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AN ESSENTIALLY COMPLETE CLASS OF MULTIPLE DECISION PROCEDURES.(U)

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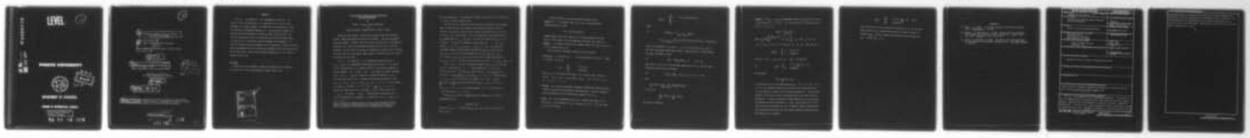

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An Essentially Complete Class of Multiple Decision Procedures*

by

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Abstract

Let $\pi_1, \ldots, \pi_k$ represent k (≥ 2) independent populations. The quality of the ith population π_i is characterized by a real-valued parameter θ_i, usually unknown. We define the best population in terms of a measure of separation between θ_i's. A selection of a subset containing the best population is called a correct selection (CS). We restrict attention to rules for which the size of the selected subset is controlled at a given point and the infimum of the probability of correct selection over the parameter space is maximized. The main theorem deals with construction of an essentially complete class of selection rules of the above type. Some classical subset selection rules are shown to belong to this class.

Key Words

Subset selection procedure, monotone likelihood ratio, monotone selection rule, normal means problem, unequal sample sizes.

An Essentially Complete Class of Multiple Decision Procedures*

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During the past decade, selection and ranking theory has developed rapidly. Many reasonable rules have been proposed. Some good properties have been studied. However, very little work has been done to consider the optimality of a selection procedure, especially in the subset selection approach. In this paper, we discuss an essentially complete class of subset selection rules (in some sense). Some classical selection rules are shown to be optimal in this sense.

Let $\pi_1,\ldots,\pi_k$ represent k ($\geq$ 2) independent populations and let $X_{i1},\ldots,X_{in_i}$ be n_i independent random observations from π_i. The quality of the ith population π_i is characterized by a real-valued parameter θ_i, usually unknown. Let $\Omega = \{\underline{\theta}|\underline{\theta} = (\theta_1,\ldots,\theta_k)\}$ denote the whole parameter space. Let $\tau_{ij} = \tau_{ij}(\underline{\theta})$ be a measure of separation between τ_i and τ_j. We assume that there exists a monotone non-increasing function h such that $\tau_{ji} = h(\tau_{ij})$. Let $\Omega_i = \{\underline{\theta}|\tau_{ij} \geq \tau_{ii},\ j\neq i\}$, $1 \leq i \leq k$. In this sequel, we assume τ_{ii} as known, $1 \leq i \leq k$. Let $\tau_i = \min\limits_{j\neq i} \tau_{ij}$, $i = 1,2,\ldots,k$. Assume that there exists an i such that $\tau_i \geq \tau_{ii}$. Thus we know that $\Omega = \bigcup\limits_{i=1} \Omega_i$. We define $\tau^* = \max\limits_{\ell} \tau_\ell$. The population associated with τ^* will be called the best population. We know that if $\underline{\theta} \in \Omega_i$, then π_i is

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the best population. A selection of a subset containing the best population is called a correct selection (CS).

We will restrict attention to those selection procedures which depend upon the observations only through a sufficient and maximal invariant statistic Z_{ij} which is based on the n_i and n_j observations from π_i and π_j $(i,j=1,2,\ldots,k)$, respectively. It is well known that the distribution of Z_{ij} depends only on τ_{ij}. For any i, let the joint denstiy of Z_{ij}, $j \neq i$, be $p_{\underline{\theta}}(\underline{z}_i)$. Let $p_{\underline{\theta}}(\underline{z}_i)$ be denoted by $p_i(\underline{z}_i)$ when $\tau_{i1} = \ldots = \tau_{ik} = \tau_{ii}$, where $\underline{z}_i = (z_{i1},\ldots,z_{i,i-1},z_{i,i+1},\ldots,z_{ik})$, $1 \leq i \leq k$. Let $F_{\underline{\theta}}$ be the continuous cumulative distribution function of $p_{\underline{\theta}}(\underline{z})$ for any $\underline{\theta}$ and let $p_i(z_{ij}|z_{i\ell},\ell \neq i,j)$ be the conditional pdf of Z_{ij}, given $Z_{i\ell} = z_{i\ell}$, $\ell \neq i,j$ and let $F_i(z_{ij}|z_{i\ell}, \ell \neq i,j)$ be the cdf of the conditional density $p_i(z_{ij}|z_{i\ell}, \ell \neq i,j)$. Let $F_i^{\circ}(y)$ be any point of the set $\{z: F_i(z|z_{i\ell}, \ell \neq i) = y\}$.

Let $\delta = (\delta_1,\ldots,\delta_k)$ be a selection procedure where $\delta_i(\underline{z})$ is the probability of selecting π_i, $1 \leq i \leq k$, having observed $\underline{z}$. Let $S(\underline{\theta},\delta) = P(CS|\delta)$ and $R(\underline{\theta},\delta) = \sum_{i=1}^{k} R^i(\underline{\theta},\delta_i)$, where $R^i(\underline{\theta},\delta_i) = \int \delta_i(\underline{z}_i)p_{\underline{\theta}}(\underline{z}_i)d\nu(\underline{z}_i)$. Let $R_j^i(\delta_i) = \int \delta_i(\underline{z}_i)p_i(z_{ij}|z_{i\ell},\ell \neq i,j)dz_{ij}$, $1 \leq i \neq j \leq k$. A decision rule $\delta_1 = (\delta_{11},\ldots,\delta_{1k})$ is said to be "as good as" $\delta_2 = (\delta_{21},\ldots,\delta_{2k})$ if $\inf_{\underline{\theta}\in\Omega} S(\underline{\theta},\delta_1) \geq \inf_{\underline{\theta}\in\Omega} S(\underline{\theta},\delta_2)$ provided that $\int \delta_{ij}p_j = \gamma_j$, $1 \leq j \leq k$, i=1,2, where γ_j, $(0 < \gamma_j < 1)$, are specified numbers. Let $\mathcal{C}$ be the class of δ such that $\int \delta_i p_i = \gamma_i$, $1 \leq i \leq k$.

A point x_0 is called a change point for a function g if in some neighborhood of x_0,

$$g(x)g(x^*) \leq 0,$$

whenever $x \leq x_0 \leq x^*$, and for some $x_1 \leq x_0 \leq x_1^*$, $g(x_1) \neq 0$ and $g(x_1^*) \neq 0$ with $x_1 \neq x_1^*$.

Karlin and Rubin [3] have proved the following result.

__Lemma__ ([3]). If φ changes sign at most once in one-dimensional Euclidean space R^1, then

$$\psi(w) = \int p(x|w)\varphi(x)d\mu(x)$$

changes sign at most once, where μ is a σ-finite measure on R^1 and $p(x|w)$ is the density of X with monotone likelihood ratio (MLR) in w.

__Remark__: It is useful to note that ψ changes sign in the same direction as φ if it changes sign at all.

Now we define a "monotone" selection rule as follows.

__Definition__: A selection rule δ is called monotone if for any i, $\delta_i(\underline{z})$ is monotone as follows:

$$\delta_i(\underline{z}) = \begin{cases} 1 & \text{if } \underline{z} \geq \underline{z}_0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\underline{z}_0$ is a fixed known vector and "$\leq$" is a partial order defined as follows: if $\underline{x}_1 = (x_{11},\ldots,x_{1m})$ and $\underline{x}_2 = (x_{21},\ldots,x_{2m})$, then $\underline{x}_1 \leq \underline{x}_2$, $\Leftrightarrow$ $x_{1i} \leq x_{2i}$ for $i = 1,\ldots,m$.

__Theorem__: Let $F_{\underline{\theta}}$ be the continuous cumulative distribution function corresponding to $p_{\underline{\theta}}(\underline{z})$ which has monotone likelihodd ratio. Then all monotone selection procedures form an essentially complete class in C.

__Proof__: Let δ be any nonmonotone rule in C. Suppose that there is an i, such that δ_i is not monotone in z_{ij} for fixed $z_{i\ell}$, $\ell \neq i,j$. For each fixed $z_{i\ell} (\ell \neq i,j)$, we define

$$\delta_i^\circ(\underline{z}_i) = \begin{cases} 1 & \text{if } z_{ij} \geq f_i^\circ(1-R_j^i(\delta_i)), \\ 0 & \quad\quad < \quad\quad , \end{cases}$$

then

(1) $$\int \delta_i^\circ p_i \, dz_{ij} = \int_{F_i^\circ(1-R_j^i(\delta_i))} p_i \, dz_{ij}$$

$$= \int_{F_i^\circ(1-R_j^i(\delta_i))} p_i(z_{ij}|z_{i\ell},\ell\neq i,j)p(z_{i\ell},\ell\neq i,j)dz_{ij} = \int \delta_i p_i(\underline{z}_i)dz_{ij}.$$

Since δ_i° is monotone in z_{ij}, thus $\delta_i - \delta_i^\circ$ is as a function of z_{ij} has at most one sign changes from plus to minus. Using this fact, the MLR property of $p_{\underline{\theta}}(\underline{z}_i)$ and Lemma, we have

(2) $$\int [\delta_i - \delta_i^\circ] p_{\underline{\theta}}(\underline{z}_i)dz_{ij} \leq 0, \quad \tau_{ij} \geq \tau_{ii}.$$

Thus from (1) and (2), δ° has the same conditional size as δ and has higher conditional power than δ as follows

(3) $$\int \delta_i p_{\underline{\theta}} \leq \int \delta_i^\circ p_{\underline{\theta}} \quad \text{for } \tau_{ij} \geq \tau_{ii},\ j=1,\ldots,k,\ j\neq i.$$

Since

$$\inf_{\underline{\theta}\in\Omega} S(\underline{\theta},\delta) = \min_{1\leq i\leq k} \inf_{\underline{\theta}\in\Omega_i} \int \delta_i p_{\underline{\theta}}(\underline{z}_i)d\nu(\underline{z}_i),$$

hence by (3),

$$\inf_{\underline{\theta}\in\Omega} S(\underline{\theta},\delta) \leq \inf_{\underline{\theta}\in\Omega} S(\underline{\theta},\delta^\circ).$$

The proof is complete.

Example: Let $X_{i1},\ldots,X_{in_i}$ be independent normally distributed with mean θ_i and variance $\sigma^2 = 1$, $i=1,2,\ldots,k$. Then the joint likelihood function of $\bar{X}_i$, $1 \le i \le k$, is

$$g_{\underline{\theta}}(\underline{x}) = \prod_{j=1}^{k} g_{\underline{\theta}_i}(\bar{x}_i),$$

where $g_{\underline{\theta}_i}(\bar{x}_i) = \frac{\sqrt{n_i}}{\sqrt{2\pi}} e^{-\frac{n_i}{2}(x_i-\theta_i)^2}$, $\bar{x}_i = \frac{1}{n_i}\sum_{\ell=1}^{n_i} x_{i\ell}$. Let $\tau_{ij} = \theta_i - \theta_j$, $1 \le j \le k$; $\tau_{ii} = 0$, $1 \le i \le k$, and $Z_{ij} = \bar{X}_i - \bar{X}_j$, $j \ne i$. Then for any i,

$$\delta_i^\circ(\underline{z}_i) = \begin{cases} 1 & \text{if } \underline{z}_i \ge \underline{d}_i, \\ 0 & \text{otherwise,} \end{cases}$$

where $\underline{d}_i = (d_{i1},\ldots,d_{i,k-1},d_{i,k+1},\ldots,d_{ik})$. Equivalently,

$$\delta_i^\circ(\underline{x}) = \begin{cases} 1 & \text{if } \bar{x}_i \ge \max_{j\ne i}(\bar{x}_j + d_{ij}) \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

We know that

$$P(\bar{X}_i \ge \max_{j\ne i}(\bar{X}_j + d_{ij}))$$

is nondecreasing in θ_i and nonincreasing in θ_j, $j=1,\ldots,k$, $j\ne i$. In this case the, the monotone selection rule also has the above property of monotone behavior in terms of the selection probability. This monotone property is the same as used in the definition of the usual selection procedures. It should be pointed out that when all d_{ij}'s are negative, the monotone selection procedure $\delta^\circ = (\delta_1^\circ,\ldots,\delta_k^\circ)$ given in (4) is the usual Gupta type procedure (cf. [1]) to select a subset containing the best population associated with the largest population associated with the largest θ_i's as follows:

$$\delta_i^\circ(\underline{x}) = \begin{cases} 1 & \text{if } \bar{x}_i \geq \max\limits_{1 \leq j \leq k} (\bar{x}_j - (-d_{ij})), \\ 0 & \text{otherwise.} \end{cases}$$

Gupta and Huang [2] have studied the selection rule for the k normal means problem with a common known variance σ^2 based on samples of unequal sizes. In their solution the monotone rules are given by $d_{ij} = -d\sigma\sqrt{\frac{1}{n_i} + \frac{1}{n_j}}$, $d > 0$.

References

[1] Gupta, S. S. (1965). On multiple decision (selection and ranking) rules. Technometrics 7, 225-245.

[2] Gupta, S. S. and Huang, D. Y. (1976). Subset selection procedures for the means and variances of normal populations: unequal sample sizes case. Sankhyā, 38, Ser. B, 112-128.

[3] Karlin, S. and Rubin, H. (1956). The theory of decision procedures for distributions with monotone likelihood ratio. Ann. Math. Statist. 27, 272-245.

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