/in. = pounds-force per inch

lbf · s/in. = pounds-force x seconds per inch

lbm = pounds-mass

s = seconds

s/in. = seconds per inch

s^2/in. = seconds squared per inch

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TRANSPORTATION VIBRATION EFFECTS ON UNITIZED CORRUGATED CONTAINERS.

By

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U.S. Department of Agriculture

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12 27p.

Introduction

Shipments of like packages have come increasingly to be unitized for reasons of economy. Mechanically arranging and stacking containers on a single pallet or other platform offers the advantages of mechanized transfer and storage with protection from the hazards of manual handling. This new environment for the package has shifted its probability of transportation damage from the shock and impact mode common with manual handling to the vibration mode.

Even where dropping or impacting a package does not occur, the product is still exposed to transit vehicle vibrations enroute between the manufacturer and recipient. And this most probable source of damage becomes an environment over which the package designer has little control; his option is to design vibration protection into the package system.

Shocks and impacts acting on single packages in simulated small parcel shipping environments have been well analyzed in numerous reports (5).[2] Some studies (3,4) have also examined the damage susceptibility of a product's container due to vibrations. But all these publications, although accurate documentations of the pertinent vibration theory, were still aimed at the single package environment and are limited to a single-degree-of-freedom analysis.

Where quantities of similar packages are shipped as a unitized lot, a new approach to the vibration analysis is demanded. The vibration theory developed for the single parcel environment may grossly underestimate the severity of acceleration levels in a unitized load. For example,

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1/ Maintained at Madison, Wis., in cooperation with the University of Wisconsin.

2/ Underlined numbers in parentheses refer to Literature Cited at the end of this report.

the dynamics in a stack of containers ten high on a pallet may approach a modeled ten-degree-of-freedom system with ten critical frequencies, each being a potentially damaging resonant.

Because corrugated boxes yield in compression due to their contained weights, they act like springs and the resulting stack natural frequencies may fall within the range of the transportation environment (3). The weight of the product supported by the resilient container behaves like an analogous spring-mass system to amplify or attenuate the vibratory motion delivered to its base. Often a product's component subassemblies will necessitate avoiding acceleration levels within certain frequency bands. Or frictional holding forces must be maintained to insure protection from load disarrangement and subsequent stack toppling and product impact damage. Also, the lower containers may require protection from dynamic crushing.

This report was written to demonstrate for package engineers the application of the fundamental theory for analyzing vibration forces in a unitized load. Because charts or tables are more revealing to the package designer than their underlying equations, the theory was extended to a computer program for plotting the containers' responses. Graphs produced via this program make apparent the shifts in product damage susceptibility among unitized load options with containers differing in mechanical properties and contained weights. Such graphs thus highlight the trends in response levels over a parameter change.

I. Design Consideration

The hazards of transportation reveal themselves with toppled stacks and crushed containers. Sometimes the damage is concealed until the packaged product is put into service. Given enough resources a shipper can employ trial and error adjustments to the package system until a protective design emerges. But analysis rather than trial and error would improve efficiency. Solving a typical problem serves to illustrate the knowledge gained from an analytical treatment. The appendix carries the problem through the detailed mathematics; only the results are presented here.

A manufacturer finds his containers totally disarrayed when unitized and shipped via a particular carrier. The transportation environment is monitored and he learns that a significant input occurs at 0.25 G acceleration at 5 cycles per second. Before employing a new design and suggested antiskid treatment, he requests an analysis to learn if the new approach will indeed solve the problem.

The new design calls for vertically alined boxes stacked four high with each box containing a rigid, fixed, nonload-supporting content $\bar{W}$ of 55.3 pounds and an antiskid treatment applied to the top and bottom flaps of each box. All boxes are identically constructed, and from top-to-bottom compression tests on similar boxes the box stiffnesses are estimated relative to their equilibrium supporting loads. Also, from vibration test observations it is estimated that the box material contributes about 30 percent of critical damping. The four high stack (fig. 1) calls for a 3 degree of freedom model since the weight in the bottom box does not affect the stack. Once the stiffnesses are linearized from the static compression test curves at

$$K_1 = 771 \text{ lbf/in.} ,$$

$$K_2 = 607 \text{ lbf/in.} ,$$

$$K_3 = 500 \text{ lbf/in.},$$

they may be combined with the weights of the contents to predict the natural frequencies of the system.

$$\omega_1 = 4.84 \text{ cycles/s},$$

$$\omega_2 = 12.7 \text{ cycles/s},$$

$$\omega_3 = 18.5 \text{ cycles/s}.$$

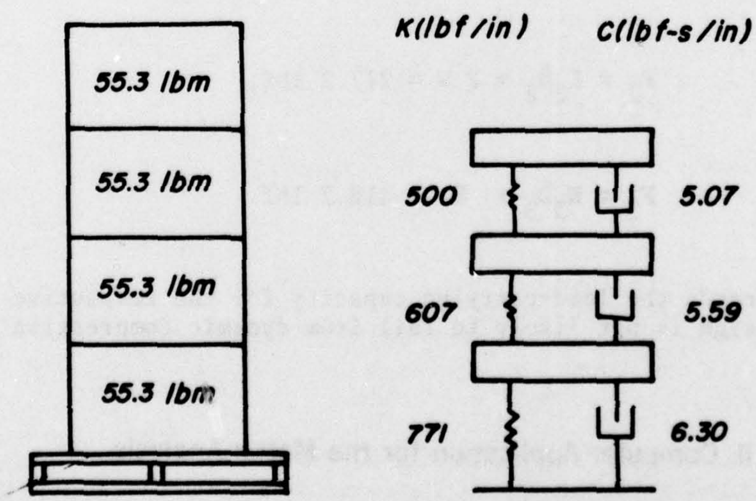

Figure 1.--Schematic diagram of a four-high stack arrangement, showing the relationship between a column of boxes and the vibration model. (M 146 029)

The excitation frequency at 5 cycles per second is close to the first natural frequency of the box stack. It is, therefore, warranted to investigate if the damping is sufficient to inhibit a damaging resonant response. A solution to the equations of motion with damping considered yields the acceleration level in each container in units of G's.

$$\ddot{X}_1 = 175.5e^{j(31.4t-1.168)} = 0.46\text{ G}\cos(31.4t-1.168),$$

$$\ddot{X}_2 = 329.4e^{j(31.4t-1.501)} = 0.85\text{ G}\cos(31.4t-1.501),$$

$$\ddot{X}_3 = 434.9e^{j(31.4t-1.609)} = 1.13\text{ G}\cos(31.4t-1.609).$$

The response in the top box exceeds 1.0 G in acceleration and bouncing would occur, thus making the antiskid treatment ineffective. The design must either be altered, or more effective but costly methods of unitizing be employed.

One can also determine if any box has been loaded beyond its maximum compression strength by examining the compressing load $\underline{F}$ on a box due to dynamic compression plus the equilibrium supporting weight. With knowledge of the relative displacements, $\underline{\bar{D}}$, between adjacent containers

$$F_1 = K_1\bar{D}_1 + 3\text{ W} = 293.9\text{ lbf},$$

$$F_2 = K_2\bar{D}_2 + 2\text{ W} = 217.2\text{ lbf},$$

$$F_3 = K_3\bar{D}_3 + \text{W} = 112.7\text{ lbf}.$$

If no force exceeds the load-carrying capacity for the respective container, the design is not likely to fail from dynamic compression overloading.

II. Computer Application for the Matrix Analysis

One can examine the response in a container stack by repeatedly solving the equations of motion with changing frequencies. However, it can be more useful for design purposes to note trends as one or more of the parameters vary. Computer-plotted graphs can show varying levels of product damage susceptibility across a range of packaging options--for instance, with containers differing in mechanical properties and contained weights. Vibrational analysis computer programs which are presently commercially available (6) are uneconomical for this purpose. (They must be made comprehensive beyond the present need so as to have broad spectrums of adaptability.) For this reason, we have developed a specialized program for the multiple-degree-of-freedom analysis described in this paper. (Due to limitations of space, our program is not reproduced in this article, but may be requested from the author along with all necessary definitions and subroutines.) The following discussion illustrates the utility of our program and may suggest the benefits to be derived from this and similar computer programs.

To illustrate the computer program in a quantitative comparison, consider again an example where a package designer desires to weigh the advantages between two palletized loads for protection against a 5-cycles/s input at 0.25 G acceleration. The situation is similar to that in the previous example--that is, four vertically alined boxes, each containing 55.3 pounds. Another alternative is to package the product in stronger, larger boxes able to contain 73.7 pounds, but vertically alined in a three-high stack. Therefore, the pallet's loaded weight is conserved although the product is packaged and unitized differently.

The designer tests the two box types in top-to-bottom compression to establish their stiffnesses relative to their equilibrium supporting loads, as suggested by Godshall (3); figure 2 illustrates the results. On the three-tier pallet, the boxes support 73.7 pounds at the second layer and 147.4 pounds at the bottom layer. Tangents drawn to the solid line curve in figure 2 at these ordinate values suggest relative stiffnesses of 666 and 810 lbf/in. For the boxes on the four-tier pallet supporting 55.3, 110.6, and 165.9 pounds from top to bottom, their stiffnesses may be similarly assumed from the dashed line curve in figure 2 to be 500, 607, and 771 lbf/in. Each box is roughly estimated to absorb energy at 0.3 times its critical damping ratio, and the analysis is supplemented with a comparison between 0.1 and 0.7 times the critical damping ratios.

These physical parameters thus define the computer program input and are subsequently organized on cards following two executions.

The plotted output from the two analyses is produced in figures 3 and 4. The damping ratio used in an analysis may be recognized by extending a horizontal line from the last point in a dashed-line curve to intersect the bottom of the characters "DR." Some significant trends are observed regarding the major amplification and attenuation ranges for the 0.3 critical damping analysis and are summarized in table I.

The potentially damaging frequencies, those amplifying the input by at least two, have broadened from a band of 3.23 to 6.23 cycles/s for the four-tier pallet to a band of 4.40 to 7.63 cycles/s for the three-tier pallet. Reducing the stack levels has made the load sensitive to higher frequencies. The attenuation region has also shifted to the higher

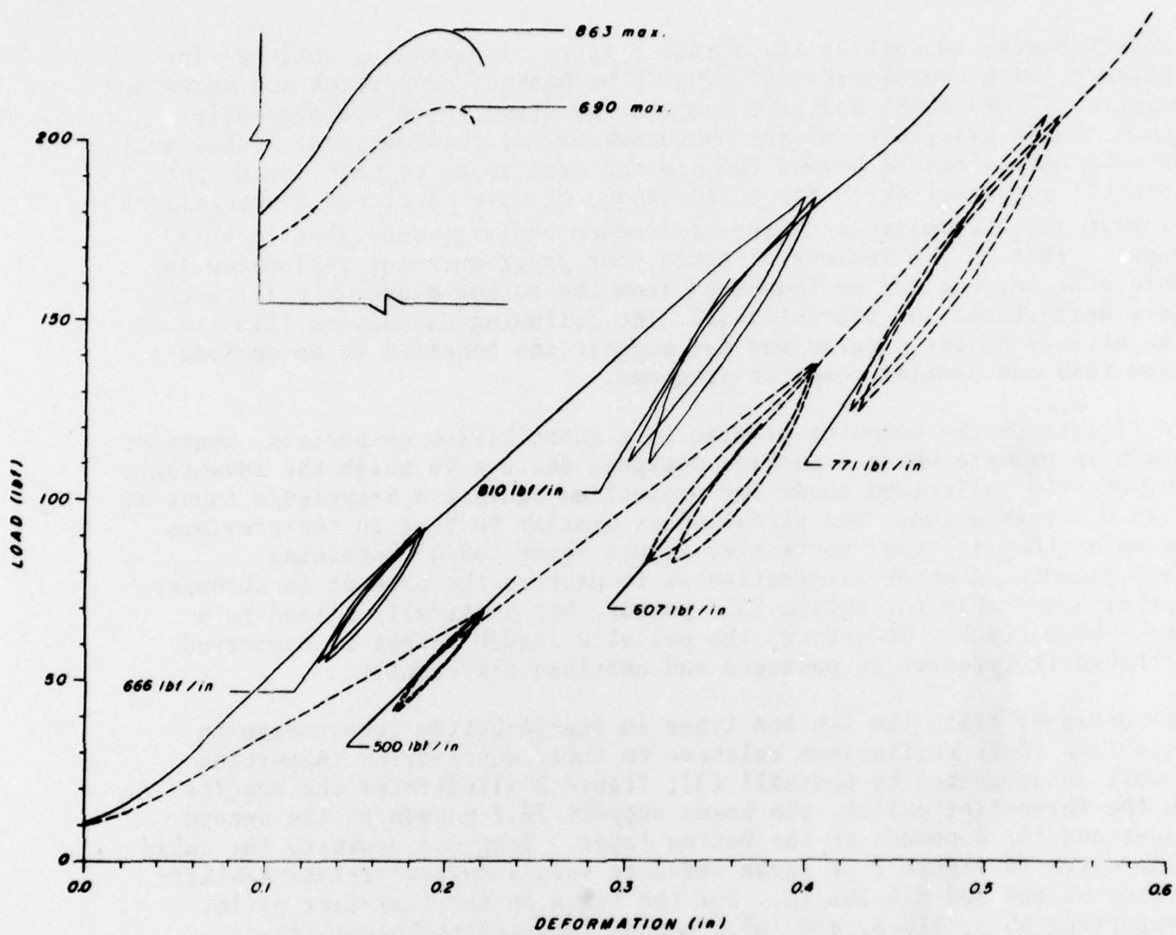

Figure 2.--Top-bottom compression curves for two box types. Solid curve is for a box designed to contain 73.7 pounds in a three-high stack; dashed line is for a box designed to contain 55.3 pounds in a four-high stack. (M 146 134)

frequencies for the three-level stack. However, the transmissibilities at the resonant levels are changed. Although the three-high stack is sensitive to a wider frequency range, a greater input magnitude would be necessary to cause damage.

The decision to choose between the two designs would be based on experience with present designs and inferences from this experience regarding anticipated transportation inputs. If the designer feels confident that the 5-cycles/s vibration is the most prominent, a decision to accept the three-high stack to avoid high-level transmissibilities would be logical.

The significance of damping becomes apparent at the higher frequencies. At these frequencies, damping, which is proportional to the base velocity, dissipates energy at an increasing rate. It can be seen from the plots for all masses that even for a lightly damped system of 0.1 critical, the responses at natural frequencies beyond the first do not even approach

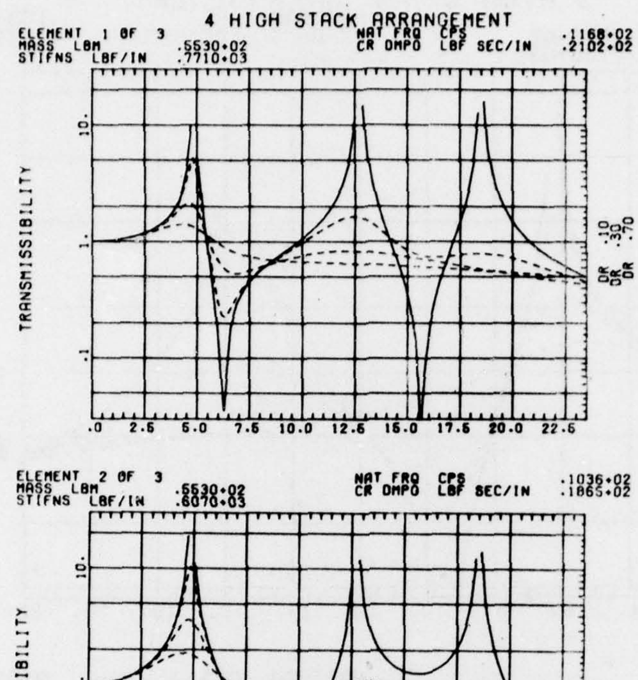

Figure 3.--Vibrational analysis of a four-high stack arrangement using three containers with different stiffness values and equal masses. (Actual computer plots, but with redundant labels deleted.)

(M 145 684)

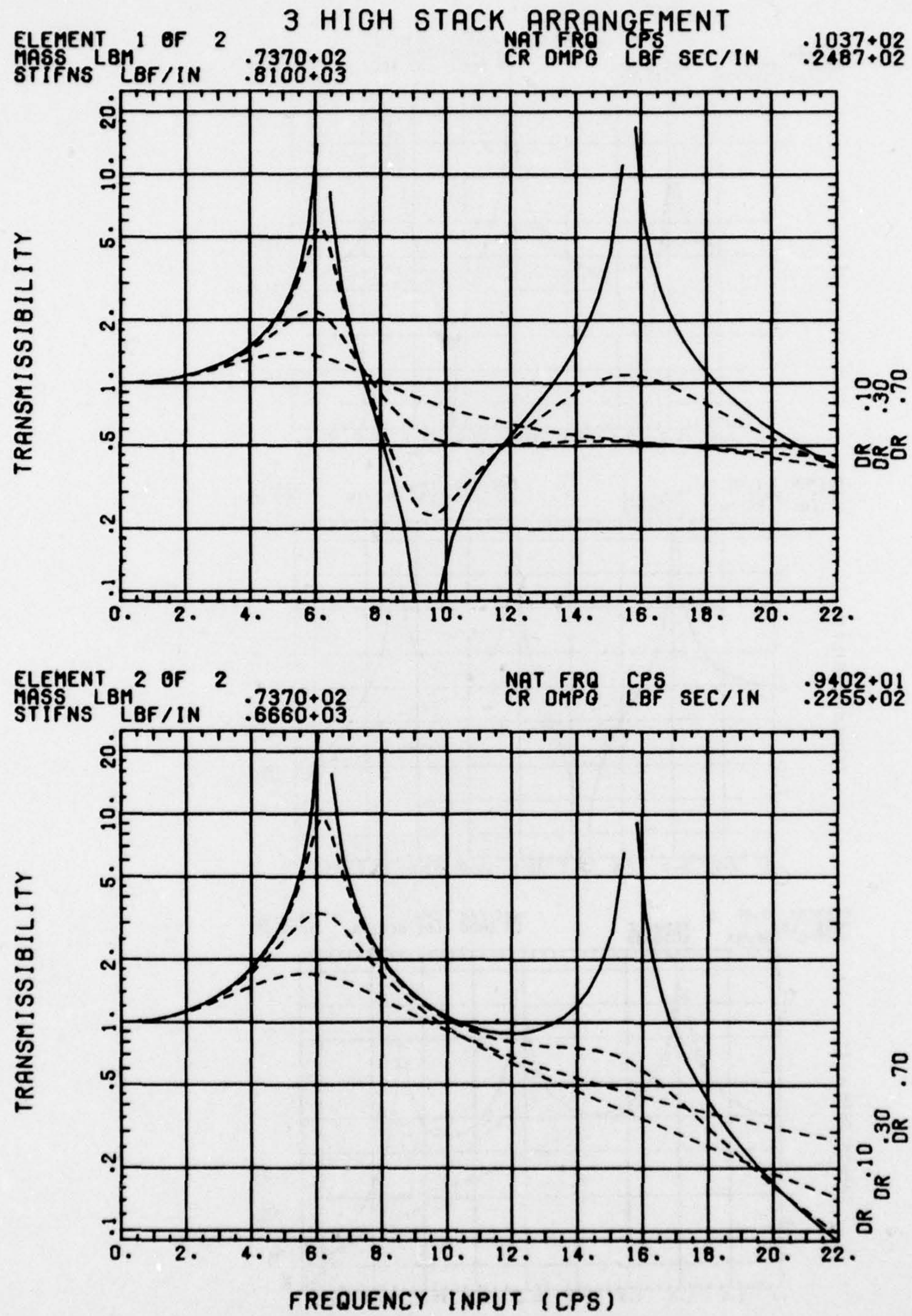

Figure 4.--Vibrational analysis of a three-high stack arrangement using two containers with different stiffness values and equal masses. (Actual computer plots, but with redundant labels deleted.)

(M 145 685)

Table I.--Summary of 0.3 critical damping analysis (figures 3 and 4)

Stack height	Element No.	Amplification ≥2(cycles/s)	Attenuation ≥1(cycles/s)	Maximum transmissibility
3 boxes	1	5.23-6.46	7.73	2.20
3 boxes	2	4.40-7.63	9.80	3.32
4 boxes	1	4.26-4.90	5.59	2.15
4 boxes	2	3.58-5.64	6.61	3.63
4 boxes	3	3.23-6.23	8.23	4.72

the severity of the first resonant response. To the designer this suggests that he may safely abbreviate his analysis by examining only the first natural frequency.

Further analyses can be performed by determining if the bottom containers can withstand the dynamic compression-loading condition. The user may do this by adjusting certain subroutines in the computer program to generate the relative transmissibilities between adjacent masses. The plotted output would then be interpreted as the factors by which to multiply a displacement input magnitude to obtain container compressions.

The dynamic compression loading value can be conservatively determined by considering the limiting case of relative displacement between the base and mass one. Adding this to the statically supporting weight gives the maximum force experienced by the bottom box. For an input magnitude $\underline{\ddot{Y}}$ in G's at a frequency $\underline{f}$ in cycles/s the critical load $\underline{CL}$ can be determined from the transmissibility in the bottom mass $\underline{Tr_1}$.

$$CL = \frac{386\, K_1\, \ddot{Y}\, (Tr_1 + 1)}{(2\pi f)^2} + \Sigma W.$$

If this value is greater than the load-carrying capacity of the bottom box, it will obviously cause failure and the design would be rejected.

For the four-high stack example, the transmissibility at 5 cycles/s and 0.3 critical damping is 1.82. Thus the critical load is

$$CL = \frac{386 \cdot 771 \cdot 0.25\ (1.82 + 1)}{(2 \cdot \pi \cdot 5)^2} + 165.5 = 212.6 + 165.5$$

$$= 378.1 \text{ lbf.}$$

This compares safely with the exact value of 293.9 lbf determined in the previous example. Because the maximum compressive strength of this box (fig. 2) is at least 690 lbf, the critical load of 378.1 lbf is well below the failure level.

III. Multiple-Degree-of-Freedom Vibration Theory

A mechanical structure may often be modeled as a multiple-degree-of-freedom system of lumped masses with adjacent linear couplings. The solution to the system becomes an expression for the displacement of each element relative to time t in terms of some known input. For the dynamic system shown in figure 5, the input is a base displacement with a harmonic motion of a constant amplitude $\bar{Y}$ at the frequency ω.

$$Y = \bar{Y} \cos (\omega t) . \qquad (1)$$

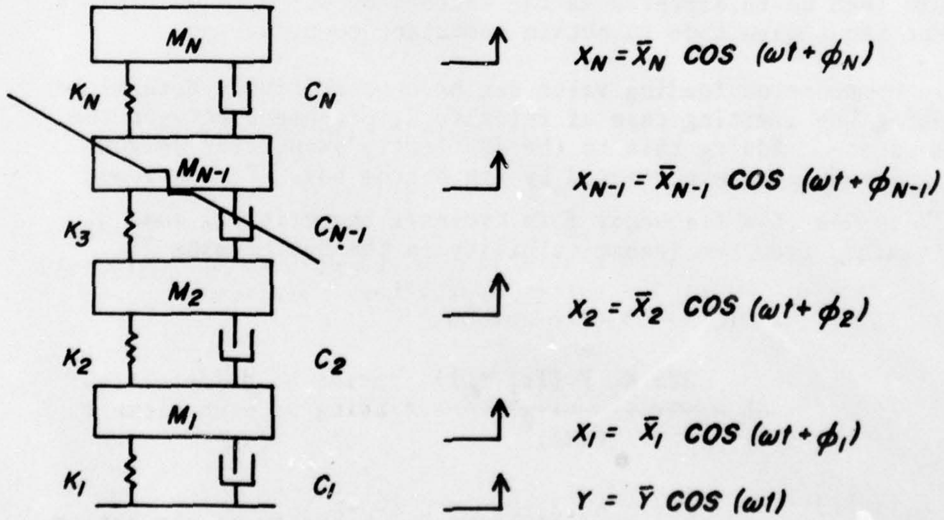

Figure 5.--Schematic diagram of a box stack modeled as a vibration system with N degrees of freedom. (M 146 030)

The analysis will consider only the steady state response that occurs after the input has been applied long enough for the transients to dissipate. If linear stiffness and viscous damping are assumed, the output at each element $\underline{i}$ will be a similar harmonic displacement at the same frequency $\underline{\omega}$ with an amplitude $\underline{\bar{X}_i}$ and phase $\underline{\Phi_i}$ from the input.

$$X_i = \bar{X}_i \cos(\omega t + \Phi_i) \ . \tag{2}$$

For design considerations it is usually desirable to express the solution in terms of the transmissibility $\underline{Tr_i}$ at each element, where

$$Tr_i = \bar{X}_i / \bar{Y} \ . \tag{3}$$

By differentiating equations (1) and (2) with respect to time, it can be shown that the displacement transmissibility is equal to both the velocity $\underline{(\dot{X}_i/\dot{Y})}$ and acceleration $\underline{(\ddot{X}_i/\ddot{Y})}$ transmissibilities. Therefore, the response ratios developed from a displacement input define also the response ratios when the input is expressed in its acceleration mode. The modal shape or pattern may also be of interest. This becomes

$$(\Phi_1/m, \ \Phi_2/m, \ \ldots, \ \Phi_N/m)$$

where $\underline{m}$ is selected to conveniently normalize the series of numbers.

Of primary significance are the natural frequencies of the system. For $\underline{N}$ degrees of freedom there will be $\underline{N}$ frequencies where, if damping is neglected, the input would produce an infinite response in each element. Where damping is considered, the system approaches a maximum response near these frequencies and is said to resonate.

The dynamics of the system may be modeled by a series of differential equations expressing the summation of forces existing at each element,

$$M_1\ddot{X}_1 = K_1(Y-X_1) + C_1(\dot{Y}-\dot{X}_1) - K_2(X_1-X_2) - C_2(\dot{X}_1-\dot{X}_2) \ ,$$

$$M_i\ddot{X}_i = K_i(X_{i-1}-X_i) + C_i(\dot{X}_{i-1}-\dot{X}_i) - K_{i+1}(X_i-X_{i+1}) + C_i(\dot{X}_i-\dot{X}_{i+1}) \ ,$$

$$M_N\ddot{X}_N = K_N(X_{N-1}-X_N) + C_N(\dot{X}_{N-1}-\dot{X}_N) .$$

Here, $\underline{K_i}$ is the resisting force due to a unit compression, $\underline{C_i}$ is the resisting force due to compressing at a unit velocity, and $\underline{M_i}$ is the mass lumped at an infinitely small point.

Rearranging and collecting terms into a more desirable form produces

$$(K_1+K_2)X_1 + M_1\ddot{X}_1 + (C_1 + C_2)\dot{X}_1 - K_2X_2 - C_2\dot{X}_2 = K_1Y + C_1\dot{Y} ,$$

$$-K_iX_{i-1} - C_i\dot{X}_{i-1} + (K_i+K_{i+1})X_i + M_i\ddot{X}_i + (C_i+C_{i+1})\dot{X}_i$$

$$-K_{i+1}X_{i+1} - C_{i+1}\dot{X}_{i+1} = 0 ,$$

$$-K_NX_{N-1} - C_N\dot{X}_{N-1} + K_NX_N + M_N\ddot{X}_N + C_NX_N = 0 .$$

Expressing the harmonics in the real components of their complex notation equivalents (7) makes the system solution more readily attainable.

Accordingly:

$$Y = \bar{Y}e^{j\omega t} ,$$

$$\dot{Y} = j\omega\bar{Y}e^{j\omega t} ,$$

$$X_i = \bar{X}_ie^{j(\omega t+\phi_i)} ,$$

$$\dot{X}_i = j\omega\bar{X}_ie^{j(\omega t+\phi_i)} ,$$

$$\ddot{X}_i = -\omega^2\bar{X}_ie^{j(\omega t+\phi_i)} ,$$

where j denotes the imaginary unit.

With these substitutions, the system becomes

$$(K_1+K_2)\bar{X}_1 e^{j(\omega t+\Phi_1)} - \omega^2 M_1 \bar{X}_1 e^{j(\omega t+\Phi_1)}$$

$$+ j\omega(C_1+C_2)\bar{X}_1 e^{j(\omega t+\Phi_1)} - K_2\bar{X}_2 e^{j(\omega t+\Phi_2)} - j\omega C_2\bar{X}_2 e^{j(\omega t+\Phi_2)}$$

$$= K_1\bar{Y}e^{j\omega t} + j\omega C_2\bar{Y}e^{j\omega t} \quad ,$$

$$- K_i\bar{X}_{i-1}e^{j(\omega t+\Phi_{i-1})} - j\omega C_i\bar{X}_{i-1}e^{j(\omega t+\Phi_{i-1})}$$

$$+ (K_i+K_{i+1})\bar{X}_i e^{j(\omega t+\Phi_i)} - \omega^2 M_i\bar{X}_i e^{j(\omega t+\Phi_i)}$$

$$+ j\omega(C_i+C_{i+1})\bar{X}_i e^{j(\omega t+\Phi_i)}$$

$$- K_{i+1}\bar{X}_{i+1}e^{j(\omega t+\Phi_{i+1})} - j\omega C_{i+1}\bar{X}_{i+1}e^{j(\omega t+\Phi_{i+1})} = 0 \quad ,$$

$$- K_N\bar{X}_{N-1}e^{j(\omega t+\Phi_{N-1})} - j\omega C_N\bar{X}_{N-1}e^{j(\omega t+\Phi_{N-1})}$$

$$+ K_N\bar{X}_N e^{j(\omega t+\Phi_N)} - \omega^2 M_N\bar{X}_N e^{j(\omega t+\Phi_N)}$$

$$+ j\omega C_N\bar{X}_N e^{j(\omega t+\Phi_N)} = 0 \ .$$

Collecting terms again and dividing through each equation by $e^{j\omega t}$ produces the final equation system form.

$$[(K_1+K_2) - \omega^2 M_1 + j\omega(C_1+C_2)]\bar{X}_1 e^{j\Phi_1} - [K_2+j\omega C_2]\bar{X}_2 e^{j\Phi_2}$$

$$= [K_1 + j\omega C_1]Y \ ,$$

$$- [K_i + j\omega C_i]\bar{X}_{i-1}e^{j\Phi_{i-1}}$$

$$+ [(K_i + K_{i+1}) - \omega^2 M_i + j\omega(C_i + C_{i+1})]\bar{X}_i e^{j\Phi_i}$$

$$- [K_{i+1} + j\omega C_{i+1}]\bar{X}_{i+1}e^{j\Phi_{i+1}} = 0 \ ,$$

$$- [K_N + j\omega C_N]\bar{X}_{N-1}e^{j\Phi_{N-1}} + [K_N - M_N\omega^2 + j\omega C_N]\bar{X}_N e^{j\Phi_N} = 0 \ .$$

If this system of equation is expressed in its equivalent matrix notation, matrix algebra may be applied to extract the solution. The component expressions become:

The (N x 1) output displacement matrix

$$[X] = \begin{bmatrix} X_1 e j\Phi 1 \\ \bar{X}_2 e^{j\Phi_2} \\ \cdot \\ \cdot \\ \bar{X}_i e^{j\Phi_i} \\ \cdot \\ \cdot \\ \bar{X}_N e^{j\Phi_N} \end{bmatrix} ,$$

the (N x 1) input displacement matrix

$$[Y] = \begin{bmatrix} \bar{Y} \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \tag{4}$$

the (N x N) diagonal mass matrix

$$[M] = \begin{bmatrix} M_1 & & & & \\ & M_2 & & & \\ & & \ddots & & \\ & & & M_i & \\ & & & & \ddots \\ & & & & & M_N \end{bmatrix}, \tag{5}$$

the (N x N) stiffness matrix

$$[K] = \begin{bmatrix} K_1+K_2 & -K_2 & & & \\ -K_2 & K_2+K_3 & -K_3 & & \\ & & \ddots & & \\ & & -K_{i-1} & K_i+K_{i+1} & -K_{i+1} \\ & & & & \ddots \\ & & & & -K_N & K_N \end{bmatrix}, \tag{6}$$

and the (N x N) damping matrix

$$[C] = \begin{bmatrix} C_1+C_2 & -C_2 & & & \\ -C_2 & C_2+C_3 & -C_3 & & \\ & & \ddots & & \\ & & -C_{i-1} & C_i+C_{i+1} & -C_{i+1} \\ & & & & \ddots \\ & & & & -C_N & C_N \end{bmatrix}. \tag{7}$$

The final mathematical model for the dynamic response of the idealized vibrating system becomes

$$\left[[K] - \omega^2[M] + j\omega[C] \right] [X] = (K_1 + j\omega C_1)\ [Y] \tag{8}$$

where the system displacement may be expressed as

$$[X] = \left[[K] - \omega^2[M] + j\omega[C] \right]^{-1} (K_1 + j\omega C_1)\ [Y]. \tag{9}$$

Because acceleration transmissibility is equal to displacement transmissibility, an equivalent expression for the acceleration response becomes

$$[\ddot{X}] = \left[[K] - \omega^2[M] + j\omega[C] \right]^{-1} (K_1 + j\omega C_1)\ [\ddot{Y}]. \tag{10}$$

To express the response of each mass in the form of equations (2) and (3), where an element from the displacement matrix of equation (9) takes the form

$$\bar{X}_i e^{j\Phi_i} = a_i + jb_i,$$

where a and b are real numbers. The amplitude is calculated as

$$\bar{X}_i = (a_i^2 + b_i^2)^{1/2}$$

and the phase difference as

$$\Phi_i = \tan^{-1}(b_i/a_i)$$

With damping neglected in equation (9), the response becomes infinite when, according to matrix theory (1), the expression

$$[K] - \omega^2[M]$$

is equivalent to the zero-filled matrix. The values of $\underline{\omega}$ that satisfy this condition then become the natural frequencies of the system.

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Appendix

The problem presented in the main text can be analyzed once the physical parameters and system input are defined.

The 3 degree of freedom model is subjected to an input of 0.25 G acceleration at 5 cycles per second. Each element has a weight W of 55.3 pounds and it is estimated that the box material contributes about 30 percent of critical damping C_{CR} where

$$C_{i\ CR} = 2(M_i K_i)^{1/2}$$

for each model element. The stiffnesses are linearized from the static compression test curves to be

$$K_1 = 771 \text{ lbf/in.} ,$$

$$K_2 = 607 \text{ lbf/in.} ,$$

$$K_3 = 500 \text{ lbf/in.}$$

The matrices for solution may be set up with appropriate units for compatibility. The mass matrix is

$$M_i = W_i(\text{lbf})/386(\text{in./s}^2) ,$$

$$M_1 = M_2 = M_3 = 55.3/386$$

$$= 0.143 \text{ lbf} \cdot \text{s}^2/\text{in.} ,$$

$$[M] = \begin{bmatrix} 0.143 & 0 & 0 \\ 0 & 0.143 & 0 \\ 0 & 0 & 0.143 \end{bmatrix} .$$

The stiffness matrix is

$$[K] = \begin{bmatrix} 1,378 & -607 & 0 \\ -607 & 1,107 & -500 \\ 0 & -500 & 500 \end{bmatrix} .$$

The damping matrix is

$$C_1 = 0.3 \times 2(0.143 \text{ lbf} \cdot \text{s}^2/\text{in.} \times 771 \text{ lbf/in.})^{1/2}$$

$$= 6.30 \text{ lbf} \cdot \text{s/in.} ,$$

$$C_2 = 0.3 \times 2(0.143 \times 607)^{1/2} = 5.59 \text{ lbf} \cdot \text{s/in.} ,$$

$$C_3 = 0.3 \times 2(0.143 \times 500)^{1/2} = 5.07 \text{ lbf} \cdot \text{s/in.} ,$$

$$[C] = \begin{bmatrix} 11.89 & -5.59 & 0 \\ -5.59 & 10.66 & -5.07 \\ 0 & -5.07 & 5.07 \end{bmatrix} .$$

The natural frequencies of the system may be calculated from the matrix formed by

$$[K] - \omega^2[M] .$$

Equating the determinant of this matrix to zero, the values for $\underline{\omega}^2$ may be solved from one of numerous techniques (8). Accordingly,

$$\begin{vmatrix} 1,378 - 0.143\,\omega^2 & -607 & 0 \\ -607 & 1,107 - 0.143\,\omega^2 & -500 \\ 0 & -500 & 500 - 0.143\,\omega^2 \end{vmatrix} = 0$$

from which

$$-2.924 \times 10^{-3}\omega^6 + 6.104 \times 10^1\omega^4 - 3.080 \times 10^5\omega^2$$

$$+ 2.361 \times 10^8 = 0$$

where the roots $\underline{\omega^2}$ become

$$\omega_1{}^2 = 930.5 \ s^{-2} ,$$

$$\omega_2{}^2 = 6,412. \ s^{-2} ,$$

$$\omega_3{}^2 = 1.353 \times 10^5 \ s^{-2}$$

In appropriate units the natural frequencies are

$$\omega_1 = 4.84 \text{ cycles/s},$$

$$\omega_2 = 12.7 \text{ cycles/s},$$

$$\omega_3 = 18.5 \text{ cycles/s}.$$

To investigate the effect of damping, equation (10) can be solved when the values with compatible units are substituted. Continuing the analysis,

$$\omega = 5 \text{ cycles/s} \times 2\pi \text{ radians/cycle}$$

$$= 31.42 \ s^{-1} ,$$

$$\omega^2 = 987.0 \ s^{-2} ,$$

$$\ddot{Y} = 0.25 \text{ G} \times 386 \text{ in./G} \cdot \text{s}^2$$
$$= 96.5 \text{ in./s}^2. \qquad (11)$$

The input acceleration matrix becomes

$$[\ddot{Y}] = \begin{bmatrix} 96.5 \\ 0 \\ 0 \end{bmatrix}.$$

Real and imaginary components may be collected separately.

$$[K] - \omega^2[M] = \begin{bmatrix} 1{,}237 & -607 & 0 \\ -607 & 965.9 & -500 \\ 0 & -500 & 358.9 \end{bmatrix},$$

$$\omega[C] = \begin{bmatrix} 373.6 & -175.6 & 0 \\ -175.6 & 334.9 & -159.3 \\ 0 & -159.3 & 159.3 \end{bmatrix},$$

$$K_1[\ddot{Y}] = \begin{bmatrix} 7.440 \cdot 10^4 \\ 0 \\ 0 \end{bmatrix},$$

$$\omega C_1[\ddot{Y}] = \begin{bmatrix} 1.910 \cdot 10^4 \\ 0 \\ 0 \end{bmatrix}.$$

The matrix algebraic expression for the acceleration response is now formed:

$$[\ddot{X}] = \begin{bmatrix} 1{,}237 + j373.6 & -607 - j175.6 & 0 \\ -607 - j175.6 & 965.9 + j334.9 & -500 - j159.3 \\ 0 & -500 - j159.3 & 359.8 + j159.3 \end{bmatrix}^{-1}$$

$$\times \begin{bmatrix} 7.440 \cdot 10^4 + j1.910 \cdot 10^4 \\ 0 \\ 0 \end{bmatrix} .$$

Without demonstrating the calculations, the indicated matrix is inverted (2,8) and the response becomes

$$[\ddot{X}] = \begin{bmatrix} 3.456 - j22.58 & -7.594 - j41.68 & -16.16 - j54.27 \\ -7.594 - j41.68 & -14.53 - j85.42 & -31.76 - j111.4 \\ -16.16 - j54.27 & -31.76 - j111.4 & -35.14 - j153.6 \end{bmatrix}$$

$$\times \begin{bmatrix} 7.440 + j1.910 \\ 0 \\ 0 \end{bmatrix} .$$

Finally

$$[\ddot{X}] = \begin{bmatrix} 68.84 - j161.4 \\ 23.11 - j328.6 \\ -16.57 - j434.6 \end{bmatrix} , \qquad (12)$$

or

$$\ddot{X}_1 = 175.5e^{j(31.4t-1.168)} = 0.46 \text{ G} \cos (31.4t-1.168),$$

$$\ddot{X}_2 = 329.4e^{j(31.4t-1.501)} = 0.85 \text{ G} \cos (31.4t-1.501),$$

$$\ddot{X}_3 = 434.9e^{j(31.4t-1.609)} = 1.13 \text{ G} \cos (31.4t-1.609).$$

To determine if any box has been loaded beyond it's maximum compression strength, combine equations (11) and (12).

$$\ddot{X}_3-\ddot{X}_2 = 113.2e^{j(31.4t-1.929)},$$

$$\ddot{X}_2-\ddot{X}_1 = 173.2e^{j(31.4t-1.838)},$$

$$\ddot{X}_1-\ddot{Y} = 163.8e^{j(31.4t-1.741)}.$$

From these the boxes' compressions may be expressed in terms of relative displacements, $\underline{D}$. Because harmonic displacement is a constant multiple of harmonic acceleration, the constants of integration are equal to zero.

$$D_1 = X_1-Y = -0.1660e^{j(31.4t-1.741)} = -0.1660 \text{ in} \cos (31.4t-1.741),$$

$$D_2 = X_2-X_1 = -0.1756e^{j(31.4t-1.838)} = -0.1756 \text{ in} \cos (31.4t-1.838),$$

$$D_3 = X_3-X_2 = -0.1147e^{j(31.4t-1.929)} = -0.1147 \text{ in} \cos (31.4t-1.929).$$

The compressing load $\underline{F}$ on a box then becomes the force due to dynamic compression plus the equilibrium supporting weight. Accordingly:

$$F_1 = K_1\bar{D}_1 + 3\ W = 293.9\ \text{lbf},$$

$$F_2 = K_2\bar{D}_2 + 2\ W = 217.2\ \text{lbf},$$

$$F_3 = K_3\bar{D}_3 + W = 112.7\ \text{lbf}.$$

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