AD-A057 273 PACIFIC-SIERRA RESEARCH CORP SANTA MONICA CALIF F/G 20/14

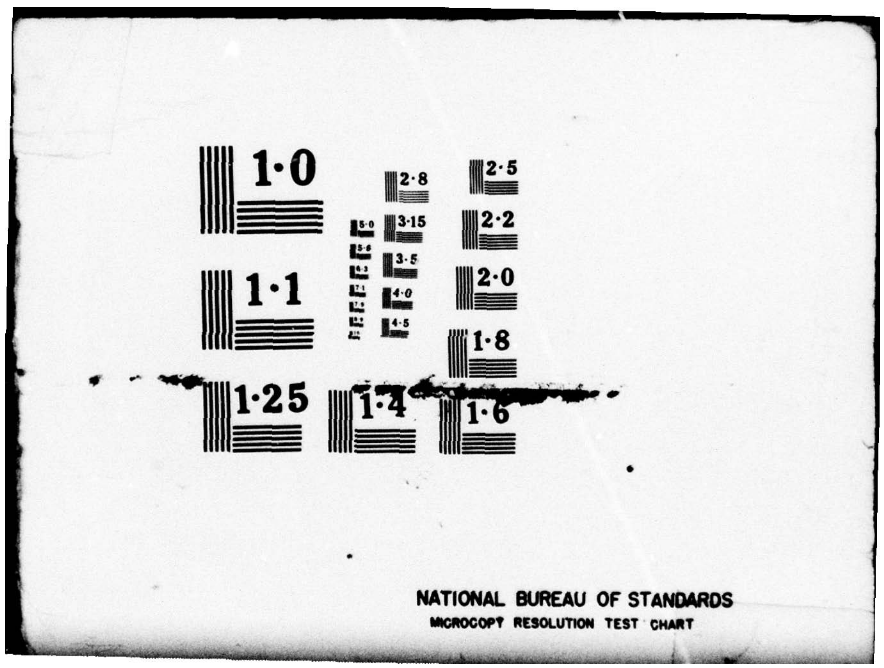
PROPAGATION OF THE LOW FREQUENCY GROUNDWAVE OVER NONUNIFORM TER--ETC(U)

MAR 78 E C FIELD, R ALLEN F19628-77-C-0120

UNCLASSIFIED PSR-717 RADC-TR-78-68 NL

1 OF 1
ADA
057273

END
DATE
FILMED
9-78
DDC


1·0
1·1
1·25
1·4
1·6
1·8
2·0
2·2
2·5
2·8
3·15
3·5
4·0
4·5
NATIONAL BUREAU OF STANDARDS
MICROCOPY RESOLUTION TEST CHART

ROPAGATION OF THE LOW-FREQUENCY
ROUNDWAVE OVER NONUNIFORM TERRAIN

acific-Sierra Research Corporation

pproved for public release; distribution unlimited.

OME AIR DEVELOPMENT CENTER
IR FORCE SYSTEMS COMMAND
RIFFISS AIR FORCE BASE, NEW YORK 13441

and is releasable to the National Technical Information Serv
At NTIS it will be releasable to the general public, includi
nations.

RADC-TR-78-68 has been reviewed and is approved for pub

APPROVED: Edward A. Lewis

EDWARD A. LEWIS, Chief
Propagation Branch (EEP)

APPROVED: Allan C. Schell

ALLAN C. SCHELL, Acting Chief
Electromagnetic Sciences Division (EE)

FOR THE COMMANDER: John P. H

JOHN P. HUSS
Acting Chief, Plans

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER	RADC-TR-78-68
2. GOVT ACCESSION NO.	
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle)	PROPAGATION OF THE LOW FREQUENCY GROUNDWAVE OVER NONUNIFORM TERRAIN
5. TYPE OF REPORT & PERIOD COVERED	Final Report, 15 Mar [redacted] - 30 Sep[redacted] 77
6. PERFORMING ORG. REPORT NUMBER	PSR-717
7. AUTHOR(s)	E. C. Field [redacted] R. Allen
8. CONTRACT OR GRANT NUMBER(s)	F19628-77-C-0120
9. PERFORMING ORGANIZATION NAME AND ADDRESS	Pacific-Sierra Research Corporation 1456 Cloverfield Boulevard Santa Monica, California 90404
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS	61102F 2304J320
11. CONTROLLING OFFICE NAME AND ADDRESS	Deputy for Electronic Technology Hanscom AFB, Massachusetts 01731 Monitor: Dr. Paul A. Kossey/EEP
12. REPORT DATE	Mar[redacted] [redacted]978
13. NUMBER OF PAGES	51
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	
15. SECURITY CLASS. (of this report)	Unclassified
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE	

16. DISTRIBUTION STATEMENT (of this Report)

Approved for Public Release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Groundwave
LORAN
Low Frequency Propagation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report examines the utility and limitations of the integral-equation representation of ground-wave propagation over nonuniform terrain. Emphasis is on frequencies between 20 kHz and 200 kHz. The one-dimensional version of the integral groundwave equation is subject to errors caused by: 1) topographic irregularities near the great-circle propagation path; 2) finite ground conductivity; 3) nonuniformities in the earth's electrical

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

407 486

properties; and 4) an approximate integration to reduce the dimension of the equation from two to one. Each of these errors is quantified, and the types of terrain to which the integral ground-wave equation is applicable are defined.

A method of numerical solution is developed and used to obtain results for the special case of a smooth, uniform, spherical earth. These results are compared in detail with numerical results obtained from the widely used residue-series representation of ground-wave propagation. The agreement between the two methods is shown to be excellent. Graphical results are given for the ground-wave attenuation function.

1. I have reviewed the Final Report by Pacific Sierra Research Corporation, titled PROPAGATION OF THE LOW FREQUENCY GROUNDWAVE OVER NONUNIFORM TERRAIN. The report describes work performed under contract to RADC/EEP, and represents the product of that work.

2. The report describes the analysis of errors associated with various formulations which mathematically describe the propagation of low frequency waves over the earth's surface. Each of the errors is quantified, and the types of terrain to which the integral ground-wave equation is applicable are defined. A method of numerical solution is developed and used to obtain results for the special case of a smooth, uniform, spherical earth. The results of this study apply directly to in-house efforts being conducted by the Propagation Branch (EEP) to support the ESD LORAN SPO's requirement for an extremely accurate means of predicting LF groundwave propagation parameters.

3. All aspects of the work contracted for by RADC/EEP have been completed and described in the subject report by Pac. Sierra Res. Corp., and meets the approval of this office. The efforts by Pacific Sierra have resulted in information of importance to our understanding of the propagation of long radio waves over non-uniform terrains.

Paul A. Kossey

Paul A. Kossey, EEP/x4265
Contract Monitor

SUMMARY

This report examines the utility and limitations of the integral-equation representation of ground-wave propagation over nonuniform terrain. Emphasis is on frequencies between 20 kHz and 200 kHz. The one-dimensional version of the integral ground-wave equation is subject to errors caused by: 1) topographic irregularities near the great-circle propagation path; 2) finite ground conductivity; 3) nonuniformities in the earth's electrical properties; 4) an approximate integration to reduce the dimension of the equation from two to one. Each of these errors is quantified, and the types of terrain to which the integral ground-wave equation is applicable are defined.

A method of numerical solution is developed and used to obtain results for the special case of a smooth, uniform, spherical earth. These results are compared in detail with numerical results obtained from the widely used residue-series representation of ground-wave propagation. The agreement between the two methods is shown to be excellent. Graphical results are given for the ground-wave attenuation function.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

List of Tables (cont'd.)

78 08 03 17

I. INTRODUCTION

The propagation velocity of low-frequency groundwaves is subject to perturbations from nonuniformities in either the topographic or electrical properties of the terrain. A sufficiently accurate theory of propagation over irregular terrain would, in principle, make it possible to correct position errors that such velocity perturbations cause on low-frequency radio navigation systems.

Historically, two theoretical treatments of groundwave propagation have evolved: the residue series of Van der Pol, Bremmer, Norton, and Fock (see e.g., *Bremmer, 1958*); and the integral equation approach (see e.g., *Hufford, 1952;* and *Feinberg, 1959*). The residue series, being more amenable to analytic solution, has been the foundation of most previous results. However, although useful for analysis of propagation over a uniform--or a piecewise uniform*--earth, the residue series is awkward for analysis of propagation over continuously varying terrain. For the latter conditions, numerical solution of the integral groundwave equation appears the most fruitful approach.

The integral equation is based on impedance boundary conditions, which are approximate. Therefore, regardless of the numerical accuracy of its solution, the classical version of the integral equation cannot provide accuracy better than that inherent in the impedance boundary condition. Beyond implementation of these boundary conditions, a number of additional approximations are usually made, with the result that the computationally simplest versions of the relevant equations are subject to the most stringent conditions on the types of terrain to which they are applicable.

Accordingly, a main purpose of this report is to quantify the accuracy of several forms of the integral ground-wave equation, thereby ascertaining the types of terrain to which they may be used to within specified error tolerances. Attention is restricted to frequencies between 20 kHz and 200 Hz, especially to the LORAN-D frequency at 100 kHz. The approach

---

*An example of a piecewise uniform earth would be two or more uniform regions separated by an abrupt boundary, such as shoreline.

taken is to derive expressions for errors due to the approximate treatment of 1) finite earth conductivity, 2) terrain curvature, and 3) nonuniformities in electrical properties. These error terms arise in two places in the derivation of the one-dimensional form of the integral equation: 1) use of impedance boundary conditions; 2) use of the method of stationary phases, or something nearly equivalent, to perform an integration over the coordinate transverse to the great-circle propagation path, thereby reducing a two-dimensional equation to a one-dimensional equation.

Use of impedance boundary conditions is essential to the derivation of the classical integral equation, and the resulting inaccuracies must be considered inherent to the formulation. The reduction from a two-dimensional to a one-dimensional integral equation, however, is a simplification that could be forgone at the expense of an order-of-magnitude increase in difficulty in obtaining numerical solutions. To determine when such an increase in difficulty would indeed provide a commensurate increase in overall accuracy, we compare the error terms due to the approximate transverse integration with the ones due to the impedance boundary conditions.

Section II gives the relevant versions of the integral ground-wave equation, and rank-orders the various error terms, which are derived in Appendices A and B. Section III gives numerical results comparing two versions of the one-dimensional integral equation with each other and with the residue series. Section IV presents conclusions; Appendix C derives the integral equation in polar coordinates for a smooth, round earth; and Appendix D outlines the procedures used to obtain numerical solutions.

II. INTEGRAL EQUATIONS FOR GROUNDWAVE PROPAGATION

We begin by outlining the steps required to derive the classical one-dimensional integral equation of Hufford (*1952*). Although the final result is simply the well-known Eq. (11) of Hufford's original paper, the intermediate steps reveal somewhat more general forms. Moreover, we identify the points at which critical approximations are made, and quantify the accuracy of these approximations. Finally, we give a form of the integral equation that is somewhat more accurate than Hufford's for the special case of a smooth, round earth.

HUFFORD'S EQUATION

Although awkward for the special conditions of a smooth, round earth, rectangular coordinates (x,y,z) are the most convenient when the shape of the earth's surface cannot be given a simple analytic form. We assume 1) that the transmitter is located at the origin, and let $\zeta(x,y)$ denote the deviation of the earth's surface from the plane $z = 0$; and 2) that the receiver is located in the vertical plane, $y = 0$. Below, we also use an integration point, Q, which is on the earth's surface and has coordinates $x,y,\zeta(x,y)$, and a receiver point, P, with coordinates $x_0,\zeta(x_0)$. Other quantities used below are r_0, r_1, and r_2, which are straight-line distances between the origin and P, the origin and Q, and Q and P, respectively. (Appendix B gives expressions for r_0, r_1, and r_2.)

The refractive index of the earth, n_g, is given by

$$n_g^2 = \kappa + i\sigma/\omega\varepsilon_0 \quad , \tag{1}$$

where κ is the dielectric constant of the earth, σ is the conductivity of the earth, ω is the angular frequency of the wave, and ε_0 is the vacuum dielectric permitivity. All parameters in Eq. (1) can be spatially nonuniform, although the validity of the forthcoming equations depends on these nonuniformities falling within constraints given below.

For certain smooth, symmetric surfaces, (e.g., planes or spheres), the Hertz potential for a vertical electric dipole has only a single component oriented normal to the surface. In such instances, only three (E_x,E_z,H_y) field components are excited that can be calculated from a single potential function, ψ. For arbitrary rough surfaces, the Hertz vector is not oriented in the normal direction, and all six field components are excited. To avoid this complexity, the derivation of the integral ground-wave equation is based on the assumption that the surface is "so smooth" that the Hertz vector is composed of essentially a single component oriented in the z direction.

Quantification of the error involved in this approximation is difficult. Intuitively, one would expect it to be valid provided the inclinations of the earth's surface with respect to its average level are small. More rigorously, if γ is the angle between the z axis and the normal to the earth's surface, the error will be roughly the amount by which

$$\cos\gamma = \frac{1}{\sqrt{1+\left(\frac{\partial\zeta}{\partial x}\right)^2+\left(\frac{\partial\zeta}{\partial y}\right)^2}} \tag{2}$$

differs from unity.

Subject to the error given by Eq. (2), ψ satisfies the wave equation

$$(\nabla^2+k^2)\psi = \tau \quad , \tag{3}$$

where τ is the source function, k is the free-space wave number, and a time dependence $e^{-i\omega t}$ has been assumed.

The second major approximation is use of impedance boundary conditions, which can be stated in the form

$$\frac{\partial\psi}{\partial n} = -ik\delta\psi \quad , \tag{4}$$

where n is the upward normal to the earth's surface and

$$\delta \approx 1/n_g \quad , \tag{5}$$

provided that n_g is large. The accuracy of the impedance boundary condition (discussed in detail in Appendix A) is summarized below. Use of the impedance boundary conditions is essential to the derivation. It permits $\partial\psi/\partial n$ to be expressed in terms of ψ on the surface, which, in turn, permits use of Green's Theorem to convert Eq. (3) to an integral equation involving integration over the earth's surface.* The procedure, described by Hufford, gives the result

$$\psi(P) = 2\psi_0(P) + \frac{ik}{2\pi}\int_A d^2Q\psi(Q)\,\frac{e^{ikr_2}}{r_2}\left[\delta + \left(1 + \frac{i}{kr_2}\right)\frac{\partial r_2}{\partial n}\right] \quad , \tag{6}$$

where ψ_0 is the potential that would exist if the earth were not present. By letting

$$\psi_0(Q) = \text{Const.}\,\frac{e^{ikr_1}}{r_1}$$

and defining W by

$$\psi(Q) = 2W(Q)\psi_0(Q) \quad ,$$

we find

$$W(P) = 1 + \frac{ik}{2\pi}\int_A \frac{d^2QW(Q)r_0}{r_1 r_2}\, e^{ik(r_1+r_2-r_0)}\left[\delta + \left(1 + \frac{i}{kr_2}\right)\frac{\partial r_2}{\partial n}\right] \quad . \tag{7}$$

---

*More accurate, albeit much more complicated, versions of Eq. (4) can be derived from the results of Rytov (*1940*).

In Eq. (7), W is an attenuation function that accounts for the fact that the earth is not flat and does not have infinite conductivity. Note that $W = 1$ if $\overline{\delta} = 0$ $(\sigma = \infty)$ and $\partial r_2/\partial n = 0$.

Equation (7) is the most general form of the integral equation, being subject only to the limitations of the impedance boundary conditions and the assumption of gentle departures from a plane earth. Being a two-dimensional integral equation, however, Eq. (7) is quite expensive to solve numerically. A major simplification (carried out in Appendix B) transforms--subject to some restrictions--Eq. (7) into a one-dimensional integral equation. Formally, this transformation involves using the method of stationary phases to perform analytically the integration over the coordinate transverse to the propagation path. Physically, this transformation implies that only regions within the first Fresnel zone significantly affect the received signal. The resulting equation is

$$W(x_0) = 1 - e^{-\pi i/4}\left[\frac{k}{2\pi}\right]^{1/2}\int_0^{x_0} dx\left[\frac{x_0}{x(x_0-x)}\right]^{1/2} W(x)(\delta + \partial r_2/\partial n)e^{ik(r_1+r_2-r_0)}, \tag{8}$$

which is the classic form derived by Hufford (*1952*).

ACCURACY OF HUFFORD'S EQUATION

Equation (8) is an order-of-magnitude simpler to solve than Eq. (7), but is less accurate because of errors incurred in the approximate transverse integration. It is therefore important to quantify the accuracy of these equations to determine whether Eq. (7) is sufficiently more accurate (or more general) than Eq. (8) to warrant the considerable additional computational complexity. Moreover, it is important to establish the limitations on Eqs. (7) and (8), thereby determining the types of terrain to which each may be applied to achieve some specified accuracy. Accordingly, Table 1 summarizes the first-order correction terms to each of the main approximations. These correction terms (derived in Appendices

A and B) denote the order of magnitude of the errors* involved in each approximation. In Table 1, R_0 denotes the local radius of curvature of the boundary.

Table 1

ACCURACY OF MAIN APPROXIMATIONS

Approximation	Fractional Error
I. Hertz vector normal to surface	$(\partial\zeta/\partial x)^2 + (\partial\zeta/\partial y)^2$
II. Impedance boundary conditions	
a. Finite conductivity	$\omega\epsilon_0/\sigma$
b. Surface curvature	$[\pi\mu_0\sigma f]^{-1/2}/R_0$
c. Nonuniform conductivity	$[\pi\mu_0\sigma f]^{-1/2}\left[\frac{1}{\sigma}\frac{\partial\sigma}{\partial z}\right]$
III. Stationary phase integration	
a. Stationary point at $y \approx 0$	$(\partial\zeta/\partial y)^2\big\vert_{y=0}$
b. Asymptotic series	$1/kx_0$

Table 1 shows that errors due to use of 1) impedance boundary conditions for finitely conducting media (IIa) and of 2) the asymptotic expansion of the stationary phase integration (IIIb) are the most fundamental in the sense that they are nonzero even for a plane, uniform earth. The other error terms depend on the degree of terrain nonuniformity.

A simple conclusion regarding the relative accuracy of approximations IIa and IIIb cannot be made, because one depends on ground conductivity, whereas the other depends on the length of the propagation path. Table A1 (p. 30) gives numerical values for the term IIa, and shows that better than 1-percent accuracy is obtained at LF/VLF provided that $\sigma \gtrsim 10^{-3}$ mhos/m.

---

*Roughly speaking, the percentage error associated with each approximation can be estimated by multiplying the fractional errors of Table 1 by 100.

Poor accuracy is obtained for Greenland ice, i.e., where $\sigma \approx 10^{-5}$ mhos/m. If the stationary-phase error-term IIIb is less than the impedance error-term IIa, no degradation in accuracy (for a plane earth) is caused by the transformation of Eq. (7) to Eq. (8), and any additional accuracy achieved by solving the two-dimensional equation would be spurious. Conversely, if the term IIIb exceeds IIa, additional accuracy is obtained by dealing with the complexity of the two-dimensional equation. Comparison of these terms shows that the stationary phase integration causes no substantial degradation in accuracy provided that x_0 exceeds a characteristic distance, ℓ, given by

$$\ell = \frac{\sigma}{\omega k \varepsilon_0} . \qquad (9)$$

Table 2 gives ℓ for various conductivities and a frequency of 100 kHz. These results show that if the conductivity is 10^{-3} mhos/m or less, the one-dimensional integral equation is essentially as accurate as the two-dimensional equation, provided that the pathlength exceeds about 100 km. For a conductivity of 10^{-2} mhos/m, the two-dimensional equation is more accurate than the one-dimensional one, unless the pathlength exceeds about 900 km. For seawater (σ = 4 mhos/m), the two-dimensional equation is far more accurate than the one-dimensional one for all realistic pathlengths.

Table 2

VALUES OF ℓ AT 100 kHz FOR SEVERAL CONDUCTIVITIES

σ(mhos/m)	4	10^{-2}	10^{-3}	10^{-5}
ℓ(km)	3.5×10^5	8.6×10^2	90	0.9

Before evaluating the other expressions in Table 1, note that their derivation involves power series expansions, and that these expansions are valid only when their magnitude is less than unity. Caution must also be exercised in interpreting the error-term I for a spherical earth. For a smooth, round earth, it is easy to show that

$$(\partial\rho/\partial x)^2 + (\partial\rho/\partial y)^2 = \frac{(x/a)^2}{1-(x/a)^2} , \qquad (10)$$

which correctly indicates that Eq. (8) becomes very inaccurate as the propagation pathlength, x, approaches an earth radius, a. This unnecessary inaccuracy, however, is due to Hufford's treatment of the earth's curvature as a perturbation to a plane earth. The situation is remedied by deriving the integral equation in spherical coordinates (Appendix C and below), which causes the appropriate Hertz vector to be rigorously normal to the surface for a smooth, round earth. Thus, the proper interpretation of such error terms as I and IIIa should treat ζ as the departure of the terrain contour from the average surface contour of the earth; i.e., ζ should include hills, etc., but not the earth's curvature, which can be accurately accounted for.

For a frequency of 100 kHz, Table A2 (p. 31) shows that errors due to surface curvature (expression IIa in Table 1) are about an order-of-magnitude greater than those due to finite conductivity (expression IIb), for conductivities of 10^{-3} mhos/m or more, and $R_0 = 1$ km. For $R_0 = 100$ m, the error terms due to curvature effects exceed 10 percent for normal ground conductivities. Nontheless, in this case, it is better to account for hills via Eqs. (7) or (8) than to leave them out of the analysis entirely. Note that by setting $R_0 = a$ in expression IIb, it follows that the impedance error caused by normal earth curvature is extremely small.

For Greenland ice, the results of Appendix A show that errors due to finite conductivity effects at 100 kHz are so large that discussion of other error sources is academic. We do not have adequate data to evaluate the term IIIc, which accounts for nonuniformities in conductivity.

It is easily shown, by evaluating expressions I or IIIa, that errors caused by terrain gradients should be less than 1 percent for grades of 5 percent or less, and less than 10 percent for grades of 15 percent or less. Much steeper grades would essentially totally destroy the accuracy of either Eq. (7) or Eq. (8).

In summary, for very smooth terrain where the error terms (Table 1) I, IIb, c, and IIIa are much smaller than the terms IIa and IIIb, the two-dimensional integral equation is much more accurate than the one-dimensional equation only for ground conductivities of 10^{-2} mhos/m or more. For much lower ground conductivities, the accuracy of the impedance boundary conditions is sufficiently poor that nothing additional is really lost by resorting to the approximate, one-dimensional integral equation. Once grades of 5 percent or more, or terrain features with radius of curvature of 1 km or less are encountered, the one-dimensional equation might as well be used, since the accuracy of the stationary phase integration is no worse that that of the other approximations.

INTEGRAL EQUATION FOR SMOOTH, ROUND EARTH

Equation (8) can be used to calculate W for the case of a smooth, round earth provided that the propagation path does not exceed a megameter or so. As shown by Eq. (10), the errors caused by departure of the earth's surface from the plane $z = 0$ can be substantial for longer propagation paths. Such errors can be avoided by rederiving the integral equation in spherical coordinates, and using the radial component--rather than the z-component--of the Hertz potential. In this instance, the error terms I and IIIa in Table 1 vanish, whereas the term IIb is extremely small for $R_0 = a$. The result is that an integral equation can be obtained that is essentially as accurate for a uniform spherical earth as is Eq. (8) for a uniform plane earth. In addition, this "spherical" integral equation provides a consistent basis of comparison with results calculated using the residue series, which rigorously accounts for a spherical earth.

The resulting one-dimensional integral equation* is

*See Appendix C for detailed derivation.

$$W(s_0) = 1 - \left[\frac{k}{2\pi}\right]^{1/2} e^{-\pi i/4} \int_0^{s_0} \frac{ds}{\sqrt{2a}} W(s) \left[\frac{\sin \frac{s_0}{2a}}{\sin s/2a} \left(\frac{\sin s/a}{\sin s_0/a \sin \frac{s_0 - s}{2a}}\right)^{1/2}\right.$$

$$\left.\left[\delta + \sin \frac{s_0 - s}{2a}\right] e^{2ika\left[\sin s/2a - \sin s_0/2a + \sin \frac{s_0 - s}{2a}\right]}\right], \quad (11)$$

where s and s_0 denote great-circle distances on the earth's surface.

By using $r_0 = 2a \sin s_0/2a$, etc., in the exponent of Eq. (11), and noting that

$$\partial r_2/\partial n = \sin\left[\frac{s_0 - s}{2a}\right] ,$$

it follows that Eqs. (8) and (11) agree to the extent that the approximation

$$\sin \frac{s_0}{a} \approx \frac{x_0}{a}$$

is valid; i.e., the fractional disagreement between Eq. (8) and Eq. (11) is of order $(s_0/a)^2$, which arises because Eq. (8) is subject to errors of the order of magnitude indicated by Eq. (10), whereas Eq. (11) is not.

III. NUMERICAL RESULTS FOR A UNIFORM, SPHERICAL EARTH

Equation (11) (p. 11) is the most accurate form of the one-dimensional integral equation for a smooth, spherical earth, and is therefore solved numerically to obtain the results given in this section. Specifically, the attenuation function, W, is computed as a function of distance for frequencies between 20 kHz and 200 kHz, and conductivities between 4 mhos/m and 2×10^{-5} mhos/m. (Appendix D outlines the numerical methods used.)

Although somewhat less accurate than Eq. (11) for a uniform, spherical earth, Hufford's integral equation (Eq. (8)) is convenient for analyzing propagation over irregular terrain. Accordingly, as a partial check on relative accuracy, we also solve Eq. (8) (p. 6) numerically and compare the results with those obtained from Eq. (11). In solving Eq. (8), we used the full expressions for r_0, r_1, r_2, and $\partial r_2/\partial n$ (e.g., $r_0 = 2a \sin s_0/2a$, etc.) rather than the expression in powers of $s/a \approx x/a$ used in Hufford's *(1952)* example.

As an accuracy check on both Eqs. (8) and (11), detailed comparisons with results given for the residue series by Wait and Howe *(1956)* are made. Care must be exercised in making these comparisons, because we have defined the attenuation function by (see Eq. (C-9))

$$\psi(s_0) \propto W(s_0) e^{ikr_0} \quad , \tag{12}$$

whereas, a different attenuation function, $\tilde{W}$, is defined by

$$\psi(s_0) \propto \tilde{W}(s_0) e^{iks_0} \quad , \tag{13}$$

in the residue series. Thus, since the Hertz potential, ψ, must be the same in both treatments, the phase of $\tilde{W}$ given by Wait and Howe must be corrected by a factor

$$k(s_0 - r_0) = k\left(s_0 - 2a \sin \frac{s_0}{2a}\right) \quad , \tag{14}$$

before comparison with our results. This adjustment in the residue series results has been made in the comparisons given below.

To make the comparisons as quantitative as possible, we use a digital rather than a graphical format. Tables 3 through 16 give the results; the number of significant figures given corresponds to the *numerical** accuracy that we used in solving Eqs. (8) and (11). The results labeled "spherical int. eq." correspond to Eq. (11); those labeled "Hufford int. eq." correspond to Eq. (8); those labeled "residue series" are taken from Wait and Howe, adjusted according to Eq. (14). Following Wait and Howe, we used an effective earth radius of 4a/3 (e.g., 8500 km) to account for atmospheric refraction. The sensitivity of the results to the choice of effective radius, which is crude at low frequencies, is examined below. For purposes of comparison with Wait and Howe, we used $\kappa = 0$ in the calculations, but do not advocate doing so in general. Also, the rather unusual distances at which the results are given were chosen so that comparison with Wait and Howe could be made.

We discuss first the results at 100 kHz, since this frequency is of more practical interest than the others. Tables 3 through 6 give these results, and show that both the Hufford equation and Eq. (11) agree with the residue-series results to within *one tenth of a degree* of phase for distances out to 600 kilometers. To put this accuracy in context, note that, at 100 kHz, one tenth of a degree of phase corresponds to a distance of *less than a meter*. Moreover, even this minute disagreement is due to roundoff, and would have been smaller had more significant figures been presented in the tables.

At distances of 1200 km or more, small--but noticeable--differences appear between the results of Eq. (8) and Eq. (11) and the residue series. Equation (11) agrees somewhat more closely with the residue series than does Eq. (8). This behavior is to be expected, because Eq. (8) is accurate only to order $(s_0/a)^2$, which becomes appreciable (0.08 at 2400 km) at the larger distances. We have no way of knowing whether the disagreement between Eq.(11) and the residue series at 2420 km (Table 3)

---

*Numerical accuracy pertains to the precision of the methods used to solve the equations, and has nothing whatever to do with the accuracy of the equations themselves, which is discussed in Sec. II and Appendices A and B.

Table 3

W AS COMPUTED FROM EQ. (11) FOR f = 100 kHz AND σ = 4 mhos/m

Distance (km)	60.6	121	242	606	1211	2420
Phase W (deg) Residue series	2.0	4.3	10.9	47.8	199.5	1148.2
Phase W (deg) Spherical int eq	2.0	4.3	10.8	47.8	199.2	1151.8
Difference	0	0	0.1	0	0.3	(3.6)
Amplitude W Residue series	0.983	0.952	0.869	0.576	0.223	0.024
Amplitude W Spherical int eq	0.982	0.951	0.868	0.575	0.223	0.023
Difference	0.001	0.001	0.001	0.001	0	0.001

Table 4

W AS COMPUTED FROM EQ. (8) FOR f = 100 kHz AND σ = 4 mhos/m

Distance (km)	60.6	121	242	606	1211	2420
Phase W (deg) Residue series	2.0	4.3	10.9	47.8	199.5	1148.2
Phase W (deg) Hufford int eq	2.0	4.3	10.9	47.9	198.8	1140.6
Difference	0	0	0	(0.1)	0.7	7.6
Amplitude W Residue series	0.983	0.952	0.869	0.576	0.223	0.024
Amplitude W Hufford int eq	0.982	0.951	0.868	0.575	0.222	0.026
Difference	0.001	0.001	0.001	0.001	0.001	(0.002)

Table 5

W AS COMPUTED FROM EQ. (11) FOR f = 100 kHz AND $\sigma = 10^{-2}$ mhos/m

Distance (km)	60.6	121	242	606	1211
Phase W (deg) Residue series	20.1	30.1	47.7	109.2	297.0
Phase W (deg) Spherical int eq	20.1	30.0	47.6	109.2	296.9
Difference	0	0.1	0.1	0	0.1
Amplitude W Residue series	0.969	0.927	0.828	0.531	0.206
Amplitude W Spherical int eq	0.969	0.927	0.828	0.531	0.206
Difference	0	0	0	0	0

Table 6

W AS COMPUTED FROM EQ. (8) FOR f = 100 kHz AND $\sigma = 10^{-2}$ mhos/m

Distance (km)	60.6	121	242	606	1211
Phase W (deg) Residue series	20.1	30.1	47.7	109.2	297.0
Phase W (deg) Hufford int eq	20.1	30.0	47.7	109.3	296.9
Difference	0	0.1	0	(0.1)	0.1
Amplitude W Residue series	0.969	0.927	0.828	0.531	0.206
Amplitude W Hufford int eq	0.968	0.926	0.829	0.531	0.206
Difference	0.001	0.001	(0.001)	0	0

Table 7

W AS COMPUTED FROM EQ. (11) FOR f = 20 kHz AND σ = 4 mhos/m

Distance (km)	60.6	121	242	606	1211
Phase W (deg) Residue series	0.6	1.6	4.2	17.5	61.6
Phase W (deg) Spherical int eq	0.6	1.5	4.1	17.4	61.5
Difference	0	0.1	0.1	0.1	0.1
Amplitude W Residue series	0.992	0.978	0.939	0.779	0.497
Amplitude W Spherical int eq	0.991	0.977	0.937	0.777	0.495
Difference	0.001	0.001	0.002	0.002	0.002

Table 8

W AS COMPUTED FROM EQ. (8) FOR f = 20 kHz AND σ = 4 mhos/m

Distance (km)	60.6	121	242	606	1211
Phase W (deg) Residue series	0.6	1.6	4.2	17.5	61.6
Phase W (deg) Hufford int eq	0.6	1.6	4.2	17.5	61.9
Difference	0	0	0	0	(0.3)
Amplitude W Residue series	0.992	0.978	0.939	0.779	0.497
Amplitude W Hufford int eq	0.992	0.977	0.938	0.778	0.495
Difference	0	0.001	0.001	0.001	0.002

Table 9

W AS COMPUTED FROM EQ. (11) FOR f = 20 kHz AND $\sigma = 10^{-2}$ mhos/m

Distance (km)	60.6	121	242	606	1211
Phase W (deg) Residue series	4.3	6.7	11.5	29.3	79.5
Phase W (deg) Spherical int eq	4.2	6.7	11.4	29.3	79.3
Difference	0.1	0	0.1	0	0.2
Amplitude W Residue series	0.993	0.978	0.938	0.780	0.504
Amplitude W Spherical int eq	0.991	0.976	0.936	0.778	0.502
Difference	0.002	0.002	0.002	0.002	0.002

Table 10

W AS COMPUTED FROM EQ. (11) FOR f = 20 kHz AND $\sigma = 10^{-3}$ mhos/m

Distance (km)	60.6	121	242	606	1211
Phase W (deg) Residue series	12.5	18.4	28.1	56.2	119.8
Phase W (deg) Spherical int eq	12.5	18.4	28.1	56.2	119.5
Difference	0	0	0	0	0.3
Amplitude W Residue series	0.987	0.967	0.920	0.750	0.477
Amplitude W Spherical int eq	0.987	0.967	0.920	0.749	0.477
Difference	0	0	0	0.001	0

Table 11

W AS COMPUTED FROM EQ. (11) FOR f = 50 kHz AND σ = 4 mhos/m

Distance (km)	60.6	121	242	606	1211
Phase W (deg) Residue series	1.2	2.8	7.1	30.5	117.5
Phase W (deg) Spherical int eq	1.2	2.8	7.1	30.5	117.4
Difference	0	0	0	0	0.1
Amplitude W Residue series	0.988	0.966	0.905	0.675	0.338
Amplitude W Spherical int eq	0.987	0.965	0.904	0.674	0.337
Difference	0.001	0.001	0.001	0.001	0.001

Table 12

W AS COMPUTED FROM EQ. (11) FOR f = 50 kHz AND $\sigma = 10^{-2}$ mhos/m

Distance (km)	60.6	121	242	606	1211
Phase W (deg) Residue series	10.3	15.6	25.0	60.8	164.4
Phase W (deg) Spherical int eq	10.3	15.6	25.0	60.8	164.2
Difference	0	0	0	0	0.2
Amplitude W Residue series	0.985	0.960	0.896	0.667	0.342
Amplitude W Spherical int eq	0.985	0.960	0.896	0.667	0.342
Difference	0	0	0	0	0

Table 13

W AS COMPUTED FROM EQ. (11) FOR f = 50 kHz AND $\sigma = 10^{-3}$ mhos/m

Distance (km)	60.6	121	242	606	1211
Phase W (deg) Residue series	30.8	44.6	66.4	126.5	263.5
Phase W (deg) Spherical int eq	30.8	44.6	66.4	126.5	263.5
Difference	0	0	0	0	0
Amplitude W Residue series	0.952	0.899	0.790	0.505	0.213
Amplitude W Spherical int eq	0.957	0.903	0.795	0.509	0.211
Difference	(0.005)	(0.004)	(0.005)	(0.004)	0.002

Table 14

W AS COMPUTED FROM EQ. (11) FOR f = 200 kHz AND σ = 4 mhos/m

Distance (km)	60.6	121	242	606	1211
Phase W (deg) Residue series	3.4	7.0	17.0	77.3	350.9
Phase W (deg) Spherical int eq	3.4	7.0	17.0	77.4	351.1
Difference	0	0	0	(0.1)	(0.2)
Amplitude W Residue series	0.976	0.932	0.820	0.462	0.130
Amplitude W Spherical int eq	0.975	0.931	0.819	0.461	0.129
Difference	0.001	0.001	0.001	0.001	0.001

Table 15

W AS COMPUTED FROM EQ. (8) FOR f = 200 kHz AND σ = 4 mhos/m

Distance (km)	60.6	121	242	606	1211
Phase W (deg) Residue series	3.4	7.0	17.0	77.3	350.9
Phase W (deg) Hufford int eq	3.4	7.0	17.0	77.5	353.9
Difference	0	0	0	(0.2)	(3.0)
Amplitude W Residue series	0.976	0.932	0.820	0.462	0.130
Amplitude W Hufford int eq	0.975	0.931	0.819	0.461	0.129
Difference	0.001	0.001	0.001	0.001	0.001

Table 16

W AS COMPUTED FROM EQ. (11) FOR f = 200 kHz AND $\sigma = 10^{-2}$ mhos/m

Distance (km)	60.6	121	242	606	1211
Phase W (deg) Residue series	39.5	58.1	89.8	199.0	546.7
Phase W (deg) Spherical int eq	39.5	58.1	89.8	199.0	544.7
Difference	0	0	0	0	2.0
Amplitude W Residue series	0.921	0.834	0.668	0.307	0.068
Amplitude W Spherical int eq	0.922	0.836	0.670	0.307	0.066
Difference	(0.001)	(0.002)	(0.002)	0	0.002

is due to the approximations made in this report, or numerical imprecisions in Wait and Howe's results. Also, Wait and Howe did not give the exact value that they used for the earth's radius. However, even for the worst case shown (σ = 4 mhos/m, s_o = 2420 km), the disagreement in phase is 3.6°, which corresponds to a distance of only 30 meters.

The remaining tables (7 through 16) further confirm the general conclusions drawn above. For distances up to 600 km, the agreement among Eq. (8), Eq. (11), and the residue series is virtually exact. For greater distances, the agreement is still excellent but, as expected, Eq. (11) agrees slightly more closely with the residue-series results than does Eq. (8).

The results of Tables 3 to 16 are sufficiently close to those of Wait and Howe that a graphical presentation here would add nothing new. Wait and Howe, however, did not present results for propagation over ice, nor were they able to obtain satisfactory convergence of the residue series for $\sigma = 10^{-3}$ mhos/m and frequencies of 100 kHz and 200 kHz. For these conditions, therefore, we present graphical results (Figs. 1 through 4).

Figure 1 gives the amplitude and phase of W for a frequency of 100 kHz and a ground conductivity of 10^{-3} mhos/m. Results are shown for both the "4/3" earth used by Wait and Howe, and a "normal earth" of radius 6372 km. The results for these two assumed earth radii agree quite closely, although noticeable differences do occur at ranges of several hundreds of kilometers. For example, at a range of 600 km, the selection of effective earth radius can influence the calculated phase by more than 20 degrees, which corresponds to a position uncertainty of about 170 meters. Since the use of an effective earth radius to account for atmospheric refraction is crude, this 20-degree difference between the "4/3" and "normal" earths must be regarded as a sort of uncertainty, which far exceeds the mathematical uncertainties associated with Eqs. (8) and (11). In other words, the accuracy of the equations seems to be far better than this input to the equations.

Figure 2 gives the amplitude and phase of W versus distance for a frequency of 200 kHz and $\sigma = 10^{-3}$ mhos/m; and Figs. 3 and 4, the amplitude and phase of W for various VLF/LF frequencies, and electrical properties

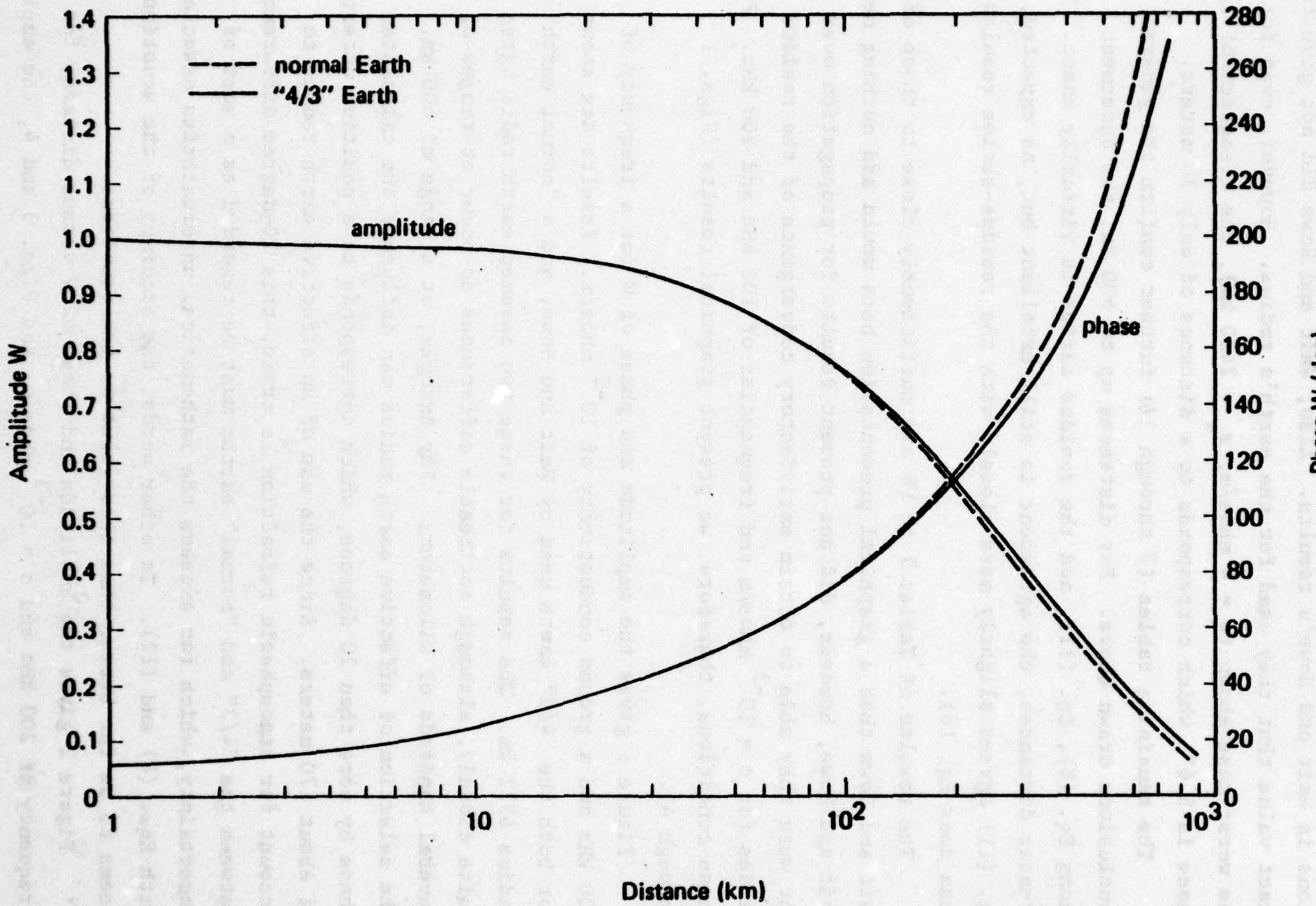


Fig. 1--Amplitude and phase of W for f = 100 kHz and $\sigma = 10^{-3}$ mhos/m

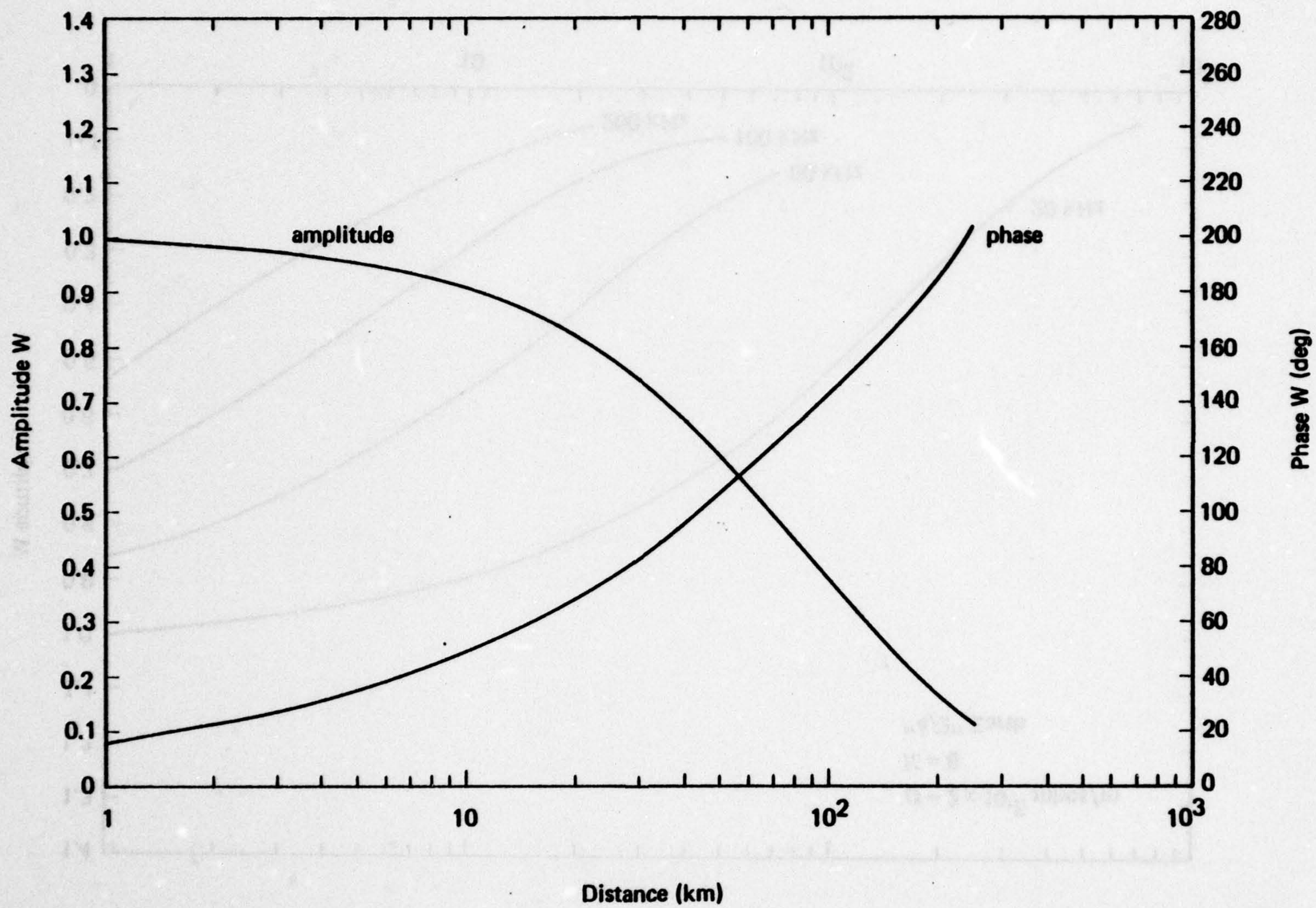


Fig. 2--Amplitude and phase of W for f = 200 kHz and $\sigma = 10^{-3}$ mhos/m; "4/3" earth

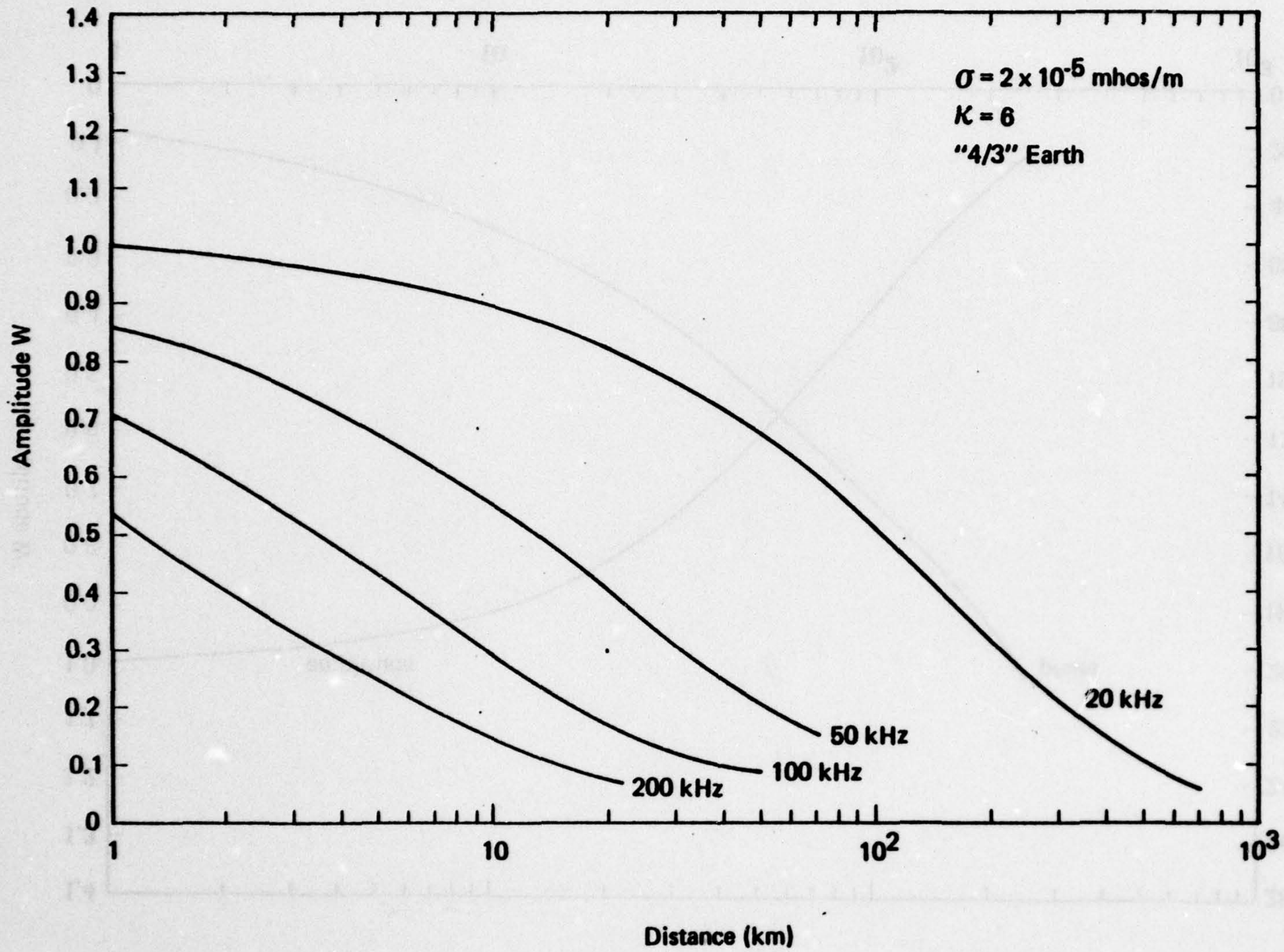


Fig. 3--Amplitude of W for propagation over Greenland ice

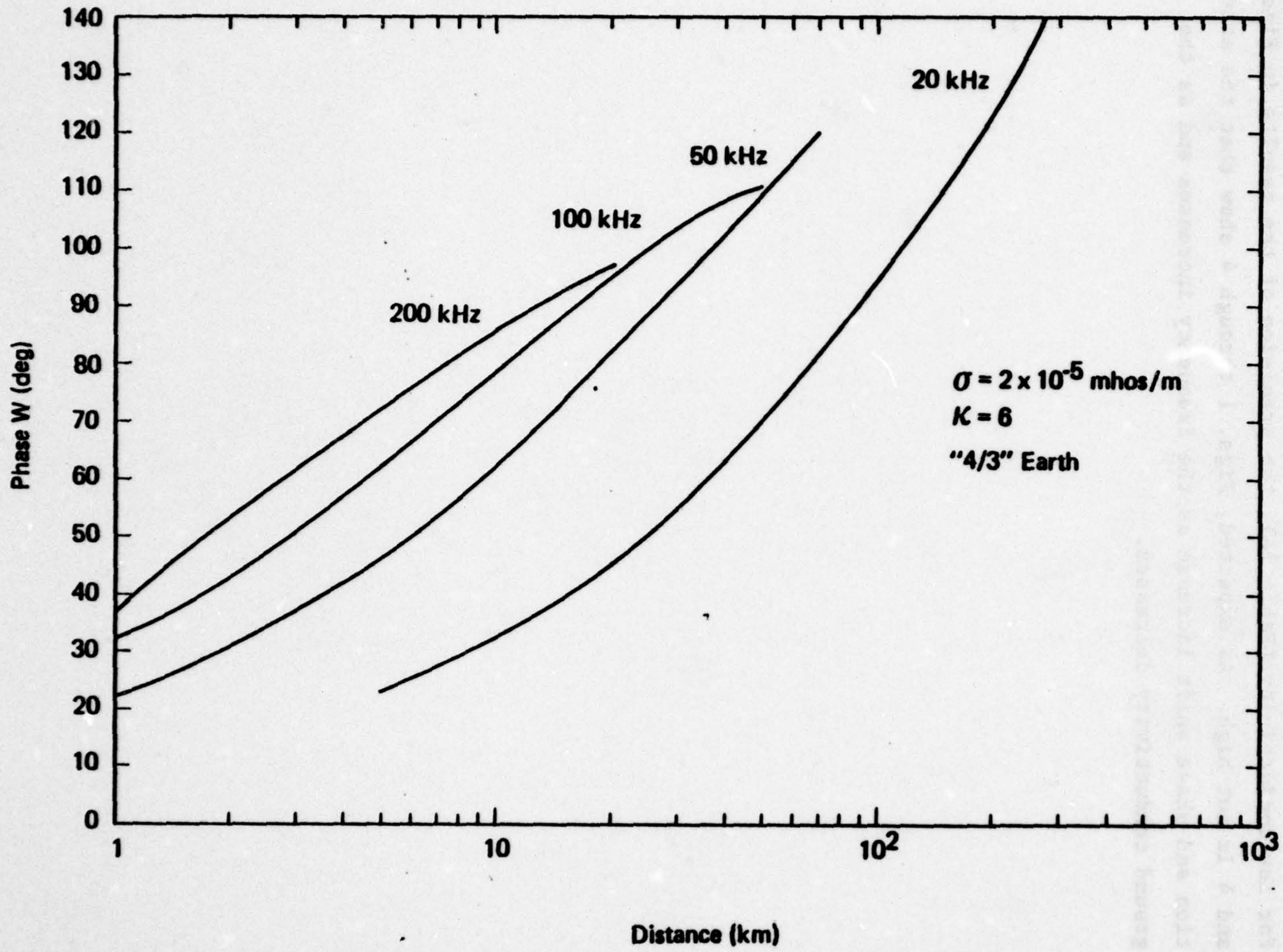


Fig. 4--Phase of W for propagation over Greenland ice

corresponding to nominal Greenland ice (e.g., $\kappa = 6$, and $\sigma = 2 \times 10^{-5}$ mhos/m). Given the poor accuracy of the impedance boundary conditions for low conductivity (Table 1A), the precision of the results in Figs. 3 and 4 is not high. As expected, Figs. 1 through 4 show that the attenuation and phase shift increase as the frequency increases and as the ground conductivity decreases.

IV. CONCLUSIONS

For a smooth, uniform earth, Hufford's one-dimensional integral equation gives essentially exact agreement with the results of the residue series for frequencies between 20 kHz and 200 kHz, and propagation distances up to several hundreds of kilometers. For propagation distances greater than 1000 km to 1500 km, the accuracy of Hufford's equation degrades somewhat, and a modified one-dimensional equation--expressed in polar coordinates--provides slightly better agreement with the residue series.

For a nonuniform earth having terrain undulations, the two-dimensional integral equation exhibits errors due to: 1) assuming that the Hertz vector is essentially normal to the surface; 2) use of impedance boundary conditions. The one-dimensional integral equation incurs additional errors due to the approximate evaluation of an integral over the coordinate transverse to the propagation path.

For ground conductivities greater than about 10^{-2} mhos/m, or for pathlengths less than about 100 km, the two-dimensional integral equation is inherently much more accurate than the one-dimensional one, provided that the earth is fairly smooth. However, this additional accuracy could be unnecessary, because--as was the case for the perfectly smooth, spherical earth--the accuracy of the one-dimensional equation could be adequate.

For ground conductivities less than about 10^{-3} mhos/m, or for relatively rough terrain, the inherent accuracy of the two-dimensional equation is really no better than that of the simpler one-dimensional version. This behavior occurs because errors due to the assumption of impedance boundary conditions and a normally oriented Hertz vector are at least as large as those due to the approximations made in reducing the two-dimensional equation to the classical one-dimensional form. More specifically, for these unfavorable terrain characteristics, the accuracy of the two-dimensional equation is degraded to the extent that no additional penalty is paid for performing the approximate transverse integration.

REFERENCES

Bremmer, H., "Propagation of Electromagnetic Waves," *Handb. der Phys.*, *16*, 423-639, 1958.

Collatz, L., *The Numerical Treatment of Differential Equations*, Springer-Verlag, 1960.

Erdelyi, A., *Asymptotic Expansions*, Dover Publications, New York, 1956, p. 51.

Feinberg, E. L., "Propagation of Radio Waves Along An Inhomogeneous Surface," *Del Nuovo Cimento, Suppl. V.1. XI, Ser. X*, 60-91, 1959.

Gradshteyn, I. S., and I. M. Ryzhik, *Tables of Integrals Series and Products*, Academic Press, New York, 1965.

Hufford, G. A., "An Integral Equation Approach to the Problem of Wave Propagation Over An Irregular Surface," *Quart. Appl. Math*, *9*, 391-403, 1952.

Krylov, V. I., *Approximate Calculation of Integrals*, MacMillian, New York, 1962.

Leontovich, M. A., "Approximate Boundary Conditions for the Electromagnetic Field on the Surface of a Good Conductor," *Bull. Acad. Sci. URSS, Sec. Phys. 9*, 16, 1944.

Rytov, S. M., "The Calculation of the Skin-Effect By the Method of Perturbation," *J. Expt. and Theor. Phys., No. 2, Vol. 10*, 180, 1940 (in Russian). Translated, R. L. Allen, PSR N176, Pacific-Sierra Research Corporation, Santa Monica, Calif., December 1977.

Senior, T. B. A., "Impedance Boundary Conditions for Imperfectly Conducting Surfaces," *Appl. Sci. Res. B9*, 418-436, 1961.

Wait, J. R., and H. H. Howe, *Amplitude and Phase Curves for Ground Wave Propagation in the Band 200 c/s to 500 kc*, NBS Circular 574, May 1956.

Appendix A

ACCURACY OF IMPEDANCE BOUNDARY CONDITIONS

Use of impedance boundary conditions (Eq. (4), p. 4) is essential to the derivation of the integral equation for the groundwave attenuation function. Therefore, the accuracy of even an exact solution of the full-fledged two-dimensional integral equation (Eq. (7), p. 5) is no better than the accuracy of the impedance boundary conditions. The applicability of these boundary conditions has received detailed attention by numerous authors (e.g., *Rytov, 1940; Leontovich, 1944; Bremmer, 1958; Feinberg, 1959; Senior, 1961*), and the details of their treatments need not be repeated here. However, to quantify the inherent accuracy of Eq. (8) for the frequencies and terrain of interest here, this appendix briefly summarizes the formulas for the correction terms to the impedance approximation.

Impedance boundary conditions are accurate for highly conducting, uniform media having flat boundaries. Errors are thus incurred if 1) the medium is imperfectly conducting, 2) the electrical properties of the medium are spatially nonuniform, and 3) the boundary of the medium is not flat. We consider each of these errors below.

ERRORS DUE TO FINITE CONDUCTIVITY

Validity of impedance boundary conditions rests on the fact that for a highly conducting earth, the refracted wave is, according to Snell's law, propagated in a direction nearly normal to the earth's surface. Even for a plane earth, deviation of the wave normal in the earth from the normal to the surface causes computational errors. Rytov (*1940*) and Leontovich (*1944*) showed that, for a vertically polarized wave and a uniform plane earth, the fractional error* incurred by use of impedance boundary conditions is of order

$$1/n_g^2 \approx \frac{\omega\epsilon_0}{\sigma} \quad , \tag{A-1}$$

*"Fractional error" is defined as the ratio of the neglected terms to the retained term in a series representation.

where we have assumed that $\sigma/\omega\epsilon_0 >> \kappa$, where κ is the dielectric constant of the ground. For situations where $\sigma/\omega\epsilon_0$ is not much greater than κ, the error is of order $1/\kappa$. Since κ is only of order 10 or less for most types of ground, Eq. (A-1) applies for all situations where the accuracy of impedance boundary conditions is better than about 10 percent.

To quantify the error term (A-1), we give its numerical values for conductivities and frequencies of interest in Table A1. These values show that, for most cases, use of impedance boundary conditions is valid in the sense that the fractional error incurred is much less than unity; i.e., the percent error is no greater than about 10 percent. If very high accuracy--say, 1 percent (fractional error of 10^{-2}) or better--is required, however, the impedance boundary conditions are inadequate in the 20 kHz to 200 kHz range for Greenland ice, which has a conductivity of about 10^{-5} mhos/m.

Table A1

VALUES OF $\omega\epsilon_0/\sigma$ FOR VARIOUS FREQUENCIES AND CONDUCTIVITIES

f(kHz) \ σ(mhos/m)	4	10^{-2}	10^{-3}	10^{-5}
20	2.8×10^{-7}	1.2×10^{-4}	1.1×10^{-3}	1.1×10^{-1}
50	6.9×10^{-7}	2.9×10^{-4}	2.9×10^{-3}	2.8×10^{-1}
100	1.4×10^{-6}	5.8×10^{-4}	5.6×10^{-3}	5.6×10^{-1}
200	2.9×10^{-6}	1.1×10^{-3}	1.2×10^{-2}	1.2

ERRORS DUE TO SURFACE CURVATURE

The errors caused by curvature of the boundary have been calculated by Rytov (*1940*) and Senior (*1961*). Physically, the validity condition is that the local radius of curvature, R_0, of the surface be large compared with the skindepth of the wave in the earth. Mathematically, the expression for the fractional error incurred by using impedance boundary conditions in the presence of a curved boundary is

$$(\pi\mu_0\sigma f)^{-1/2}/R_0 \quad . \qquad \text{(A-2)}$$

To quantify this error term, Table A2 gives values for the expression (A-2) for various conductivities and frequencies and $R_0 = 10^3$m. One way to interpret Table A2 is that the values shown are the fractional errors due to hills (for example) having a radius of curvature of 1 km. The error caused by hills with a 100-m radius of curvature would be 10 times as large as shown in Table A2, whereas that due to hills with a 10-km curvature would be one tenth as large.

Table A2

VALUES OF $(\pi\mu_0\sigma f)^{-1/2}/R_0$ FOR VARIOUS CONDUCTIVITIES AND FREQUENCIES AND $R_0 = 10^3$m

f(kHz) \ σ(mhos/m)	4	10^{-2}	10^{-3}	10^{-5}
20	1.8×10^{-3}	3.6×10^{-2}	1.1×10^{-1}	1.1
50	1.1×10^{-3}	2.2×10^{-2}	7.1×10^{-2}	7.1×10^{-1}
100	8.0×10^{-4}	1.6×10^{-2}	5.0×10^{-2}	5.0×10^{-1}
200	5.6×10^{-4}	1.1×10^{-2}	3.6×10^{-2}	3.6×10^{-1}

Table A2 also shows that, with regard to curvature effects, the accuracy of the impedance boundary condition degrades as the frequency decreases. This behavior is different than that shown for finite-conductivity effects (Table A1), where the accuracy degrades as the frequency is increased. Thus, if the frequency is too high, the impedance approximation fails because the ground refractive index is too small to refract the wave into a nearly normal direction; whereas if the frequency is too low, the approximation fails because the skindepth can become comparable with the radius of curvature of the surface. Comparison of Tables A1 and A2 shows that, except for very smooth ground, curvature effects induce larger computational errors than finite-conductivity effects.

The errors shown in Table A2 apply where the entire propagation path is characterized by undulations having characteristic dimensions of a kilometer, and would be smaller if only part of the path contained such irregularities. Also, so long as the fractional errors shown in Table A2 are less than a few tenths, it is better to account for surface undulations using the impedance method, than to neglect them entirely. On this basis, we estimate that for a frequency of 100 kHz, and the average ground ($\sigma = 10^{-3}$ to 10^{-2} mhos/m), the impedance method of treating curved terrain should be useful, provided that the characteristic dimensions of the undulations are at least 100 meters. For smaller values of R_0, the error terms become comparable with the retained terms.

ERRORS DUE TO NONUNIFORM ELECTRICAL PROPERTIES

Rytov (*1940*), Leontovich (*1944*), and Senior (*1962*) have evaluated the errors incurred by using impedance boundary conditions for media having nonuniform electrical properties. Specifically, for κ and σ that are functions of x, y, and z, where z is the vertical coordinate, they showed that the fractional error is given by

$$\frac{1}{\kappa}\frac{\partial\delta}{\partial z} \sim (\pi\mu_0 f\sigma)^{-1/2}\left[\frac{1}{\sigma}\frac{\partial\sigma}{\partial z}\right] \quad . \qquad \text{(A-3)}$$

Since $(\pi\mu_0 f\sigma)^{-1/2}$ is simply the skindepth of the wave in the ground, the error term (A-3) will be small if the conductivity undergoes only a small fractional change within a skindepth of the surface. The expression (A-3) is obviously extremely small for seawater because the skindepth is small and the medium is nearly uniform. At a frequency of 100 kHz, we see--by multiplying the values in Table A2 by 1000--that the skindepth in normal ground is several tens of meters. Thus, for Eq. (A-3) to be small, the ground must be nearly homogeneous to a depth of tens of meters. Precise evaluation of Eq. (A-3) must use data on shallow conductivity-depth profiles for terrain of interest.

As pointed out by Senior (*1962*) and Leontovich (*1944*), the condition (A-3) depends only on vertical variations in conductivity, even through σ was assumed to have lateral inhomogeneities as well. Thus, to first order, nonuniformities in electrical properties introduce errors only to the extent that the wave can penetrate to a depth where the conductivity departs from its surface value. Errors due to lateral nonuniformities are of higher order, and are appreciable only where the quantity

$$\frac{1}{\sigma}\left(\frac{\partial\sigma}{\partial x} + \frac{\partial\sigma}{\partial y}\right) \quad , \qquad \text{(A-4)}$$

is very large; e.g., at a coastline.

Appendix B

VALIDITY CRITERIA FOR ONE-DIMENSIONAL INTEGRAL EQUATION

Application of impedance boundary conditions and Green's theorem leads to the following type of integral equation for the attenuation function, W(P)

$$W(P) = 1 + \int_A d^2Q W(Q) K(Q,P) \quad . \qquad \text{(B-1)}$$

In Eq. (B-1), the integration is taken over the surface, A, of the earth, Q is an integration point on this surface, and P is the receiver point, which we assume is also on the surface. By making certain approximations, the two-dimensional integration over the surface, A, can be converted into a one-dimensional integration along the terrain between transmitter and receiver. This appendix determines the accuracy of these approximations.

The detailed form of the right-hand side of Eq. (B-1) depends on the coordinate system used. However, the validity criteria for the approximate integration of the right-hand side of Eq. (B-1) do not depend on the coordinate system. Therefore, to keep the algebra as simple as possible, we establish the accuracy of the approximate integration in rectangular coordinates. Appendix C rederives the integral equation in spherical coordinates, which are more natural to propagation over a spherical earth.

Consider a rectangular coordinate system with the transmitter at the origin, and with the plane defined by $z = 0$ oriented perpendicular to the vertical electric-dipole transmitting antenna. We let the deviation of the surface of the earth from the $z = 0$ plane be given by $\zeta(x,y)$, and assume the receiving antenna to be located in the vertical plane defined by $y = 0$. The integration point, Q, has the coordinates x, y, $\zeta(x,y)$, and the following relationships hold:

$$r_0^2 = x_0^2 + \zeta_0^2 \quad ; \quad \zeta_0 = \zeta(x_0, 0) \tag{B-2}$$

$$r_1^2 = x^2 + y^2 + \zeta^2(x,y) \quad , \tag{B-3}$$

$$r_2^2 = \left[x_0 - x\right]^2 + y^2 + \left[\zeta_0 - \zeta(x,y)\right]^2 \quad , \tag{B-4}$$

$$d^2Q = dxdy \sqrt{1 + \left(\frac{\partial\zeta}{\partial x}\right)^2 + \left(\frac{\partial\zeta}{\partial y}\right)^2} \quad . \tag{B-5}$$

In rectangular coordinates, Eq. (B-1) can be written (see Sec. II or *Hufford, 1952*)

$$W(P) = 1 + \int_A \frac{d^2Q}{r_1 r_2} F(Q)\, e^{ikr_0\left[\frac{r_1 + r_2}{r_0} - 1\right]} \quad , \tag{B-6}$$

where

$$F(Q) = \frac{ikr_0}{2\pi} W(Q) \left[\delta + \left(1 + \frac{i}{kr_2}\right) \frac{\partial r_2}{\partial n}\right] \quad . \tag{B-7}$$

By using the relationships (B-2) through (B-5), Eq. (B-6) can be rewritten

$$W(x_0) = 1 + \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \Gamma(x,y) e^{ikr_0\left[\frac{r_1 + r_2}{r_0} - 1\right]} \quad , \tag{B-8}$$

where

$$\Gamma(x,y) = \frac{ik}{2\pi}\left[\frac{r_0}{r_1 r_2}\right] \cdot W(x,y) \sqrt{1 + \left(\frac{\partial\zeta}{\partial x}\right)^2 + \left(\frac{\partial\zeta}{\partial y}\right)^2}$$

$$\cdot \left[\delta + \left(1 + \frac{i}{kr_2}\right)\frac{\partial r_2}{\partial n}\right] \quad . \tag{B-9}$$

Thus, to reduce Eq. (B-9) to a one-dimensional integral equation, our task is to evaluate the integral

$$I(x) = \int_{-\infty}^{\infty} dy\Gamma(x,y)e^{ikr_0h(x,y)} \quad , \tag{B-10}$$

where

$$h = \frac{r_1 + r_2}{r_0} - 1 \quad . \tag{B-11}$$

For transmission paths much greater than $\lambda/2\pi$ (about 0.5 km at 100 kHz), $kr_0 >> 1$, and the integral (B-10) is of the classic form amenable to approximate evaluation by the method of stationary phases (e.g., *Erdelyi, 1956*). At this point of the derivation, other authors (e.g., *Hufford, 1952*) have correctly argued that the integral (B-10) is approximately given by the stationary-phase formula, which is well-known and can be written down by inspection. This formula is, in fact, the leading term in an asymptotic series representation of the integral (B-10). To quantify the accuracy of this term, we must retain and evaluate the second-order correction terms.

We assume that h has a stationary point at $y = y_0$, given by the equation

$$h'(y_0) = \left.\frac{\partial h}{\partial y}\right|_{y=y_0} = 0 \quad . \tag{B-12}$$

The value of y_0 will be found below; but, for now, we need only assume that such a stationary point exists. Because of Eq. (B-12), the power series for h becomes

$$h(y) = h + \frac{h''}{2}(y-y_0)^2 + \frac{h'''}{6}(y-y_0)^3 + \frac{h''''}{24}(y-y_0)^4 + \ldots , \tag{B-13}$$

where the prime denotes differentiation with respect to y, and all derivatives are evaluated at $y = y_0$. In the conventional stationary phase method, only the first two terms in Eq. (B-13) are retained--the others being correction terms that are small if $kr_0 >> 1$. We retain these higher-order terms and assume--subject to *a posteriori* justification--that they are small enough to permit the following series expansion of the exponent in Eq. (B-10):

$$e^{ikr_0h} \approx e^{ikr_0h(y_0)} e^{ikr_0h''(y-y_0)^2/2} \cdot \left\{1 + ikr_0\left[\frac{h'''}{6}(y-y_0)^3 + \frac{h''''}{24}(y-y_0)^4\right]\right\} . \tag{B-14}$$

By similar reasoning, and subject to similar *a posteriori* justification, we write

$$\Gamma(x,y) \approx \Gamma(x,y_0)\left[1 + \frac{\Gamma'(x,y_0)(y-y_0)}{\Gamma(x,y_0)}\right] . \tag{B-15}$$

By inserting Eqs. (B-14) and (B-15) into Eq. (B-10), and noting that odd powers of $(y-y_0)$ vanish due to odd symmetry, the integral becomes

$$I(x) \approx e^{ikr_0h(y_0)}\,\Gamma(x,y_0)\int_{-\infty}^{\infty} dy\, e^{ikr_0h''(y-y_0)^2/2} \cdot \left\{1 + \left[\frac{ikr_0h'''\Gamma'}{6\Gamma} + \frac{ikr_0h''''}{24}\right](y-y_0)^4\right\} , \tag{B-16}$$

which can be immediately integrated to give

$$I(x) \approx e^{\pi i/4}\, e^{ikr_0h(y_0)}\left(\frac{2\pi}{kr_0h''}\right)^{1/2}\Gamma(x,y_0) \cdot \left[1 - \frac{ih'''\Gamma'}{2kr_0(h'')^2\Gamma} - \frac{ih''''}{8kr_0(h'')^2}\right] , \tag{B-17}$$

where, again, a prime denotes y-differentiation and all quantities are evaluated at $y = y_0$.

Aside from the question of the limits on the x-integration, and one or two minor constraints, insertion of Eq. (B-17) into Eqs. (B-10) and (B-8), will give the classical one-dimensional integral equation (e.g., *Hufford, 1952; Bremmer, 1958*) provided that the following two conditions are satisfied:

$$h(y_0) \approx h(0); \quad r_1(y_0) \approx r_1(0) \text{ etc.,} \tag{B-18}$$

$$\frac{h'''}{2kr_0(h'')^2}\frac{\Gamma'}{\Gamma} << 1 \quad , \tag{B-19a}$$

$$\frac{h''''}{8kr_0(h'')^2} << 1 \quad . \tag{B-19b}$$

Condition (B-18) simply states that the stationary phase point, y_0, is sufficiently close to the plane defined by $y = 0$, that $y_0 \approx 0$ may be substituted in all relations. This condition results in an integration along the line between transmitter and receiver. Conditions (B-19) require that kr_0 be large, which is the validity requirement for the stationary phase integration. To quantify the accuracy of these approximations, we further simplify and evaluate the correction terms.

By inserting Eq. (B-11) into Eq. (B-12), performing the differentiation, and using a perturbation expansion, we obtain the following equation for the stationary phase point:

$$y_0 \approx \left. \frac{[r_1(\zeta_0 - \zeta) + r_2\zeta]\zeta'}{r_1 + r_2} \right|_{y=0} \quad . \tag{B-20}$$

If the transverse derivative of the terrain contour, ζ', vanishes at $y = 0$, then $y_0 = 0$, and we recapture the classical result that the integrand be evaluated on the line $z = 0$, $y = 0$. In fact, Eq. (B-20) shows that the stationary point is displaced from the line $z = 0$, $y = 0$ by an amount proportional to the lateral gradient of the terrain contour, ζ', evaluated on the line between transmitter and receiver.

It follows from Eq. (B-17) that the phase of $I(x)$ is governed by

$$r_0 h(y_0) = r_1 + r_2 - r_0 \quad . \tag{B-21}$$

By using Eqs. (B-2) through (B-4), and keeping leading terms in a power series expansion of r_1, r_2, and r_0, it follows that

$$r_0 h(y_0) \approx \frac{y_0^2 + \zeta^2}{2x} + \frac{y_0^2 + (\zeta_0 - \zeta)^2}{2(x_0 - x)} - \frac{\zeta_0^2}{2x_0} \quad , \tag{B-22}$$

for $0 \leq x \leq x_0$.

From Eq. (B-20), it follows that the order of magnitude of y_0 is $\zeta\zeta'$, because $r_1/(r_1 + r_2)$ and $r_2/(r_1 + r_2)$ are of order of unity. Therefore, the fractional phase error incurred by setting $y_0 = 0$ is

$$y_0^2/\zeta^2 \sim O\left[\left(\frac{\partial\zeta}{\partial y}\bigg|_{y=0}\right)^2\right] \quad . \tag{B-23}$$

To the same order of accuracy, the y-derivative term in Eq. (B-5) may also be neglected.

Tedious, but straightforward differentiation of $h(y)$, and insertion of $y = 0$, show that the error incurred by neglecting the terms given by (B-19) is of order

$$\frac{h''''}{8kr_0(h'')^2}\bigg|_{y=0} \sim O\,(1/kr_0) \quad . \tag{B-24}$$

It also follows from direct differentiation that, to the accuracy indicated by Eqs. (B-23) and (B-24),

$$r_0 h''(y=0) = \frac{r_1 + r_2}{r_1 r_2} \quad , \tag{B-25}$$

whence Eq. (B-17) can be written

$$I(x) = e^{\pi i/4} e^{ik(r_1+r_2-r_0)} \left[\frac{2\pi r_1 r_2}{k(r_1 + r_2)}\right]^{1/2} \Gamma(x,y_0) \quad , \tag{B-26}$$

where all quantities are evaluated at $y = 0$.

Comparison of Eq. (B-26) with Eq. (B-10) shows that the stationary-phase integration is equivalent to multiplying the integrand of Eq. (B-10) by a lateral distance, Δy, whose magnitude is given by

$$\Delta y = \left[\frac{2\pi r_1 r_2}{k(r_1 + r_2)}\right]^{1/2} = \left[\frac{\lambda r_1 r_2}{(r_1 + r_2)}\right]^{1/2} \quad , \tag{B-27}$$

which is essentially the width of the first Fresnel zone. Therefore, the stationary-phase integration is physically equivalent to retaining contributions from transverse distances of the order of a Fresnel zone. Satisfaction of condition (B-18) (or, equivalently, if the error term (B-23) is small) guarantees that the terrain will not vary appreciably across this zone. Combination of Eq. (B-26) with Eqs. (B-8) through (B-11) gives the following integral equation for the attenuation function

$$W(x_0) = 1 - e^{-\pi i/4} \left(\frac{k}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} dx \sqrt{1 + (\partial\zeta/\partial x)^2} \left[\frac{r_0^2}{r_1 r_2 (r_1 + r_2)}\right]^{1/2}$$

$$\cdot\, W(x) e^{ik(r_1+r_2-r_0)} \left[\delta + \frac{\partial r_2}{\partial n}\right] \quad . \tag{B-28}$$

Eq. (B-22) shows that the exponent, $k(r_1+r_2-r_0)$, is of order $k\zeta^2/x$ provided that $0 \leq x \leq x_0$. Thus, for terrain irregularities that are comparable to or smaller than a wavelength, the exponent in Eq. (B-24) is small and oscillates slowly. For $x < 0$ or $x > x_0$, however, the exponent is of order $2kx$ or $2k(x-x_0)$, which oscillates very rapidly. Thus, to within the same order of accuracy as the stationary-phase integration (e.g., Eq. (B-24)), the integration limits in Eq. (B-28) can be taken equal to 0 and x_0.

Use of these finite limits in the x-integration converts Eq. (B-28) into virtually Hufford's classic form. However, Hufford made two additional approximations, which--although consistent with those described above and in Sec. II--are not really necessary. First, Hufford assumed that $d^2Q = dxdy$, which is tantamount to neglecting $(\partial\zeta/\partial x)^2$ in Eq. (B-28). Second, he assumed that terrain irregularities were sufficiently gentle that, except in the exponent, it is permissible to set

$$\begin{aligned} r_0 &= x_0 \\ r_1 &= x \\ r_2 &= x_0 - x \end{aligned} \qquad \text{(B-29)}$$

The approximation (B-29) is accurate to order ζ_0^2/x_0^2. To within the accuracy of these approximations, Eq. (B-24) becomes Eq. (8) (p. 6), which is identcal to Eq. (11) in Hufford's *(1952)* original paper.

Appendix C

INTEGRAL EQUATION FOR UNIFORM, SPHERICAL EARTH

Equation (8) (p. 6) can be used to calculate the attenuation function for a smooth, spherical earth of uniform conductivity, σ. Somewhat better accuracy could be obtained by using Eq. (B-27) (p. 40) and retaining the full expressions for r_0, r_1, and r_2, rather than the approximate ones given by Eq. (B-28) (p. 40). In each instance, the curvature of the earth is accounted for in the function $\zeta(x)$, which represents the deviation of the earth's surface from a plane.

The above comments nothwithstanding, certain of the procedures and approximations used in Appendix B to derive Eq. (B-28) were necessitated by the fact that the shape of the surface was not specified. If one assumes a smooth, round earth at the outset, a much more direct--and slightly more accurate--derivation of the appropriate integral equation can be given. The major improvement is that spherical coordinates are used, thereby treating the earth's curvature in a natural way rather than as a perturbation to a plane surface.

In spherical coordinates (r,θ,ϕ), the three field components generated by a radially oriented (vertical) electric dipole located at $r = a$, $\theta = 0$, are given by

$$E_r = [k^2 + \partial^2/\partial r^2]\,[r\psi] \quad , \tag{C-1}$$

$$E_\theta = \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\,[r\psi] \quad , \tag{C-2}$$

and

$$H_\phi = i\varepsilon\omega\,\partial\psi/\partial\theta \quad , \tag{C-3}$$

where ε is the complex dielectric constant and the Hertz potential, ψ, satisfies

$$(\nabla^2 + k^2)\psi = 0 \quad , \tag{C-4}$$

except at the transmitter location. In the above equations, r is the distance from the origin--in this instance, the center of the earth--and therefore differs qualitatively from r_0, r_1, and r_2, which are linear distances between transmitter and receiver, transmitter and some integration point, and integration point to receiver.

Continuity of E and H at $r = a$ implies that

$$\left.\frac{\frac{\partial}{\partial r}[r\psi]}{\varepsilon_0 [r\psi]}\right|_{r=a^+} = \left.\frac{\frac{\partial}{\partial r}[r\psi]}{\varepsilon[r\psi]}\right|_{r=a^-} \quad . \tag{C-5}$$

Subject to the validity conditions for the impedance boundary conditions given in Appendix A,

$$\partial/\partial r\,[r\psi] \approx \frac{-ik}{n_g}[r\psi] \text{ if } r < a \quad ,$$

and the boundary condition becomes

$$\frac{\partial}{\partial r}[r\psi] = -ik\delta[r\psi] \quad , \tag{C-6}$$

which is simply a spherical coordinate version of Eq. (4) (p. 4).

Equations (C-4) and (C-6) may be used in conjunction with Green's theorem to obtain a two-dimensional integral equation. The steps are identical to those used by Hufford (*1952*) for the plane earth, except that here the volume is bounded by two disconnected surfaces--one a large sphere with a radius approaching infinity, the other coincident with the earth's surface except for an infinitesimal hemisphere about the receiver location. The resulting integral equation is

$$\psi(r_0) = 2\psi_0 + \frac{ik}{2\pi}\int dA\psi(r_1)\frac{e^{ikr_2}}{r_2}\left[\delta - \frac{i}{kr} + \left(1 + \frac{i}{kr_2}\right)\frac{\partial r_2}{\partial r}\right], \tag{C-7}$$

where the integral is taken over the surface of the earth and ψ_0 is the free-space Hertz potential.

We let

$$\psi_0(r_1) = \text{Const.}\ \frac{e^{ikr_1}}{r_1} , \qquad \text{(C-8)}$$

and

$$\psi(r_1) = 2W(r_1)\ \psi_0(r_1) , \qquad \text{(C-9)}$$

where, as above, W denotes an attenuation function accounting for departures from the situation where the earth is flat and of infinite conductivity. Insertion of Eqs. (C-8) and (C-9) into Eq. (C-7) gives the following integral equation for W.

$$W(r_0) = 1 + \frac{ik}{2\pi} \int dA W(r_1)\ \frac{r_0}{r_1 r_2}\ e^{ik(r_1+r_2-r_0)} \left[\delta - \frac{i}{kr} + \left(1 + \frac{i}{kr_2}\right)\frac{\partial r_2}{\partial r}\right] . \qquad \text{(C-10)}$$

Equation (C-10) is formally nearly identical to Eqs. (B-6) and (B-7), the extra term in the square brackets of Eq. (C-10) resulting from the polar spherical coordinate system. For a smooth, round earth, however, explicit functional forms can be given for dA, r_1, r_2, etc., rather than expressing them in terms of the unspecified terrain function, ζ, and its derivatives. These function forms permit the transverse part of the surface integral to be performed without having to use a full-fledged stationary-phase approximation.

We thus write (after considerable rearrangement)

$$dA = a^2 \sin\theta\ d\theta\ d\phi , \qquad \text{(C-11)}$$

$$r_0 = 2a\ \sin\theta_0/2 , \qquad \text{(C-12)}$$

$$r_1 = 2a\ \sin\theta/2 , \qquad \text{(C-13)}$$

$$r_2 = \sqrt{2}\, a\, [1 - \cos\theta \cos\theta_0 - \sin\theta \sin\theta_0 \cos\phi]^{1/2} \quad , \tag{C-14}$$

$$\left.\frac{\partial r_2}{\partial r}\right|_{r=a} = \frac{1}{\sqrt{2}} [1 - \cos\theta \cos\theta_0 - \sin\theta \sin\theta_0 \cos\phi]^{1/2} \quad , \tag{C-15}$$

$$\left.\frac{1}{r_2} \frac{\partial r_2}{\partial r}\right|_{r=a} = \frac{1}{2a} \quad , \tag{C-16}$$

where $[\theta_0, 0]$ and $[\theta, \phi]$ are the angular locations of the receiver and the integration point on the surface of the earth. Substituting Eqs. (C-11) through (C-15) into Eq. (C-10) gives the following form for the integral equation

$$W(\theta_0) = 1 + \left[\frac{ik}{2\pi}\right] \sqrt{2}\, a \int_0^{\pi} d\theta \sin\theta\, W(\theta) \left[\frac{\sin\theta_0/2}{\sin\theta/2}\right] e^{2ika[\sin\theta/2 - \sin\theta_0/2]}$$

$$\cdot \left[\phi - i/2ka + \frac{1}{2ia} \frac{\partial}{\partial k}\right] I(k,\theta) \quad , \tag{C-17}$$

where

$$I(k,\theta) = \int_0^{\pi} d\phi \frac{e^{i\sqrt{2}\, ka[A - B\cos\phi]^{1/2}}}{[A - B\cos\phi]^{1/2}} \quad , \tag{C-18}$$

and

$$A = 1 - \cos\theta \cos\theta_0 \quad , \tag{C-19}$$

$$B = \sin\theta \sin\theta_0 \quad . \tag{C-20}$$

Note that although the earth's surface is azimuthally symmetric, r_2 depends on ϕ and a transverse integration (C-18) must be performed.

Equation (C-17) is much simpler, however, than a full-fledged two-dimensional integral equation because the unknown function, W, depends only on θ. The transverse integral, I(k,θ), depends only on known quantities and could be tabulated via numerical integration, i.e., I(k,θ) is a known function of θ that need not be determined as part of the solution process. Further, note that the only approximation used to derive Eq. (C-17) is the application of the impedance boundary conditions, and that numerical solution of Eq. (C-17)--in conjunction with numerical tabulation of I(k,θ)--would give W(θ) to the accuracy of the impedance boundary conditions. However, as is evident from the close agreement of the numerical results given in Sec. III with those computed by other authors using the residue series, an asymptotic approximation to I(k,θ) gives good accuracy, and removes the need for numerical tabulation.

It is well-known, and was shown in Appendix B, that only the first Fresnel zone contributes significantly to the received signal, provided that the surface contains no abrupt nonuniformities in either shape or electrical properties. The angular width, Δϕ, of this zone is of order

$$\Delta\phi \approx [\lambda/r_0]^{1/2} \quad , \tag{C-21}$$

which is small provided that the transmission pathlength is at least several wavelengths. Thus, subject to *a posteriori* justification we use

$$\cos\phi \approx 1 - \phi^2/2 \quad , \tag{C-22}$$

in Eq. (C-18), which results in the following form of I(k,θ):

$$I \approx \sqrt{2/B} \int_0^{\pi} d\phi \, \frac{e^{ika\sqrt{B}[2(A-B)/B+\phi^2]^{1/2}}}{[2(A-B)/B+\phi^2]^{1/2}} \quad . \tag{C-23}$$

Further, since large values of ϕ are unimportant, the upper limit can be changed from π to ∞, and Eq. (C-23) can be approximated by

$$I \approx \lim_{b\to 0} \sqrt{2/B} \int_0^\infty d\phi \, \frac{e^{ika\sqrt{B}[2(A-B)/B+\phi^2]^{1/2}}}{[2(A-B)/B+\phi^2]^{1/2}} \cos b\phi \quad , \qquad \text{(C-24)}$$

which is of the standard form (*Gradshteyn and Ryzhik, 1965*)

$$\int_0^\infty d\phi \, \frac{e^{ip[m^2+\phi^2]^{1/2}}}{[m^2+\phi^2]^{1/2}} \cos b\phi \, d\phi = \frac{\pi i}{2} H_0^1\left(m\sqrt{p^2-b^2}\right) \quad , \qquad \text{(C-25)}$$

where the positive square root is taken and H_0^1 is the Hankel function. It follows that

$$I(k,\theta) \approx \frac{\pi i}{2} \sqrt{2/B} \, H_0^1\left(\sqrt{2} \, ka \, \sqrt{A-B}\right) \qquad \text{(C-26)}$$

or, after using Eqs. (C-19) and (C-20),

$$I \approx \frac{\pi i}{2} \sqrt{\frac{2}{\sin\theta \, \sin\theta_0}} \, H_0^1\left(2 \, ka \, \sin\frac{\theta_0-\theta}{2}\right) \quad \theta_0 > \theta$$

$$I \approx \frac{\pi i}{2} \sqrt{\frac{2}{\sin\theta \, \sin\theta_0}} \, H_0^1\left(2 \, ka \, \sin\frac{\theta-\theta_0}{2}\right) \quad \theta > \theta_0 \quad . \qquad \text{(C-27)}$$

The argument of the Hankel function is kr_2 (see Eq. C-14), evaluated at $\phi = 0$, and may be assumed large enough to use the asymptotic expansion of the Hankel function. Thus, to order $1/ka \sin\theta$, substitution of Eq. (C-27) into Eq. (C-17) gives

$$W(s_0) \approx 1 - \left[\frac{k}{2\pi}\right]^{1/2} e^{-\pi i/4} \int_0^{\pi a} \frac{ds}{\sqrt{2a}} W(s) \left[\frac{\sin s_0/2a}{\sin s/2a} \left(\frac{\sin s/a}{\sin s_0/a \left|\sin\left(\frac{s_0-s}{2a}\right)\right|}\right)^{1/2}\right]$$

$$\cdot \left[\delta + \sin\left(\frac{s_0-s}{2a}\right)\right] e^{2ika(\sin s/2a - \sin s_0/2a + |\sin (s-s_0)/2a|)} \quad , \quad \text{(C-28)}$$

where s denotes great-circle distances along the earth, and we have used $\theta = s/a$, etc. For $s \leq s_0$, the integrand oscillates very slowly, as can be seen by the fact that the exponent vanishes to first order in $s_0/2a$. For $s > s_0$, however, the integrand oscillates very rapidly, as can be seen from the fact that the exponent is $2ik(s-s_0)$ to first order in $s_0/2a$. Thus, we write for the final form of the integral equation for a smooth, round earth

$$W(s_0) \approx 1 - \left[\frac{k}{2\pi}\right]^{1/2} e^{-\pi i/4} \int_0^{s_0} \frac{ds}{\sqrt{2a}} W(s) \left[\frac{\sin s_0/2a}{\sin s/2a} \left(\frac{\sin s/a}{\sin (s_0/a) \sin\left(\frac{s_0-s}{2a}\right)}\right)^{1/2}\right]$$

$$\cdot \left[\delta + \sin\left(\frac{s_0-s}{2a}\right)\right] e^{2ika(\sin s/2a - \sin s_0/2a + \sin (s_0-s)/2a)} \quad , \quad \text{(C-29)}$$

which is Eq. (11) on p. 11 of the main text. The correction terms to Eq. (C-19) are extremely tedious to derive, and are not really needed, because the excellent agreement between numerical solutions of Eq. (C-29) and available results from the residue series is an adequate accuracy check. Nonetheless, we point out that by following steps analogous to Eqs. (B-13) through (B-17), we find after considerable algebra, that the fractional error of Eq. (C-29) is of the same order of magnitude as the error term given by Eq. (B-24).

Appendix D

NUMURICAL SOLUTION OF EQS. (8) AND (11)

METHOD

Both Eq. (8) and Eq. (11) (pp. 6 and 11) are instances of the Volterra integral equation of the second kind:

$$W(x) = g(x) + \int_0^x W(s)K(x,s)ds \quad . \qquad \text{(D-1)}$$

We solve this equation iteratively using the well-known method of Picard. Starting with the initial guess $W_0(x) = g(x)$, successive approximations W_1, W_2, ... are found such that

$$W_{j+1}(x) = g(x) + \int_0^x W_j(s)K(x,s)ds \quad . \qquad \text{(D-2)}$$

Iteration continues until two successive approximations differ by less than some specified tolerance. Volterra integral equations and their solution by Picard's iterative method are discussed by Collatz (*1960*).

The difficulty with the solution by Picard's method lies in the repeated evaluations of the integral in Eq. (D-1). From Eqs. (8) and (11), it is seen that the kernal, $K(x,s)$, viewed as a function of s with x fixed, is singular at the endpoints of the interval (0,x). Both Eq. (8) and Eq. (11) can be rewritten in the form*

$$W(x) = g(x) + \int_0^x \frac{W(s)H(x,s)}{\sqrt{x(x-s)}} ds \quad , \qquad \text{(D-3)}$$

---

*This procedure is possible for Eq. (11) because $\sin(s/2a)$ and $\sin\left(\frac{x-s}{2a}\right)$ have only first-order zeros near $s=0$ and $s=x$.

where H(x,s) has derivatives of all orders. A Chebyshev-Gauss quadrature is used to evaluate the integral in Eq. (D-3). For each iteration, W_j, we take

$$\int_0^x \frac{W_j(s)H(x,s)}{\sqrt{s(x-s)}}\,ds = \sum_{i=1}^{n} \frac{\pi}{n} f_j(s_i) + R_N \quad , \qquad \text{(D-4)}$$

where

$$f_j(s) = W_j(s)H(x,s)$$

$$s_i = \frac{x}{2} + \frac{x}{2}\,\frac{\cos(2i-1)\pi}{2n} \quad , \qquad \text{(D-5)}$$

and

$$R_n = \frac{2\pi}{2^{2n}(2n)!}\, f^{(2n)}(\xi) \quad 0 < \xi < x \quad . \qquad \text{(D-6)}$$

Here, R_n is the remainder term corresponding to the computational error incurred by use of a finite number, n, quadratures. Note that this remainder is zero, and the integration exact, when f is a polynomial of degree less than 2n. The Chebsyhev-Gauss quadrature is described by Krylov (*1962*).

IMPLEMENTATION

The code used to solve Eq. (8) and Eq. (11) is written in FORTRAN. The function W_j, represented as a vector array of chosen length, mathematically corresponds to representing the successive approximations to the solution as a piece-wise linear function. The code user selects the number and widths of the steps in the approximating piece-wise linear function. Iteration according to Eq. (D-2) continues until a pair W_j and W_{j+1} are found that satisfy the inequality

$$\frac{\left\{\sum_{N=1}^{N_{max}} \left[W_j(x_N) - W_{j+1}(x_N)\right]^2\right\}^{1/2}}{N_{max} - 1} < T \quad , \qquad \text{(D-7)}$$

where N_{max} is the number of values in the vector array approximating W, x_N is the value of x at the N^{th} step, and T is a chosen tolerance. Thus, our convergence criterion demands that the rms difference between two successive iterations be less than the chosen tolerance.

Most of the results given in the main text are obtained with a convergence tolerance of 10^{-4}, and were run on a CDC 7600 computer. The running time required to calculate W as a function of distance depends on the propagation pathlength, ground conductivity, wave frequency, number of quadratures, N_{max}, and T. The running time tended to increase as the frequency increased or the conductivity decreased. For example, for $T=10^{-4}$, 11 sec of running time was required to calculate W out to a distance of 2000 km for a conductivity of 4 mhos/m and a frequency of 100 kHz. Reduction of the frequency to 20 kHz reduced the running time to 8 sec; but, for 100 kHz, decreasing σ to 2×10^{-5} mhos/m caused 18 sec of running time to be required to calculate W out to only 50 km. Of course, at 100 kHz, about as much attenuation occurs for 50 km of propagation over ice (2×10^{-5} mhos/m) as for more than 1000 km over seawater (4 mhos/m). Thus, the running time required to compute a given amount of attenuation does not vary drastically.

MISSION

of

Rome Air Development Center

RADC plans and conducts research, exploratory and advanced development programs in command, control, and communications (C^3) activities, and in the C^3 areas of information sciences and intelligence. The principal technical mission areas are communications, electromagnetic guidance and control, surveillance of ground and aerospace objects, intelligence data collection and handling, information system technology, ionospheric propagation, solid state sciences, microwave physics and electronic reliability, maintainability and compatibility.