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ACCELERATED LIFE TESTING - A PRAGMATIC BAYESIAN APPROACH

by

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Serial T-375
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In this paper we ~~propose~~ as proposed a model for accelerated life testing which differs from those considered in the past. Our model does not require the usual assumptions about the failure distributions and the accelerated functions. Our approach is Bayesian, and depends on the estimation of ordered failure rate functions at different stress levels. Since our model is non-parametric, and our approach is quite pragmatic, our results should be of use to engineers working on accelerated life tests, and biometricians involved

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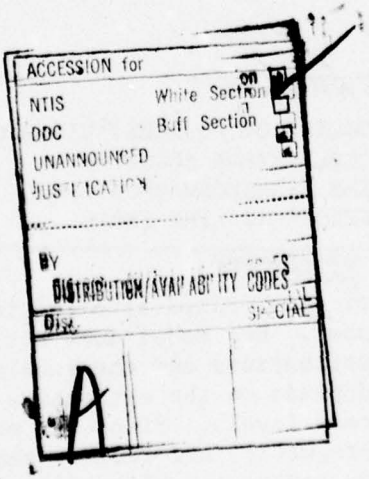

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ACCELERATED LIFE TESTING - A PRAGMATIC BAYESIAN APPROACH

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Frank Proschan
Nozer D. Singpurwalla

In this paper we propose a model for accelerated life testing which differs from those considered in the past. Our model does not require the usual assumptions about the failure distributions and the *acceleration functions*. Our approach is Bayesian, and depends on the estimation of ordered failure rate functions at different stress levels. Since our model is nonparametric, and our approach is quite pragmatic, our results should be of use to engineers working on accelerated life tests, and biometricians involved in carcinogenic experiments.

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0. Introduction and Overview

In practice, long life items are often subjected to larger than normal stresses (doses) in order to obtain failure data in a short amount of test time. It is common to test the items at more than one stress level, and typically, more items are tested at the higher stress levels than those at the lower stress levels; such tests are known as *accelerated* or *overstress* life tests. The basic aim is to make inferences about the life distribution of the items at the normal stress levels using the failure data from accelerated tests.

The current approach to this important problem makes inferences under parametric assumptions. This may be valid in some situations, and does yield results which are appealing from a statistical point of view. However, an engineer or a statistician working under less well-defined conditions may find this approach too restrictive. For example, it is common to assume that at all stress levels, failure times are governed by exponential distributions or Weibull distributions. In addition, a functional relationship between the parameters

of the failure distribution and the applied stress is assumed. Such a relationship is known as an *acceleration function* or a *time transformation function*; examples can be found in Mann, Schafer, and Singpurwalla (Chapter 9, (1974)). In two recent papers, Phatak, Zimmer, and Williams (1977) and Shaked, Zimmer and Ball (1977), the distributional assumption is dropped, but the requirement is retained that the (unknown) failure distribution be of the same form at the use and at all the accelerated stress levels.

Clearly, there are many situations where the above assumptions may not be appropriate. Of particular concern is the assumption that the failure distribution is of the same form at all the stress levels. One reason for this concern is that different stress levels may have different effects on the mechanism which causes failure, and thus from a physical point of view, it may be more realistic to allow for different forms of the failure distribution at the different stress levels.

The approach that we propose in this paper requires neither distributional assumptions nor the specification of a time transformation function. Rather, our approach is Bayesian, and is prompted by what is actually done in practice. The Bayesian point of view allows us to incorporate some *a priori* information which is available in accelerated life tests. We would like this paper to be construed as preliminary and pragmatic, and thus have not attempted to give a full theoretical justification. Consequently, we would like to invite mathematical statisticians to resolve the statistical problems posed.

1. Preliminaries

As stated before, in accelerated life tests the items are tested under different stress environments. A stress environment may be characterized by a single stress such as voltage or temperature, or by multiple stresses, each of a different type. We denote a stress environment by E, and the set of all E's by $\mathcal{E}$. We assume that the elements of $\mathcal{E}$ may be ordered according to the magnitude of their severity. Thus, for any two elements E_i and E_j

belonging to $\mathcal{E}$, $E_i \succ E_j$ denotes the fact that E_i is more severe than E_j. Let us denote the k accelerated stress environments by $E_1, E_2, \ldots, E_k$, and the normal or use conditions stress environment by E_u. We shall assume that the set $\mathcal{E}$ is completely ordered with respect to the relationship $\succ$ and that

$$E_1 \succ E_2 \succ \ldots \succ E_k \succ E_u .$$

The basic problem in accelerated life testing is to make inferences about the life distribution of the item under use environment E_u using failure data obtained under accelerated stress environments $E_1, E_2, \ldots, E_k$, where $k \geq 1$. In some situations, it is possible to obtain a limited amount of failure data under the use conditions environment E_u; however, in practice, these situations do not appear to be very common. Although having failure data under the use conditions environment E_u has advantages, its absence is in no way detrimental to our approach. However, in order to obtain results which are useful, it is necessary that k be moderate to large, and this is what we require. This requirement does not impose any practical difficulties in many situations of interest, especially those involving the accelerated life testing of electronic components and in bioassay experiments on animal populations.

In order to introduce some notation, let us denote the failure distribution of the items which are tested under environment E_j by F_j, where $F_j(0^-) \equiv 0$ for all values of j. We assume that F_j is *absolutely continuous* and thus $f_j(x)$, its probability density function, exists for $x \in [0,\infty)$. If we denote $1 - F_j(x)$ by $\bar{F}_j(x)$, then $\lambda_j(x)$, the *failure rate* of $F_j(x)$, is defined by

$$\lambda_j(x) = \frac{f_j(x)}{\bar{F}_j(x)} , \quad \bar{F}_j(x) > 0 .$$

We find it convenient and reasonable to assume that $\lambda_j(x)$ is continuous in x for $x\varepsilon[0,\infty)$. We use the terms *failure rate* and *hazard rate* interchangeably. Since $E_1 \succ E_2 \succ \cdots \succ E_k \succ E_u$, it is logical to assume that

$$\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_k(x) \geq \lambda_u(x) \tag{1.1}$$

for all $x\varepsilon[0,\infty)$.

Using failure data obtained under $E_1, E_2, \ldots, E_k$, we would like to obtain estimators $\hat{\lambda}_j(x)$ of $\lambda_j(x)$, $j = 1,2,..,k$, such that for some $0 \leq L < \infty$ and all $x\varepsilon[0,L]$,

$$\hat{\lambda}_1(x) \overset{st}{\geq} \hat{\lambda}_2(x) \overset{st}{\geq} \cdots \overset{st}{\geq} \hat{\lambda}_k(x) . \tag{1.2}$$

The notation $X \overset{st}{\geq} Y$ denotes the fact that X is stochastically larger than Y; i.e.,

$$P[X \geq x] \geq P[Y \geq x] \quad \text{for all values of } x .$$

In order to obtain estimators of $\lambda_j(x)$, $j = 1,2,\ldots,k$, which satisfy Equation (1.2), we shall use a Bayesian approach. Under this approach, Condition (1.1) is incorporated as a prior assumption. Our approach is in contrast with that of Brunk, Franck, Hanson and Hogg (1966), who embody a similar but weaker condition than (1.1) in their likelihood function. Specifically, Brunk et al. assume that the distributions are ordered, i.e.,

$$F_1(x) \geq F_2(x) \geq \cdots \geq F_k(x) \qquad \forall x .$$

The above condition is a consequence of Condition (1.1), but is not equivalent to it. Also, the approach of Brunk et al. is not Bayesian.

We shall first present a methodology for a Bayesian estimation of the individual hazard rate $\lambda_j(x)$ unconstrained by (1.1).

2. Bayesian Estimation of a Single Hazard Rate Function

For estimating an individual hazard rate $\lambda_j(t)$, we use the following Bayesian procedure.

Let $N_j(t)$ be the number of items exposed to the environment E_j at time t. Thus, $N_j(0)$ is the number of items that are initially put on test in environment E_j.

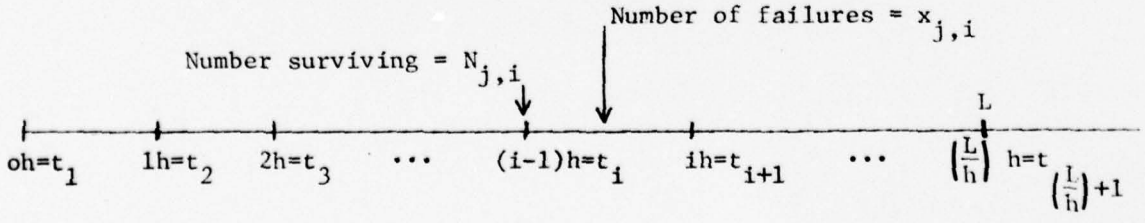
We hypothesize that the failure times are governed by a time-dependent Poisson process with the probability of failure in $[t,t+h]$ given by

$$N_j(t)\lambda_j(t)h + o(h) ,$$

where $\lambda_j(t)$ is the failure rate at time t under environment E_j.

For purposes of analysis, we divide the time interval $[o,L]$ into intervals of length $h > o$, where h is chosen to make L a multiple of h. For convenience, we denote $(i-1)h$ by t_i, $N_j(t_i)$ by $N_{j,i}$, and $\lambda_j(t_i)$ by $\lambda_{j,i}$, for $i = 1,2,\ldots,(L/h)$.

Let $x_{j,i}$ denote the number of failures in $[t_i,t_i+h)$: if there are no withdrawals, removals, censoring, etc., then $N_{j,i+1} = N_{j,i} - x_{j,i}$. It is helpful to clarify the above terms by the following diagram:



The joint distribution of $x_{j,1}, x_{j,2}, \ldots, x_{j,(L/h)+1}$, is given by the multinomial distribution

$$(N_{j1})! \prod_{i=1}^{(L/h)+1} (p_{j,i})^{x_{j,i}} \Big/ x_{j,i}! \ , \tag{2.1}$$

where $p_{j,i}$ is the probability of failure of a specified unit in $[t_i, t_{i+1})$, with

$$p_{j,(L/h)+1} = 1 - \sum_{i=1}^{(L/h)} p_{j,i} \ , \text{ and } \ x_{j,(L/h)+1} = N_{j,1} - \sum_{i=1}^{(L/h)} x_{j,i} \ .$$

Denoting $p_{j,i} \Big/ [1 - \sum_{\ell=1}^{i-1} p_{j,\ell}]$ by $p^*_{j,i}$, Equation (2.1) can be expressed [Wilks (1962), p. 151, Problem 6.13] as

$$\prod_{i=1}^{(L/h)} \frac{(N_{j,i})!}{(x_{j,i})!\ (N_{j,i}-x_{j,i})!} (p^*_{j,i})^{x_{j,i}} (1-p^*_{j,i})^{N_{j,i}-x_{j,i}} \ . \tag{2.2}$$

The $p^*_{j,i}$'s can be interpreted as the *average failure rate* over the interval $[t_i, t_{i+1})$; that is, $p^*_{j,i}$ is the conditional probability that an item which survived to time t_i will fail by time t_{i+1}. In terms of the $\lambda_{j,i}$, we can write

$$\lambda_{j,i}\, h = p^*_{j,i} + o(h) \ .$$

Equation (2.2) provides us with a starting point for a Bayesian analysis.

2.1 Prior Distributions for $p^*_{j,i}$.

We shall confine the discussion in this section to the jth environmental condition.

Suppose that the $p^*_{j,1}$, $i = 1,2,\ldots,(L/h)$, are *a priori* independent beta random variables with marginal densities

$$f(p^*_{j,i}) = \left[\frac{\Gamma(\alpha_{j,i})\Gamma(\beta_{j,i})}{\Gamma(\alpha_{j,i}+\beta_{j,i})}\right]^{-1} (p^*_{j,i})^{\alpha_{j,i}-1} (1-p^*_{j,i})^{\beta_{j,i}-1} .$$

It is of interest to note [cf. Lochner (1975)] that the above prior density leads us to a *generalized Dirichlet density* for the $\underset{\sim}{p}_j = (p_{j,1}, p_{j,2}, \ldots, p_{j,(L/h)+1})$.

Given the $N_{j,i}$'s and the $x_{j,i}$'s , the *posterior* density of $(p^*_{j,1}, p^*_{j,2}, \ldots, p^*_{j,(L/h)})$ is

$$\prod_{i=1}^{(L/h)} \frac{\Gamma(\alpha_{j,i}+\beta_{j,i}+N_{j,i})}{\Gamma(\alpha_{j,i}+x_{j,i})\,\Gamma(\beta_{j,i}+N_{j,i}-x_{j,i})} (p^*_{j,i})^{\alpha_{j,i}+x_{j,i}-1} (1-p^*_{j,i})^{\beta_{j,i}+N_{j,i}-x_{j,i}-1} \tag{2.3}$$

see DeGroot (1970) p. 160.

3. A Bayesian Estimation of Ordered Average Failure Rates

In a Bayesian context, Condition (1.1) leads us to the requirement that for every fixed value of i ,

$$p^*_{j-1,i} \overset{st}{\geq} p^*_{j,i} \quad \text{for} \quad j = 2,3,\ldots,k \tag{3.1}$$

Thus, our prior distributions on $p^*_{j-1,i}$ and $p^*_{j,i}$ will have to be chosen such that

$$P[p^*_{j-1,i} \geq p] \geq P[p^*_{j,i} \geq p] \quad \text{for all} \quad p \geq 0 . \tag{3.2}$$

One way of achieving Condition (3.2) is to assume that the parameters of the prior distributions of $p^*_{j-1,i}$ and $p^*_{j,i}$ satisfy

the following conditions for every fixed value of i :

$$\alpha_{j,i} \leq \alpha_{j-1,i}$$

and (3.3)

$$\beta_{j,i} \geq \beta_{j-1,i} \quad \text{for} \quad j = 2,3,\ldots,k \ .$$

For a proof of the above statement, see Appendix A.

In order to be assured that Condition (3.1) is also satisfied with respect to the posterior distributions of $p^*_{j-1,i}$ and $p^*_{j,i}$, it is sufficient (see Appendix A) that, for every fixed value of i

$$\alpha_{j,i} + x_{j,i} \leq \alpha_{j-1,i} + x_{j-1,i}$$

and (3.4)

$$\beta_{j,i} + N_{j,i} - x_{j,i} \geq \beta_{j-1,i} + N_{j-1,i} - x_{j-1,i} \quad \text{for} \quad j = 2,3,\ldots,k \ .$$

In order to make our Bayesian analysis more practical, we will have to reduce the number of prior parameters. One way of doing this is to assume that

$$\alpha_{j,i} = \alpha \quad \text{for all values of} \quad i \quad \text{and} \quad j \ ,$$

and that (3.5)

$$\beta_{j,i} = \beta_j \quad \text{for} \quad i = 1,2,\ldots,(L/h) \ .$$

Thus, the prior distribution of each $p^*_{j,i}$, $i = 1,2,\ldots,(L/h)$, is a beta with parameters α and β_j. This plus Conditions (3.3) and (3.4) lead us to the following remarks.

The first part of Condition (3.4) will be satisfied whenever $x_{j,i} \leq x_{j-1,i}$ for $j = 2,3,\ldots,k$. That is, the number of failures in the interval $[t_i, t_i+h)$ under environment E_j must *not be greater than* the number of failures in the same interval under environment

E_{j-1} , for all values of j . Because Condition (3.4) will have to be satisfied for every fixed value of i , $i = 1,2,\ldots,(L/h)$, a reasonable strategy is to have $N_{j,1} < N_{j-1,}$ for $j = 2,3,\ldots,k$. That is, we shall put more items on test (initially) under the more severe environment E_{j-1} , than under the environment E_j .

Since $N_{j,i} = N_{j,i-1} - x_{j,i-1}$, $i = 1,2,\ldots,(L/h)$, the second part of Condition (3.4) can be written as

$$\beta_{j-1} + N_{j-1,i+1} \leq \beta_j + N_{j,i+1} \quad \text{for} \quad j = 2,3,\ldots,k . \tag{3.6}$$

An interpretation of the above condition is that, for every fixed value of i , the number surviving at the start of the (i+1)st interval plus the prior parameter β_{j-1} for the environment E_{j-1} , must be smaller than the corresponding sum for the environment E_j . Thus, whether the second part of Condition (3.4) is satisfied or not, depends not only on the number of failures in the ith interval and the number surviving, but also on the values of the prior parameters β_j and β_{j-1} . Since the number of failures in a particular interval is a function of the severity of the environment and the number on test, and since β_{j-1} has to be less than or equal to β_j (see Conditions (3.5) and (3.6)), it is reasonable to have $\beta_{j-1} < \beta_j$ whenever $E_j \prec E_{j-1}$. That is, the values of the prior parameters β_{j-1} and β_j are indicative of the relative severity of the environmental conditions E_{j-1} and E_j . Since $E_1 \succ E_2 \succ \ldots \succ E_k$, we will choose the β_j's such that $\beta_1 < \beta_2 < \ldots < \beta_k$, and the values of the β_j's will be indicative of the severity of the environmental conditions E_j , $j = 1,2,\ldots,k$.

If the prior parameters β_j , $j = 1,2,\ldots,k$, and the data from the accelerated life test, $x_{j,i}$ and $N_{j,i}$, $i = 1,2,\ldots,(L/h)$, are such that Condition (3.4) is satisfied for every fixed value of i , then the stochastic ordering Condition (3.1) will be automatically satisfied with respect to the posterior distribution of

$(p^*_{j,1}, p^*_{j,2}, \ldots, p^*_{j,(L/h)})$. If the above is not the case, then we will have to pool the adjacent violators using the pooling procedure described in the next section.

4.0 The Pooling of Adjacent Violators

The procedure for pooling adjacent violators described here is commonly used in *isotonic regression*; see Barlow, Bartholomew, Bremner and Brunk (1972). The pooling is between the violators of the assumed ordering; that is, whenever $x_{j-1,i} < x_{j,i}$, then $x_{j-1,i}$ and $x_{j,i}$ are pooled.

Consider the time interval $[(i-1)h, ih)$; by Condition (3.4) we require

$$x_{1,i} \geq x_{2,i} \geq \cdots \geq x_{j-1,i} \geq x_{j,i} \geq \cdots \geq x_{k,i}$$

and

$$\beta_1 + N_{1,i} - x_{1,i} \leq \beta_2 + N_{2,i} - x_{2,i} \leq \cdots \leq \beta_{j-1} + N_{j-1,i} - x_{j-1,i}$$

$$\leq \beta_j + N_{j,i} - x_{j,i} \leq \cdots \leq \beta_k + N_{k,i} - x_{k,i}$$

If a reversal occurs, that is, if either

$$x_{j-1,i} < x_{j,i}$$

or if

$$\beta_{j-1} + N_{j-1,i} - x_{j-1,i} > \beta_j + N_{j,i} - x_{j,i} ,$$

then we pool the violators and replace them as shown below.

Replace both $x_{j-1,i}$ and $x_{j,i}$ by $\frac{1}{2}(x_{j-1,i} + x_{j,i})$ and

$\beta_{j-1} + N_{j-1,i} - x_{j-1,i}$ and $\beta_j + N_{j,i} - x_{j,i}$ by

$\frac{1}{2}(\beta_{j-1} + \beta_j + N_{j-1,i} + N_{j,i} - x_{j-1,i} - x_{j,i})$.

We now test to see if the new sequence is properly ordered, i.e.,

$$x_{1,i} \geq x_{2,i} \geq \cdots \geq \frac{1}{2}(x_{j-1,i} + x_{j,i}) = \frac{1}{2}(x_{j-1,i} + x_{j,i}) \geq$$

$$x_{j+1,i} \geq \cdots \geq x_{k,i} ,$$

and

$$\beta_1 + N_{1,i} - x_{1,i} \leq \cdots \leq \frac{1}{2}(\beta_{j-1} + \beta_j + N_{j-1,i} + N_{j,i} - x_{j-1,i} - x_{j,i})$$

$$= \frac{1}{2}(\beta_{j-1} + \beta_j + N_{j-1,i} + N_{j,i} - x_{j-1,i} - x_{j,i}) \leq \cdots \leq \beta_k + N_{k,i} - x_{k,i} .$$

If a reversal exists in either of the new sequences, then we replace again by appropriate averages. Thus, if

$$\frac{1}{2}(x_{j-1,i} + x_{j,i}) = \frac{1}{2}(x_{j-1,i} + x_{j,i}) < x_{j+1,i} ,$$

or if

$$\frac{1}{2}(\beta_{j-1} + \beta_j + N_{j-1,i} + N_{j,i} - x_{j-1,i} - x_{j,i}) > \beta_{j+1} + N_{j+1,i} - x_{j+1,i} ,$$

then we replace each one of the three by the corresponding average

$$\frac{1}{3}(x_{j-1,i} + x_{j,i} + x_{j+1,i})$$

(in the first sequence) and

$$\frac{1}{3}(\beta_{j-1} + \beta_j + \beta_{j+1} + N_{j-1,i} + N_{j,i} + N_{j+1,i} - x_{j-1,i} - x_{j,i} - x_{j+1,i})$$

(in the second sequence).

We continue the above procedure until all reversals in the interval $[(i-1)h, ih)$ are eliminated. We use this pooling scheme for each of the (L/h) time intervals to achieve the desired ordering.

4.1 Some General Comments Regarding the Pooling Procedure

The following comments regarding the pooling of adjacent violators will be helpful.

1. An excessive amount of pooling occurs if the relationship specified by Condition (1.1) is incorrect, or if the environmental conditions $E_1, E_2, \ldots, E_k$ are too similar to each other.

2. If $\lambda_j(0^+) \equiv 0$ for $j = 1,2,..,k$, then it is reasonable to expect that some pooling will be necessary at the lower values of i, $i = 1,2,\ldots,(L/h)$.

For many practical situations, it is reasonable to assume that the failure distributions F_j, $j = 1,2,..,k,u$, have increasing failure rate (see Barlow and Proschan (1975)). If we wish to make such an assumption, and incorporate it into our analysis, then, for each value of j, we must have in addition to Condition (3.1),

$$\lambda_{j,i-1} \leq \lambda_{j,i} ,$$

or that

$$p^*_{j,i-1} \overset{st}{\geq} p^*_{j,i} \quad i = 1,2,\ldots,(L/h) .$$

The above condition will further complicate our pooling procedure; we shall therefore *not* assume that the F_j are increasing failure rate.

5. The Posterior Distributions of the $p^*_{j,i}$'s

It is apparent from the discussion in Sections 3 and 4 that the posterior distributions of the $p^*_{j,i}$'s depend on the outcome of our pooling procedure. The posterior analysis is straightforward if no pooling is necessary, for then, the posterior distribution of $(p^*_{j,1}, p^*_{j,2}, \ldots, p^*_{j,(L/h)+1})$ is simply

$$\prod_{i=1}^{(L/h)} \frac{\Gamma(\alpha+\beta_j+N_{j,i})}{\Gamma(\alpha+x_{j,i})\Gamma(\beta_j+N_{j,i}-x_{j,i})} (p^*_{j,i})^{\alpha+x_{j,i}-1} (1-p^*_{j,i})^{\beta_j+N_{j,i}-x_{j,i}-1}$$

for $j = 1,2,\ldots,k$.

Under the assumption of a squared error loss function, a Bayes estimator of $p^*_{j,i}$, $i = 1,2,\ldots,(L/h)$, $j = 1,2,\ldots,k$, is simply the posterior mean; that is

$$\hat{p}^*_{j,i} = \frac{\alpha+x_{j,i}}{(\alpha+\beta_j+N_{j,i})} ,$$

(see DeGroot (1970) p.40) .

If pooling is necessary, then some or all of the $x_{j,i}$'s and the $(\beta_j+N_{j,i})$'s will be replaced by their appropriately pooled averages. In any case, the general expression for $\hat{p}^*_{j,i}$ will be of the form given above.

6. A Model for Extrapolations to Use Conditions Stress

Our analysis leads us to an array of Bayes estimators of the average failure rates over intervals of length h , for each environmental condition. Because of our pooling strategy, the Bayes estimators will be correctly ordered. That is, for each fixed value of i ,

$$\hat{p}^*_{1,i} \geq \hat{p}^*_{2,i} \geq \cdots \geq \hat{p}^*_{k,i} . \tag{6.1}$$

Given the $\hat{p}^*_{j,i}$'s , $j = 1,2,\ldots,k$ and $i = 1,2,\ldots,(L/h)$, our goal is to predict $p^*_{u,1}$, $p^*_{u,2}$, $\cdots$, $p^*_{u,(L/h)}$, the average failure rates over the time intervals $[(i-1)h,ih)$, $i = 1,2,\ldots,(L/h)$, respectively, under the use conditions environment E_u .

In the absence of any knowledge about a relationship between the average failure rates and the values of the various stresses which constitute an environment, some form of an assumption is essential. This is particularly crucial if k, the number of distinct environmental conditions of interest, is small. If, however, k is large, then a relationship between the average failure rates and the stresses can be empirically obtained; this is what is often done in practice.

When k is small, we shall *postulate* the following simple but reasonable relationship between the average failure rate estimates.

For some k *unknown* constants $w_0, w_1, \cdots, w_{k-1}$, we assume that for each value of i, $i = 1,2,\ldots,(L/h)$,

$$\hat{p}^*_{k,i} = w_0 + w_1 \hat{p}^*_{k-1,i} + w_2 \hat{p}^*_{k-2,i} + \cdots + w_{k-1} \hat{p}^*_{1,i} . \quad (6.2)$$

The above relationship states that the average failure rate over a particular time interval under the environmental condition E_k, is a weighted sum of the average failure rates over the same time interval under the conditions $E_{k-1}, E_{k-2}, \cdots, E_1$. This is reminiscent of an *autoregressive process* of order $k-1$ which has found useful applications in forecasting (see Box and Jenkins (1975)).

In order to make the above relationship more meaningful we shall require that the environmental conditions $E_k \prec E_{k-1} \prec \cdots \prec E_2 \prec E_1$ increase in magnitude of severity by the same fixed amount. For example, if E_j represents a single stress, say a voltage stress V_j, then we shall require that $V_{j-1} - V_j = C$, where C is some suitable constant, $j = 1,2,\cdots,k$.

Since Equation (6.2) holds for $i = 1,2,\cdots,(L/h)$, the least squares estimators of $w_0, w_1, \cdots, w_{k-1}$ can be obtained in a routine manner. These estimators are denoted by $\hat{w}_0, \hat{w}_1, \cdots, \hat{w}_{k-1}$.

If the increase in the magnitude of the severity of the environment from E_u to E_k is the same as that from E_j to E_{j-1}, $j = 2,3,\cdots,k$, then an estimator of the average failure rate under E_u is given by

$$\hat{p}^*_{u,i} = \hat{w}_0 + \hat{w}_1 \hat{p}^*_{k,i} + \cdots + \hat{w}_{k-1,i}\, \hat{p}^*_{2,i} \tag{6.3}$$

for $i = 1,2,\cdots,(L/h)$, and $\hat{p}^*_{u,(L/h)+1} = 1 - \sum_{i=1}^{(L/h)} \hat{p}^*_{u,i}$.

If the increase in magnitude of the severity of the environment from E_u to E_k is two times that from E_j to E_{j-1}, $j = 2,3,\ldots,k$, then we iterate upon Equation (6.3) one more time to obtain the desired result. Thus, in principle, we can iterate upon Equation (6.3) as many times as is necessary, depending upon the separation between E_u and E_k.

By the definition of $p^*_{u,i}$, the probability of an item surviving to time t^* (assumed a multiple of h) under environment E_u is

$$\bar{F}_u(t^*) = \prod_{i=1}^{t^*/h} (1-p^*_{u,i}) .$$

Thus, we have the following as an estimator of $\bar{F}_u(t^*)$:

$$\hat{\bar{F}}_u(t^*) = \prod_{i=1}^{t^*/h} (1-\hat{p}^*_{u,i}) ,$$

where the $\hat{p}^*_{u,i}$ are given by (6.3).

The properties of this estimator have not been studied. The estimation procedure has been proposed as a practical method for providing answers in the difficult area of accelerated testing.

Appendix A

In this appendix we determine sufficient conditions under which two beta random variables are stochastically ordered. The results may already be known; however, they are included for completeness.

The notation and terminology are taken from Barlow and Proschan (1975).

We first note that a gamma density $f(x,\alpha) = \frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)}$ is TP_2 in (x,α). Therefore, $\bar{F}(x,\alpha) = \int_x^\infty \frac{y^{\alpha-1}}{\Gamma(\alpha)} e^{-x}dx$ is TP_2 in (x,α) since $\bar{F}(x,\alpha) = \int f(y,\alpha)\, H(x,y)dy$, where $H(x,y) = 1$ for $y \geq x$ and $H(x,y) = 0$ for $y < x$. (Note that $\begin{vmatrix} H(x_1,y_1) & H(x_1,y_2) \\ H(x_2,y_1) & H(x_2,y_2) \end{vmatrix} \geq 0$ for $x_1 < x_2$, $y_1 < y_2$. Thus, $H(x,y)$ is TP_2, and so the composition of f, H is TP_2.)

Since $\bar{F}(x,\alpha)$ is TP_2, then for $x_1 < x_2$, $\alpha_1 < \alpha_2$, $\begin{vmatrix} \bar{F}(x_1,\alpha_1) & \bar{F}(x_1,\alpha_2) \\ \bar{F}(x_2,\alpha_1) & \bar{F}(x_2,\alpha_2) \end{vmatrix} \geq 0$. Choose $x_1 = 0$. Then $\begin{vmatrix} 1 & 1 \\ \bar{F}(x,\alpha_1) & \bar{F}(x,\alpha_2) \end{vmatrix} \geq 0$ for $0 < x < \infty$, $\alpha_1 < \alpha_2$. Thus, $\bar{F}(x,\alpha_2) \geq \bar{F}(x,\alpha_1)$ for $\alpha_1 < \alpha_2$ and $x > 0$.

We have shown:

1. Lemma. Let X_α be a gamma r.v. with shape parameter α. Then X_α is increasing stochastically in α.

2. Proposition. Let Y_{α_i} have density $\frac{y^{\alpha_i-1}}{\Gamma(\alpha_i)} e^{-y}$, $i = 1,2$, and Y_{α_1}, Y_{α_2} be independent. Then $X_{\alpha_1,\alpha_2} \equiv \frac{Y_{\alpha_1}}{Y_{\alpha_1} + Y_{\alpha_2}}$ has beta density $f(\alpha_1,\alpha_2,x) = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x^{\alpha_1-1} (1-x)^{\alpha_2-1}$ for $0 \le x \le 1$. (Hogg & Craig, 1970, p. 134).

3. Theorem. Let X_{α_1,α_2} have beta density $f(\alpha_1,\alpha_2,x)$. Then X_{α_1,α_2} is increasing stochastically in α_1 and decreasing stochastically in α_2.

Proof. Write $X_{\alpha_1,\alpha_2} = \frac{Y_{\alpha_1}}{Y_{\alpha_1} + Y_{\alpha_2}}$, $X_{\alpha_1,\alpha_2'} = \frac{Y_{\alpha_1}}{Y_{\alpha_1} + Y_{\alpha_2'}}$, where $\alpha_2 < \alpha_2'$. For fixed Y_{α_1}, $X_{\alpha_1,\alpha_2} \overset{st}{\ge} X_{\alpha_1,\alpha_2'}$. By unconditioning $X_{\alpha_1,\alpha_2} \overset{st}{>} X_{\alpha_1,\alpha_2'}$ for $\alpha_2 < \alpha_2'$.

By similar reasoning, we can show $X_{\alpha_1,\alpha_2} \overset{st}{\le} X_{\alpha_1',\alpha_2}$ for $\alpha_1 < \alpha_1'$. ||

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