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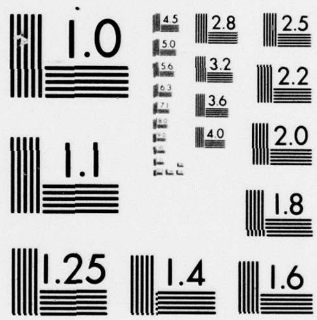

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THE STRANGE BEHAVIOR OF ELECTROMAGNETIC WAVES IN CONDUCTING MAGNETO-DIELECTRIC MEDIA

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1. Introduction

Reflection and transmission of electromagnetic waves at boundaries between dielectric media, and reflection at highly conducting boundaries have been treated extensively in the literature for years. Usually approximations have been made to fit the special case under discussion. When dealing with good dielectrics, the conductivity is usually neglected and for highly conducting materials the dielectric constant is neglected. Almost always, the permeability is considered to be equal to that of free space. Between these limiting cases, there are a host of materials in which all three constitutive parameters, permeability, permittivity, and conductivity should be considered.

The fact that any one or all three of these parameters can be complex generates the following interesting situation. Assume $\varepsilon = \varepsilon'-j\varepsilon''$, $\sigma = \sigma'-j\sigma''$ and $\mu = \mu'-j\mu''$. In a conducting medium the apparent dielectric constant is $\varepsilon_T = \varepsilon-j\sigma/\omega = (\varepsilon'-\sigma''/\omega)-j(\varepsilon''+\sigma'/\omega)$, (1) where $\omega = 2\pi f$. Thus, if σ' and σ'' are of the same sign as ε' and ε'', the real part of dielectric constant appears to be decreased. Further, Landau and Lifshitz[1] have shown that μ'' and ε'' must always be positive, but that there is no physical restriction on the sign of μ' or ε'. It can also be shown that σ' must also always be positive by noting that the total heat dissipated per unit time and volume, in a conducting medium is

$$Q = \omega/2 \left|\mu''\tilde{H}^2+\varepsilon''\tilde{E}^2+ \frac{\sigma'}{\omega}\tilde{E}^2\right| \quad (2)$$

where $\sim$ indicates the time average of the fields. Hence, σ', μ'', and ε'' must all be positive to insure that the medium heats up rather than cools. In this paper the following constitutive parameters will be used.

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$$\mu = \mu_o(\mu_r' - j\mu_r'') = \mu_o|\mu_r|e^{j\Theta\mu} \tag{3}$$

$$\varepsilon_T = \varepsilon_o(\varepsilon_r - j\sigma/\omega\varepsilon_o) = \varepsilon_o|\varepsilon_r|e^{j\Theta\varepsilon} \tag{4}$$

$$= \varepsilon_o\varepsilon_r(1 - j\tan\delta) \tag{5}$$

Here $\mu_o = 4\pi x 10^{-7}$ H/m, and $\varepsilon_o = (1/36\pi) x 10^{-9}$ F/m in MKSQ dimensions and units. The term tan δ is the loss tangent. The subscript r denotes the dimensionless relative value. In many dielectric materials ε_r and tan δ are well documented, but little is known about μ'' in magnetic materials. The term ε_r, where necessary, will be taken as $(\varepsilon_r' - \sigma''/\omega\varepsilon_o)$ which may mean that ε_r can appear to be negative in some materials. This occurs in some materials such as gold at infrared wave lengths, as will be shown later. When ε_r is negative it is usually not possible to determine whether ε' or $\sigma''\omega\varepsilon_o$ is the predominant factor.

Equations A(8) and A(9) in the appendix define the propagation constant in conducting media. Setting $\mu_{rn} = 1+j0$ and $\sigma_n = 0$ then

$$k = \omega^2\mu_o\varepsilon_o = \omega^2/c^2 = \beta \text{ rad/m} \tag{6}$$

where a wavelength is defined as

$$\lambda = 2\pi/\beta = 2\pi/k \tag{7}$$

in which all quantities are real. If the medium is conducting and k becomes complex, do we define λ as $2\pi/\beta$, or as $2\pi/k$? In this paper the latter definition is used which dictates that the wavelength (λ), index of refraction (n), and phase velocity of propagation (υ_p) must all be complex quantities. Also, since λ is complex, all distances must be complex so that the total phase change along any path length can be a real number of radians.

2. Weakly Conducting Dielectrics at Extremely Low Frequencies (ELF)

Although the title of this paper refers specifically to conducting media it is necessary to begin with two semi-infinite non-conducting dielectrics in contact in order to describe several interesting angles of incidence as defined by Lytle and Lager[2]. To begin with, a vertically polarized plane wave incident on the boundary (Figure 5) between free space and a perfect dielectric is assumed. When the medium of incidence (medium 1) is free space, there is an angle of incidence (Θ_i) such that total transmission into a medium of refraction (medium 2) occurs, and the reflection coefficient $R_{12}^H = 0$(εqn.A(23)). This is the well known Brewster angle Θ_B which occurs when:

$$\Theta_B = \Theta_{t2} = \sin^{-1} |\varepsilon_{r2}/(\varepsilon_{r2}+1)|^{-\frac{1}{2}}; \ \varepsilon_{r1}>\varepsilon_{r1}, \varepsilon_{r1} = 1. \tag{8}$$

When the situation is reversed such that the medium of incidence (medium 1) has a large dielectric constant and the medium of refraction is free space (medium 2), three more interesting angles of incidence occur as pointed out by Lytle and Lager:

$$\Theta_B' = \sin^{-1}|1/(\varepsilon_{r1}+1)|^{\frac{1}{2}}; \ \varepsilon_{r2} = 1 \tag{9}$$

at which total transmission into free space occurs ($R_{12}^H = 0$) and the angle of transmission (Θ_{t2}) is equal to the Brewster angle (Θ_B) above,

$$\Theta_c = \sin^{-1}(1/\varepsilon_{r1})^{\frac{1}{2}}; \ \varepsilon_{r2} = 1 \tag{10}$$

called the critical angle, which occurs when the reflection coefficients for both vertical and horizontal polarization are both equal to unity (i.e. $R_{12}^H = R_{12}^E = 1$), and

$$\Theta_D = \sin^{-1}|(\varepsilon_{r1}+1)/(\varepsilon_{r1}^2+1)|^{\frac{1}{2}}; \ \varepsilon_{r2} = 1 \tag{11}$$

called the "Devil" angle, which occurs when the reflection coefficient is purely imaginary (i.e. $R_{12}^H = -j$). The name was coined by Lytle and Lager because of their devilment at the unusual occurrences at this angle.

This latter angle is treated, as example 1, in detail in Figure 1 where the medium of incidence has constitutive parameters of $\varepsilon_{r1} = 25$, and $\sigma_1 = 0$. The frequency (f) used in this example is one Hertz since this represents a frequency near the low end of the spectrum. However, the results here and in subsequent examples will be obtained at any frequency provided the loss tangent ($\tan \delta = \sigma/\omega\varepsilon_T$) Eqn. A(11) remains constant as a function of frequency. The H-mode transmission and reflection coefficients (T_{12}^H and R_{12}^H) are calculated using equations A(22) through A(25) in the appendix where the "Devil" angle of evidence is 11.759°. The E-mode coefficients are not calculated since they are uniquely determined by equations A(26) through A(30). The principle points of interest to be observed are as follows:

(a) The reflected H-field vector is equal in magnitude to the incident field but lags the latter in time phase by 90°.

(b) The transmitted H-field is 1.414 times greater than the incident field and lags the latter in time phase by 45°.

(c) The angle of refraction (Θ_{t2}) is a complex angle of $90^\circ + j\ 11.07^\circ$ which is quite contrary to what would be expected.

(d) The planes of constant phase and constant amplitude in free space (medium 2) are not coincident. The constant phase plane is perpendicular to the interface, while the constant amplitude is plane parallel to the interface.

(e) The incident angles Θ_B', Θ_c, and Θ_D all occur within the range of $\Theta_{i1} = 11^o$-12^o such that $\Theta_B<\Theta_c<\Theta_D$(Figure 3). For all angles below the critical angle (Θ_c) the transmitted H-field (H_{t2}) is in phase with the incident H-field (H_{i2}). Between Θ_B' and Θ_c the reflected H-field (H_{r1}) is out of time phase with the incident field by 180^o.

(Note 1 - When the angle of refraction is complex, the planes of constant phase and amplitude, in the medium of refraction (medium 2), are not usually coincident. We assume in all examples that they are coincident in the medium of incidence (medium 1). The angle (ψ) of constant phase plane can be calculated from

$$\cos\psi = \frac{\beta_2 R\cos + \alpha_2 I\cos}{|(\beta_2 R\cos + \alpha_2 I\cos)^2 + (\beta_2 R\sin + \alpha_2 I\sin)^2|^{\frac{1}{2}}}, \quad (9)$$

where $\cos\Theta_{t2}$ = Rcos+jIcos, and $\sin\Theta_{t2}$ = Rsin+jIsin are the complex cosine and sine of the complex angle of refraction (Θ_{t2}). The phase constant (β_2), and attenuation constant (α_2) are defined by equation A(9). Similarly, the angle ($\emptyset$) of the constant amplitude plane in medium 2 can be determined by

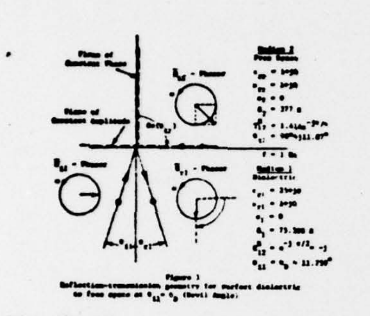
$$\cos\emptyset = \frac{\beta_2 I\cos - \alpha_2 R\cos}{|(\beta_2 I\cos - \alpha_2 R\cos)^2 + (\beta_2 I\sin - \alpha_2 R\sin)^2|^{\frac{1}{2}}}, \quad (10)$$

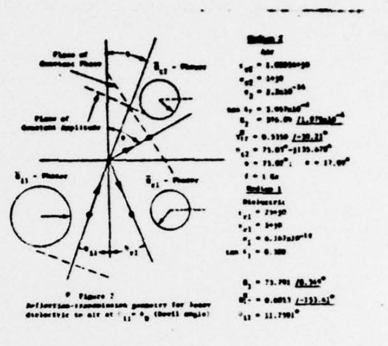
Both of these angles are measured from the plane of incidence (Figure 5) and can be considered as the refraction angles for constant phase and constant amplitude propagation directions. The respective planes are normal to these directions.)

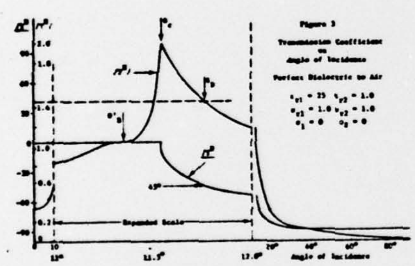
When the two media are allowed to become weakly conducting, as shown in Figure 2, the situation changes quite drastically. The principal points to note are:

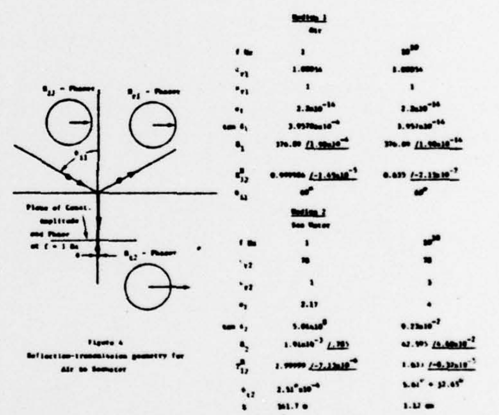
(a) The angle of refraction (Θ_{t2}) has changed, but is still complex. The real part of Θ_{t2} still corresponds to the refraction angle (ψ) of constant phase planes.

(b) The reflected H-field has decreased in magnitude and now lags the incident field by 153.4^o in time phase. The transmitted H-field has also decreased in magnitude but lags in time phase by only 30.2^o. It should be noted that if the conductivity of medium 1 is increased (say $\sigma = 10^{-6}$ S/m), the reflected field amplitude becomes unity and lags by 180^o. Hence the transmitted field becomes zero in magnitude so that the boundary behaves like a near perfect magnetic sheet reflector.









3. Strongly Conducting Media

Sea water is a well known example of a strongly conducting medium. Much interest has been shown in the reflection-transmission coefficients when an electromagnetic wave is incident on the surface of an ocean (medium 2) from air (medium 1). The constitutive parameters for both media at one Hertz and 10 GHz were estimated from an extensive literature search and are shown in figure 4.

It comes as no surprise that sea water is a good reflector of electromagnetic energy. However, it is not well known that the refracted H-field (H_{t2}) is nearly twice that of the incident field for angles of incidence up to 60°. The reason for this is that the incident, reflected, and refracted H-fields are all in phase. This does not mean that a lot of energy is refracted, however, since the refracted E-field amplitude is reduced by a factor of $|Z_2/Z_1|$ which is

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of the order of $5x10^{-6}$ at one Hertz. This suggests that, if one desired to communicate with a submerged submarine at one Hertz, an H-field sensor, such as a magnetometer, may be more effective than an E-field sensor such as a trailing wire antenna.

It is also interesting to note that the angle of refraction is nearly 90^{o} for both the constant phase and constant amplitude propagation at most angles of incidence at low frequencies. This is true at all frequencies for the constant amplitude planes, however at 10 GHz the constant phase plane angle of refraction is 5.6^{o}. Thus, the energy travels straight down in the sea water.

There is one exception to the fact that the reflection coefficient (R^{H}_{12}) is nearly unity at all angles of incidence. This exception is at an angle of incidence of 88^{o}-89^{o}, depending on frequency, where a quasi-Brewster angle occurs such that the reflection coefficient becomes quite small and the transmission is nearly unity. This effect is more pronounced at the higher frequencies.

Figure 4 also shows that the skin depth(s), where $H/H_{o}=e^{-1}$, is 30 000 times greater at one Hertz than at 10 GHz. At fifty Hertz the skin depth(s) is 48.3 meters as compared to 341.7 m at one Hertz.

4. Layered Conducting and Permeable Media

The subject of multi-layered sandwiches is of extreme interest in applications such as radar radomes, radar attenuating paints, radar attenuating aerosols, and electromagnetic wave shielding. Born and Wolf[3]; Cady, Karelitz, and Turner[4]; and others have treated this subject in great detail. Born and Wolf, on the one hand, were primarily interested in optics so several approximations were made. On the other hand, Cady, et al. were interested in microwaves through good dielectrics, so somewhat different approximations were made. Both approaches to the problem involved chain matrix multiplication of a succession of 2x2 matrices. If one follows this approach but makes no approximations, a matrix equation such as equation A(36) results. Using this, the reflection coefficient at the incident face of a multilayered sandwich and the transmission coefficient through the last face can be calculated by virtue of equations A(37) through A(47) at any angle of incidence. These equations are also applicable for any value of ε_r, μ_r, and σ providing ε_r and σ are real. The permeability can be complex as indicated by equations A(7) and A(8). The geometry of a single layer sandwich is shown in figure 5. It is obvious how to add additional layers up to m = n-1 where n is the number of faces.

Consider as the reflection coefficient of a radar attenuating paint on aluminum for normal incidence at frequencies of 9 GHz, 10 GHz, 11 GHz, and 20 GHz. The constitutive parameters of the paint, used in the problem, are shown in figure 6. It is not known whether such a paint having these characteristics can be compounded, but, at least, here is one set of parameters which will produce the results shown. They were selected such that $|(\mu_r/(\varepsilon_r - j\sigma/\omega_{to})|^{\frac{1}{2}} = 1$, so that the intrinsic impedance at normal incidence and $f = 10^{10}$ Hz matches that of air. The reflection coefficients, at normal incidence, are plotted as a function of paint thickness. Note that at a thickness of 5.08 mm (0.02"), the magnitude of the reflection coefficient is 40 dB below that of a perfect reflector at a frequency of 10 GHz.

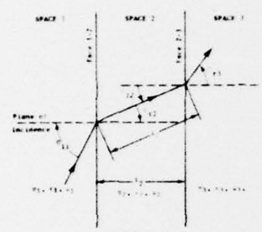
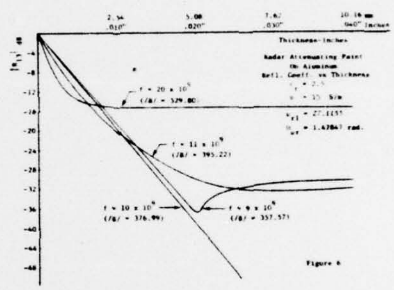


Figure 5
Layered Media Geometry



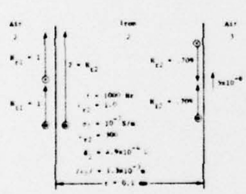


Figure 7
Transmission-Reflection Through an Iron Sheet

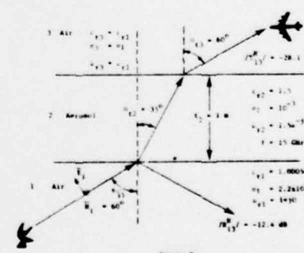


Figure 8
Radar Attenuating Aerosol

Now consider a thin sheet of iron, as an electromagnetic shield at one KHz, as shown in figure 7. The most salient points to note are as follows:

(a) The refracted H-field, at the incident face, is twice the incident field as occurs in all highly conducting fields for the same reasons stated earlier.

(b) The H-field exiting through the output face is negligi-

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ble because the reflected field lags incident field by 180°. They therefore cancel each other at the output interface.

As a final example consider a one meter thick aerosol between a radar, operating at 15 GHz, and an aircraft flying overhead. The constitutive properties of this aerosol are shown in figure 8. It is not known whether such an aerosol could be compounded but the principle is demonstrated that, with such an aerosol at an angle of incidence of 60°, the overall transmission coefficient magnitude is -28.1 dB per meter thickness. A signal reflected from the aircraft, upon returning to the radar would be subject to the same degree of attenuation which could render an aircraft nearly undetectable.

5. Highly Conducting Media

In addition to the basic metals there are many highly conducting materials such as graphite loaded epoxy and conducting polymers. Little is known about their constitutive parameters so that the equations included herein cannot be used to investigate various practical examples. Examination of the published data on many materials, however, leads one to suspect that conductivities become significantly complex at far infrared frequencies. The complex index of refraction of the basic metals are well documented so that this suspicion can be verified and at least demonstrate that complex conductivities do indeed exist.

We consider the case of gold which has a complex index of refraction of $n = n' - jn'' = 25.2 - j55.9$ at $\lambda = 9.9\ \mu$m. Noting that $n = c/\omega\ (\beta - j\alpha)$, the propagation constant ($k = \beta - j\alpha$) can be calculated as $k = 2.52\times10^{7} - j3.41\times10^{7}$ at a frequency of 3.03×10^{13} Hertz. Finding the square of k, where $k^2 = \omega^2\mu\varepsilon - j\omega\mu\sigma$, allows us to calculate σ if we assume values of unity for μ_r and ε_r. Thus

$$k^2 = -5.31\times10^{14} - j1.718\times10^{14} = \frac{\omega^2}{c^2}\,\mu_r\varepsilon_r\left(\frac{-\sigma''}{\omega\varepsilon_o} - j\,\frac{\sigma'}{\omega\varepsilon_o}\right)$$

from which σ is determined as

$$\sigma = \sigma' - j\sigma'' = 4.274\times10^{6} - j1.605\times10^{6}$$

$$= 4.565\times10^{6}\ \angle -0.359 \text{ rad S/m}$$

which is an order of magnitude smaller than the direct current value of $(\sigma \approx 4\times10^{7})$S/m for gold.

6. Conclusions

Unfortunately, in a paper of this length, it is impossible to include derivations of the equations used to make the calculations of the quantities shown in the various figures. In fact, it is not possible to show all of the quantities calculated on each figure. However, the most significant quantities, which explain the strange behaviors discussed, are included. Also all equations used are included.

This by no means exhausts the subject of strange behaviors under other circumstances. However, the author has found that when all parameters are included, no matter how small, many surprises can occur under the most ordinary of circumstances.

Acknowledgements

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Appendix
Mathematical Equations

Snell's Laws

$$\cos \Theta_{i1} = \cos \Theta_{r1} \qquad \text{A (1)}$$

$$k_1 \sin \Theta_{i1} = k_2 \sin \Theta_{t2} \qquad \text{A (2)}$$

Fresnel's Equations

T. E. mode (horizontal polarization)

$$\overline{E}_t = \frac{2\mu_2 k_1 \cos \Theta_i}{\mu_2 k_1 \cos \Theta_i + \mu_1 (k_2^2 - k_1^2 \sin \Theta_i)^{1/2}} \overline{E}_i \qquad \text{A (3)}$$

$$\overline{E}_r = \frac{\mu_2 k_1 \cos \Theta_i - \mu_1 (k_2^2 - k_1^2 \sin^2 \Theta_i)^{1/2}}{\mu_2 k_1 \cos \Theta_i + \mu_1 (k_2^2 - k_1^2 \sin^2 \Theta_i)^{1/2}} \overline{E}_i \qquad \text{A (4)}$$

T. M. mode (vertical polarization)

$$\overline{H}_t = \frac{2\mu_1 k_2^2 \cos \Theta_i}{\mu_1 k_2^2 \cos \Theta_i + \mu_2 k_1 (k_2^2 - k_1^2 \sin^2 \Theta_i)^{1/2}} \overline{H}_i \qquad \text{A (5)}$$

$$\overline{H}_r = \frac{\mu_1 k_2^2 \cos \Theta_i - \mu_2 k_1 (k_2^2 - k_1^2 \sin^2 \Theta_i)^{1/2}}{\mu_1 k_2^2 \cos \Theta_i + \mu_2 k_1 (k_2^2 - k_1^2 \sin^2 \Theta_i)^{1/2}} \overline{H}_i \qquad \text{A (6)}$$

The subscripts i, r, and t, denote incidence, reflection, and transmission respectively. The numerical subscripts 1, 2, ---, n identify the medium in which the wave is traveling. Where the subscript r is used with the permitivity (ε), and permeability (μ) it denotes the relative value such that $\varepsilon_n = \varepsilon_{rn} \varepsilon_o$, and $\mu_n = \mu_{rn} \mu_o$ where

$\varepsilon_o = \frac{1}{36\pi} \times 10^{-9}$ F/m, and $\mu_o = 4\pi \times 10^{-7}$ H/m in rationalized MKS units.

The relative values are dimensionless. The conductivity (σ_n) has the dimensions of Siemens/meter (S/m).

Horizontal polarization (T. E. mode) equations (3), and (4) apply when the incident $\overline{E}$-field is parallel to the boundary, as shown in figure 1. For vertical polarization (T. M. mode), equations (5), and (6) apply when the incident $\overline{H}$-field is parallel to the boundary. In these equations, the relative permeabilities are allowed to be complex so that

$$\mu_n = \mu_{rn}\mu_o = |\mu_{rn}|\, e^{j\Theta_u}\, \mu_o \qquad \text{A (7)}$$

The square of the propagation constants which result from the wave equations are

$$\begin{aligned} k_n^2 &= \omega^2 \mu_n \epsilon_n - j\omega\mu_n\sigma_n \\ &= \omega^2 \mu_o \epsilon_o \left[\mu_{rn}\left(\epsilon_{rn} - j\frac{\sigma_n}{\omega\epsilon_o}\right)\right] \end{aligned} \qquad \text{A (8)}$$

where $\omega = 2\pi f$, and f is the frequency in Hertz. The propagation constant is expressible as

$$k_n = \beta_n - j\alpha_n \qquad \text{A (9)}$$

where β_n is the phase constant in radians per meter, and α_n is the attenuation constant in nepers per meter. The quantity ε_{rn} can be complex as well, such that

$$\begin{aligned} \epsilon_{rn} &= \epsilon'_{rn} - j\left(\epsilon''_{rn} + \frac{\sigma_n}{\omega\epsilon_o}\right) \\ &= |\epsilon_{rn}|\, e^{j\tan\delta} \end{aligned} \qquad \text{A (10)}$$

In this report the quantities in parentheses are treated as one quantity attributable to a single equivalent conductivity. The loss tangent then is

$$\tan\delta = \frac{\sigma_n}{\omega\epsilon_T} \qquad \text{A (11)}$$

where

$$\epsilon_T = \epsilon_o |\epsilon_{rn}| \left(1 - j\frac{\sigma_n}{\omega|\epsilon_{rn}|\epsilon_o}\right) \qquad \text{A (12)}$$

If ε_{rn} and tan δ are known at a given frequency, the conductivity can be calculated by

$$\sigma_n = \frac{1}{18} f_m |\epsilon_{rn}| \tan\delta \times 10^{-9} \qquad \text{A (13)}$$

where f_m is the reported frequency at which tan δ was measured. Here, σ_n is calculated from the loss tangent at the nearest reported frequency to the actual frequency of interest and then treated as a constant value.

From equations (3) and (4) the TE mode transmission (T) and reflection (R) coefficients for a positive going wave can be expressed as

$$T^E_{12} = \frac{\bar{E}_{t2}}{\bar{E}_{i1}} = \frac{2}{1 + Y^E_{12}} \qquad \text{A (14)}$$

and

$$R^E_{12} = \frac{\bar{E}_{r1}}{\bar{E}_{i1}} = \frac{1-Y^E_{12}}{1+Y^E_{12}} \qquad \text{A (15)}$$

so that

$$R^E_{12} = T^E_{12} - 1 \qquad \text{A (16)}$$

where the admittance is

$$Y^E_{12} = \frac{1}{Z^E_{12}} = \frac{\mu_1 k_2}{\mu_2 k_1} \quad \frac{\left[1-(k_1/k_2 \sin \Theta_{i1})^2\right]^{1/2}}{\cos \Theta_{i1}} \qquad \text{A (17)}$$

and Θ_{i1} is the angle of incidence from medium 1 on face 1-2. These equations hold at any interface between two media for a positive going incident wave which is from left to right in figure 1. For a negative going incident wave, these equations become

$$T^E_{21} = \frac{\bar{E}_{t1}}{\bar{E}_{i2}} = \frac{2}{1+Y^E_{21}} \qquad \text{A (18)}$$

and

$$R^E_{21} = \frac{\bar{E}_{r2}}{\bar{E}_{i2}} = \frac{1-Y^E_{21}}{1+Y^E_{21}} \qquad \text{A (19)}$$

so that

$$R^E_{21} = T^E_{21} - 1 \qquad \text{A (20)}$$

where

$$Y^E_{21} = \frac{1}{Z^E_{21}} = \frac{\mu_2 k_1}{\mu_1 k_2} \quad \frac{\left[1-(k_2/k_1 \sin \Theta_{i2})^2\right]^{1/2}}{\cos \Theta_{i2}} \qquad \text{A (21)}$$

At successive boundaries, the subscripts change from 12 and 21 to 23 and 32 at face 23 as shown in figure 1.

For the T. M. mode, similar equations result as follows.

$$T^H_{12} = \frac{\overline{H}_{t2}}{\overline{H}_{i1}} = \frac{2}{1+Y^H_{12}} \qquad A\ (22)$$

$$R^H_{12} = \frac{\overline{H}_{r1}}{\overline{H}_{i1}} = \frac{1-Y^H_{12}}{1+Y^H_{12}} \qquad A\ (23)$$

$$R^H_{12} = T^H_{12} - 1 \qquad A\ (24)$$

$$Y^H_{12} = \frac{1}{Z^H_{12}} = \frac{\mu_2 k_1}{\mu_1 k_2} \frac{\left[1-(k_1/k_2 \sin \Theta_{i1})^2\right]^{1/2}}{\cos \Theta_{i1}} \qquad A\ (25)$$

$$T^H_{21} = \frac{\overline{H}_{t1}}{\overline{H}_{i2}} = \frac{2}{1+Y^H_{21}} \qquad A\ (26)$$

$$R^H_{21} = \frac{\overline{H}_{r2}}{\overline{H}_{i2}} = \frac{1-Y^H_{21}}{1+Y^H_{21}} \qquad A\ (27)$$

$$R^H_{21} = T^H_{21} - 1 \qquad A\ (28)$$

$$Y^H_{21} = \frac{\mu_1 k_2}{\mu_2 k_1} \frac{\left[1-(k_2/k_1 \sin \Theta_{i2})^2\right]^{1/2}}{\cos \Theta_{i2}} \qquad A\ (29)$$

From these equations it is easily shown that

$$T^E_{12} = T^H_{21}\ ;\ T^E_{21} = T^H_{12}\ ;\ R^E_{12} = R^H_{21} \text{ and } R^E_{21} = R^H_{12} \qquad A\ (30)$$

If space 2 in figure 1 is a sheet between two semi-infinite media, it is necessary to determine the phase shift and attenuation of waves due to path length (l_2) in medium 2. A complex phase can be defined as

$$\phi = k_2 l_2 = \beta_2 l_2 - j\alpha_2 l_2 \qquad A\ (31)$$

When the boundaries are plane and parallel l_2 is related to the thickness (t_2) by

$$l_2 = \frac{t_2}{\sin \Theta_{t2}} \qquad A\ (32)$$

where sin Θ_{t2} can be complex by Snell's law (equation 2). Hence, l_2 becomes a complex length. The complex impedance of the medium is

$$Z = \sqrt{\frac{\mu_2}{\epsilon_{T2}}} = 377\sqrt{\frac{|\mu_{r2}|\, e^{j\Theta\mu}}{\epsilon_{r2} - j\frac{\sigma_2}{\omega\epsilon_o}}} . \qquad \text{A (33)}$$

the velocity of propagation is

$$v_{P2} = \frac{1}{\sqrt{\mu_2\epsilon_{T2}}} = \frac{1}{\sqrt{\left(|\mu_{r2}|\, e^{j\Theta\mu}\right)\left(\epsilon_{r2} - j\frac{\sigma_2}{\omega\epsilon_o}\right)}} \qquad \text{A (34)}$$

and the wavelength is

$$\lambda_2 = \frac{v_{P2}}{f} \qquad \text{A (35)}$$

We are now in such a position that we can calculate the transmission and reflection coefficients at each interface and the phase and attenuation in each space. Hence, the reflection and transmission coefficient of a multi-layered sandwich can be calculated by using successive (ABCD) matrix multiplication, as shown by Born and Wolfe[4] by assuming a positive going electromagnetic wave to be incident on the face of the sandwich and a negative going wave incident on the opposite face. Then, the TM mode (vertical polarization) wave matrix equation can be written for a multi-layer sandwich in matrix form as follows.

$$\begin{bmatrix} \overline{H}_{i1} \\ \overline{H}_{r1} \end{bmatrix} = \frac{1}{T^H_{12}} \begin{bmatrix} 1e^{jo} & -R^H_{21} \\ R^H_{12} & T^H_{12}T^H_{21} - R^H_{12}R^H_{21} \end{bmatrix} \begin{bmatrix} e^{\alpha_2 l_2} e^{j\beta_2 l_2} & o \\ o & e^{-\alpha_2 l_2} e^{-j\beta_2 l_2} \end{bmatrix} x$$

$$\frac{1}{T^H_{23}} \begin{bmatrix} 1e^{jo} & -R^H_{32} \\ R^H_{23} & T^H_{23}T^H_{32} - R^H_{23}R^H_{32} \end{bmatrix} \begin{bmatrix} e^{\alpha_3 l_3} e^{j\beta_3 l_3} & o \\ o & e^{-\alpha_3 l_3} e^{-j\beta_3 l_3} \end{bmatrix} x \qquad \text{A (36)}$$

$$\cdots\; x \frac{1}{T_{nm}} \begin{bmatrix} 1e^{jo} & -R^H_{mn} \\ R^H_{nm} & T^H_{nm}T^H_{mn} - R^H_{nm}R^H_{mn} \end{bmatrix} \begin{bmatrix} \overline{H}_{tm} \\ \overline{H}_{im} \end{bmatrix} \quad ; n = m-1$$

where l_n is the complex path length in the n-th space. All other quantities have been defined above. For the TE mode (horizontal polarization), T^E_{nm}, T^E_{mn}, R^E_{nm}, R^E_{mn}, coefficients can be obtained from equation (30).

For a three-layer sandwich, this matrix in a short from is as follows

$$\begin{bmatrix} \overline{H}_{i1} \\ \overline{H}_{r1} \end{bmatrix} = \frac{1}{\Pi T^H_{nm}} [a]^H [b]^H [c]^H [d]^H [e]^H [f]^H [g]^H \begin{bmatrix} \overline{H}_{tm} \\ \overline{H}_{im} \end{bmatrix}$$

$$= \frac{1}{\Pi T^H_{nm}} \begin{bmatrix} h_1 & h_2 \\ h_3 & h_4 \end{bmatrix} \begin{bmatrix} \overline{H}_{tm} \\ \overline{H}_{im} \end{bmatrix} \qquad \text{A (37)}$$

Setting $\overline{H}_{im} = 0$, then

$$R^H_{1m} = \frac{\overline{H}_{r1}}{\overline{H}_{i1}} = \frac{h^H_3}{h^H_1} \qquad \text{A (38)}$$

and

$$T^H_{1m} = \frac{\overline{H}_{tm}}{\overline{H}_{i1}} = \frac{\Pi T^H_{nm}}{h^H_1} \qquad \text{A (39)}$$

Additional layers can be added as desired with appropriate symbols for the overall (h) matrix. The E-mode (horizontal polarization) coefficients can be calculated likewise by substituting the T^E, R^E, quantities for the T^H, R^H, quantities in (36). Then,

$$\begin{bmatrix} \overline{E}_{i1} \\ \overline{E}_{r1} \end{bmatrix} = \frac{1}{\Pi T^E_{nm}} \begin{bmatrix} h_1 & h_2 \\ h_3 & h_4 \end{bmatrix}^E \begin{bmatrix} \overline{E}_{tm} \\ \overline{E}_{im} \end{bmatrix} \qquad \text{A (40)}$$

where

$$R^E_{1m} = \frac{\overline{E}_{r1}}{E_{i1}} = \frac{h^E_3}{h^E_1} \qquad \text{A (41)}$$

and

$$T^E_{1m} = \frac{\overline{E}_{tm}}{\overline{E}_{i1}} = \frac{\pi T^E_{nm}}{h^E_1} \qquad \text{A (42)}$$